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A Valid Quantization of the Particle in a Box Field Theory, and Well Beyond

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Abstract: The usual particle in a box is turned into a field theory, and its behavior is examined using canonical and affine quantizations. The result leads to a valid affine quantization of the particle in a box field theory, which points toward further valid quantizations of more realistic field theory models.

Keywords: expanding the particle in a box; affine quantization required; a novel replacement of conventional field theory

MSC: 53-02; 99

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1. Introduction

A recent article by the author [1] considered the half-harmonic oscillator field theory, using both canonical and affine quantization procedures. The results of that study led to a valid quantization using affine quantization. In this paper, we follow a similar path to turn the particle in a box model [2] into a field theory, and in this paper, we exploit a novel version of a 'quantum field theory'. Hereafter, we often will use CQ as an abbreviation for canonical quantization along with AQ as an abbreviation for affine quantization.

Such a simple model can also shed information on the family of traditional covariant quantum field theories. Indeed, we will illustrate a novel approach to a set of common covariant quantum field theory known as φ^p_n.

2. Establishing the Classical Story

The classical Hamiltonian is chosen as $H = p^2$, and the range of coordinates is $-b < q < b$, where $0 < b < \infty$. Evidently, the particle must bounce off the two 'walls', located at $q = \pm b$, as it travels back and forth.

2.1. Multiple Identical Particles, Finite and, Formally, Infinite

The next step is to collect many such terms which are completely independent of one another. Now the classical Hamiltonian is $H = \Sigma_{k=1}^{N}\ p_k^2$, while the coordinate space becomes $-b < q_k < b$ for every $k \in (1, 2, 3, \ldots, N)$.

Although nothing is infinitely substantive, even the number of atoms, we prepare our Hamiltonian as done traditionally, to become a spatial field. Since we do not accept this route later, we temporarily introduce that of a single coordinate y (using a to become 0 and $N \to \infty$), and demand that $Na = 100$. Now, we pass to that limit by

$$H_N = \Sigma_{k=1}^{N}\ p_k^2\ a \Rightarrow Na = 100\ ,\ \ H_{N\ \lim_{a\to 0}} \Rightarrow H_c = \int_0^{100} \pi(y)^2\, dy\ , \tag{1}$$

while we now have $-b < \varphi(y) < b$ for all $0 < y < 100$. The equation of motion for this system involves $\pi(y,t) = \dot{\varphi}(y,t) = \pm A(y)$, hence $\varphi(y,t) = \pm A(y)\, t + B(y)$, restricted so that $|\varphi(y,t)| < b$. The field value, $\varphi(y,t)$, oscillates back and forth, bouncing off each of the two 'walls' that are located so that $|\varphi(y)| = b$. (Later, in the grand summation, we lead to several dimensions simply by changing $k \Rightarrow \mathbf{k} = (k_1, k_2, \ldots, k_s)$.)

This can be good math, but it is not a very good match to physics. This statement becomes more evident in the following section.

2.2. A Path Integration Issue

An important part of canonical quantization (CQ) involves path integrations of the classical Hamiltonian in which terms like K (named for the kinetic term)

$$K = \int_0^T \{ \int_0^{100} \pi(y,t)^2 \, dy \} \, dt \; < \infty \, , \tag{2}$$

in which there are two time limits, i.e., $\pi(y,0)$ and $\pi(y,T)$, both of which are fixed. It follows that the range of paths can lead to 'integrable infinities', such as $\pi(y)^2 = R(y)/|y|^{2/3}$, for a harmless $R(y) > 0$, or for $\pi(y,t)^2 = W(y,t)/|T/2 - t|^{2/3}$, also for a harmless factor $W(y,t) > 0$. Such issues complicate conventual quantum field theories using CQ, and now, they can also complicate this 'toy model'.

We continue dealing with integrations and potentially infinite densities, aiming toward alternative formulations.

2.3. A Procedure to Eliminate 'Integrable Infinities'

We start by repeating that $\pi(y)^2$ can reach infinity in value. To eliminate this possibility we introduce the dilation field, $\kappa(y) \equiv \pi(y)\, \varphi(y)$ along with $\varphi(y) \neq 0$. We deliberately remove $\varphi(y) = 0$ so that $\kappa(y)$ should not be forced by $\varphi(y) = 0$ to be zero because that value should be reserved for $\pi(y) = 0$, which, after all, is really $\dot{\varphi}(x,t)$. Likewise, we require that $|\pi(y)| + |\varphi(y)| < \infty$ — because, any one of them being infinite forces another factor to be infinite as well, and then the third one would be helpless — which insures that $|\kappa(y)| < \infty$ as well. Since $\pi(y,t) = \dot{\varphi}(y,t)$, it is acceptable to let $0 \leq [\; |\pi(y,t)| + |\kappa(y,t)| \;] < \infty$, along with $0 < |\varphi(y,t)| < \infty$. The fact that $|\dot{\varphi}(y,t)| < \infty$ ensures that $\varphi(x,t)$ is continuous in time.

Summarizing, the properties outlined above, point to $\pi(y)^2 = \kappa(y)^2/\varphi(y)^2$, which requires that since $|\kappa(y)| < \infty$ and $0 < |\varphi(y)| < \infty$, and now, it follows that $|\pi(y)| < \infty$ ensuring that there are no 'integrable infinities'.

This feature can also apply to discrete summation expressions, prior to passing to their infinite expression for a genuine integration, which then forbids any potential 'integrable infinites' from even being created.

Still dealing with integrations, we can choose our classical Hamiltonian field, using affine variables $\kappa(x)$ and $\varphi(x) \neq 0$, as $H' = \int_0^{100} k(y)^2/\varphi(y)^2 \; dy$, as well as $K' = \int_0^T \{ \int_0^{100} [\kappa(y,t)^2/\varphi(y,t)^2] \; dy \} \; dt$. This formulation avoids any 'integrable infinities' thanks to the rules that, $0 \leq |\kappa(x,t)| < \infty$ and $0 < |\varphi, x, t)| < \infty$, which then ensures that $0 \leq |\pi(x,t)| < \infty$, and all three functions are related correctly.

3. Preparations for the Particle in a Box 'Quantum Field Theory'

3.1. A Conventional CQ Approach to Quantization

The usual field operators are $\hat{\pi}(x)$ and $\hat{\varphi}(x)$ where $x = (x_1, x_2, \ldots, x_s)$. The standard commutation term is $[\hat{\varphi}(x), \hat{\pi}(y)] = i\hbar\, \delta(x-y)\mathbb{1}$, which essentially leads to $\delta \; \varphi(y)/\delta\varphi(x) = \delta(x-y)$. The later term represents Dirac's $\delta(x)$, which is zero if $x \neq 0$ and infinity if $x = 0$, such that $\int_{x<0}^{x>0} \delta(x) \; dx = 1$. This is formally accepted, but it does not fit good math or not seem to fit good physics either. This is because $\lim_{(y \to x)} \delta \; \varphi(y)/\delta\varphi(x) = 0$ while $\delta \; \varphi(x)/\delta\varphi(x) = \infty$. Not only is the second limit not zero, it is infinity. Nature, as represented by physics, does not need to accept discontinuous limits, and should be far more careful with infinities.

The foregoing story was included to stress that nature is composed of atoms not of mathematical points of zero size. After all, we build walls with bricks not points. A deck of cards, which is roughly 1cm tall, and consists of roughly 50 cards, is not composed of infinitely many sheets of zero thickness, etc.

We continue with an effort to make field quantization more physical and far less infinite.

3.2. A Procedure to Simulate Quantum Physics Better

To begin with, every integration technically starts with the limit of a grand summation, such as $\Sigma_{n=1}^{N} f_n\, a \;\Rightarrow_{a\to 0}\Rightarrow\; \int_A^B f(x)\, dx$ provided that $Na = B - A$. For the vast amount of useful integrations, stopping at $a = 10^{-25}$ and $N = (B - A)\, 10^{25}$ would be quite sufficient. Let us introduce a similar game with quantum field operators.

We begin by introducing the symbol $\tau = a^s > 0$ as a substitute for $d^s x$, and $\sum_{\mathbf{k}}^{\mathbf{N}}$, where $\mathbf{k} = (\dots, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \dots)^s$ and $\mathbf{N} = N^s$. We approximate a^s to be an s-dimensional, 'atom sized block', with a 'volume', say $a^s = 10^{-s\times 10}$ meters^s, with 'atom blocks' that stack well together.

Since conventual integration relies on an infinite limit of a vast number of tiny terms, there is no reason not to adapt that procedure for quantum theory, especially for models that are composed of completely independent terms, like our present model, i.e., "The particle in a box, field theory". In that case, we promote each term in the classical Hamiltonian. Using CQ, we are led to the quantum Hamiltonian, given by $\mathcal{H} = \sum_{\mathbf{k}}^{\mathbf{N}} \hat{\tilde{\pi}}(\mathbf{k})^2\, \tau$, where we have renamed our 'discrete atoms', as $P_{\mathbf{k}} = \tilde{\pi}(\mathbf{k})$ and $Q_{\mathbf{k}} = \tilde{\varphi}(\mathbf{k})$. The labels of the quantum variables are now used to become labels of a 'simulated field' (hence the tilde), not by infinitely many 'points', but by a uniform stack of 'atom blocks'. (There are useful procedures that admit coordinate math-like points of quantum fields in which rescaling of the quantum operators is accepted. An example of just that procedure can be seen in [1]. However, in the present paper, we choose a different approach altogether.)

4. An AQ Formulation of the Particle in a Box 'Quantum Field Theory'

The classical Hamiltonian of multiple particles is still $H = \sum_{\mathbf{k}}^{\mathbf{N}} \tilde{\pi}(\mathbf{k})^2\, \tau$. However, now using AQ, for which $\tilde{\kappa}(\mathbf{k}) = \tilde{\pi}(\mathbf{k})\, (b^2 - \tilde{\varphi}(\mathbf{k})^2)$, it follows that

$$\hat{\tilde{\kappa}}(\mathbf{k}) = [\hat{\tilde{\pi}}(\mathbf{k})^{\dagger}\, (b^2 - \hat{\tilde{\varphi}}(\mathbf{k})^2) + (b^2 - \hat{\tilde{\varphi}}(\mathbf{k})^2)\, \hat{\tilde{\pi}}(\mathbf{k})]/2\,, \tag{3}$$

so that

$$\begin{aligned} \mathcal{H}' &= \Sigma_{\mathbf{k}}^{\mathbf{N}} \{\, \hat{\tilde{\kappa}}(\mathbf{k})\, (b^2 - \hat{\tilde{\varphi}}(\mathbf{k})^2)^{-2})\, \hat{\tilde{\kappa}}(\mathbf{k})\, \}\, \tau \\ &= \Sigma_{\mathbf{k}}^{\mathbf{N}} \{\, \hat{\tilde{\pi}}(\mathbf{k})^2 + \hbar^2 [2\hat{\tilde{\varphi}}(\mathbf{k})^2 + b^2]/[\, b^2 - \hat{\tilde{\varphi}}(\mathbf{k})^2\,]^2\, \}\, \tau\,. \end{aligned} \tag{4}$$

This special $\hbar$-term has been taken from the dilation variation study in [3].

In these expressions we choose $\tau = 10^{-s\times 10}$, which pertains to the size of typical atoms in meters, while $\mathbf{N} \times \mathbf{10}^{+\mathbf{s}\times\mathbf{10}}$ to fulfill an approximate value of the integral, in which we let the $\mathbf{N} \times \boldsymbol{\o} = \mathbf{s} \times \mathbf{100}$ while $\tau \to 0$. However, we do **NOT** do that, but instead we leave the 'grand summation' in effect for the simple reason that it carries much more physics than the final integration does.

Since we had numerous, identical, toy models of the particle in a box, their eigenfunctions are identical at every 'block', such as at any $\mathbf{k}$. In that case, eigenfunctions of our grand summation have the form

$$\Psi_n(f) = \Sigma_{\mathbf{k}}^{\mathbf{N}} f(\mathbf{k})\, \psi_n(\mathbf{k})\,, \tag{5}$$

where we require that $\Sigma_{\mathbf{k}}^{\mathbf{N}}\, |f(\mathbf{k})|^2\, \tau < \infty$, and where $n = 0, 1, 2, 3, \dots$ denotes the ordering of the individual eigenvalues by their increasing magnitude. At the present time, the basic AQ quantization of a particle in a box, i.e.,

$$\hbar^2\, [-(d^2/dx^2) + [2x^2 + b^2]/[b^2 - x^2]^2\, \psi_n(x) = E_n\, \psi_n(x)\,, \tag{6}$$

has not offered any solution to the eigenfunctions or eigenvalues. Very likely, the solutions are partially of the form $\psi_n(x) = (b^2 - x^2)^{3/2}\, (remainder_n)$. Any further information about the solutions of this differential equation would be welcome.

5. Extending the Particle in a Box 'Field Theory' That Leads to a Novel Version of Quantum Field Theory

Let us suppose that we accept AQ and its solution for a single item. Suppose we let b become huge, and that we add terms to the potential so that the new quantum Hamiltonian (restoring the usual factor $1/2$) takes the form,

$$\mathcal{H} = \Sigma_{\mathbf{k}}^{\mathbf{N}} \tfrac{1}{2} [\hat{\pi}(\mathbf{k})^2 + \hbar^2 (2\hat{\varphi}(\mathbf{k})^2 + b^2)/(b^2 - \hat{\varphi}(\mathbf{k})^2)^2 + m^2\, \hat{\varphi}(\mathbf{k})^2] + g\, \hat{\varphi}(\mathbf{k})^r\, \tau\,. \tag{7}$$

Next, we add a new term, like gradients $\Sigma_{\mathbf{k}^*}(\tilde{\varphi}(\mathbf{k}^*) - \tilde{\varphi}(\mathbf{k})^2/2\, a^2$, that represents a proper covariant term. Now, as $b \to \infty$, it leads to a familiar form for the standard expression, which leads CQ to a grand summation formulation.

Some Monte Carlo Results for the Model φ_4^4

We note that the field model, φ_4^4, using both CQ and AQ, has been involved with several Monte Carlo studies. (Note that Monte Carlo, in affect, works a lot like a 'grand summation'.)

The studies that used CQ have led to 'unacceptable results', namely, as if the interaction term was absent when, in fact, it was active [4–7] . However, adopting AQ, and using the same Monte Carlo procedures, has found 'acceptable results', by finding notable behavior from the interaction term [8,9].

Let us examine that good behavior using an AQ formulation, and by introducing a novel procedure.

6. The Introduction of the 'Anti-Box'

Our present 'box' has been created by removing the coordinate points at $q = \pm b$, and then keeping the region $|q| < b$, while discarding $|q| > b$. For our 'anti-box', the 'only change made', is that now we *keep* $|q| > b$ and *discard* $|q| < b$. To deal with large q-values, we add potential terms, such as $m^2 q^2$ and $g\, q^r$.

Jumping way ahead, and again restoring the usual factor of $1/2$ to the Hamiltonian, we introduce the grand summation for the classical Hamiltonian, given (with AB $\Rightarrow$ 'Anti-Box') by

$$H_{AB} = \Sigma_{\mathbf{k}}^{\mathbf{N}} \{ \tfrac{1}{2} [\tilde{\pi}(\mathbf{k})^2 + m^2\, \tilde{\varphi}(\mathbf{k})^2] + g\, \tilde{\varphi}(\mathbf{k})^r \}\, \tau\,, \tag{8}$$

which becomes the CQ quantum Hamiltonian

$$\mathcal{H}_{CQ-AB} = \Sigma_{\mathbf{k}}^{\mathbf{N}} \{ \tfrac{1}{2} [\hat{\pi}(\mathbf{k})^2 + m^2\, \hat{\varphi}(\mathbf{k})^2] + g\, \hat{\varphi}(\mathbf{k})^r \}\, \tau\,. \tag{9}$$

Now switching from CQ to AQ, the AQ quantum Hamiltonian is given by

$$\mathcal{H}'_{AQ-AB} = \Sigma_{\mathbf{k}}^{\mathbf{N}} \{ \tfrac{1}{2} [\hat{\pi}(\mathbf{k})^2 + \hbar^2\, (2\hat{\varphi}(\mathbf{k})^2 + b^2)/(b^2 - \hat{\varphi}(\mathbf{k})^2)^2 + m^2\, \hat{\varphi}(\mathbf{k})^2] + g\, \hat{\varphi}(\mathbf{k})^r \}\, \tau\,. \tag{10}$$

In this case, since we are in the 'Anti-Box' realm, and where $0 < b < \infty$, we do not let $b \to \infty$ again, but this time, we let $b \to 0$, which leads to a nearly complete coordinate space for which $-\infty < q \neq 0 < \infty$. This important fact leads to an AQ form of the quantization with the new coordinate space, which is given (now using AB0 $\Rightarrow$'Anti-Box-$b \to 0$') by

$$\mathcal{H}'_{AQ-AB0} = \Sigma_{\mathbf{k}}^{\mathbf{N}} \{ \tfrac{1}{2} [\hat{\pi}(\mathbf{k})^2 + 2\hbar^2/\hat{\varphi}(\mathbf{k})^2 + m^2\, \hat{\varphi}(\mathbf{k})^2] + g\, \hat{\varphi}(\mathbf{k})^r \}\, \tau\,. \tag{11}$$

6.1. Rotational Invariance

To date, we have focused on a single coordinate space rather than a multi-dimensional coordinate space. Our location of singular points can be realigned to independent linear paths that all cross over a fixed single point, which ultimately becomes the single point

that is removed from the entire space. As fully independent paths, the separate lines are made up from individual contributions. In that case our AQ quantum Hamiltonian would become

$$\mathcal{H}''_{AQ-AB0} = \Sigma_{\mathbf{k}}^{\mathbf{N}} \left\{ \tfrac{1}{2} \left[\hat{\tilde{\pi}}(\mathbf{k})^2 + 2\hbar^2 / \hat{\tilde{\varphi}}(\mathbf{k})^2 + m^2 \hat{\tilde{\varphi}}(\mathbf{k})^2 \right] + g \hat{\tilde{\varphi}}(\mathbf{k})^r \right\} \tau \,. \tag{12}$$

The vector in this equation refers to the spatial coordinates, and it has multiple values in which we can let $\mathbf{k} = (k_1, k_2, \ldots, k_s)$.

6.2. A Note Regarding Two ħ-Factors in AQ

When the active coordinate space is only $q > 0$, it has been well determined [10] that the $\hbar$-term, i.e., the $(3/4)\hbar^2$ factor, is correct. However, we now require a larger linear coordinate space, where $|q| > 0$, and there are both positive and negative contributions present. Classical mechanics allows one to just join the behavior of $q > 0$ and $q < 0$ to get the right physics for $|q| > 0$. However, that is definitely *not* good physics for quantum mechanics.

All of the present AQ Monte Carlo studies have used the $(3/4)\hbar^2$ factor while adopting both positive and negative field variables. Useful data was observed that proved that the AQ results were different from the CQ results. However, raising the factor term from 3/4 to 2, which now is proposed to correctly represent the fact that positive and negative fields need a different $\hbar$-factor. Accepting the lager $\hbar$-factor can only enhance the AQ results! (The expression of eigenfunctions for 'the particle in a box' are of the form $\zeta_n(x) = (b^2 - x^2)^{3/2}(remainder_n)$, which becomes, partially at least, an eigenfunction for (11). Now, after $b \to 0$, this changes the previous expression to become $\psi_n(x) = x^3(remainder'_n)$. This last expression enforces continuity for each eigenfunction and its first two derivatives in the region around $x = 0$).

7. Summary

We have adopted 'The particle in a box', a common toy model for beginners [1], which is now promoted to a 'field theory' that is also focussed on a 'Grand Summation' that contains a colossal, but finite, number of our 'atom sized blocks' rather than pass to the continuum limit of an integral. This formulation has been adopted after choosing a common size of atoms, which is truly tiny, but not zero. The procedure of skipping an integral to choose one of its colossal grand summations, has the virtue of none of the mathematical problems faced by conventual quantum field theory. After all, our physical world is composed of atoms and not by zero dimensional, infinitely many, mathematical points. As noted before, a grand summation has far more physics pertaining to quantum theory than the aspects inferred by mathematical integration.

The question may be asked what about wave functions involving integrations and infinities. The coordinate x is an item of the classical realm, and it has been welcomed to be used in quantum theory. The single operator pair P and Q, where, $Q|x\rangle = x|x\rangle$, and for which $[Q, P] = i\hbar\,\mathbb{1}$, offer wave functions, $\psi(x) = \langle x|\psi\rangle$, provided that $\int |\psi(x)|^2\, dx < \infty$.

Observe that we accept wave functions, such as $\psi(x)$, and do **NOT** treat them to our 'multiple atom blocks' approach. This is because wave function expressions, and their integrated results, are *not* part of nature. Instead, they represent *probabilistic features*. This feature appears as $\int_A^B |\psi(x)|^2\, d^s x$, which asserts that the location probability of their particular particle being found in the interval $B - A$, provided, of course, that $\int_{-\infty}^{\infty} |\psi(x)|^2\, d^s x = 1$, which implies 100% certainty.

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