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Hybrid Fuzzy Contraction Theorems with Their Role in Integral Inclusions

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Abstract: The focus of this paper is to establish a new concept of b-hybrid fuzzy contraction regarding the study of fuzzy fixed-point theorems in the setting of b-metric spaces. This idea harmonizes and refines several well-known results in the direction of point-valued, multivalued, and fuzzy-set-valued maps in the comparable literature. To attract new researchers to this field, some important results are shown to be corollaries. Moreover, a result is presented to establish sufficient conditions for the existence of solutions of integral inclusion of Fredholm type. Lastly, illustrations are presented to validate the suppositions of the given theorems.

Keywords: b-metric space; fuzzy set; fuzzy fixed point; hybrid contraction; integral inclusion; set-valued

MSC: 46S40; 47H10; 54H25; 34A12

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1. Introduction

A fixed-point theorem is an outcome indicating that a mapping Γ on a nonempty set has at least one fixed point (fp) under some circumstances on a nonempty set Ξ and Γ that can be stated in general terms. Results of this kind are most generally useful in almost all types of applied sciences. The Banach contraction principle (Cp) [1] is one of the prototypical, simple, and multiuse results in fp theory in a metric space (MS) structure. Many studies embraced the applications and refinements of this principle in various arms by, for example, relaxing the axioms, and employing different mappings and several kinds of metric spaces (MSs). In this context, the work of Rhoades [2] is useful in assembling multiple modifications of Banach-type contractive ideas.

Finding new spaces and their properties has been an inviting topic among mathematical investigators, and b-MSs is presently a focus in the literature. The idea began with the work of Bakhtin [3] and Bourbaki [4]. Later on, Czerwik [5] provided an assumption that was more relaxed than the known triangle inequality, and officially brought up b-MS to improve the Banach fp theorem. Meanwhile, the notion of b-MSs is gaining enormous refinements; see, for example, [6,7]. For some new surveys on the concepts of fp theory in the framework of b-MSs, we direct the interested reader to Karapinar [8]. An active field of fp theory is the investigation of hybrid contractions. The notion was considered in two ways: first, hybrid contraction concerns contractions involving single-valued and set-valued maps; second, harmonizing linear and nonlinear contractions. Hybrid fp theory plays enormous roles in functional inclusions, optimization theory, fractal graphics, discrete dynamics for set-valued operators, and other ambits of functional analysis. For a few related works, refer to [9–11].

As a refinement of the theory of crisp sets, the fuzzy set (Fs) was presented by Zadeh [12]. Since then, to avail this concept, many researchers have advanced the theory and its roles to other arms of sciences, social sciences, and engineering. In 1981, Heilpern [13] utilized the idea of Fs to initiate a class of Fs-valued maps, and coined the fp theorem for fuzzy contraction mapping, which is a fuzzy analog of the fp theorem of Nadler [14]. Later, several researchers examined the existence of fp of Fs-valued maps and related developments in Fs theory; for example, the work of Al-Mazrooei et al. [15], Azam et al. [16], Alansari et al. [17], Martino and Sessa [18], Qiu and Shu [19], and Shehu and Akbar [20].

Integral inclusions emerge in diverse domains in mathematical physics, control theory, critical point theory for nonsmooth energy functionals, differential variational inequalities, Fs arithmetic, and traffic theory (see, for instance, [21–23]). Commonly, the first most examined aspect in the study of integral inclusions is the criteria for the occurrence of their solutions. Seeveral authors applied different fp approaches and topological tools to deduce the existence results of integral inclusions in various spaces; see, for example, Appele et al. [21], Cardinali and Papageorgiou [23], Kannan and O'Regan [24], Pathak et al. [25], Sintamarian [26], and the references therein. Almost all of the results proposed in the above papers depend on the multivalued versions of the Banach, Leray–Schauder, Matelli, Schauder, and Sadovskii-type fp theorems. Moreover, the ambient space of the existence results is either a Banach space or a classical MS.

With the above discussion in mind, we propose the idea of b-hybrid fuzzy contraction (f_z-contraction) in a b-MS, and prove a fuzzy fp theorem via this contraction. Thereafter, a few corollaries are deduced that include fp theorems due to Heilpern [13], Karapinar and Fulga [11], Nadler [14], and allied ones. Moving further, we employ one of our results to present a sufficient yardstick for the existence of solutions to an integral inclusion of the Fredholm type. The latter inference was coined from Sintamarian [26]. However, our result, which was achieved through a b-hybrid f_z-contraction in the body of b-MSs, leads to a new existence principle that improves and complements existing ideas.

The rest of the paper is organized as follows: Section 2 shows basic definitions and results needed in the sequel. In Section 3, the notion of a b-hybrid f_z-contraction and corresponding fp ideas are discussed. Some consequences of our proposed concepts in the theory of multivalued and single-valued mappings are highlighted in Section 4. Section 5 establishes the new yardstick for obtaining the solutions of Fredholm-type integral inclusions. Concluding statements of the principal ideas presented here are stated in Section 6.

2. Preliminaries

Here, we collate coherent notations, specific definitions and basic results needed hereafter. Throughout this paper, $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ signify sets of positive integers, non-negative real numbers, and real numbers, respectively. Most of these preliminaries are from [5,11,14].

In 1993, Czerwik [5] launched the notion of a b-MS as follows:

Definition 1 ([5])**.** *Let Ξ be a nonempty set, and $\eta \geq 1$ be a real number. Suppose that mapping $\sigma : \Xi \times \Xi \longrightarrow \mathbb{R}_+$ satisfies the following yardstick for all $\hbar, \jmath, \ell \in \Xi$:*

(i) $\sigma(\hbar, \jmath) = 0$ *if and only if* $\hbar = \jmath$ *(self-distancy).*
(ii) $\sigma(\hbar, \jmath) = \sigma(\jmath, \hbar)$ *(symmetry).*
(iii) $\sigma(\hbar, \jmath) \leq \eta[\sigma(\hbar, \ell) + \sigma(\ell, \jmath)]$ *(weighted triangle inequality).*

Afterwards, (Ξ, σ, η) is termed a b-MS.

Example 1 ([27])**.** *Let $\Xi = \mathbb{N} \cup \{\infty\}$ and $\sigma : \Xi \times \Xi \longrightarrow \mathbb{R}_+$ be defined by*

$$\sigma(\hbar, \jmath) = \begin{cases} 0, & \text{if } \hbar = \jmath \\ \left|\frac{1}{\hbar} - \frac{1}{\jmath}\right|, & \text{if } \hbar, \ell \text{ are even or } \hbar\jmath = \infty \\ 5, & \text{if } \hbar, \jmath \text{ are odd and } \hbar \neq \jmath \\ 2, & \text{otherwise.} \end{cases}$$

After that, (Ξ, σ) is a b-MS with the parameter $\eta = 3$, but σ is not a continuous functional.

Definition 2 ([4]). *Let (Ξ, σ, η) be a b-MS. A sequence $\{\hbar_p\}_{p\in\mathbb{N}}$ is termed as follows:*

(i) Convergent if and only if we can find $\hbar \in \Xi$ for which $\sigma(\hbar_p, \hbar) \longrightarrow 0$ as $p \longrightarrow \infty$, and we write this as $\lim_{p\longrightarrow\infty} \hbar_p = \hbar$.
(ii) Cauchy if and only if $\sigma(\hbar_p, \hbar_m) \longrightarrow 0$ as $p, m \longrightarrow \infty$.
(iii) Complete if every Cauchy sequence in Ξ is convergent.

Definition 3 ([4]). *Let (Ξ, σ, η) be a b-MS. After that, a subset P of Ξ is termed:*

(i) Compact if and only if, for every sequence of elements of P, we can find a subsequence that converges to an element of P.
(ii) Closed if and only if for every sequence $\{\hbar_p\}_{p\in\mathbb{N}}$ of elements of P that converges to an element $\hbar$, we have $\hbar \in P$.

Definition 4 ([28]). *A nonempty subset P of Ξ is termed proximal if, for each $\hbar \in \Xi$, we can find $a \in P$ for which $\sigma(\hbar, a) = \sigma(\hbar, P)$.*

Throughput this paper, $\mathfrak{g}^*$, $\mathcal{P}^r(\Xi)$ and $\mathcal{K}(\Xi)$ depict the set of all nonempty closed and bounded subsets of Ξ, the class of all nonempty bounded proximal subsets of Ξ, and the class of nonempty compact subsets of Ξ, respectively.

Let (Ξ, σ, η) be a b-MS. For $P, L \in \mathcal{P}^r(\Xi)$, function $H_b : \mathcal{P}^r(\Xi) \times \mathcal{P}^r(\Xi) \longrightarrow \mathbb{R}_+$, defined by

$$H_b(P, L) = \max\left\{\sup_{\hbar\in P} \sigma(\hbar, L), \sup_{\hbar\in L} \sigma(\hbar, P)\right\}$$

is termed the Hausdorff–Pompeiu b-metric on $\mathcal{P}^r(\Xi)$ generated σ, for which

$$\sigma(\hbar, P) = \inf_{\jmath\in P} \sigma(\hbar, \jmath)$$

see [29].

Definition 5 ([11]). *A nondecreasing function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is termed:*

(i) a c-comparison function if $\varphi^p(t) \longrightarrow 0$ as $p \longrightarrow \infty$ for every $t \in \mathbb{R}_+$;
(ii) a b-comparison function if we can find $k_0 \in \mathbb{N}$, $\lambda \in (0, 1)$ and a convergent non-negative series $\sum_{p=1}^{\infty} \hbar_p$ for which $\eta^{k+1}\varphi^{k+1}(t) \le \lambda\eta^k\varphi^k(t) + \hbar_k$, for $\eta \ge 1$, $k \ge k_0$ and any $t \ge 0$, for which φ^p denotes the p^{th} iterate of φ.

Ω_φ denotes the class of functions $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ obeying the following yardstick:

(i) φ is a b-comparison function.
(ii) $\varphi(t) = 0$ if and only if $t = 0$.
(iii) φ is continuous.

Lemma 1 ([30]). *For a comparison function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, the following properties hold:*

(i) Each iterated φ^p, $p \in \mathbb{N}$ is also a comparison function.
(ii) $\varphi(t) < t$ for all $t > 0$.

Lemma 2 ([30]). *Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a b-comparison function. After that, series $\sum_{k=0}^{\infty} \eta^k\varphi^k(t)$ converges for every $t \in \mathbb{R}_+$.*

Remark 1 ([11]). *From Lemma 2, every b-comparison function is a comparison function; thus, from Lemma 1, every b-comparison function satisfies $\varphi(t) < t$.*

Lemma 3 ([29]). *Let (Ξ, σ, η) be a b-MS. For $P, L \in \mathfrak{g}^*$ and $\hbar, \hbar \in \Xi$, the following yardstick holds:*

(i) $\sigma(\hbar, L) \leq H_b(P, L),\ \hbar \in P.$
(ii) $\sigma(\hbar, P) \leq \eta[\sigma(\hbar, \jmath) + \sigma(\jmath, L)].$
(iii) $\sigma(\hbar, P) = 0 \Longleftrightarrow \hbar \in P.$
(iv) $H_b(P, L) = 0 \Longleftrightarrow P = L.$
(v) $H_b(P, L) = H_b(L, P).$
(vi) $H_b(P, L) \leq \eta[H_b(P, C) + H_b(C, L)].$

Let Ξ depict a reference set. Fs in Ξ is a function with domain Ξ and values in $[0, 1] = I$. Designed with I^{Ξ}, the class of all Fs is in Ξ. If P is a Fs in Ξ, then $P(\hbar)$ is the grade of the membership of $\hbar$ in P. The α-level set of a Fs P is depicted with $[P]_\alpha$ and defined as follows:

$$[P]_\alpha = \{\hbar \in \Xi : P(\hbar) \geq \alpha\}, \text{ if } \alpha \in (0, 1]$$

$$[P]_0 = \overline{\{\hbar \in \Xi : P(\hbar) > 0\}}.$$

A Fs P, in a metric linear space V, is an approximate quantity if and only if $[P]_\alpha$ is compact and convex in V and $\sup_{\hbar \in V} P(\hbar) = 1$.

We depict the collection of all approximate quantities in V with $W(V)$. If we can find an $\alpha \in [0, 1]$ for which $[P]_\alpha, [L]_\alpha \in \mathcal{P}^r(\Xi)$, we define

$$p_\alpha(P, L) = \inf_{\hbar \in [P]_\alpha, \jmath \in [L]_\alpha} \sigma(\hbar, \jmath)$$

$$D_\alpha(P, L) = H([P]_\alpha, [L]_\alpha).$$

$$p(P, L) = \sup_\alpha p_\alpha(P, L).$$

$$\sigma_\infty(P, L) = \sup_\alpha D_\alpha(P, L).$$

Definition 6 ([13]). *Let Ξ be an arbitrary set, and Y a MS. Mapping $P : \Xi \longrightarrow I^Y$ is termed Fs-valued map. An Fs-valued map P is a fuzzy subset of $\Xi \times Y$ with membership function $P(\hbar)(\jmath)$. Value $P(\hbar)(\jmath)$ is the grade of membership of $\jmath$ in $P(\hbar)$.*

Definition 7 ([13]). *Let $P, L : \Xi \longrightarrow I^{\Xi}$ be Fs-valued maps. Point $\flat \in \Xi$ is a fuzzy fp of P if we can find an $\alpha \in (0, 1]$ for which $\flat \in [P\flat]_\alpha$. Point $\flat$ is a common fuzzy fp of P and L if we can find an $\alpha \in (0, 1]$ for which $\flat \in [P\flat]_\alpha \cap [L\flat]_\alpha$.*

The set of all fp of P is depicted with $\mathcal{F}_{ix}(P)$, and the fp of P and L with $\mathcal{F}_{ix}(P, L)$.

Example 2. *Let $\Xi = [-4, 4]$ and $Y = [-3, 3]$. $P : \Xi \longrightarrow I^Y$ is defined with*

$$P(\hbar)(\jmath) = \frac{\sin^2 3\hbar + \cos^2 \jmath}{50}.$$

After that, P is a Fs-valued map. $P(\hbar)(\jmath) \in [0, 1]$ for all $\hbar \in \Xi$ and $\jmath \in Y$. The graphical representation of P showing all possible grades of membership of $\jmath$ in the Fs $P\hbar$ is depicted in Figure 1.

Figure 1 shows that, for instance, the grade of membership of $\jmath = -3$ in the Fs $P(4)$ was ≈ 0.03.

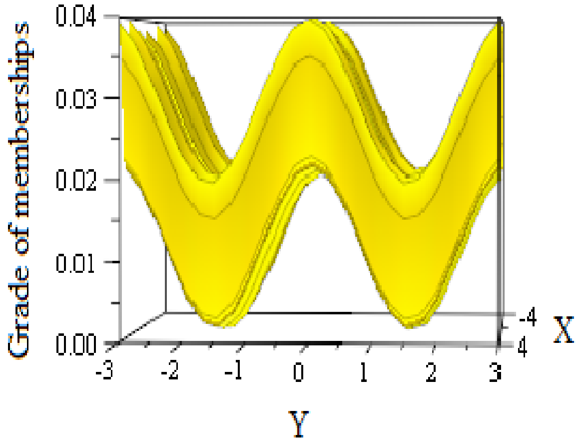

Figure 1. Graphical representation of Fs-valued map in Example 2.

Example 3. *Let* $\Xi = Y = [-4, 4]$. *Define* $P : \Xi \longrightarrow I^Y$ *by*

$$P(\hbar)(\jmath) = 1 - \frac{\hbar^2 + \jmath^2}{\hbar^2 + \jmath^2 + 10}.$$

After that, P *is a* Fs*-valued map.* $P(\hbar)(\jmath) \in [0, 1]$ *for all* $\hbar \in \Xi$ *and* $\jmath \in Y$*. Figure 2 depicts the graphical representation of the membership values of* $\jmath$ *in* $P(\hbar)$*.*

Figure 2 shows that element $\jmath = 0$ *had a full grade of membership (i.e.,* 1*) in the Fs* $P(0)$*.*

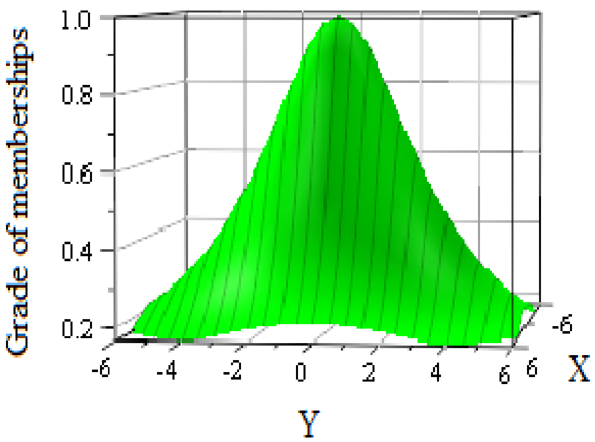

Figure 2. Graphical representation of the Fs-valued map in Example 3.

3. Main Results

First, we present the notion of b-hybrid f_z-contraction in the following manner.

Definition 8. *Let* (Ξ, σ, η) *be a b-MS and* $P, L : \Xi \longrightarrow I^\Xi$ *be Fs-valued maps. Pair* (P, L) *forms a b-hybrid* f_z*-contraction if, for each* $\hbar, \jmath \in \Xi$*, we can find* $\alpha(\hbar), \alpha(\jmath) \in (0, 1]$ *for which* $[P\hbar]_{\alpha(\hbar)}, [L\jmath]_{\alpha(\jmath)} \in \mathcal{P}^r(\Xi)$*,*

$$H_b\Big([P\hbar]_{\alpha(\hbar)}, [L\jmath]_{\alpha(\jmath)}\Big) \leq \varphi\Big(\mathcal{C}^r_{(P,L)}(\hbar, \jmath)\Big), \tag{1}$$

for which $\varphi \in \Omega_\varphi$*,* $r \geq 0$*,* $a_i \geq 0$*,* $i = 1, 2, 3, 4$ *with* $\sum_{i=1}^4 a_i = 1$ *and*

$$\mathcal{C}^r_{(P,L)}(\hbar,\jmath) = \begin{cases} \Bigg[a_1(\sigma(\hbar,\jmath))^r + a_2(\sigma(\hbar,[P\hbar]_{\alpha(\hbar)}))^r \\ +a_3(\sigma(\jmath,[L\jmath]_{\alpha(\jmath)}))^r \\ +a_4\left(\frac{\sigma(\jmath,[P\hbar]_{\alpha(\hbar)})+\sigma(\hbar,[L\jmath]_{\alpha(\jmath)})}{2\eta}\right)^r\Bigg]^{\frac{1}{r}} \\ \textit{for } \ r>0,\ \hbar,\jmath\in\Xi \\ (\sigma(\hbar,\jmath))^{a_1}(\sigma(\hbar,[P\hbar]_{\alpha(\hbar)}))^{a_2} \\ \times(\sigma(\jmath,[L\jmath]_{\alpha(\jmath)}))^{a_3} \\ \left(\frac{\sigma(\hbar,[L\jmath]_{\alpha(\jmath)})+\sigma(\jmath,[P\hbar]_{\alpha(\hbar)})}{2\eta}\right)^{a_4} \\ \text{for } r=0,\hbar,\jmath\in\Xi\setminus\mathcal{F}_{ix}(P,L). \end{cases}, \tag{2}$$

for which

$$\mathcal{F}_{ix}(P,L) = \{\hbar,\jmath\in\Xi : \hbar\in[P\hbar]_{\alpha(\hbar)},\ \jmath\in[L\jmath]_{\alpha(\jmath)}\}$$

Theorem 1. *Let (Ξ,σ,η) be a complete b-MS and $P,L:\Xi\longrightarrow I^{\Xi}$ be Fs-valued maps. Suppose that, for each $\hbar,\jmath\in\Xi$, we can find $\alpha(\hbar),\alpha(\jmath)\in(0,1]$ for which $[P\hbar]_{\alpha(\hbar)}$ and $[L\jmath]_{\alpha(\jmath)}$ are bounded proximal subsets of Ξ. If pair (P,L) forms a b-hybrid f_z-contraction, then P and L enjoy a common fuzzy fp in Ξ.*

Proof. Let $\hbar_0\in\Xi$; then, by supposition, we can find $\alpha_1\in(0,1]$ for which $[P\hbar_0]_{\alpha_1}\in\mathcal{P}^r(\Xi)$. $\hbar_1\in[P\hbar_0]_{\alpha_1}$ is chosen for which $\sigma(\hbar_0,\hbar_1)=\sigma(\hbar_0,[P\hbar_0]_{\alpha_1})$. Similarly, we can find $\alpha_2\in(0,1]$ for which $[L\hbar_1]_{\alpha_2}\in\mathcal{P}^r(\Xi)$. So, we can find $\hbar_2\in[L\hbar_1]_{\alpha_2}$ for which $\sigma(\hbar_1,\hbar_2)=\sigma(\hbar_1,[L\hbar_1]_{\alpha_2})$. We can then find a sequence $\{\hbar_p\}_{p\in\mathbb{N}}$ of elements of Ξ for which

$$\hbar_{2p+1}\in[P\hbar_{2p}]_{\alpha_{2p+1}},\ \hbar_{2p+2}\in[L\hbar_{2p+1}]_{\alpha_{2p+2}}$$

and

$$\sigma(\hbar_{2p},\hbar_{2p+1})=\sigma(\hbar_{2p},[P\hbar_{2p}]_{\alpha_{2p+1}}),$$

$\sigma(\hbar_{2p+1},\hbar_{2p+2})=\sigma(\hbar_{2p+1},[L\hbar_{2p+1}]_{\alpha_{2p+2}}),\ p\in\mathbb{N}.$

By Lemma 3 and the above results,

$$\sigma(\hbar_{2p},\hbar_{2p+1})\le H_b\Big([P\hbar_{2p}]_{\alpha_{2p+1}},[L\hbar_{2p-1}]_{\alpha_{2p}}\Big). \tag{3}$$

$\sigma(\hbar_{2p+1},\hbar_{2p+2})\le H_b\Big([P\hbar_{2p}]_{\alpha_{2p+1}},[L\hbar_{2p+1}]_{\alpha_{2p+2}}\Big).$

Suppose that $\hbar_{2p}=\hbar_{2p+1}$ for some $p\in\mathbb{N}$ and $r>0$. After that,

$$\begin{aligned} &\mathcal{C}^r_{(P,L)}(\hbar_{2p},\hbar_{2p+1}) \\ =\ &\Bigg[a_1(\sigma(\hbar_{2p},\hbar_{2p+1}))^r + a_2(\sigma(\hbar_{2p},[P\hbar_{2p}]_{\alpha_{2p+1}}))^r \\ &+a_3(\sigma(\hbar_{2p+1},[L\hbar_{2p+1}]_{\alpha_{2p+2}}))^r \\ &+a_4\Bigg(\frac{\sigma(\hbar_{2p+1},[P\hbar_{2p}]_{\alpha_{2p+1}})}{2\eta} \\ &+\frac{\sigma(\hbar_{2p},[L\hbar]_{2p+1})}{2\eta}\Bigg)^r\Bigg]^{\frac{1}{r}} \end{aligned}$$

$$
\begin{aligned}
&= \left[a_1(\sigma(\hbar_{2p},\hbar_{2p+1}))^r + a_2(\sigma(\hbar_{2p},\hbar_{2p+1}))^r \right.\\
&\quad + a_3(\sigma(\hbar_{2p+1},\hbar_{2p+2}))^r\\
&\quad \left. + a_4\left(\frac{\sigma(\hbar_{2p+1},\hbar_{2p+1}) + \sigma(\hbar_{2p},\hbar_{2p+2})}{2\eta}\right)^r\right]^{\frac{1}{r}}\\
&\leq \left[a_3(\sigma(\hbar_{2p+1},\hbar_{2p+2}))^r \right.\\
&\quad \left. + a_4\left(\eta\left(\frac{\sigma(\hbar_{2p},\hbar_{2p+1}) + \sigma(\hbar_{2p+1},\hbar_{2p+2})}{2\eta}\right)\right)^r\right]^{\frac{1}{r}}\\
&\leq \left[a_3(\sigma(\hbar_{2p+1},\hbar_{2p+2}))^r + a_4(\sigma(\hbar_{2p+1},\hbar_{2p+2}))^r\right]^{\frac{1}{r}}\\
&= (a_3+a_4)^{\frac{1}{r}}\sigma(\hbar_{2p+1},\hbar_{2p+2})\\
&= \sigma(\hbar_{2p+1},\hbar_{2p+2}) \text{ as } r \longrightarrow \infty.
\end{aligned}
$$

Hence, by availing the continuity of φ and Lemma 1, we have

$$
\begin{aligned}
\sigma(\hbar_{2p+1},\hbar_{2p+2}) &\leq H_b([P\hbar_{2p}]_{\alpha_{2p+1}}, [L\hbar_{2p+1}]_{\alpha_{2p+2}})\\
&\leq \varphi(\sigma(\hbar_{2p+1},\hbar_{2p+2}))\\
&< \sigma(\hbar_{2p+1},\hbar_{2p+2}),
\end{aligned}
$$

a contradiction. It follows that $\sigma(\hbar_{2p+1},\hbar_{2p+2}) = 0$. Whence,

$$\hbar_{2p} = \hbar_{2p+1} \in [P\hbar_{2p}]_{\alpha_{2p+1}} \text{ and}$$

$$\hbar_{2p} = \hbar_{2p+1} = \hbar_{2p+2} \in [L\hbar_{2p+1}]_{\alpha_{2p+2}} = [L\hbar_{2p}]_{\alpha_{2p+1}}.$$

So, $\hbar_{2p}$ happens to emerge the common fuzzy fp of P and L.

Again, for $r = 0$ and $\hbar_{2p} = \hbar_{2p+1}$, $\mathcal{C}^r_{(P,L)}(\hbar_{2p},\hbar_{2p+1}) = 0$. Hence, with property (ii) of Ω_φ, we obtain $\sigma(\hbar_{2p+1},\hbar_{2p+2}) = 0$ from which, on a related reasoning, the same deduction follows that $\hbar_{2p} \in [P\hbar_{2p}]_{\alpha_{2p+1}} \cap [L\hbar_{2p}]_{\alpha_{2p+1}}$. For this purpose, we hypothesize that, for all $p \in \mathbb{N}$,

$$\hbar_{p+1} \neq \hbar_p \text{ if and only if } \sigma(\hbar_{p+1},\hbar_p) > 0.$$

Now, regarding (1), we set $\hbar = \hbar_{2p}$ and $\jmath = \hbar_{2p-1}$, $\mathcal{C}^r_{(P,L)}(\hbar_{2p},\hbar_{2p-1}) =$

$$
\begin{cases}
\left[a_1(\sigma(\hbar_{2p},\hbar_{2p-1}))^r \right.\\
+a_2(\sigma(\hbar_{2p},[P\hbar_{2p}]_{\alpha_{2p+1}}))^r\\
+a_3(\sigma(\hbar_{2p-1},[L\hbar_{2p-1}]_{\alpha_{2p}}))^r\\
\left. +a_4\left(\frac{\sigma(\hbar_{2p-1},[P\hbar_{2p}]_{\alpha_{2p+1}})+\sigma(\hbar_{2p},[L\hbar_{2p-1}]_{\alpha_{2p}})}{2\eta}\right)^r\right]^{\frac{1}{r}}\\
\text{for } r > 0,\\
(\sigma(\hbar_{2p},\hbar_{2p-1}))^{a_1}(\sigma(\hbar_{2p},[P\hbar_{2p}]_{\alpha_{2p+1}}))^{a_2}\\
\times(\sigma(\hbar_{2p-1},[L\hbar_{2p-1}]_{\alpha_{2p}}))^{a_3}\\
\times\left(\frac{\sigma(\hbar_{2p},[L\hbar_{2p-1}]_{\alpha_{2p}})+\sigma(\hbar_{2p-1},[P\hbar_{2p}]_{\alpha_{2p+1}})}{2\eta}\right)^{a_4}\\
\text{for } r = 0.
\end{cases}
$$

That is,

$$
\mathcal{C}^r_{(P,L)}(\hbar_{2p},\hbar_{2p-1}) = \begin{cases} \Big[a_1(\sigma(\hbar_{2p},\hbar_{2p-1}))^r + a_2(\sigma(\hbar_{2p},\hbar_{2p+1}))^r \\ +a_3(\sigma(\hbar_{2p-1},\hbar_{2p}))^r \\ +a_4\left(\frac{\sigma(\hbar_{2p-1},\hbar_{2p+1})+\sigma(\hbar_{2p},\hbar_{2p})}{2\eta}\right)^r \Big]^{\frac{1}{r}} \\ \text{for } r>0, \\ (\sigma(\hbar_{2p},\hbar_{2p-1}))^{a_1}(\sigma(\hbar_{2p},\hbar_{2p+1}))^{a_2} \\ \times(\sigma(\hbar_{2p-1},\hbar_{2p}))^{a_3} \\ \times\left(\frac{\sigma(\hbar_{2p},\hbar_{2p})+\sigma(\hbar_{2p-1},\hbar_{2p+1})}{2\eta}\right)^{a_4} \\ \text{for } r=0. \end{cases} \tag{4}
$$

Consider the two following cases:
Case 1: $r>0$. Suppose that $\sigma(\hbar_{2p},\hbar_{2p+1}) \geq \sigma(\hbar_{2p-1},\hbar_{2p})$. Then, from (4), we have

$$
\begin{aligned}
&\mathcal{C}^r_{(P,L)}(\hbar_{2p},\hbar_{2p-1}) \\
&\leq \Bigg[a_1(\sigma(\hbar_{2p+1},\hbar_{2p}))^r + a_2(\sigma(\hbar_{2p+1},\hbar_{2p}))^r \\
&+ a_3(\sigma(\hbar_{2p+1},\hbar_{2p}))^r \\
&+ a_4\left(\eta\left(\frac{\sigma(\hbar_{2p+1},\hbar_{2p})+\sigma(\hbar_{2p},\hbar_{2p-1})}{2\eta}\right)\right)^r\Bigg]^{\frac{1}{r}} \\
&\leq \Bigg[a_1(\sigma(\hbar_{2p+1},\hbar_{2p}))^r + a_2(\sigma(\hbar_{2p+1},\hbar_{2p}))^r \\
&+ a_3(\sigma(\hbar_{2p+1},\hbar_{2p}))^r \\
&+ a_4\left(\eta\left(\frac{\sigma(\hbar_{2p+1},\hbar_{2p})+\sigma(\hbar_{2p+1},\hbar_{2p})}{2\eta}\right)^r\right)\Bigg]^{\frac{1}{r}} \\
&\leq \Bigg[a_1(\sigma(\hbar_{2p+1},\hbar_{2p}))^r + a_2(\sigma(\hbar_{2p+1},\hbar_{2p}))^r \\
&+ a_3(\sigma(\hbar_{2p+1},\hbar_{2p}))^r + a_4(\sigma(\hbar_{2p+1},\hbar_{2p}))^r\Bigg]^{\frac{1}{r}} \\
&= \Big[(a_1+a_2+a_3+a_4)\sigma(\hbar_{2p+1},\hbar_{2p})^r\Big]^{\frac{1}{r}} \\
&= \sigma(\hbar_{2p+1},\hbar_{2p})\left(\sum_{i=1}^{4} a_i\right)^{\frac{1}{r}} = \sigma(\hbar_{2p+1},\hbar_{2p}).
\end{aligned} \tag{5}
$$

From (1) and (5), we have

$$
\sigma(\hbar_{2p+1},\hbar_{2p}) \leq \varphi\big(\sigma(\hbar_{2p+1},\hbar_{2p})\big). \tag{6}
$$

Employing the given property of φ, (6) provides that

$$
\sigma(\hbar_{2p+1},\hbar_{2p}) < \sigma(\hbar_{2p+1},\hbar_{2p}),
$$

which is a contradiction. Hence, it follows that $\sigma(\hbar_{2p+1}, \hbar_{2p}) < \sigma(\hbar_{2p}, \hbar_{2p-1})$. Thus, from (6), we obtain

$$\sigma(\hbar_{2p+1}, \hbar_{2p}) \leq \varphi(\sigma(\hbar_{2p}, \hbar_{2p-1})). \tag{7}$$

Setting $p = 2p \in \mathbb{N}$ in (7), we have

$$\begin{aligned} \sigma(\hbar_{p+1}, \hbar_p) \leq & \varphi(\sigma(\hbar_p, \hbar_{p-1})) \\ \leq & \varphi^2(\sigma(\hbar_{p-1}, \hbar_{p-2})) \\ \leq & \varphi^3(\sigma(\hbar_{p-2}, \hbar_{p-3})) \\ & \vdots \; \vdots \\ \leq & \varphi^p(\sigma(\hbar_1, \hbar_0)). \end{aligned} \tag{8}$$

From (8), utilizing Lemma 2 and the triangle inequality with respect to (Ξ, σ, η), for every $k \geq 1$,

$$\begin{aligned} \sigma(\hbar_{p+k}, \hbar_p) \leq & \eta(\sigma(\hbar_{p+k}, \hbar_{p+1}) + \sigma(\hbar_{p+1}, \hbar_p)) \\ \leq & \frac{1}{\eta^{p-1}} \sum_{i=p}^{p+k-1} \eta^k \sigma(\hbar_i, \hbar_{i+1}) \\ \leq & \frac{1}{\eta^{p-1}} \sum_{i=p}^{p+k-1} \eta^k \varphi^k(\sigma(\hbar_1, \hbar_0)) \\ \leq & \frac{1}{\eta^{p-1}} \sum_{i=p}^{\infty} \eta^i \varphi^i(\sigma(\hbar_1, \hbar_0)) \\ \longrightarrow & \, 0 \text{ as } p \longrightarrow \infty. \end{aligned}$$

Hence, $\{\hbar_p\}_{p \in \mathbb{N}}$ is a Cauchy sequence, and we can find $\flat \in \Xi$ for which

$$\lim_{p \longrightarrow \infty} \sigma(\hbar_p, \flat) = 0. \tag{9}$$

Now, we demonstrate that $\flat$ is the expected common fuzzy fp of P and L. First, assume that $\flat \notin [P\flat]_{\alpha(\flat)}$. After that, with Lemma 3 and considering case $r > 0$ in Contractive Inequality (1), we have

$$\begin{aligned} & \sigma(\flat, [P\flat]_{\alpha(\flat)}) \\ \leq & \; \sigma(\flat, \hbar_p) + \sigma(\hbar_p, [P\flat]_{\alpha(\flat)}) \\ \leq & \; \sigma(\flat, \hbar_p) + H_b\Big([P\flat]_{\alpha(\flat)}, [L\hbar_{p-1}]_{\alpha(\hbar_{p-1})}\Big) \\ \leq & \; \sigma(\flat, \hbar_p) + \varphi\Big(\mathcal{C}^r_{(P,L)}(\flat, \hbar_{p-1})\Big) \\ = & \; \sigma(\flat, \hbar_p) + \varphi\Bigg(\Bigg[a_1(\sigma(\flat, \hbar_{p-1}))^r \\ & + a_2(\sigma(\flat, [P\flat]_{\alpha(\flat)}))^r + a_3(\sigma(\hbar_{p-1}, [L\hbar_{p-1}]_{\alpha_{\hbar_p}}))^r \\ & + a_4\left(\frac{\sigma(\hbar_{p-1}, [P\flat]_{\alpha(\flat)} + \sigma(\flat, [L\hbar_{p-1}]_{\alpha(\hbar_p)}))}{2\eta}\right)^r\Bigg]^{\frac{1}{r}}\Bigg) \\ = & \; \sigma(\flat, \hbar_p) + \varphi\Bigg(\Bigg[a_1(\sigma(\flat, \hbar_{p-1}))^r \\ & + a_2(\sigma(\flat, [P\flat]_{\alpha(\flat)}))^r + a_3(\sigma(\hbar_{p-1}, \hbar_p))^r \\ & + a_4\left(\frac{\sigma(\hbar_{p-1}, [P\flat]_{\alpha(\flat)}) + \sigma(\flat, \hbar_p)}{2\eta}\right)^r\Bigg]^{\frac{1}{r}}\Bigg). \end{aligned} \tag{10}$$

Letting $p \longrightarrow \infty$ in (10) and availing the fact that $\varphi \in \Omega_\varphi$ give

$$\sigma(\flat, [P\flat]_{\alpha(\flat)}) < \sigma(\flat, [P\flat]_{\alpha(\flat)})(a_2 + a_4)^{\frac{1}{r}},$$

and $r \longrightarrow \infty$,

$$\sigma(\flat, [P\flat]_{\alpha(\flat)}) < \sigma(\flat, [P\flat]_{\alpha(\flat)})$$

yields a contradiction. Hence, $\flat \in [P\flat]_{\alpha(\flat)}$. On a related reasoning, by assuming that $\flat$ is not a fuzzy fp of L, and noting

$$\begin{aligned} \sigma(\flat, [L\flat]_{\alpha(\flat)}) &\leq \sigma(\flat, \hbar_p) + \sigma(\hbar_p, [L\flat]_{\alpha(\flat)}) \\ &\leq \sigma(\flat, \hbar_p) + H_b\Big([P\hbar_{p-1}]_{\alpha(\hbar_p)}, [L\flat]_{\alpha(\flat)}\Big) \\ &\leq \sigma(\flat, \hbar_p) + \varphi\Big(\mathcal{C}^r_{(P,L)}(\hbar_{p-1}, \flat)\Big), \end{aligned}$$

we can demonstrate that $\flat \in [L\flat]_{\alpha(\flat)}$. Hence, for $r > 0$, $\flat$ is a common fuzzy fp of P and L.

Case 2: $r = 0$. employing Inequality (1) on the recognition of b-comparison of φ,

$$\begin{aligned} &\sigma(\hbar_{2p}, \hbar_{2p-1}) \\ &\leq H_b([P\hbar_{2p-1}]_{\alpha_{2p}}, [L\hbar_{2p-2}]_{\alpha_{2p-1}}) \\ &\leq \varphi\left(\mathcal{C}^r_{(P,L)}(\hbar_{2p-1}, \hbar_{2p-2})\right) \\ &< (\sigma(\hbar_{2p-1}, \hbar_{2p-2}))^{a_1}(\hbar_{2p-1}, [P\hbar_{2p-1}]_{\alpha_{2p}})^{a_2} \\ &\times (\sigma(\hbar_{2p-2}, [L\hbar_{2p-2}]_{\alpha_{2p-2}}))^{a_3} \\ &\times \left(\frac{\sigma(x_{2p-1}, [L\hbar_{2p-2}]_{\alpha_{2p-1}} + \sigma(\hbar_{2p-2}, [P\hbar_{2p-1}]_{\alpha_{2p}}))}{2\eta}\right)^{a_4} \\ &= (\sigma(\hbar_{2p-1}, \hbar_{2p-2}))^{a_1}(\sigma(\hbar_{2p-1}, \hbar_{2p}))^{a_2} \\ &\times (\sigma(\hbar_{2p-2}, \hbar_{2p-1}))^{a_3} \\ &\times \left(\frac{\sigma(\hbar_{2p-1}, \hbar_{2p-1}) + \sigma(\hbar_{2p-2}, \hbar_{2p})}{2\eta}\right)^{a_4} \\ &\leq (\sigma(\hbar_{2p-1}, \hbar_{2p-2}))^{a_1}(\sigma(\hbar_{2p-1}, \hbar_{2p}))^{a_2} \\ &\times (\sigma(\hbar_{2p-2}, \hbar_{2p-1}))^{a_3} \\ &\times \left(\frac{\sigma(\hbar_{2p}, \hbar_{2p-1}) + \sigma(\hbar_{2p-1}, \hbar_{2p-2})}{2}\right)^{a_4}. \\ &= (\sigma(\hbar_{2p-1}, \hbar_{2p-2}))^{a_1+a_3}(\sigma(\hbar_{2p-1}, \hbar_{2p}))^{a_2} \\ &\times \left(\frac{\sigma(\hbar_{2p}, \hbar_{2p-1}) + \sigma(\hbar_{2p-1}, \hbar_{2p-2})}{2}\right)^{1-a_1-a_2-a_3}. \end{aligned} \tag{11}$$

Assuming that $\sigma(\hbar_{2p-1}, \hbar_{2p-2}) \leq \sigma(\hbar_{2p}, \hbar_{2p-1})$, (11) gives

$$\begin{aligned} &\sigma(\hbar_{2p}, \hbar_{2p-1}) \\ &\leq \varphi\left(\mathcal{C}^r_{(P,L)}(\hbar_{2p-1}, \hbar_{2p-2})\right) \\ &< \left(\sigma(\hbar_{2p}, \hbar_{2p-1})\right)^{a_1+a_2+a_3} \\ &\times \left(\sigma(\hbar_{2p}, \hbar_{2p-1})\right)^{1-a_1-a_2-a_3} \end{aligned}$$

$$= \sigma(\hbar_{2p}, \hbar_{2p-1}),$$

a contradiction. Hence,

$$\sigma(\hbar_{2p}, \hbar_{2p-1}) < \sigma(\hbar_{2p-1}, \hbar_{2p-2}). \tag{12}$$

by employing (11) and (12),

$$\sigma(\hbar_{2p}, \hbar_{2p-1}) \leq \varphi(\sigma(\hbar_{2p-1}, \hbar_{2p-2})). \tag{13}$$

The (13) is equivalent to (8). So, we infer that iterative sequence $\{\hbar_p\}_{p\in\mathbb{N}}$ is Cauchy in (Ξ, σ, η). Thus, the completeness of this space guarantees that $\sigma(\hbar_p, \flat) \longrightarrow 0$ as $p \longrightarrow \infty$, for some $\flat \in \Xi$.

To realize that $\flat$ is a common fuzzy fp of L and P, we apply Lemma 3 and Inequality (1) as follows:

$$\begin{aligned}
\sigma(\flat, [L\flat]_{\alpha_(\flat)}) &\leq \sigma(\flat, \hbar_p) + \sigma(\hbar_p, [L\flat]_{\alpha_(\flat)}) \\
&\leq \sigma(\flat, \hbar_p) + H_b([P\hbar_{p-1}]_{\alpha_p}, [L\flat]_{\alpha(\flat)}) \\
&\leq \sigma(\flat, \hbar_p) + \varphi\left(\mathcal{C}^r_{(P,L)}(\hbar_{p-1}, \flat)\right),
\end{aligned} \tag{14}$$

for which

$$\begin{aligned}
&\mathcal{C}^r_{(P,L)}(\hbar_{p-1}, \flat) \\
&= (\sigma(\hbar_{p-1}, \flat))^{a_1} (\sigma(\hbar_{p-1}, [P\hbar_{p-1}]_{\alpha_p}))^{a_2} \\
&\times (\sigma(\flat, [L\flat]_{\alpha(\flat)}))^{a_3} \\
&\times \left(\frac{\sigma(\hbar_{p-1}, [L\flat]_{\alpha(\flat)}) + \sigma(\flat, [P\hbar_{p-1}]_{\alpha_p})}{2\eta}\right)^{a_4} \\
= \; &\times (\sigma(\hbar_{p-1}, \flat))^{a_1} (\sigma(\hbar_{p-1}, \hbar_p))^{a_2} \\
&\times (\sigma(\flat, [L\flat]_{\alpha(\flat)}))^{a_3} \\
&\times \left(\frac{\sigma(\hbar_{p-1}, [L\flat]_{\alpha(\flat)}) + \sigma(\flat, \hbar_p)}{2\eta}\right)^{a_4}.
\end{aligned}$$

$\lim_{p\longrightarrow\infty} \mathcal{C}^r_{(P,L)}(\hbar_{p-1}, \flat) = 0$. Hence, under this limiting case, (14) becomes

$$\sigma(\flat, [L\flat]_{\alpha(\flat)}) \leq \varphi(0). \tag{15}$$

Via criterion (ii) of φ, (15) implies that $\sigma(\flat, [L\flat]_{\alpha(\flat)}) = 0$. Hence, $\flat \in [L\flat]_{\alpha(\flat)}$. On a related reasoning, one can demonstrate that $\flat \in [P\flat]_{\alpha(\flat)}$. Hence, $\flat$ is the common fuzzy fp of P and L. $\square$

Corollary 1. *Let (Ξ, σ, η) be a b-MS and $P : \Xi \longrightarrow I^\Xi$ be a Fs-valued map. Assume that, for each $\hbar \in \Xi$, we can find $\alpha(\hbar) \in (0, 1]$ for which $[P\hbar]_{\alpha(\hbar)}$ is a bounded proximal subset of Ξ, and*

$$H_b\left([P\hbar]_{\alpha(\hbar)}, [P\jmath]_{\alpha(\jmath)}\right) \leq \varphi\left(\mathcal{C}^r_{(P)}(\hbar, \jmath)\right),$$

for which $\varphi \in \Omega_\varphi$, $r \geq 0$, $a_i \geq 0$, $i = 1, 2, 3, 4$ with $\sum_{i=1}^4 a_i = 1$ and

$$\mathcal{C}^r_{(P)}(\hbar,\jmath) = \begin{cases} \Bigg[a_1(\sigma(\hbar,\jmath))^r + a_2(\sigma(\hbar,[P\hbar]_{\alpha(\hbar)}))^r \\ +a_3(\sigma(\ell,[P\jmath]_{\alpha(\jmath)}))^r \\ +a_4\left(\frac{\sigma(\jmath,[P\hbar]_{\alpha(\hbar)})+\sigma(\hbar,[P\jmath]_{\alpha(\jmath)})}{2\eta}\right)^r \Bigg]^{\frac{1}{r}} \\ \textit{for } r>0,\ \hbar,\jmath \in \Xi \\ (\sigma(\hbar,\jmath))^{a_1}(\sigma(\hbar,[P\hbar]_{\alpha(\hbar)}))^{a_2}(\sigma(\jmath,[P\jmath]_{\alpha(\jmath)}))^{a_3} \\ \times\left(\frac{\sigma(\hbar,[P\jmath]_{\alpha(\jmath)})+\sigma(\jmath,[P\hbar]_{\alpha(\hbar)})}{2\eta}\right)^{a_4} \\ \text{for } r=0, \hbar,\jmath \in \Xi \setminus \mathcal{F}_{ix}(P), \end{cases} \tag{16}$$

for which

$$\mathcal{F}_{ix}(P) = \{\hbar \in \Xi : \hbar \in [P\hbar]_{\alpha(\hbar)}.\}$$

After that, $\mathcal{F}_{ix}(P) \neq \emptyset$.

Proof. Put $P = L$ in Theorem 1. □

Example 4. *Set* $\Xi = [1,5]$ *and* $\sigma : \Xi \times \Xi \longrightarrow \mathbb{R}_+$ *are defined by* $\sigma(\hbar,\jmath) = (\hbar - \jmath)^2$, $\hbar,\jmath \in \Xi$. *After that,* (Ξ,σ) *is a complete b-MS with parameter* $\eta = 2$. (Ξ,σ) *is not an MS; for example, taking* $\hbar = 1$, $\jmath = 5$ *and* $\ell = 3$,

$$\sigma(1,5) = 16 > 8 = \sigma(1,3) + \sigma(3,5).$$

For $\hbar \in \Xi$, *consider two Fs-valued maps* $P, L : \Xi \longrightarrow I^{\Xi}$, *defined as follows:*

$$P(\hbar)(t) = \begin{cases} \frac{1}{2}, & \textit{if } t \in [1,2] \\ \frac{1}{6}, & \textit{if } t \in (2,3] \\ \frac{2}{11}, & \textit{if } t \in (3,4] \\ \frac{1}{8}, & \textit{if } t \in (4,5]. \end{cases}$$

$$L(\hbar)(t) = \begin{cases} \frac{4}{7}, & \textit{if } t \in [1,2] \\ \frac{4}{13}, & \textit{if } t \in (2,3] \\ \frac{3}{10}, & \textit{if } t \in (3,4] \\ \frac{1}{9}, & \textit{if } t \in (4,5]. \end{cases}$$

If we take $\alpha(\hbar) = \frac{1}{3}$ *and* $\alpha(\jmath) = \frac{2}{5}$ *for all* $\hbar,\jmath \in \Xi$, *then*

$$[P\hbar]_{\alpha(\hbar)} = [1,2] = [L\jmath]_{\alpha(\jmath)}.$$

Clearly, $[P\hbar]_{\alpha(\hbar)}, [L\jmath]_{\alpha(\jmath)} \in \mathcal{P}^r(\Xi)$. $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is defined with* $\varphi(t) = \frac{t}{2}$. *Then,* $\varphi \in \Omega_\varphi$. *For* $a_1 = a_2 = \frac{1}{5}$, $a_3 = \frac{3}{5}$ *and* $a_4 = 0$ *via elementary calculation, we have*

$$H_b([P\hbar]_{\alpha(\hbar)}, [L\jmath]_{\alpha(\jmath)}) \leq \varphi\Big(\mathcal{C}^r_{(P,L)}(\hbar,\jmath)\Big).$$

In this case, $\mathcal{F}_{ix}(P,L) = \{t \in \Xi : 1 \leq t \leq 2\}$ *and* $\Xi \setminus \mathcal{F}_{ix}(P,L) = \{t \in \Xi : 2 < t \leq 5\}$. *Thus, all the suppositions of Corollary 1 are obeyed.*

Corollary 2. *Let* (Ξ,σ,η) *be a complete b-MS, and* $P : \Xi \longrightarrow I^{\Xi}$ *be a Fs-valued map. Suppose that, for* $\hbar \in \Xi$, *we can find* $\alpha(\hbar) \in (0,1]$ *for which* $[P\hbar]_{\alpha(\hbar)}$ *is a bounded proximal subset of* Ξ. *If*

$$H_b([P\hbar]_{\alpha(\hbar)}, [P\jmath]_{\alpha(\jmath)}) \leq \varphi\left(\frac{1}{4}\mathcal{C}^r_{(P)}(\hbar,\jmath)\right),$$

for all $\hbar,\jmath \in \Xi$, *for which* $\varphi \in \Omega_\varphi$ *and*

$$\mathcal{C}^r_{(P)} = \sigma(\hbar, \jmath) + \sigma(\hbar, [P\hbar]_{\alpha(\hbar)}) + \sigma(\Xi, [P\jmath]_{\alpha(\jmath)}) + \frac{\sigma(\jmath, [P\hbar]_{\alpha(\hbar)}) + \sigma(\hbar, [P\jmath]_{\alpha(\jmath)})}{2\eta}.$$

After that, we can find $\flat \in \Xi$ for which $\flat \in [P\flat]_{\alpha(\flat)}$.

Proof. $P = L$, $r = 1$ and $a_1 = a_2 = a_3 = a_4 = \frac{1}{4}$ are taken in Theorem 1. □

Corollary 3. *Let (Ξ, σ, η) be a complete b-MS and $P, L : \Xi \longrightarrow I^{\Xi}$ be Fs-valued maps. Assume that, for each $\hbar, \jmath \in \Xi$, we can find $\alpha(\hbar), \alpha(\jmath) \in (0,1]$ and $\lambda \in [0,1)$ for which*

$$\begin{aligned} &H_b([P\hbar]_{\alpha(\hbar)}, [L\jmath]_{\alpha(\jmath)}) \\ &\leq \lambda \left(\sqrt[4]{(\sigma(\hbar, \jmath))\sigma(\hbar, [P\hbar]_{\alpha(\hbar)})(\sigma(\jmath, [L\jmath]_{\alpha(\jmath)}))} \right) \\ &\times \left(\sqrt[4]{\left(\frac{\sigma(\hbar, [L\jmath]_{\alpha(\jmath)}) + \sigma(\jmath, [P\hbar]_{\alpha(\hbar)})}{2\eta} \right)} \right), \end{aligned}$$

for which $[P\hbar]_{\alpha(\hbar)}, [L\jmath]_{\alpha(\jmath)} \in \mathcal{P}^r(\Xi)$. After that, P and L have a common fuzzy fp in Ξ.

Proof. $a_1 = a_2 = a_3 = a_4 = \frac{1}{4}$, $\varphi(t) = \lambda t \geq 0$ and $r = 0$ are taken in Theorem 1. □

Corollary 4. *Let (Ξ, σ, η) be a complete b-MS and $P : \Xi \longrightarrow I^{\Xi}$ be a Fs-valued map. Further, assume that, for all $\hbar \in \Xi$, we can find $\alpha(\hbar) \in (0,1]$ for which $[P\hbar]_{\alpha(\hbar)}$ is a bounded proximal subset of Ξ, and*

$$H_b([P\hbar]_{\alpha(\hbar)}, [P\jmath]_{\alpha(\jmath)}) \leq \lambda\sigma(\hbar, \jmath)$$

for all $\hbar, \jmath \in \Xi$. After that, we can find $\flat \in \Xi$ for which $\flat \in [P\flat]_{\alpha(\flat)}$.

Proof. $P = L$, $a_1 = r = 1$, $a_2 = a_3 = a_4 = 0$ and $\varphi(t) = \lambda t \geq 0$ are put in Theorem 1. □

Corollary 5. *Let (Ξ, σ, η) be a complete b-MS and $P, L : \Xi \longrightarrow W(\Xi)$ be Fs-valued maps for which*

$$\sigma_\infty(P(\hbar), L(\jmath)) \leq \varphi\left(\mathcal{C}^r_{(P,L)}(\hbar, \jmath)\right)$$

for which $\varphi \in \Omega_\varphi$, $r \geq 0$, $a_i \geq 0$, $i = 1,2,3,4$ with $\sum_{i=1}^4 a_i = 1$ and

$$\mathcal{C}^r_{(P,L)}(\hbar, \jmath) = \begin{cases} \Big[a_1(\sigma(\hbar, \jmath))^r + a_2(p(\hbar, P(\hbar)))^r \\ +a_3(p(\jmath, L(\jmath)))^r \\ +a_4\left(\frac{p(\jmath, P(\hbar)) + p(\hbar, L(\jmath))}{2\eta}\right)^r \Big]^{\frac{1}{r}} \\ \textit{for } \; r > 0, \; \hbar, \jmath \in \Xi \\ (\sigma(\hbar, \jmath))^{a_1}(p(\hbar, P(\hbar)))^{a_2}(p(\jmath, L(\jmath)))^{a_3} \\ \times \left(\frac{p(\hbar, L(\jmath)) + p(\jmath, P(\hbar))}{2\eta}\right)^{a_4} \\ \text{for } r = 0, \hbar, \jmath \in \Xi \setminus \mathcal{F}_{ix}(P, L), \end{cases}$$

for which

$$\mathcal{F}_{ix}(P, L) = \{\hbar, \jmath \in \Xi : \{\hbar\} \subset P(\hbar), \; \{\jmath\} \subset L(\jmath)\}$$

After that, we can find $\flat \in \Xi$ for which $\{\flat\} \subset P(\flat)$ and $\{\flat\} \subset L(\flat)$.

Proof. Take $\hbar \in \Xi$ and $\alpha(\hbar) = 1$. Then, by supposition, $[P\hbar]_1$ and $[L\hbar]_1$ are nonempty compact subsets of Ξ. Now, via the definition of D_α and σ_∞-metric for Fs, we have

$$\begin{aligned} D_1(P(\hbar), L(\jmath)) &\leq \sigma_\infty(P(\hbar), P(\jmath)) \\ &\leq \varphi\Big(\mathcal{C}^r_{(P,L)}\Big). \end{aligned}$$

Since $[P\hbar]_1 \subseteq [P\hbar]_{\alpha(\hbar)}$ for each $\alpha(\hbar) \in (0,1]$, then $\sigma(\hbar, [P\hbar]_{\alpha(\hbar)}) \leq \sigma(\hbar, [P\hbar]_1)$ for each $\alpha(\hbar) \in (0,1]$. It follows that $p(\hbar, P(\hbar)) \leq \sigma(\hbar, [P\hbar]_1)$. Similarly, $p(\hbar, L(\hbar)) \leq \sigma(\hbar, [L\hbar]_1)$. Furthermore, this implies that, for all $\hbar, \jmath \in \Xi$,

$$H_b([P\hbar]_1, [L\jmath]_1) \leq \varphi\Big(\mathcal{C}^r_{(P,L)}(\hbar, \flat)\Big).$$

Theorem 1 can, thus, be applied to obtain $\flat \in \Xi$ for which $\flat \in [P\flat]_1 \cap [L\flat]_1$, that is, $\{\flat\} \subset P(\flat)$ and $\{\flat\} \subset L(\flat)$. □

Corollary 6. *Let (Ξ, σ, η) be a complete b-MS and $P : \Xi \longrightarrow W(\Xi)$ be a Fs-valued map. Assume that we can find $\lambda \in (0,1)$ for which*

$$\sigma_\infty(P(\hbar), P(\hbar)) \leq \lambda\sigma(\hbar, \jmath)$$

for each $\hbar, \jmath \in \Xi$. After that, we can find $\jmath \in \Xi$ for which $\{\jmath\} \subset P(\jmath)$.

Proof. Place $P = L$, $a_1 = 1$, $a_2 = a_3 = a_4 = r = 0$ and $\varphi(t) = \lambda t$, $t \geq 0$ in Corollary 5. □

Example 5. *$\Xi = [0,1]$ is endowed with metric $\sigma(\hbar, \jmath) = |\hbar - \jmath|^2$ for all $\hbar, \jmath \in \Xi$. After that, Ξ is a complete b-MS with parameter $\eta = 2$. (Ξ, σ) is not a metric; to see this, take $\hbar = 0$, $\jmath = 1$ and $\ell = \frac{1}{4}$. After that,*

$$\sigma(\hbar, \jmath) = 1 > \frac{5}{8} = \sigma(\hbar, \ell) + \sigma(\ell, \jmath).$$

Now, define $P : \Xi \longrightarrow W(\Xi)$ by

$$P(\hbar)(t) = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq t \leq \frac{\hbar}{3} \\ \frac{2}{7}, & \text{if } \frac{\hbar}{3} < t \leq 1. \end{cases}$$

Let $\alpha(\hbar) = \frac{2}{5}$. Then,

$$[P\hbar]_{\frac{2}{5}} = \{t \in \Xi : P(\hbar)(t) \geq \alpha(\hbar)\} = \left[0, \frac{\hbar}{3}\right].$$

Suppose that, without loss of generality, $\hbar \leq \jmath$ for all $\hbar, \jmath \in \Xi$. If $\hbar = \jmath$, then $[P\hbar]_{\alpha(\hbar)} = [P\jmath]_{\alpha(\jmath)}$, so $\sigma_\infty(P(\hbar), P(\jmath)) = 0 \leq \lambda\sigma(\hbar, \jmath)$ for all $\lambda \in (0,1)$. Otherwise, for all $\hbar < \jmath$, we have

$$\begin{aligned} \sigma_\infty(P(\hbar), P(\jmath)) &= \sup_{\frac{2}{5}} H_b\Big([P\hbar]_{\frac{2}{5}}, [P\jmath]_{\frac{2}{5}}\Big) \\ &= \frac{1}{9}|\hbar - \jmath|^2 \leq \lambda\sigma(\hbar, \jmath). \end{aligned}$$

Thus, all the yardsticks of Corollary 6 hold. In this case, we can find $u = 0 \in \Xi$ for which $\{0\} \subset P(0)$.

Corollary 7. *Let (Ξ, σ) be a complete MS, and $P, L : \Xi \longrightarrow I^\Xi$ be Fs-valued maps. Assume that, for each $\hbar, \jmath \in \Xi$, we can find $\alpha(\hbar), \alpha(\jmath) \in (0,1]$ for which $[P\hbar]_{\alpha(\hbar)}$ and $[L\jmath]_{\alpha(\jmath)}$ are nonempty compact subsets of Ξ, and*

$$H\Big([P\hbar]_{\alpha(\hbar)}, [L\jmath]_{\alpha(\jmath)}\Big) \leq \varphi\Big(\mathcal{C}^r_{(P,L)}(\hbar, \jmath)\Big),$$

for which $\varphi \in \Omega_\varphi$, $r \geq 0$, $a_i \geq 0$, $i = 1,2,3,4$ with $\sum_{i=1}^4 a_i = 1$ and

$$\mathcal{C}^r_{(P,L)}(\hbar,\jmath)=\begin{cases}\Bigg[a_1(\sigma(\hbar,\jmath))^r+a_2(\sigma(\hbar,[P\hbar]_{\alpha(\hbar)}))^r\\ +a_3(\sigma(\jmath,[L\jmath]_{\alpha(\jmath)}))^r\\ +a_4\left(\frac{\sigma(\jmath,[P\hbar]_{\alpha(\hbar)})+\sigma(\hbar,[L\jmath]_{\alpha(\jmath)})}{2}\right)^r\Bigg]^{\frac{1}{r}}\\ \textit{for}\ \ r>0,\ \hbar,\jmath\in\Xi\\ (\sigma(\hbar,\jmath))^{a_1}(\sigma(\hbar,[P\hbar]_{\alpha(\hbar)}))^{a_2}\\ \times(\sigma(\jmath,[L\jmath]_{\alpha(\jmath)}))^{a_3}\left(\frac{\sigma(\hbar,[L\jmath]_{\alpha(\jmath)})+\sigma(\jmath,[P\hbar]_{\alpha(\hbar)})}{2}\right)^{a_4}\\ \text{for } r=0,\hbar,\jmath\in\Lambda\setminus\mathcal{F}_{ix}(P,L),\end{cases}$$

for which

$$\mathcal{F}_{ix}(P,L)=\{\hbar,\jmath\in\Xi:\hbar\in[P\hbar]_{\alpha(\hbar)},\jmath\in[L\jmath]_{\alpha(\jmath)}.\}$$

After that, P and L have a common fuzzy fp in Ξ.

Proof. Take $\eta=1$ in Theorem 1. □

Corollary 8. *(Heilpern (Theorem 3.1 [13])) Let (Ξ,σ) be a complete MS and $P:\Xi\longrightarrow W(\Xi)$ be a Fs-valued map. In addition, suppose that we can find $\lambda\in(0,1)$ for which*

$$\sigma_\infty(P(\hbar),P(\jmath))\leq\lambda\sigma(\hbar,\jmath)$$

for each $\hbar,\jmath\in\Xi$. After that, we can find $\flat\in\Xi$ for which $\{\flat\}\subset P(\flat)$.

Proof. Take $\eta=1$ in Corollary 6. □

4. Applications and Significance in the Theory of Multivalued and Single-Valued Mappings

Let (Ξ,σ) be an MS and $\mathcal{N}(\Xi)$ the class of nonempty subsets of Ξ. A set-valued mapping $T:\Xi\longrightarrow\mathcal{N}(\Xi)$ is a multivalued map. Point $\hbar\in\Xi$ is a fp of T if $\hbar\in T\hbar$. For a single-valued mapping $T:\Xi\longrightarrow\Xi$, if $\hbar=T\hbar$, then $\hbar$ is a fp of T.

In 1969, Nadler [14] first gave a refinement of the Cp for multivalued map by availing the Hausdorff metric. Since then, a number of refinements in various frames of Nadler's fp theorem have been observed by several authors; see, for example, [6,25,31]. Following this advancement, we obtain some consequences of the corresponding results of the previous section in the setting of multivalued and single-valued mappings.

Corollary 9. *Let (Ξ,σ,η) be a complete b-MS and $F,G:\Xi\longrightarrow\mathcal{P}^r(\Xi)$ be multivalued maps. Assume that, for each $\hbar,\jmath\in\Xi$,we can find $\varphi\in\Omega_\varphi$ for which*

$$H_b(F\hbar,G\jmath)\leq\varphi\Big(\mathcal{C}^r_{(F,G)}(\hbar,\jmath)\Big)$$

for which $\varphi\in\Omega_\varphi$, $r\geq0$, $a_i\geq0$, $i=1,2,3,4$ with $\sum_{i=1}^4a_i=1$ and

$$\mathcal{C}^{r}_{(F,G)}(\hbar,\jmath)=\begin{cases}\Big[a_1(\sigma(\hbar,\jmath))^r+a_2(\sigma(\hbar,F\hbar))^r\\ +a_3(\sigma(\jmath,G\jmath))^r\\ +a_4\left(\frac{\sigma(\jmath,F\hbar)+\sigma(\hbar,G\jmath)}{2\eta}\right)^r\Big]^{\frac{1}{r}}\\ \textit{for } r>0,\ \hbar,\jmath\in\Xi\\ (\sigma(\hbar,\jmath))^{a_1}(\sigma(\hbar,F\hbar))^{a_2}(\sigma(\jmath,G\jmath))^{a_3}\\ \times\left(\frac{\sigma(\hbar,G\jmath)+\sigma(\jmath,F\hbar)}{2\eta}\right)^{a_4}\\ \text{for } r=0,\hbar,\jmath\in\Xi\setminus\mathcal{F}_{i\hbar}(F,G),\end{cases} \tag{17}$$

for which

$$\mathcal{F}_{i\hbar}(F,G)=\{\hbar,\jmath\in\Xi:\hbar\in F\hbar,\jmath\in G\jmath\}.$$

After that, we can find $\flat\in\Xi$ for which $\flat\in F\flat\cap G\flat$.

Proof. Consider a mapping $\Theta:\Xi\longrightarrow(0,1]$ and a pair of Fs-valued maps $P,L:\Xi\longrightarrow I^{\Xi}$ defined by

$$P(\hbar)(t)=\begin{cases}\Theta\hbar, & \text{if } \hbar\in F\hbar\\ 0, & \text{if } \hbar\notin F\hbar.\end{cases}$$

and

$$L(\hbar)(t)=\begin{cases}\Theta\hbar, & \text{if } \hbar\in G\hbar\\ 0, & \text{if } \hbar\notin G\hbar.\end{cases}$$

After that, for $\alpha(\hbar)\in(0,1]$,

$$[P\hbar]_{\alpha(\hbar)}=\{t\in\Xi:P(\hbar)(t)\geq\alpha\}=F\hbar$$

and

$$[L\hbar]_{\alpha(\hbar)}=\{t\in\Xi:L(\hbar)(t)\geq\alpha\}=G\hbar.$$

Hence, Theorem 1 can be applied to obtain $\flat\in\Xi$ for which

$$\flat\in[P\flat]_{\alpha(\flat)}\cap[L\flat]_{\alpha(\flat)}=F\flat\cap G\flat.$$

□

Corollary 10. *Let (Ξ,σ) be a complete MS and $F,G:\Xi\longrightarrow\mathcal{K}(\Xi)$ be multivalued maps. Assume that, for each $\hbar,y\in\Xi$, we can find $\varphi\in\Omega_\varphi$ for which*

$$H(F\hbar,G\jmath)\leq\varphi\Big(\mathcal{C}^{r}_{(F,G)}(\hbar,\jmath)\Big)$$

for which $\varphi\in\Omega_\varphi$, $r\geq0$, $a_i\geq0$, $i=1,2,3,4$ with $\sum_{i=1}^{4}a_i=1$ and

$$\mathcal{C}^r_{(F,G)}(\hbar,\jmath)=\begin{cases}\Big[a_1(\sigma(\hbar,\jmath))^r+a_2(\sigma(\hbar,F\hbar))^r\\ +a_3(\sigma(y,Gy))^r\\ +a_4\left(\frac{\sigma(\jmath,F\hbar)+\sigma(\hbar,G\jmath)}{2}\right)^r\Big]^{\frac{1}{r}}\\ \text{for } r>0,\ \hbar,\jmath\in\Xi\\ (\sigma(\hbar,\jmath))^{a_1}(\sigma(\hbar,F\hbar))^{a_2}(\sigma(\jmath,G\jmath))^{a_3}\\ \times\left(\frac{\sigma(\hbar,G\jmath)+\sigma(\jmath,F\hbar)}{2}\right)^{a_4}\\ \text{for } r=0,\hbar,\jmath\in\Xi\setminus\mathcal{F}_{i\hbar}(F,G),\end{cases} \tag{18}$$

for which

$$\mathcal{F}_{i\hbar}(F,G)=\{\hbar,\jmath\in\Xi:\hbar\in F\hbar,\jmath\in G\jmath,\}.$$

After that, we can find $\flat\in\Xi$ for which $\flat\in F\flat\cap G\flat$.

Proof. Take $\eta=1$ in Corollary 9. □

Corollary 11. *Let (Ξ,σ,η) be a complete b-MS and $F:\Xi\longrightarrow\mathcal{K}(\Xi)$ be a multivalued map. Assume that we can find $\lambda\in[0,1)$ for which*

$$H_b(F\hbar,F\jmath)\le\lambda\sigma(\hbar,\jmath)$$

for each $\hbar,\jmath\in\Xi$. After that, we can find $\flat\in\Xi$ for which $\flat\in F\flat$.

Proof. Set $F=G$, $a_1=1$, $a_2=a_3=a_4=r=0$ and $\varphi(t)=\lambda t,\ t\ge0$ in Corollary 9. □

The following example supports the suppositions of Corollary 11.

Example 6. *Let $\Xi=\left\{1,\frac{1}{4},\frac{1}{5}\right\}$ be equipped with the metric $\sigma(\hbar,\jmath)=|\hbar-\jmath|^2$ for all $\hbar,\jmath\in\Xi$. After that, Ξ is a complete b-MS with parameter $\eta=2$. However, (Ξ,σ) is not a metric; for instance, take $\hbar=\frac{1}{5}$, $\jmath=1$ and $\ell=\frac{1}{4}$. After that,*

$$\sigma(\hbar,\jmath)=\frac{16}{15}>\frac{113}{200}=\sigma(\hbar,\ell)+\sigma(\ell,\jmath).$$

Now, define $F:\Xi\longrightarrow\mathcal{K}(\Xi)$ by

$$F\hbar=\begin{cases}\left\{\frac{1}{5}\right\}, & \text{if } \hbar\in\left\{\frac{1}{5},\frac{1}{4}\right\}\\ \left\{\frac{1}{4},\frac{1}{5}\right\}, & \text{if } \hbar=1.\end{cases}$$

Without loss of generality, let $\hbar\le\jmath$ for all $\hbar,\jmath\in\Xi$. If $\hbar=\jmath$, then $F\hbar=F\jmath$, Hence, $H_b(F\hbar,F\jmath)=0\le\lambda\sigma(\hbar,\jmath)$ for all $\lambda\in[0,1)$. Otherwise, for all $\hbar<\jmath$ (that is, $\hbar\in\left\{\frac{1}{5},\frac{1}{4}\right\}$ and $\jmath=1$), consider the following cases:

Case 1: If $\hbar=\frac{1}{5}$ and $\jmath=1$, then $\sigma(\hbar,\jmath)=\frac{16}{25}$ and

$$H_b(F\hbar,F\jmath)=\frac{1}{400}\le\frac{48}{125}=\frac{3}{5}.\frac{16}{25}=\lambda\sigma(\hbar,\jmath).$$

Case 2: If $\hbar=\frac{1}{4}$ and $\jmath=1$, then $\sigma(\hbar,\jmath)=\frac{9}{16}$ and

$$H_b(F\hbar,F\jmath)=\frac{1}{400}\le\frac{27}{80}=\frac{3}{5}.\frac{9}{16}=\lambda\sigma(\hbar,\jmath).$$

Hence, we infer that all the suppositions of Corollary 11 are obeyed for all $\hbar,\jmath\in\Xi$. Here, $\mathcal{F}_{ix}(F)=\left\{\frac{1}{5},\frac{1}{4}\right\}$.

Corollary 12. *(Nadler [14] Theorem 5) Let (Ξ, σ) be a complete MS and $F : \Xi \longrightarrow \mathcal{K}(\Xi)$ be a multivalued map. Assume that we can find $\lambda \in [0, 1)$ for which*

$$H(F\hbar, F\jmath) \leq \lambda\sigma(\hbar, \jmath)$$

for each $\hbar, \jmath \in \Xi$. After that, we can find $\flat \in \Xi$ for which $\flat \in F\flat$.

Proof. Take $\eta = 1$ in Corollary 11. □

Corollary 13 ([11] Theorem 1). *Let (Ξ, σ, η) be a complete b-MS, and $F : \Xi \longrightarrow \Xi$ be single-valued mapping. Assume that, for each $\hbar, \jmath \in \Xi$, we can find $\varphi \in \Omega_\varphi$ for which*

$$\sigma(F\hbar, F\jmath) \leq \varphi(\mathcal{C}^r_F(\hbar, \jmath))$$

for which $\varphi \in \Omega_\varphi$, $r \geq 0$, $a_i \geq 0$, $i = 1, 2, 3, 4$ with $\sum_{i=1}^4 a_i = 1$ and

$$\mathcal{C}^r_F(\hbar, \jmath) = \begin{cases} \Bigg[a_1(\sigma(\hbar, \jmath))^r + a_2(\sigma(\hbar, F\hbar))^r \\ +a_3(\sigma(\jmath, F\jmath))^r \\ +a_4\left(\frac{\sigma(\jmath, F\hbar) + \sigma(\hbar, F\jmath)}{2\eta}\right)^r \Bigg]^{\frac{1}{r}} \\ \textit{for } \ r > 0, \ \hbar, \jmath \in \Xi \\ (\sigma(\hbar, \jmath))^{a_1}(\sigma(\hbar, F\hbar))^{a_2}(\sigma(\jmath, F\jmath))^{a_3} \\ \times \left(\frac{\sigma(\hbar, F\jmath) + \sigma(\jmath, F\hbar)}{2\eta}\right)^{a_4} \\ \text{for } r = 0, \hbar, \jmath \in \Xi \setminus \mathcal{F}_{ix}(F), \end{cases} \tag{19}$$

for which

$$\mathcal{F}_{ix}(F) = \{\hbar \in \Xi : \hbar = F\hbar\}.$$

After that we can find $\flat \in \Xi$ for which $\flat = F\flat$.

Proof. $\{\hbar\} \in \mathcal{K}(\Xi)$ for every $\hbar \in \Xi$. Consider a mapping $\Upsilon : \Xi \longrightarrow \mathcal{K}(\Xi)$ defined as $\Upsilon x = \{F\hbar\}, \hbar \in \Xi$. After that, all the yardsticks of Corollary 9 are reduced to the yardstick of Corollary 13 with $F = G$ and $H_b(F\hbar, F\jmath) = \sigma(F\hbar, F\jmath)$, for all $\hbar, \jmath \in \Xi$. Thus, by applying Corollary 9, we can find $\flat \in \Xi$ for which $\{\flat\} = \Upsilon\flat$. The definition of Υ implies that $\Upsilon\flat = \{F\flat\}$. Hence, $\flat = F\flat$. □

5. Applications to Fredholm Integral Inclusions

Integral inclusions, the multivalued version of integral equations, play an important role in many fields of applied sciences. Fixed-point results for contractive inequalities are commonly investigated and have had enormous applications in the study of differential inclusions and equations (e.g., see [32,33]).

Recently, Abdou and Ahmad [34] discussed the solution of a Fredholm integral inclusion by availing fixed-point results for F-contraction. Motivated by this result, we applied one of our results to examine adequate yardsticks for the existence of the solutions of a Fredholm Integral inclusion. For the rudiments of the integral inclusions, we refer the interested reader to [21,22,28].

We have

$$\hbar(t) \in \left[g(t) + \int_a^b K(t, s, \hbar(s))\sigma s \right] \tag{20}$$

for $t \in [a, b]$ for which $\hbar \in C([a, b], \mathbb{R})$ is unknown, $g \in C([a, b], \mathbb{R})$ is a provided real-valued function, and $K : [a, b] \times [a, b] \times \mathbb{R} \longrightarrow \digamma_{cv}(\mathbb{R})$ is a provided multivalued map for which we depict the class of nonempty compact and convex subsets of $\mathbb{R}$ by $\mathcal{F}_{cv}(\mathbb{R})$. The set of all real-valued continuous functions on $[a,b]$ are represented by $C([a,b],\mathbb{R})$.

Theorem 2. *Suppose that:*

(C_1) *Multivalued map* $K : [a,b]\times[a,b]\times\mathbb{R} \longrightarrow \mathcal{F}_{cv}(\mathbb{R})$ *is for which for every* $x \in C([a,b],\mathbb{R})$, *map* $K_\hbar := K(t,s,\hbar(s))$ *is lower semicontinuous.*
(C_2) $g \in C([a,b],\mathbb{R})$.
(C_3) *we can find a b-comparison function* $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *for which, for all* $\hbar, \jmath \in C([a,b],\mathbb{R})$,

$$\begin{aligned}&H_b(K_\hbar(t,s),K_\hbar(t,s))\\&\leq\ \pi(t,s)\varphi\Bigg(\frac{1}{4}\Bigg(|\hbar(s)-\jmath(s)|^2+|\hbar-K(t,s,\hbar(s))|^2\\&+|\jmath(s)-K(t,s,\jmath(s))|^2\\&\ +\frac{|\jmath(s)-K(t,s,\hbar(s))|^2+|\hbar(s)-K(t,s,\jmath(s))|^2}{2\eta}\Bigg)\Bigg)\end{aligned}$$

for each $t,s \in [a,b]$, *and*
$\sup\left(\int_a^b \pi(t,s)\right)^2 \sigma s \leq 1$, *for which* $\pi(t,.) \in L^1[a,b]$ *and* $\eta \geq 1$.
After that, Integral Inclusion (20) *has at least one solution in* $C([a,b],\mathbb{R})$.

Proof. Let $\Xi = C([a,b],\mathbb{R})$ and $\sigma : \Xi\times\Xi \longrightarrow \mathbb{R}_+$ be defined by

$$\sigma(\hbar,\jmath) = |\hbar-\jmath|^2, \text{ for all } \hbar,\jmath \in \Xi.$$

After that, (Ξ,σ,η) is a complete b-MS with parameter $\eta = 2$. The Ξ endowed with this metric σ is not a classical MS. Let $P : \Xi \longrightarrow I^\Xi$ be a Fs-valued map. Consider the α-level set of P, defined as follows:

$$[P\hbar]_{\alpha(\hbar)} = \Bigg\{\jmath \in \Xi : \jmath(t) \in g(t) + \int_a^b K(t,s,\hbar(s))\sigma s,\\ t \in [a,b]\Bigg\}.$$

The set of solutions of (20) coincided with the set of fuzzy fp of P. Thus, we must demonstrate that, under the given suppositions, P has at least one fuzzy fp in Ξ. For this, we check that all the suppositions of Corollary 2 are obeyed.

Let $\hbar \in \Xi$ be arbitrary. For multivalued map $K_\hbar : [a,b]\times[a,b] \longrightarrow \mathcal{F}_{cv}(\mathbb{R})$, it follows from Michael's selection theorem (Theorem 1 [35]) that we can find a continuous map $\rho_\hbar : [a,b]\times[a,b] \longrightarrow \mathbb{R}$ for which $\rho_\hbar(t,s) \in K_\hbar(t,s)$, for each $(t,s) \in [a,b]\times[a,b]$. Hence, $g(t) + \int_a^b \rho_\hbar(t,s)\sigma s \in [P\hbar]_{\alpha(\hbar)}$. So, $[P\hbar]_{\alpha(\hbar)}$ is nonempty. One can easily see that $[P\hbar]_{\alpha(\hbar)}$ is a closed subset of Ξ. Further, since $g \in C([a,b])$ and $K_\hbar(t,s)$ is continuous on $[a,b]\times[a,b]$, their range sets are compact. Hence, $[P\hbar]_{\alpha(\hbar)}$ is also compact.

Take $\hbar_1,\hbar_2 \in \Xi$; then, we can find $\alpha(\hbar_1),\alpha(\hbar_2) \in (0,1]$ for which $[P\hbar_1]_{\alpha(\hbar_1)}$ and $[P\hbar_2]_{\alpha(\hbar_2)}$ are nonempty compact subsets of Ξ. Let $\jmath_1 \in [P\hbar_1]_{\alpha(\hbar_1)}$ be arbitrary for which

$$\jmath_1(t) \in g(t) + \int_a^b K(t,s,\hbar_1(s))\sigma s,\ t \in [a,b].$$

This means that, for each $(t,s) \in [a,b]\times[a,b]$, we can find $\rho_{\hbar_1} \in K_{\hbar_1}(t,s)$ for which

$$\jmath_1(t) = g(t) + \int_a^b \rho_{\hbar_1}(t,s)\sigma s,\ t \in [a,b].$$

Since, from (C_2),

$$
\begin{aligned}
& H_b(K(t,s,\hbar_1(s)),K(t,s,\hbar_2(s))) \\
& \le\ \pi(t,s)\varphi\Bigg(\frac{1}{4}\Bigg(|\hbar_1(s)-\hbar_2(s)|^2+|\hbar_1(s)-K(t,s,\hbar_1(s))|^2 \\
& +|\hbar_2(s)-K(t,s,\hbar_2(s))|^2 \\
& \quad +\frac{|\hbar_2(s)-K(t,s,\hbar_1(s))|^2+|\hbar_1-K(t,s,\hbar_2(s))|^2}{2\eta}\Bigg)\Bigg),
\end{aligned}
$$

for each $t,s\in[a,b]$ and $\eta\ge 1$. So, we can find $\gamma(t,s)\in K_{\hbar_2}(t,s)$ for which

$$
\begin{aligned}
& |\rho_{\hbar_1}(t,s)-\gamma(t,s)|^2 \\
& \le\ \pi(t,s)\varphi\Bigg(\frac{1}{4}\Bigg(|\hbar_1(s)-\hbar_2(s)|^2+|\hbar_1(s)-K(t,s,\hbar_1(s))|^2 \\
& +|\hbar_2(s)-K(t,s,\hbar_2(s))|^2 \\
& \quad +\frac{|\hbar_2(s)-K(t,s,\hbar_1(s))|^2+|\hbar_1-K(t,s,\hbar_2(s))|^2}{2\eta}\Bigg)\Bigg),
\end{aligned}
$$

for all $(t,s)\in[a,b]\times[a,b]$. Now, consider the multivalued map $\mathfrak{M}$ defined by

$$
\begin{aligned}
\mathfrak{M}(t,s) = K_{\hbar_2}(t,s)\cap\Bigg\{ & \omega\in\mathbb{R}: \\
& |\rho_{\hbar_1}(t,s)-\omega|^2\le\pi(t,s)\varphi\Bigg(\frac{1}{4}|\hbar_1(s)-\hbar_2(s)|^2\Bigg)\Bigg\}
\end{aligned}
$$

Taking into account the fact that, from (C_1), $\mathfrak{M}$ is lower semicontinuous, we can find a continuous map $\rho_{\hbar_2}:[a,b]\times[a,b]\longrightarrow\mathbb{R}$ for which $\rho_{\hbar_2}(t,s)\in\mathfrak{M}(t,s)$, for all $(t,s)\in[a,b]\times[a,b]$. After that,

$$
\begin{aligned}
\jmath_2(t) &= g(t)+\int_a^b\rho_{\hbar_2}(t,s)\sigma s \\
&\in g(t)+\int_a^b K(t,s,\hbar_2(s))\sigma s,\ t\in[a,b].
\end{aligned}
$$

Thus, $\jmath_2\in[P\hbar_2]_{\alpha(\hbar_2)}$, and

$$
\begin{aligned}
& |\jmath_1(t)-\jmath_2(t)|^2 \\
& \le\ \left(\int_a^b|\rho_{\hbar_1}(t,s)-\rho_{\hbar_2}(t,s)|\sigma s\right)^2 \\
& \le\ \sup\left(\int_a^b\pi(t,s)\sigma s\right)^2 \\
& \times\ \varphi\Bigg(\frac{1}{4}\Bigg(|\hbar_1(s)-\hbar_2(s)|^2 \\
& +|\hbar_1(s)-K(t,s,\hbar_1(s))|^2 \\
& +|\hbar_2(s)-K(t,s,\hbar_2(s))|^2 \\
& +\frac{|\hbar_2(s)-K(t,s,\hbar_1(s))|^2}{2\eta}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{|\hbar_1(s) - K(t,s,\hbar_2(s))|^2}{2\eta}\Bigg)\Bigg) \\
&\leq \varphi\Bigg(\frac{1}{4}\Bigg(|\hbar_1(s) - \hbar_2(s)|^2 \\
&+ |\hbar_1(s) - K(t,s,\hbar_1(s))|^2 \\
&+ |\hbar_2(s) - K(t,s,\hbar_2(s))|^2 \\
&+ \frac{|\hbar_2(s) - K(t,s,\hbar_1(s))|^2}{2\eta} \\
&+ \frac{|\hbar_1(s) - K(t,s,\hbar_2(s))|^2}{2\eta}\Bigg)\Bigg) \\
&\leq \varphi\Bigg(\frac{1}{4}\Bigg(\sigma(\hbar_1,\hbar_2) + \sigma(\hbar_1,[P\hbar_1]_{\alpha(\hbar_1)}) \\
&+ \sigma(\hbar_2,[P\hbar_2]_{\alpha(\hbar_2)}) \\
&+ \frac{\sigma(\hbar_2,[P\hbar_1]_{\alpha(\hbar_1)})}{2\eta} \\
&+ \frac{\sigma(\hbar_1,[P\hbar_2]_{\alpha(\hbar_2)})}{2\eta}\Bigg)\Bigg).
\end{aligned}
$$

Whence, $\sigma(\jmath_1,\jmath_2) \leq \varphi\left(\frac{1}{4}C^r_{(P)}(\hbar_1,\hbar_2)\right)$, that is,

$$H_b\left([P\hbar_1]_{\alpha(\hbar_1)},[P\hbar_2]_{\alpha(\hbar_2)}\right) \leq \varphi\left(\frac{1}{4}C^r_{(P)}(\hbar_1,\hbar_2)\right).$$

Hence, for $\hbar = \hbar_1$ and $\jmath = \hbar_2$,

$$
\begin{aligned}
C^r_P(\hbar,\jmath) =&\sigma(\hbar,\jmath) + \sigma(\hbar,[P\hbar]_{\alpha(\hbar)}) + \sigma(\jmath,[P\jmath]_{\alpha(\jmath)}) \\
&+ \frac{\sigma(\jmath,[P\hbar]_{\alpha(\hbar)}) + \sigma(\hbar,[P\jmath]_{\alpha(\jmath)})}{2\eta},
\end{aligned}
$$

and

$$H_b\left([P\hbar]_{\alpha(\hbar)},[P\jmath]_{\alpha(\jmath)}\right) \leq \varphi\left(\frac{1}{4}C^r_{(P)}(\hbar,\jmath)\right),$$

Hence, all the yardsticks of Corollary 2 are obeyed. So, the conclusion of Theorem 2 consequently holds. □

6. Conclusions

The basic notion of Banach's fp theorem is understood as a modification of the successive approximation method that was initially used by Cauchy, Liouville, Lipschitz, Picard, and Poincaré in the context of classical *MSs*. However, in certain spaces, the triangle inequality cannot be obeyed. However, by taking the product of parameter $\eta \geq 1$ with the right-hand side of the inequality, we can derive a more versatile abstract frame, namely, the *b-MS*. Following this advancement, in this work, the idea of a b-hybrid f_z-contraction was proposed in the setting of *b-MSs*. The results suggest merging several ideas in a theorem. A few of these particular cases are mentioned. We then established an existence theorem for the solutions of an integral inclusion of the Fredholm type by utilizing one of the presented results. The main ideas of this paper, discussed in a fuzzy setting, are fundamental. As possible future work, the paper can be examined in the setting of refined *Fs* such as soft and rough sets, and related domains. In addition, the *b-MS* component of this work can be extended to other dislocated or quasimetric spaces.

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References

1. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund. Math.* **1922**, *3*, 133–181. [CrossRef]
2. Rhoades, B.E. A comparison of various definitions of contractive mappings. *Trans. Am. Math. Soc.* **1977**, *226*, 257–290. [CrossRef]
3. Bakhtin, I.A. The contraction mapping principle in almost metric spaces. *Funct. Anal.* **1989**, *30*, 26–37.
4. Boriceanu, M. Fixed point theory for multivalued generalized contraction on a set with two b-metrics. *Stud. Univ.-Babes-Bolyai Math.* **2009**, *5*, 3–12.
5. Czerwik, S. Contraction mappings in b-metric spaces. *Acta Math. Inform. Univ. Ostrav.* **1993**, *1*, 5–11.
6. Kutbi, M.A.; Karapınar, E.; Ahmad, J.; Azam, A. Some Fixed-point results for multi-valued mappings in b-metric spaces. *J. Inequal. Appl.* **2014**, *2014*, 126. [CrossRef]
7. Rao, K.P.R.; Swamy, P.R.; Prasad, J.R. A common fixed point theorem in complex valued b-metric spaces. *Bul. Math. Stat. Res.* **2013**, *1*, 1-8.
8. Karapinar, E. A Short Survey on the Recent Fixed-point results on b-Metric Spaces. *Constr. Math. Anal.* **2018**, *1*, 15-44. [CrossRef]
9. Ahmad, J.; Al-Mezel, S.A.; Agarwal, R.P. Fixed-point results for Perov–Ćirić–Prešić-Type Θ-Contractions with Applications. *Mathematics* **2022**, *10*, 2062. [CrossRef]
10. Al-Mazrooei, A.E.; Ahmad, J. Fixed-point results in Controlled Metric Spaces with Applications. *Mathematics* **2022**, *10*, 490. [CrossRef]
11. Karapinar, E.; Fulga, A. New Hybrid Contractions on b-Metric Spaces. *Mathematics* **2019**, *7*, 578. [CrossRef]
12. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
13. Heilpern, S. Fuzzy mappings and fixed point theorem. *J. Math. Anal. Appl.* **1981**, *83*, 566–569. [CrossRef]
14. Nadler, S.B. Multi-valued contraction mappings. *Pac. J. Math.* **1969**, *30*, 475–488. [CrossRef]
15. Al-Mazrooei, A.E.; Ahmad, J. Fixed point theorems for fuzzy mappings with applications. *J. Intell. Fuzzy Syst.* **2019**, *36*, 3903–3909. [CrossRef]
16. Azam, A.; Arshad, M.; Beg, I. Fixed points of fuzzy contractive and fuzzy locally contractive maps. *Chaos Solitons Fractals* **2009**, *42*, 2836–2841. [CrossRef]
17. Alansari, M.; Shehu, S.M.; Akbar, A. Fuzzy Fixed-point results in $\mathcal{F}$-metric spaces with applications. *J. Funct. Spaces* **2020**, *2020*, 5142815. [CrossRef]
18. Di Martino, F.; Sessa, S. A Multilevel Fuzzy Transform Method for High Resolution Image Compression. *Axioms* **2022**, *11*, 551. [CrossRef]
19. Qiu, D.; Shu, L. Supremum metric on the space of fuzzy sets and common fixed point theorems for fuzzy mappings. *Inf. Sci.* **2008**, *178*, 3595–3604. [CrossRef]
20. Shehu, S.M.; Akbar, A. An algorithm for fuzzy soft set based decision making approach. *Yugosl. J. Oper. Res.* **2020**, *30*, 59–70. [CrossRef]
21. Appelle, J.; Depascale, E.; Nguyen, H.T.; Zabreiko, P.P. Nonlinear integral inclusions of Hammerstein type. *Topol. Methods Nonlinear Anal.* **1995**, *5*, 111–124. [CrossRef]
22. Aubin, J.P.; Cellina, A. *Differential Inclusions*; Springer: Berlin/Heidelberg, Germany, 1984.
23. Cardinali, T.; Papageorgiou, N. Hammerstein integral inclusions in reflexive Banach spaces. *Proc. Am. Math. Soc.* **1999**, *127*, 95–103. [CrossRef]
24. Kannan, R.; O'Regan, D. A note on the solutions of integral inclusions. *J. Integral Equ. Appl.* **2000**, *12*, 85–94. [CrossRef]
25. Pathak, H.K.; Agarwal, R.P.; Cho, Y.J. Coincidence and fixed points for multi-valued mappings and its application to nonconvex integral inclusions. *J. Comput. Appl. Math.* **2015**, *283*, 201–217.
26. Sîntamarian, A. Integral inclusions of Fredholm type relative to multivalued φ-contractions. *Semin. Fixed Point Theory-Cluj-Napoca* **2002**, *3*, 361–368.
27. Hussain, N.; Doric, D.; Kadelburg, Z.; Radenovic, S. Suzuki-type Fixed-point results in metric type spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 126. [CrossRef]

28. Kaneko, H.; Sessa, S. Fixed point theorems for compatible multi-valued and single-valued mappings. *Int. J. Math. Math. Sci.* **1989**, *12*, 257–262.
29. Czerwik, S. Nonlinear set-valued contraction mappings in *b*-metric spaces. *Atti Sem. Mat. Fis. Univ. Modena* **1998**, *46*, 263-276.
30. Rus, I.A. *Generalized Contractions and Applications*; Cluj University Press: Cluj-Napoca, Romania, 2001.
31. Azam, A.; Mehmood, N. Multivalued fixed point theorems in tvs-cone metric spaces. *Fixed Point Theory Appl.* **2013**, *2013*, 184. [CrossRef]
32. Mohammed, S.S.; Imam, A.T.; Danbaba, U.A.; Yahaya, J.; Oni, M.O.; Tijjani, A.A. Existence of fixed points via C^*-algebra-valued simulation functions with applications. *J. Anal.* **2022**, 1–21. [CrossRef]
33. Monairah, A.; Mohammed, S.S. Analysis of fractional differential inclusion model for COVID-19 via Fixed-point results in metric space. *J. Funct. Spaces* **2022**, *2022*, 8311587. [CrossRef]
34. Abdou, A.A.; Ahmad, J. Solving Fredholm integral inclusion for L-fuzzy mappings. *Int. J. Comput. Math.* **2021**, *98*, 2393–2405. [CrossRef]
35. Michael, E. A selection theorem. *Proc. Am. Math. Soc.* **1966**, *17*, 1404–1406. [CrossRef]