

Article

The Laplace Transform of Composed Functions and Bivariate Bell Polynomials

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Abstract: The problem of computing the Laplace transform of composed functions has not found its way into the literature because it was customarily believed that there were no suitable formula to solve it. Actually, it has been shown in previous work that by making use of Bell polynomials, efficient approximations can be found. Moreover, using an extension of Bell's polynomials to bivariate functions, it is also possible to approximate the Laplace transform of composed functions of two variables. This topic is solved in this paper and some numerical verifications, due to the first author using the computer algebra system Mathematica[®], are given proving the effectiveness of the proposed method.

Keywords: Laplace transform of composed functions; bivariate Bell polynomials

MSC: 44A15; 11B83; 30-08



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1. Introduction

In the current literature, a large number of papers falsely claimed to have generalized the classical Laplace transform and the s -multiplied (or the Laplace–Carson) transform by making some obviously trivial changes in the parameter s and the argument t . Most (if not all) of such trivialities and inconsequential variations of the classical Laplace transform and the Laplace–Carson transform, which appeared and continue to appear in the literature, were pointed out and documented by H. M. Srivastava (see [1], pp. 1508–1510, and [2], pp. 36–38).

In order not to create misunderstandings, we want to say right away that the purpose of this article is not to provide generalizations of the Laplace transform (LT), but only to extend the tables of LTs often used in applied mathematics. The well known tables of Oberhettinger and Badii cited among the references [3] does not include the LT of functions considered in this paper.

We want to underline the fact that in literature it is usually considered that there are no formulas for the computation of the LT of composed functions. This belief has been refuted in a previous article [4] and the possibility of obtaining approximations in this field is shown in an even more general situation in this paper. As far as we know, there are no other articles on this topic and our approach seems to be the first in this regard.

More precisely, we show how to approximate the Laplace transform of a 2-variable composed analytic function $f[\varphi(t), \psi(t)]$, where $\varphi(t)$, $y = \psi(t)$, $0 \leq t < +\infty$ are analytic functions, by using the bivariate Bell polynomials, a suitable set of Bell polynomials, introduced in a preceding paper [5].

The classical Bell polynomials are exploited in very different frameworks, which range from number theory [6–10] to operators theory [11,12], and from differential equations [13]

to integral transforms [14,15]. Applications of the Laplace Transform (LT) in Analysis and Mathematical Physics problems are well known [16].

We use the classical definition:

$$\mathcal{L}(f) := \int_0^\infty \exp(-st) f(t) dt = F(s).$$

The LT converts a function of a real variable t (representing the time) to a function of a complex variable s (representing the complex frequency).

The LT can be applied to locally integrable functions on $[0, +\infty)$. It converges in each half plane $Re(s) > a$, where a is a constant (called the convergence abscissa), depending on the growth behavior of $f(t)$.

Owing to the importance of this transformation in the solution of the most diverse differential problems, a large number of LT, together with the respective anti-transforms, are reported in literature (see, e.g., [3,17]).

Given a 2-variable composed function $f[\varphi(t), \psi(t)]$ it is natural to define the relevant LT by putting:

$$\mathcal{L}_c(f) := \int_0^\infty \exp(-st) f[\varphi(t), \psi(t)] dt = F_c(s).$$

In previous articles, we have shown how to compute the LT of higher-order nested functions (see [4] and the references therein). In this article, we apply a similar method to approximate the LT of a composed analytic function of 2 variables, taking advantage of the bivariate Bell polynomials introduced in [5]. The results obtained demonstrate the correctness of the method considered, as can be seen in the numerical verifications obtained by the first author using the Mathematica[®] computer algebra system.

The first bivariate Bell polynomials are given in the Appendix A at the end of this article.

2. Recalling the Bell’s Polynomials

The Bell’s polynomials [18] express the n th derivative of a composed function $\Phi(t) := f(g(t))$ in terms of the successive derivatives of the (sufficiently smooth) component functions $x = g(t)$ and $y = f(x)$. More precisely, if

$$\Phi_m := D_t^m \Phi(t), \quad f_h := D_x^h f(x)|_{x=g(t)}, \quad g_k := D_t^k g(t),$$

then the n th derivative of $\Phi(t)$ is represented by

$$\Phi_n = B_n(f_1, g_1; f_2, g_2; \dots; f_n, g_n), \tag{1}$$

where B_n denotes the n th Bell polynomial.

The first few Bell polynomials are given by:

$$\begin{aligned} B_1(f_1, g_1) &= f_1 g_1 \\ B_2(f_1, g_1; f_2, g_2) &= f_1 g_2 + f_2 g_1^2 \\ B_3(f_1, g_1; f_2, g_2; f_3, g_3) &= f_1 g_3 + f_2 (3g_2 g_1) + f_3 g_1^3 \\ &\dots \end{aligned} \tag{2}$$

More general results can be found in [19], p. 49.

The Bell polynomials [6] are given by the equation

$$B_n(f_1, g_1; f_2, g_2; \dots; f_n, g_n) = \sum_{k=1}^n B_{n,k}(g_1, g_2, \dots, g_{n-k+1}) f_k, \tag{3}$$

where the $B_{n,k}$ satisfy the recursion [6]

$$B_{n,k}(g_1, g_2, \dots, g_{n-k+1}) = \sum_{h=0}^{n-k} \binom{n-1}{h} B_{n-h-1,k-1}(g_1, g_2, \dots, g_{n-h-k+1}) g_{h+1}. \tag{4}$$

The $B_{n,k}$ functions for any $k = 1, 2, \dots, n$ are polynomials in the g_1, g_2, \dots, g_n variables homogeneous of degree k and isobaric of weight n (i.e., they are linear combinations of monomials $g_1^{k_1} g_2^{k_2} \dots g_n^{k_n}$ whose weight is constantly given by $k_1 + 2k_2 + \dots + nk_n = n$).

Therefore, we have the equations

$$B_{n,k}(\alpha\beta g_1, \alpha\beta^2 g_2, \dots, \alpha\beta^{n-k+1} g_{n-k+1}) = \alpha^k \beta^n B_{n,k}(g_1, g_2, \dots, g_{n-k+1}), \tag{5}$$

and

$$B_n(f_1, \beta g_1; f_2, \beta^2 g_2; \dots; f_n, \beta^n g_n) = \beta^n B_n(f_1, g_1; f_2, g_2; \dots; f_n, g_n). \tag{6}$$

3. Bivariate Bell’s Polynomials

Consider the composed function $f(\varphi(t), \psi(t))$, under the standard assumptions on the domains of definition and differentiability. Compute the the partial derivative of the function f , h -times with respect to x and k -times with respect to y and then put $x = \varphi(t), y = \psi(t)$. Indicate the result with $f_{x^h y^k} := \frac{d^{(h+k)}}{dx^h dy^k} f(x, y)|_{(x=\varphi(t), y=\psi(t))}$, and also set $D_t^m \varphi(t) =: \varphi_m, D_t^m \psi(t) =: \psi_m$.

Hence, the bivariate Bell polynomials are defined as follows:

$$\mathcal{B}_n^{(2)}(f; \varphi; \psi) := D_t^n f(\varphi(t), \psi(t)). \tag{7}$$

Note that

$$D_t f(\varphi(t), \psi(t)) = f_x \varphi_1 + f_y \psi_1 = B_1(f_x, \varphi_1) + B_1(f_y, \psi_1), \tag{8}$$

where B_n denotes the n th ordinary Bell polynomial defined in (1).

Putting

$$\begin{aligned} B_1^2(f_x; \varphi_1) &:= B_2(f_x, f_{x^2}; \varphi_1, \varphi_2), & B_1^2(f_y; \psi_1) &:= B_2(f_y, f_{y^2}; \psi_1, \psi_2), \\ B_1(f_x; \varphi_1) \circ B_1(f_y; \psi_1) &= f_x \varphi_1 \circ f_y \psi_1 := f_{xy} \varphi_1 \psi_1, \end{aligned} \tag{9}$$

we find

$$\begin{aligned} D_t^2 f(\varphi(t), \psi(t)) &= f_{x^2}(\varphi_1)^2 + f_x \varphi_2 + f_{y^2}(\psi_1)^2 + f_y \psi_2 + 2f_{xy} \varphi_1 \psi_1 = \\ &= B_2(f_x, f_{x^2}; \varphi_1, \varphi_2) + 2B_1(f_x; \varphi_1) \circ B_1(f_y; \psi_1) + B_2(f_y, f_{y^2}; \psi_1, \psi_2) = \\ &= B_1^2(f_x; \varphi_1) + 2B_1(f_x; \varphi_1) \circ B_1(f_y; \psi_1) + B_1^2(f_y; \psi_1) = \\ &= [B_1(f_x; \varphi_1) + B_1(f_y; \psi_1)]^2. \end{aligned} \tag{10}$$

In the above formula, for the formal multiplication symbol \circ , we assume the definition below

$$\begin{aligned} f_{x^h} \circ f_{y^k} &:= f_{x^h y^k}; & B_1^m(f_x; \varphi_1) &= B_m(f_x, \dots, f_{x^m}; \varphi_1, \dots, \varphi_m) \\ & & & \text{(and } B_1^m(f_y; \psi_1) = B_m(f_y, \dots, f_{y^m}; \psi_1, \dots, \psi_m)). \end{aligned} \tag{11}$$

Therefore, we find the result [5].

Theorem 1. Let $f(x, y), \varphi(t), \psi(t)$ be analytical functions. The n th polynomial of the system $\{\mathcal{B}_n^{(2)}(f; \varphi; \psi)\}_{n \in \mathbf{N}}$, can be computed with the following rule

$$\mathcal{B}_n^{(2)}(f; \varphi; \psi) = [B_1(f_x; \varphi_1) + B_1(f_y; \psi_1)]^n. \tag{12}$$

By the Leibniz rule, we can write

$$\begin{aligned} D_t^{n+1} f(\varphi(t), \psi(t)) &= D_t^n D_t f(\varphi(t), \psi(t)) = \\ &= D_t^n [f_x \varphi_1 + f_y \psi_1] = \sum_{k=0}^n \binom{n}{k} D_t^{n-k} f_x \varphi_{k+1} + \sum_{k=0}^n \binom{n}{k} D_t^{n-k} f_y \psi_{k+1}. \end{aligned} \tag{13}$$

Then, for the bivariate Bell polynomials, the recurrence relation holds

Theorem 2.

$$\begin{aligned} \mathcal{B}_0^{(2)}(f_x; \varphi; \psi) &:= f_x; \quad \mathcal{B}_0^{(2)}(f_y; \varphi; \psi) := f_y; \\ \mathcal{B}_{n+1}^{(2)}(f; \varphi; \psi) &= \sum_{k=0}^n \binom{n}{k} [\mathcal{B}_{n-k}^{(2)}(f_x; \varphi; \psi) \varphi_{k+1} + \mathcal{B}_{n-k}^{(2)}(f_y; \varphi; \psi) \psi_{k+1}]. \end{aligned} \tag{14}$$

4. LT of a 2-Variable Composed Function

Let $f(\varphi(t), \psi(t))$ be a composed function, analytic in a neighborhood of the origin, so that it is expressed by the Taylor’s expansion:

$$f(\varphi(t), \psi(t)) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}, \quad a_n = D_t^n [f(\varphi(t), \psi(t))]_{t=0}. \tag{15}$$

According to the preceding equations, it results

$$\begin{aligned} a_0 &= f(\overset{\circ}{\varphi}_0, \overset{\circ}{\psi}_0) = \overset{\circ}{f}_0, \\ a_n &= D_t^n [f(\varphi(t), \psi(t))]_{t=0} = \mathcal{B}_n^{(2)}(\overset{\circ}{f}; \overset{\circ}{\varphi}; \overset{\circ}{\psi}), \quad (n \geq 1), \end{aligned} \tag{16}$$

where

$$\overset{\circ}{f}_k := D_t^k f(x, y)|_{[x=\varphi(0), y=\psi(0)]}, \quad \overset{\circ}{\varphi}_h = D_t^h \varphi(t)|_{t=0}, \quad \overset{\circ}{\psi}_h = D_t^h \psi(t)|_{t=0}, \tag{17}$$

where f_k denotes the k th derivative of the function $f(x(t), y(t))$ with respect to t .

Using the previous formulas we are able to approximate the calculation of the LT of the function $f(\varphi(t), \psi(t))$ with that of a series of elementary LT of powers. In fact, we have the following Theorem.

Theorem 3. Considering a composed function $f(\varphi(t), \psi(t))$, expressed by the Taylor’s expansion in Equation (15), for its LT the following equation holds

$$\int_0^{+\infty} f(\varphi(t), \psi(t)) e^{-ts} dt = \frac{\overset{\circ}{f}_0}{s} + \sum_{n=1}^{\infty} [B_1(\overset{\circ}{f}_x; \overset{\circ}{\varphi}_1) + B_1(\overset{\circ}{f}_y; \overset{\circ}{\psi}_1)]^n \frac{1}{s^{n+1}}. \tag{18}$$

where the symbolic power in the above equation must be computed using the definitions in (11).

Proof. Representing the coefficients of the Taylor expansion in (15) in terms of bivariate Bell polynomials, and using the uniform convergence of series, we find

$$\int_0^{+\infty} f(\varphi(t), \psi(t)) e^{-ts} dt = \frac{\overset{\circ}{f}_0}{s} + \int_0^{+\infty} \sum_{n=1}^{\infty} \mathcal{B}_n^{(2)}(\overset{\circ}{f}; \overset{\circ}{\varphi}; \overset{\circ}{\psi}) \frac{t^n}{n!} e^{-ts} dt =$$

$$= \frac{\overset{\circ}{f}_0}{s} + \sum_{n=1}^{\infty} [B_1(\overset{\circ}{f}_x; \overset{\circ}{\varphi}_1) + B_1(\overset{\circ}{f}_y; \overset{\circ}{\psi}_1)]^n \int_0^{+\infty} \frac{t^n}{n!} e^{-ts} dt.$$

Then, the result follows from elementary calculation of the LT of powers. \square

5. Examples

5.1. The Particular Case of Exponential Functions

We start considering the case of the nested exponential function

Example 1

- Let $f(x, y) = e^{x+y}$ and $\varphi(t) = \sin t; \psi(t) = -t$.

Then, $f(\varphi(t), \psi(t)) = \exp(\sin t - t)$, and for the relevant LT, using the above described method, we find

$$\int_0^{+\infty} e^{\sin t - t - ts} dt = \frac{1}{s} - \frac{1}{s^4} + \frac{1}{s^6} + \frac{10}{s^7} - \frac{1}{s^8} - \frac{56}{s^9} - \frac{279}{s^{10}} + \frac{246}{s^{11}} + O\left(\frac{1}{s^{12}}\right). \tag{19}$$

The graphical display of our approximation is shown in Figures 1–3.

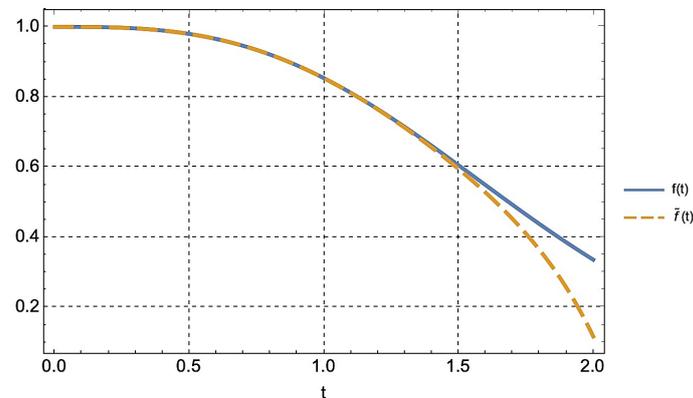


Figure 1. Distribution of $l(t) = e^{\sin t - t}$ and the relevant approximant $\tilde{l}(t)$.

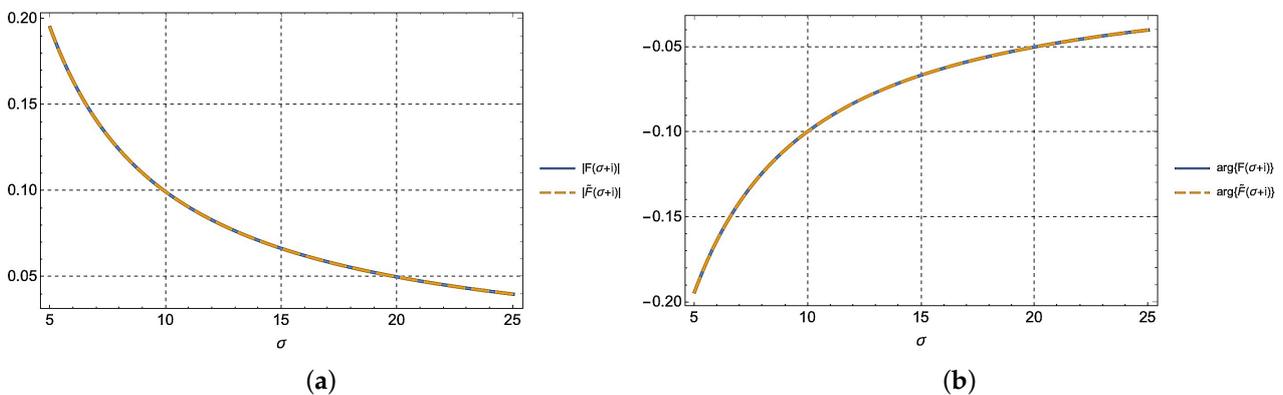


Figure 2. Magnitude (a) and argument (b) of the Laplace transform of $l(t) = e^{\sin t - t}$ as evaluated through the approximant $\tilde{L}(s)$ and the rigorous analytical expression $L(s)$.

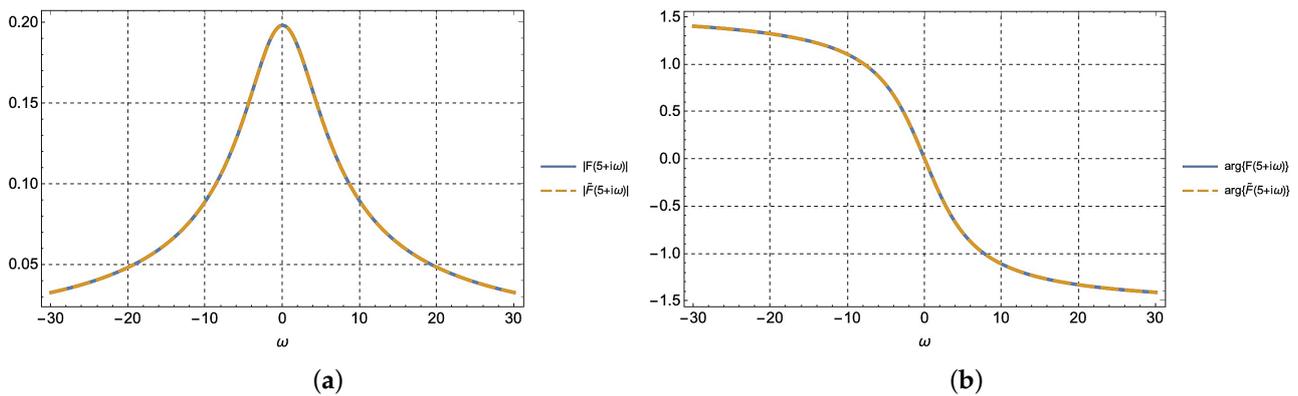


Figure 3. Magnitude (a) and argument (b) of the Laplace transform of $l(t) = e^{\sin t - t}$ as evaluated through the approximant $\tilde{L}(s)$ and the rigorous analytical expression $L(s)$ for $s = 5 + i\omega$.

5.2. The General Case

Examples of the above method for approximating the LT of general nested functions are reported in what follows.

5.2.1. Example 2

- Assuming $f(x, y) = \log(xy)$, $\varphi(t) = (\cosh t)^{10}$, $\psi(t) = \frac{1}{1 + \frac{t}{10}}$, we have $f(\varphi(t), \psi(t)) = \log\left[\frac{(\cosh t)^{10}}{1 + \frac{t}{10}}\right]$. By using the above described method, for the relevant LT we find the approximation:

$$\int_0^{+\infty} \log\left[\frac{(\cosh t)^{10}}{1 + \frac{t}{10}}\right] e^{-ts} dt = -\frac{1}{10s^2} + \frac{1001}{100s^3} - \frac{1}{500s^4} - \frac{99997}{5000s^5} - \frac{3}{12500s^6} + \frac{4000003}{25000s^7} - \frac{9}{125000s^8} - \frac{3399999937}{1250000s^9} - \frac{63}{1562500s^{10}} + O\left(\frac{1}{s^{11}}\right). \tag{20}$$

The graphical display of our approximation is shown in Figures 4–6.

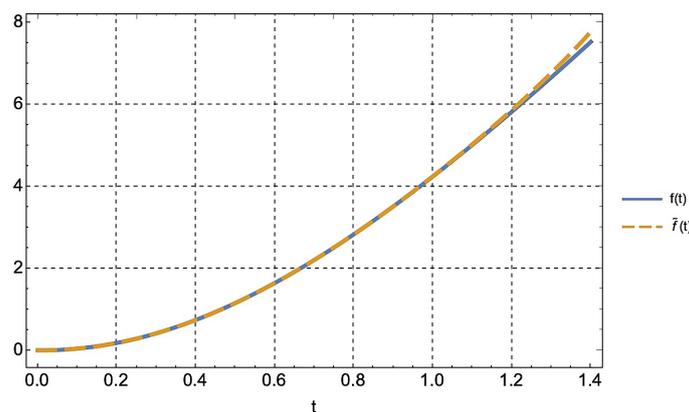


Figure 4. Distribution of $l(t) = \log\left[\frac{(\cosh t)^{10}}{1 + \frac{t}{10}}\right]$ and the relevant approximant $\tilde{l}(t)$.

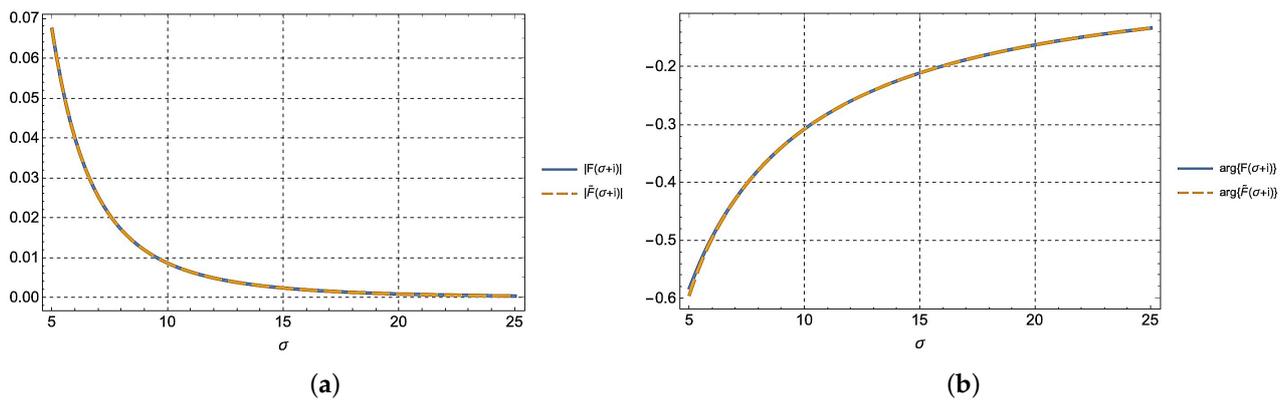


Figure 5. Magnitude (a) and argument (b) of the Laplace transform of $l(t) = \log \left[\frac{(\cosh t)^{10}}{1 + \frac{t}{10}} \right]$ as evaluated through the approximant $\tilde{L}(s)$ and the rigorous analytical expression $L(s)$.

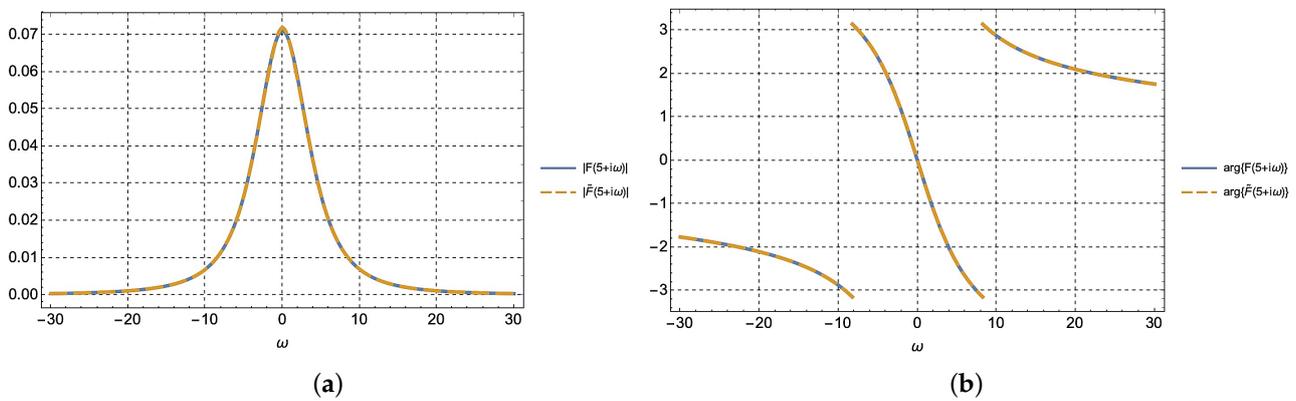


Figure 6. Magnitude (a) and argument (b) of the Laplace transform of $l(t) = \log \left[\frac{(\cosh t)^{10}}{1 + \frac{t}{10}} \right]$ as evaluated through the approximant $\tilde{L}(s)$ and the rigorous analytical expression $L(s)$ for $s = 5 + i\omega$.

5.2.2. Example 3

- Assuming $f(x, y) = \cos\left(\frac{x+y}{5}\right)$, $\varphi(t) = \sin t$, $\psi(t) = \arctan t$, we have $f(\varphi(t), \psi(t)) = \cos([\sin t + \arctan t]/5)$. By using the above described method, for the relevant LT we find the approximation:

$$\int_0^{+\infty} \cos\left(\frac{\sin t + \arctan t}{5}\right) e^{-ts} dt = \frac{1}{s} - \frac{4}{25s^3} + \frac{616}{625s^6} - \frac{255814}{15625s^7} + \frac{259309656}{390625s^9} - \frac{472466309274}{9765625s^{11}} + O\left(\frac{1}{s^{13}}\right). \tag{21}$$

The graphical display of our approximation is shown in Figures 7–9.

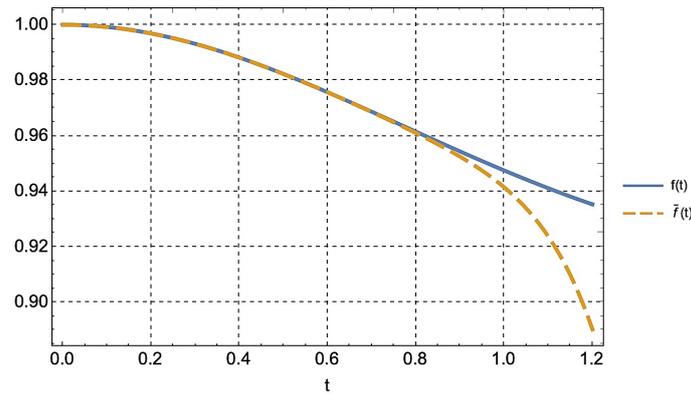


Figure 7. Distribution of $l(t) = l(t) = \cos([\sin t + \arctan t]/5)$ and the relevant approximat $\tilde{l}(t)$.

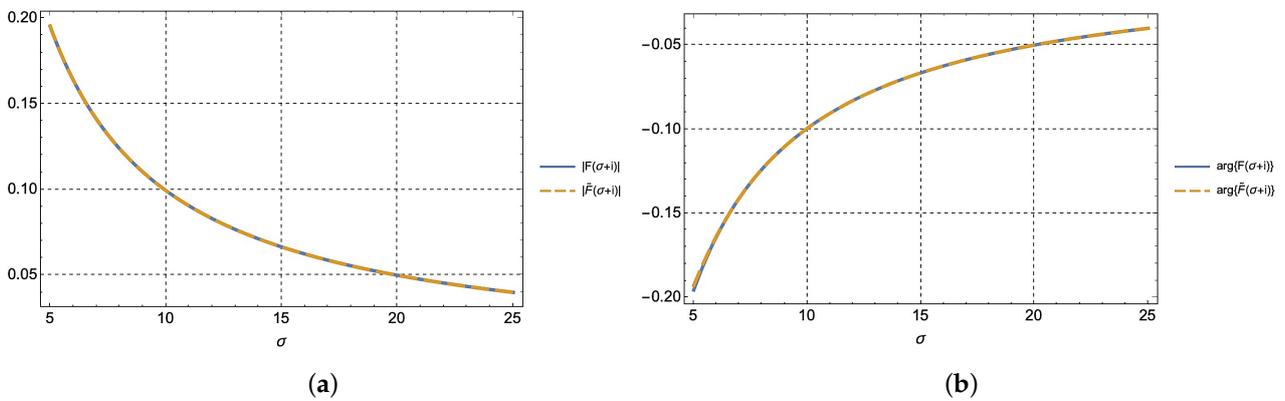


Figure 8. Magnitude (a) and argument (b) of the Laplace transform of $l(t) = \cos([\sin t + \arctan t]/5)$ as evaluated through the approximat $\tilde{L}(s)$ and the rigorous analytical expression $L(s)$.

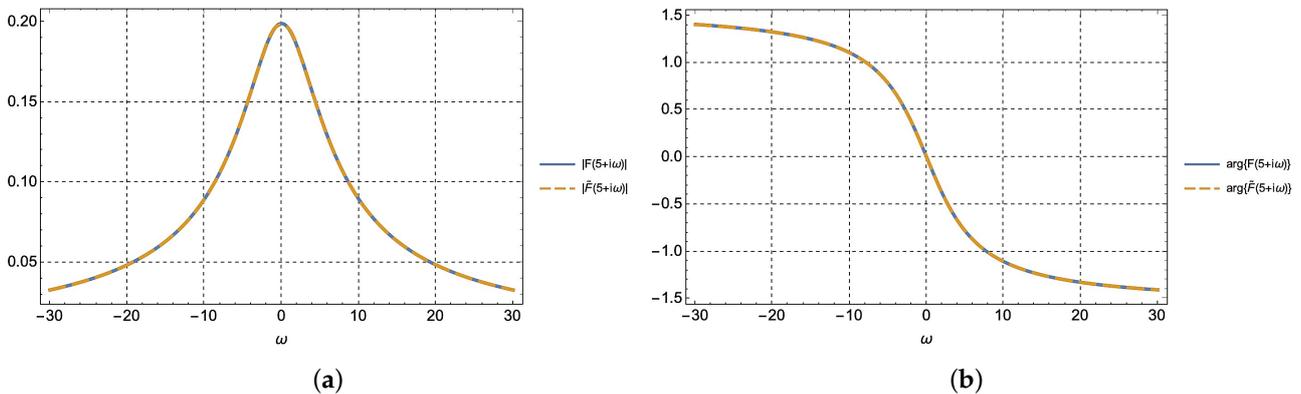


Figure 9. Magnitude (a) and argument (b) of the Laplace transform of $l(t) = \cos([\sin t + \arctan t]/5)$ as evaluated through the approximat $\tilde{L}(s)$ and the rigorous analytical expression $L(s)$ for $s = 5 + i\omega$.

6. Conclusions

We have shown a method for approximating the LT of a 2-variable composed function $f[\varphi(t), \psi(t)]$, where $\varphi(t), \psi(t), 0 \leq t < +\infty$, are analytic functions, by using the bivariate Bell polynomials. Starting from the Taylor expansion in a neighborhood of the origin of the function $f[\varphi(t), \psi(t)]$, since the coefficients can be expressed in terms of the bivariate Bell polynomials, the integral is reduced to the computation of an approximating series, which obviously converges if the integral is convergent.

We, thus, showed that just as it was possible to approximate the LT of composed functions using Bell polynomials [4], so it can be done for the LT of composed functions in two variables, using the bivariate Bell polynomials, introduced in [5].

$$\begin{aligned}
 \mathcal{B}_8^{(2)} [f; \varphi, \psi] = & f_{x^8} \varphi_1^8 + 28 f_{x^7} \varphi_1^6 \varphi_2 + 210 f_{x^6} \varphi_1^4 \varphi_2^2 + 420 f_{x^5} \varphi_1^2 \varphi_2^3 + 105 f_{x^4} \varphi_1^4 + 56 f_{x^6} \varphi_1^5 \varphi_3 + 560 f_{x^5} \varphi_1^3 \varphi_2 \varphi_3 + 840 f_{x^4} \varphi_1 \varphi_2^2 \varphi_3 + 280 f_{x^4} \varphi_1^2 \varphi_3^2 + 280 f_{x^3} \varphi_2^2 \varphi_3^2 + \\
 & 70 f_{x^5} \varphi_1^4 \varphi_4 + 420 f_{x^4} \varphi_1^2 \varphi_2 \varphi_4 + 210 f_{x^3} \varphi_2^2 \varphi_4 + 280 f_{x^3} \varphi_1 \varphi_3 \varphi_4 + 35 f_{x^2} \varphi_4^2 + 56 f_{x^4} \varphi_1^3 \varphi_5 + 168 f_{x^3} \varphi_1 \varphi_2 \varphi_5 + 56 f_{x^2} \varphi_3 \varphi_5 + 28 f_{x^3} \varphi_1^2 \varphi_6 + 28 f_{x^2} \varphi_2 \varphi_6 + \\
 & 8 f_{x^2} \varphi_1 \varphi_7 + f_x \varphi_8 + 8 f_{x^7} \varphi_1^7 \psi_1 + 168 f_{x^6} \varphi_1^5 \varphi_2 \psi_1 + 840 f_{x^5} \varphi_1^3 \varphi_2^2 \psi_1 + 840 f_{x^4} \varphi_1 \varphi_2 \varphi_3 \psi_1 + 280 f_{x^5} \varphi_1^4 \varphi_3 \psi_1 + 1680 f_{x^4} \varphi_1^2 \varphi_2 \varphi_3 \psi_1 + 840 f_{x^3} \varphi_2^2 \varphi_3 \psi_1 + \\
 & 560 f_{x^3} \varphi_1 \varphi_2^2 \psi_1 + 280 f_{x^4} \varphi_1^3 \varphi_4 \psi_1 + 840 f_{x^3} \varphi_1 \varphi_2 \varphi_4 \psi_1 + 280 f_{x^2} \varphi_3 \varphi_4 \psi_1 + 168 f_{x^3} \varphi_1^2 \varphi_5 \psi_1 + 168 f_{x^2} \varphi_2 \varphi_5 \psi_1 + 56 f_{x^3} \varphi_1 \varphi_2 \varphi_6 \psi_1 + 8 f_{x^2} \varphi_7 \psi_1 + \\
 & 28 f_{x^6} \varphi_1^6 \psi_1^2 + 420 f_{x^5} \varphi_1^4 \varphi_2 \psi_1^2 + 1260 f_{x^4} \varphi_1^2 \varphi_2^2 \psi_1^2 + 420 f_{x^3} \varphi_2^2 \varphi_3 \psi_1^2 + 560 f_{x^4} \varphi_1^3 \varphi_3 \psi_1^2 + 1680 f_{x^3} \varphi_1 \varphi_2 \varphi_3 \psi_1^2 + 280 f_{x^2} \varphi_3^2 \psi_1^2 + 420 f_{x^3} \varphi_1^2 \varphi_4 \psi_1^2 + \\
 & 420 f_{x^2} \varphi_2^2 \varphi_4 \psi_1^2 + 168 f_{x^2} \varphi_1 \varphi_5 \psi_1^2 + 28 f_{x^3} \varphi_2 \varphi_6 \psi_1^2 + 56 f_{x^3} \varphi_1^2 \psi_1^3 + 560 f_{x^4} \varphi_1^3 \varphi_2 \psi_1^3 + 840 f_{x^3} \varphi_1 \varphi_2^2 \psi_1^3 + 560 f_{x^2} \varphi_3^2 \psi_1^3 + 560 f_{x^2} \varphi_2 \varphi_3 \psi_1^3 + \\
 & 280 f_{x^2} \varphi_1 \varphi_4 \psi_1^3 + 56 f_{x^3} \varphi_2 \varphi_5 \psi_1^3 + 70 f_{x^4} \varphi_1^4 \psi_1^3 + 420 f_{x^3} \varphi_1^2 \varphi_4 \psi_1^3 + 210 f_{x^2} \varphi_3 \varphi_4 \psi_1^3 + 280 f_{x^2} \varphi_1 \varphi_5 \psi_1^3 + 70 f_{x^3} \varphi_2 \psi_1^4 + 56 f_{x^3} \varphi_1^2 \psi_1^5 + 168 f_{x^2} \varphi_3 \psi_1^5 + \\
 & 56 f_{x^2} \varphi_2 \psi_1^5 + 28 f_{x^2} \varphi_1 \psi_1^6 + 8 f_{x^3} \varphi_2 \psi_1^6 + 420 f_{x^4} \varphi_1^4 \psi_2 + 1260 f_{x^3} \varphi_1^2 \varphi_2 \psi_2 + 1260 f_{x^4} \varphi_1^3 \psi_2 + 420 f_{x^3} \varphi_1 \varphi_2^2 \psi_2 + 420 f_{x^2} \varphi_3 \psi_2 + \\
 & 560 f_{x^4} \varphi_1^2 \varphi_3 \psi_2 + 1680 f_{x^3} \varphi_1 \varphi_2 \varphi_3 \psi_2 + 280 f_{x^2} \varphi_3^2 \psi_2 + 420 f_{x^3} \varphi_1 \varphi_4 \psi_2 + 420 f_{x^2} \varphi_2 \varphi_4 \psi_2 + 168 f_{x^2} \varphi_1 \varphi_5 \psi_2 + 28 f_{x^3} \varphi_2 \psi_2 + 168 f_{x^2} \varphi_1 \varphi_6 \psi_2 + 168 f_{x^2} \varphi_1 \psi_2^2 + \\
 & 1680 f_{x^4} \varphi_1^4 \psi_2^2 + 2520 f_{x^3} \varphi_1^2 \varphi_2 \psi_2^2 + 1680 f_{x^3} \varphi_1 \varphi_3 \psi_2^2 + 1680 f_{x^2} \varphi_2 \varphi_3 \psi_2^2 + 840 f_{x^2} \varphi_1 \varphi_4 \psi_2^2 + 168 f_{x^2} \varphi_1 \varphi_5 \psi_2^2 + 420 f_{x^4} \varphi_1^4 \psi_2^2 + \\
 & 2520 f_{x^3} \varphi_1^2 \varphi_2 \psi_2^2 + 1260 f_{x^2} \varphi_3 \varphi_2 \psi_2^2 + 1680 f_{x^2} \varphi_1 \varphi_3 \psi_2^2 + 420 f_{x^3} \varphi_1 \varphi_4 \psi_2^2 + 560 f_{x^3} \varphi_1^2 \psi_2^2 + 1680 f_{x^2} \varphi_1 \varphi_2 \psi_2^2 + 560 f_{x^4} \varphi_1^4 \psi_3 + 560 f_{x^3} \varphi_1^2 \varphi_3 \psi_2^2 + \\
 & 420 f_{x^2} \varphi_3^2 \psi_2^2 + 420 f_{x^3} \varphi_1 \varphi_2 \psi_2^2 + 168 f_{x^2} \varphi_1 \psi_1^2 \psi_2 + 28 f_{x^3} \varphi_2 \psi_1^2 \psi_2 + 210 f_{x^4} \varphi_1^4 \psi_2^2 + 1260 f_{x^3} \varphi_1^2 \varphi_2 \psi_2^2 + 630 f_{x^2} \varphi_3^2 \psi_2^2 + 840 f_{x^2} \varphi_1 \varphi_3 \psi_2^2 + \\
 & 210 f_{x^3} \varphi_1 \varphi_4 \psi_2^2 + 840 f_{x^3} \varphi_1^2 \psi_1^2 \psi_2^2 + 2520 f_{x^2} \varphi_3 \varphi_1 \varphi_2 \psi_1^2 \psi_2^2 + 840 f_{x^3} \varphi_1 \varphi_3 \psi_1^2 \psi_2^2 + 1260 f_{x^2} \varphi_3 \varphi_1 \varphi_2 \psi_1^2 \psi_2^2 + 840 f_{x^3} \varphi_1 \varphi_2 \psi_1^2 \psi_2^2 + 210 f_{x^2} \varphi_3 \psi_1^2 \psi_2^2 + \\
 & 420 f_{x^2} \varphi_1 \varphi_4 \psi_1^2 \psi_2^2 + 420 f_{x^3} \varphi_1^2 \varphi_2 \psi_1^2 \psi_2^2 + 840 f_{x^4} \varphi_1^4 \psi_3 + 420 f_{x^3} \varphi_1^2 \varphi_3 \psi_3 + 840 f_{x^2} \varphi_3^2 \psi_3 + 560 f_{x^3} \varphi_1 \varphi_2 \varphi_3 \psi_3 + \\
 & 560 f_{x^2} \varphi_2 \varphi_3 \psi_3 + 280 f_{x^2} \varphi_1 \varphi_4 \psi_3 + 56 f_{x^3} \varphi_2 \psi_3 + 280 f_{x^4} \varphi_1^4 \psi_1 \psi_3 + 1680 f_{x^3} \varphi_1^2 \varphi_2 \psi_1 \psi_3 + 840 f_{x^2} \varphi_3^2 \psi_1 \psi_3 + 1120 f_{x^2} \varphi_1 \varphi_3 \psi_1 \psi_3 + \\
 & 280 f_{x^3} \varphi_1 \varphi_4 \psi_1 \psi_3 + 560 f_{x^3} \varphi_1^2 \varphi_3 \psi_1 \psi_3 + 1680 f_{x^2} \varphi_3 \varphi_1 \varphi_2 \psi_1 \psi_3 + 560 f_{x^4} \varphi_1^4 \psi_2 \psi_3 + 560 f_{x^3} \varphi_1^2 \varphi_3 \psi_2 \psi_3 + 280 f_{x^2} \varphi_3^2 \psi_2 \psi_3 + \\
 & 560 f_{x^2} \varphi_1 \varphi_4 \psi_2 \psi_3 + 1680 f_{x^2} \varphi_1 \varphi_2 \varphi_3 \psi_2 \psi_3 + 1680 f_{x^2} \varphi_1 \varphi_3 \psi_2 \psi_3 + 1680 f_{x^2} \varphi_1 \varphi_4 \psi_2 \psi_3 + 1680 f_{x^2} \varphi_1 \varphi_5 \psi_2 \psi_3 + 560 f_{x^4} \varphi_1^4 \psi_3 + \\
 & 840 f_{x^3} \varphi_1^2 \varphi_3 \psi_3 + 840 f_{x^3} \varphi_1 \varphi_4 \psi_3 + 280 f_{x^2} \varphi_3^2 \psi_3 + 280 f_{x^2} \varphi_1 \varphi_4 \psi_3 + 560 f_{x^3} \varphi_1 \varphi_5 \psi_3 + 70 f_{x^4} \varphi_1^4 \psi_4 + 420 f_{x^3} \varphi_1^2 \varphi_4 \psi_4 + \\
 & 210 f_{x^2} \varphi_4^2 \psi_4 + 280 f_{x^2} \varphi_1 \varphi_3 \psi_4 + 70 f_{x^3} \varphi_2 \psi_4 + 280 f_{x^3} \varphi_1 \varphi_4 \psi_4 + 840 f_{x^2} \varphi_3 \varphi_1 \varphi_2 \psi_1 \psi_4 + 280 f_{x^2} \varphi_3 \varphi_1 \psi_1 \psi_4 + 420 f_{x^2} \varphi_3 \varphi_2 \psi_1 \psi_4 + 420 f_{x^2} \varphi_3 \varphi_3 \psi_1 \psi_4 + \\
 & 280 f_{x^2} \varphi_3 \varphi_4 \psi_1 \psi_4 + 70 f_{x^3} \varphi_2 \psi_1^2 \psi_4 + 420 f_{x^2} \varphi_3 \varphi_2 \psi_1^2 \psi_4 + 420 f_{x^2} \varphi_3 \varphi_3 \psi_1^2 \psi_4 + 840 f_{x^3} \varphi_1 \varphi_2 \psi_1 \psi_2 \psi_4 + 420 f_{x^2} \varphi_3 \varphi_1 \psi_1 \psi_2 \psi_4 + 210 f_{x^3} \varphi_2 \psi_1 \psi_2 \psi_4 + 280 f_{x^2} \varphi_3 \varphi_1 \psi_1 \psi_2 \psi_4 + \\
 & 280 f_{x^2} \varphi_3 \varphi_2 \psi_1 \psi_2 \psi_4 + 35 f_{x^2} \varphi_4^2 \psi_5 + 56 f_{x^3} \varphi_1 \varphi_3 \psi_5 + 168 f_{x^2} \varphi_3 \varphi_1 \varphi_2 \psi_5 + 56 f_{x^2} \varphi_3 \varphi_2 \psi_5 + 168 f_{x^2} \varphi_3 \varphi_1 \psi_1 \psi_5 + 168 f_{x^2} \varphi_3 \varphi_2 \psi_1 \psi_5 + 168 f_{x^3} \varphi_1 \varphi_2 \psi_1^2 \psi_5 + 56 f_{x^4} \varphi_1^4 \psi_5 + \\
 & 168 f_{x^2} \varphi_3 \varphi_1 \psi_2 \psi_5 + 168 f_{x^2} \varphi_3 \varphi_2 \psi_5 + 56 f_{x^2} \varphi_3 \psi_5 + 28 f_{x^2} \varphi_3 \psi_6 + 8 f_{x^2} \varphi_3 \psi_7 + 8 f_{x^2} \varphi_3 \psi_7 + f_y \psi_8
 \end{aligned}$$

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