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Approximating Solutions of Optimization Problems via Fixed Point Techniques in Geodesic Spaces

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Abstract: The principal objective of this paper is to find the solution to a constrained minimization problem and the zero of the monotone operator in geodesic spaces. The basic tool in our study is a nonexpansive mapping. Further, we employ the general Picard–Mann iterative method to approximate fixed points of nonexpansive mappings under various conditions. We obtain certain theorems concerning Δ and strong convergence.

Keywords: nonexpansive mapping; monotone operator; geodesic space

MSC: 47H10; 47H09; 47H05



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1. Introduction

Finding a minimizer of a function defined on a Hilbert space, which is convex lower semicontinuous (in short, lsc), is an important problem in optimization theory. See the early works of Martinet [1], Rockafellar [2], and Brézis and Lions [3]. Further, monotone operator theory is a pivotal topic in nonlinear analysis (in particular, convex analysis). More precisely, a zero of a monotone operator is a solution of a variational inequality problem governed by the monotone operator and an equilibrium point of an evolution equation.

Gradually, many of the iterative methods for solving optimization problems have been generalized from linear spaces (Banach, Hilbert, and Euclidean) into nonlinear spaces (Riemannian manifolds and geodesic metric spaces of nonpositive curvature), see [4–7]. For instance, Bačák [4] generalized the proximal point method (in terms of the Moreau–Yosida resolvent) from Hilbert spaces to geodesic spaces. Further, Khatibzadeh and Ranjbar [6] used the duality theory and considered monotone operators and their resolvents in Hadamard spaces. For the most recent results dealing with monotone operators, see [8,9] and references therein.

On the other hand, nonexpansive mappings are those that have a Lipschitz constant equal to one. This class of mappings need not admit a fixed point in a complete space. For the results ensuring the fixed point of nonexpansive mappings in Banach spaces, see the early works of Browder [10], Göhde [11], and Kirk [12]. This class of mappings has a strong connection with transition operators for initial value problems (of differential inclusion), monotone operators, accretive operators, equilibrium problems, and variational inequality problems. Takahashi [13] endowed the metric space with a convex structure and obtained theorems concerning the existence of a fixed point of nonexpansive mappings. Goebel and Kirk [14] considered the Krasnosel'skiĭ–Mann iterative method to approximate fixed points of nonexpansive mappings in nonlinear spaces. Over the last few years, a number of papers have been published dealing with the important fixed point results in the setting of geodesic spaces, see [15–24]. Indeed, Ariza-Ruiz et al. [18] generalized some well known theorems on firmly nonexpansive mappings (even asymptotic behaviour of Picard iterative method) in linear spaces to geodesic spaces. Leuştean [16] extended celebrated fixed point theory results in geodesic spaces, for example, the monotone modulus of uniform convexity, asymptotic centers, and the asymptotic regularity for the Ishikawa iterative method.

Motivated by the above developments, we approximate fixed points of nonexpansive mappings in nonlinear spaces (in particular, geodesic spaces). We extend the general Picard–Mann iterative method from Banach spaces to geodesic spaces and obtain Δ and strong convergence theorems under certain assumptions. Thereafter, we use these findings to obtain the solution of a constrained minimization problem and the zero of the monotone operator. Our results generalize, extend, and complement several results from [7,25].

2. Preliminaries

Let (Y, Γ) be a metric space, and $[0, 1] \subset \mathbb{R}$. Given a pair of points $\sigma, \varsigma \in Y$, a path $\xi : [0, 1] \rightarrow Y$ joins σ and ς if $\xi(0) = \sigma$ and $\xi(1) = \varsigma$. A path ξ is called a geodesic if

$$\Gamma(\xi(s), \xi(t)) = \Gamma(\xi(0), \xi(1))|s - t|, \text{ for all } s, t \in [0, 1].$$

A given metric space (Y, Γ) is called a geodesic space if any pair of points $\sigma, \varsigma \in Y$ are connected by a geodesic. The geodesic segment joining σ and ς is not necessarily unique. The following precise formulation of hyperbolic spaces was introduced by Kohlenbach [26].

Definition 1 ([26]). A triplet (Y, Γ, W) is called a hyperbolic metric space (or W -hyperbolic space) if (Y, Γ) is a metric space, and the function $W : Y \times Y \times [0, 1] \rightarrow Y$ satisfies the following conditions for all $\sigma, \varsigma, z, w \in Y$ and $\mu, \theta \in [0, 1]$

$$(W1) \Gamma(z, W(\sigma, \varsigma, \mu)) \leq (1 - \mu)\Gamma(z, \sigma) + \mu\Gamma(z, \varsigma);$$

$$(W2) \Gamma(W(\sigma, \varsigma, \mu), W(\sigma, \varsigma, \theta)) = |\mu - \theta|\Gamma(\sigma, \varsigma);$$

$$(W3) W(\sigma, \varsigma, \mu) = W(\varsigma, \sigma, 1 - \mu);$$

$$(W4) \Gamma(W(\sigma, z, \mu), W(\varsigma, w, \mu)) \leq (1 - \mu)\Gamma(\sigma, \varsigma) + \mu\Gamma(z, w).$$

Any Busemann space is uniquely geodesic; that is, for any pair of points $\sigma, \varsigma \in Y$, there exists a unique geodesic segment that joins σ and ς , see [27]. The following spaces are some well-known examples of W -hyperbolic spaces: all normed spaces, Hadamard manifolds, the CAT(0)-spaces, and the Hilbert open unit ball equipped with the hyperbolic metric (cf. [18,26]).

Remark 1. If $W(\sigma, \varsigma, \mu) = (1 - \mu)\sigma + \mu\varsigma$ for all $\sigma, \varsigma \in Y, \mu \in [0, 1]$, then it follows that all normed linear spaces are W -hyperbolic spaces.

We shall write

$$W(\sigma, \varsigma, \mu) := (1 - \mu)\sigma \oplus \mu\varsigma$$

to denote a point $W(\sigma, \varsigma, \mu)$ in a W -hyperbolic space. For $\sigma, \varsigma \in Y$, we denote

$$[\sigma, \varsigma] = \{(1 - \mu)\sigma \oplus \mu\varsigma : \mu \in [0, 1]\}$$

as a geodesic segment. A nonempty subset $\mathcal{C}$ of W -hyperbolic space (Y, Γ, W) is said to be convex if $[\sigma, \varsigma] \subset \mathcal{C}$ for all $\sigma, \varsigma \in \mathcal{C}$.

Definition 2 ([19]). A W hyperbolic space (Y, Γ) is uniformly convex (in short, UCW-hyperbolic space) if for $\varepsilon \in (0, 2]$ and any $t > 0$, there exists a $\delta \in (0, 1]$ in such a way that

$$\left. \begin{array}{l} \Gamma(\sigma, z) \leq t \\ \Gamma(\varsigma, z) \leq t \\ \Gamma(\sigma, \varsigma) \geq \varepsilon t \end{array} \right\} \Rightarrow \Gamma\left(\frac{1}{2}\sigma \oplus \frac{1}{2}\varsigma, z\right) \leq (1 - \delta)t$$

for all $\sigma, \varsigma, z \in Y$.

Remark 2. Leuştean [16] proved that complete CAT(0) spaces are complete uniformly convex hyperbolic spaces (or UCW-hyperbolic spaces).

Let $\{\sigma_n\}$ be a bounded sequence in a hyperbolic space (Y, Γ, W) and $\mathcal{C}$ a nonempty subset of Y . A functional $r(\cdot, \{\sigma_n\}) : Y \rightarrow [0, +\infty)$ can be defined as follows:

$$r(\zeta, \{\sigma_n\}) = \limsup_{n \rightarrow +\infty} \Gamma(\zeta, \sigma_n).$$

The asymptotic radius of $\{\sigma_n\}$ with respect to $\mathcal{C}$ is described as

$$r(\mathcal{C}, \{\sigma_n\}) = \inf\{r(\zeta, \{\sigma_n\}) : \zeta \in \mathcal{C}\}.$$

A point σ in $\mathcal{C}$ is called as an asymptotic center of $\{\sigma_n\}$ with respect to $\mathcal{C}$ if

$$r(\sigma, \{\sigma_n\}) = r(\mathcal{C}, \{\sigma_n\}).$$

$A(\mathcal{C}, \{\sigma_n\})$ is denoted as the set of all asymptotic centers of $\{\sigma_n\}$ with respect to $\mathcal{C}$.

Definition 3 ([28]). Let $\{\sigma_n\}$ be a bounded sequence in a W hyperbolic space (Y, Γ) . The sequence $\{\sigma_n\}$ Δ -converges to σ if σ is the unique asymptotic center for every subsequence $\{\rho_n\}$ of $\{\sigma_n\}$.

Let (Y, Γ) be a W hyperbolic space and $\mathcal{C} \subset Y$ such that $\mathcal{C} \neq \emptyset$. A sequence $\{\sigma_n\}$ in Y is said to be Fejér monotone with respect to $\mathcal{C}$ if

$$\Gamma(\sigma^+, \sigma_{n+1}) \leq \Gamma(\sigma^+, \sigma_n), \text{ for all } n \geq 0, \text{ for all } \sigma^+ \in \mathcal{C}.$$

A mapping $\Psi : Y \rightarrow Y$ is nonexpansive if $\Gamma(\Psi(\sigma), \Psi(\zeta)) \leq \Gamma(\sigma, \zeta)$, for all $\sigma, \zeta \in Y$. We denote $F(\Psi) := \{\sigma \in Y : \Psi(\sigma) = \sigma\}$.

Definition 4 ([29]). A mapping $\Psi : \mathcal{C} \rightarrow \mathcal{C}$ with $F(\Psi) \neq \emptyset$ satisfies Condition (I), if there exists a function $f : [0, +\infty) \rightarrow [0, +\infty)$ with the following conditions:

1. $f(r) > 0$ for $r \in (0, +\infty)$ and $f(0) = 0$.
2. $\Gamma(\sigma, \Psi(\sigma)) \geq f(\Gamma(\sigma, F(\Psi)))$ for all $\sigma \in \mathcal{C}$,

where $\Gamma(\sigma, F(\Psi)) = \inf\{\Gamma(\sigma, \zeta) : \zeta \in F(\Psi)\}$.

Definition 5. Let (Y, Γ) be a metric space, and $\mathcal{C} \subset Y$ such that $\mathcal{C} \neq \emptyset$. A mapping $\Psi : \mathcal{C} \rightarrow \mathcal{C}$ is compact if $\Psi(\mathcal{C})$ has a compact closure.

Proposition 1 ([16]). Let (Y, Γ, W) be a complete UCW-hyperbolic space, $\mathcal{C} \subset Y$ such that $\mathcal{C} \neq \emptyset$, and $\mathcal{C}$ be closed convex. If a sequence $\{\sigma_n\}$ in Y is bounded, then $\{\sigma_n\}$ has a unique asymptotic center with respect to $\mathcal{C}$.

Lemma 1 ([16]). Let $\{\sigma_n\}$ be a bounded sequence in (Y, Γ, W) , and $A(\mathcal{C}, \{\sigma_n\}) = \{z\}$. Let $\{r_n\}$ and $\{s_n\}$ be two sequences in $\mathbb{R}$ such that $r_n \in [0, +\infty)$ for all $n \in \mathbb{N}$, $\limsup r_n \leq 1$ and $\limsup s_n \leq 0$. Suppose that $\zeta \in \mathcal{C}$, and there exist $m, N \in \mathbb{N}$ such that

$$\Gamma(\zeta, \sigma_{n+m}) \leq r_n \Gamma(z, \sigma_n) + s_n, \text{ for all } n \geq N.$$

Then, $\zeta = z$.

Lemma 2 ([18]). Let (Y, Γ, W) be a W -hyperbolic space, $\mathcal{C} \subset Y$ such that $\mathcal{C} \neq \emptyset$. If $\{\sigma_n\}$ is Fejér monotone with respect to $\mathcal{C}$, $A(\mathcal{C}, \{\sigma_n\}) = \{\sigma\}$, and $A(Y, \{\rho_n\}) \subseteq \mathcal{C}$ for every subsequence $\{\rho_n\}$ of $\{\sigma_n\}$. Then, the sequence $\{\sigma_n\}$ Δ -converges to $\sigma \in \mathcal{C}$.

The following Lemma is motivated by [16] (Lemma 2.1):

Lemma 3 ([17]). Let (Y, Γ, W) be a complete UCW-hyperbolic space. Let $w \in Y$ and $\{\alpha_n\}$ be a sequence such that $0 < \alpha \leq \alpha_n \leq \beta < 1$. If $\{\sigma_n\}$ and $\{\zeta_n\}$ are sequences in Y such that for some $r \geq 0$, we have

$$\limsup_{n \rightarrow +\infty} \Gamma(\sigma_n, w) \leq r, \quad \limsup_{n \rightarrow +\infty} \Gamma(\zeta_n, w) \leq r \quad \text{and} \quad \lim_{n \rightarrow +\infty} \Gamma(\alpha_n \zeta_n \oplus (1 - \alpha_n) \sigma_n, w) = r.$$

Then, $\lim_{n \rightarrow +\infty} \Gamma(\zeta_n, \sigma_n) = 0$.

3. Main Results

In [25], Shukla et al. considered the following iterative method (known as GPM). Let $\mathcal{B}$ be a Banach space and $\mathcal{C} \subset \mathcal{B}$ such that $\mathcal{C} \neq \emptyset$, and $\mathcal{C}$ is convex. Let $\Psi : \mathcal{C} \rightarrow \mathcal{C}$ be a mapping.

$$\begin{cases} \sigma_1 = \sigma \in \mathcal{C} \\ \sigma_{n+1} = \Psi^k\{(1 - \alpha_n)\sigma_n + \alpha_n \Psi(\sigma_n)\}, \quad n \in \mathbb{N}, \end{cases} \quad (1)$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$, and k is a fixed natural number.

In the setting of geodesic space, the above method can be defined. Let (Y, Γ, W) be a W -hyperbolic space and $\mathcal{C} \subset Y$ such that $\mathcal{C} \neq \emptyset$, and $\mathcal{C}$ is convex. Let $\Psi : \mathcal{C} \rightarrow \mathcal{C}$ be a mapping.

$$\begin{cases} \sigma_1 = \sigma \in \mathcal{C} \\ \sigma_{n+1} = \Psi^k\{(1 - \alpha_n)\sigma_n \oplus \alpha_n \Psi(\sigma_n)\}, \quad n \in \mathbb{N}, \end{cases} \quad (2)$$

where k is a fixed natural number, and $\{\alpha_n\}$ is a sequence in $[0, 1]$.

Lemma 4. Let (Y, Γ, W) be a complete UCW-hyperbolic space and $\mathcal{C} \subset Y$ such that $\mathcal{C} \neq \emptyset$, $\mathcal{C}$ be closed convex. Let $\Psi : \mathcal{C} \rightarrow \mathcal{C}$ be a nonexpansive mapping with $F(\Psi) \neq \emptyset$. For a given $\sigma_1 \in \mathcal{C}$ and $\alpha_n \in [\alpha, \beta]$ with $\alpha, \beta \in (0, 1)$, the sequence $\{\sigma_n\}$ is defined by (2). Then, the following results hold.

- (1) $\lim_{n \rightarrow +\infty} \Gamma(\sigma_n, \sigma^\dagger)$ exists for all $\sigma^\dagger \in F(\Psi)$.
- (2) $\lim_{n \rightarrow +\infty} \Gamma(\sigma_n, \Psi(\sigma_n)) = 0$.

Proof. Let $\sigma^\dagger \in F(\Psi)$ and from (W1), we have

$$\begin{aligned} \Gamma(\sigma_{n+1}, \sigma^\dagger) &= \Gamma(\Psi^k\{(1 - \alpha_n)\sigma_n \oplus \alpha_n \Psi(\sigma_n)\}, \sigma^\dagger) \\ &\leq \Gamma((1 - \alpha_n)\sigma_n \oplus \alpha_n \Psi(\sigma_n), \sigma^\dagger) \\ &\leq (1 - \alpha_n)\Gamma(\sigma_n, \sigma^\dagger) + \alpha_n \Gamma(\Psi(\sigma_n), \sigma^\dagger) \\ &\leq (1 - \alpha_n)\Gamma(\sigma_n, \sigma^\dagger) + \alpha_n \Gamma(\sigma_n, \sigma^\dagger) \\ &= \Gamma(\sigma_n, \sigma^\dagger). \end{aligned}$$

Thus, the sequence $\{\Gamma(\sigma_n, \sigma^\dagger)\}$ is monotone nonincreasing. Hence, $\lim_{n \rightarrow +\infty} \Gamma(\sigma_n, \sigma^\dagger)$ exists. Let

$$\lim_{n \rightarrow +\infty} \Gamma(\sigma_n, \sigma^\dagger) = h > 0. \quad (3)$$

By the nonexpansiveness of Ψ ,

$$\lim_{n \rightarrow +\infty} \Gamma(\Psi(\sigma_n), \sigma^\dagger) \leq h. \quad (4)$$

From (3),

$$\begin{aligned} h &= \lim_{n \rightarrow +\infty} \Gamma(\sigma_{n+1}, \sigma^\dagger) = \limsup_{n \rightarrow +\infty} \Gamma(\Psi^k\{(1 - \alpha_n)\sigma_n \oplus \alpha_n\Psi(\sigma_n)\}, \sigma^\dagger) \\ &\leq \limsup_{n \rightarrow +\infty} \Gamma((1 - \alpha_n)\sigma_n \oplus \alpha_n\Psi(\sigma_n), \sigma^\dagger) \\ &\leq \lim_{n \rightarrow +\infty} \Gamma(\sigma_n, \sigma^\dagger) = h. \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow +\infty} \Gamma((1 - \alpha_n)\sigma_n \oplus \alpha_n\Psi(\sigma_n), \sigma^\dagger) = h. \quad (5)$$

From (3)–(5) and Lemma 3,

$$\lim_{n \rightarrow +\infty} \Gamma(\sigma_n, \Psi(\sigma_n)) = 0. \quad (6)$$

□

Theorem 1. Let Y , C , Ψ , and $\{\sigma_n\}$ be the same as in Lemma 4. Then, the sequence $\{\sigma_n\}$ Δ -converges to a point in $F(\Psi)$.

Proof. In view of Lemma 4, the sequence $\{\Gamma(\sigma_n, z^\dagger)\}$ is monotone nonincreasing for all $z^\dagger \in F(\Psi)$. The sequence $\{\sigma_n\}$ is Fejér monotone with respect to $F(\Psi)$. It is noted that $F(\Psi)$ is closed and convex [18]. From Proposition 1, the sequence $\{\sigma_n\}$ has a unique asymptotic center $w^\dagger$ with respect to $F(\Psi)$. Suppose $\{\rho_n\}$ is a subsequence of $\{\sigma_n\}$, then from Proposition 1, $\{\rho_n\}$ has a unique asymptotic center $\rho^\dagger$ with respect to $F(\Psi)$. Now,

$$\begin{aligned} \Gamma(\rho_n, \Psi(\rho^\dagger)) &\leq \Gamma(\Psi(\rho_n), \Psi(\rho^\dagger)) + \Gamma(\Psi(\rho_n), \rho_n) \\ &\leq \Gamma(\rho_n, \rho^\dagger) + \Gamma(\Psi(\rho_n), \rho_n) \end{aligned}$$

From (6) and Lemma 1, it follows that $\Psi(\rho^\dagger) = \rho^\dagger$. From Lemma 2, the sequence $\{\sigma_n\}$ Δ -converges to a point in $F(\Psi)$. □

Theorem 2. Let Y , C , Ψ , and $\{\sigma_n\}$ be same as in Lemma 4. If the mapping Ψ has condition (I), then the sequence $\{\sigma_n\}$ strongly converges to a point in $F(\Psi)$.

Proof. By Lemma 4, the sequences $\{\Gamma(\sigma_n, z^\dagger)\}$ are monotone nonincreasing for all $z^\dagger \in F(\Psi)$. Thus, the sequence $\{\Gamma(\sigma_n, F(\Psi))\}$ is monotone nonincreasing. Hence, $\lim_{n \rightarrow +\infty} \Gamma(\sigma_n, F(\Psi))$ exist. From Lemma 4,

$$\lim_{n \rightarrow +\infty} \Gamma(\sigma_n, \Psi(\sigma_n)) = 0. \quad (7)$$

Since Ψ satisfies condition (I),

$$\Gamma(\sigma_n, \Psi(\sigma_n)) \geq f(\Gamma(\sigma_n, F(\Psi))).$$

From (7), $\lim_{n \rightarrow +\infty} f(\Gamma(\sigma_n, F(\Psi))) = 0$, and

$$\lim_{n \rightarrow +\infty} \Gamma(\sigma_n, F(\Psi)) = 0. \quad (8)$$

Now, one can verify that the sequence $\{\sigma_n\}$ is Cauchy. For a given $\varepsilon > 0$, from (8), there exists a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\Gamma(\sigma_n, F(\Psi)) < \frac{\varepsilon}{4}$$

and

$$\inf\{\Gamma(\sigma_{n_0}, z^\dagger) : z^\dagger \in F(\Psi)\} < \frac{\varepsilon}{4},$$

and there exists $z^\dagger \in F(\Psi)$ such that

$$\Gamma(\sigma_{n_0}, z^\dagger) < \frac{\varepsilon}{2}.$$

Therefore, for all $m, n \geq n_0$,

$$\Gamma(\sigma_{n+m}, \sigma_n) \leq \Gamma(\sigma_{n+m}, z^\dagger) + \Gamma(z^\dagger, \sigma_n) \leq 2\Gamma(\sigma_{n_0}, z^\dagger) < 2 \cdot \frac{\varepsilon}{2} = \varepsilon,$$

and the sequence $\{\sigma_n\}$ is Cauchy. By the closedness of the set $\mathcal{C}$ of Y , the sequence $\{\sigma_n\}$ converges to a point $\sigma^\dagger \in \mathcal{C}$. Now,

$$\begin{aligned} \Gamma(\sigma^\dagger, \Psi(\sigma^\dagger)) &\leq \Gamma(\sigma^\dagger, \sigma_n) + \Gamma(\sigma_n, \Psi(\sigma_n)) + \Gamma(\Psi(\sigma_n), \Psi(\sigma^\dagger)) \\ &\leq 2\Gamma(\sigma^\dagger, \sigma_n) + \Gamma(\sigma_n, \Psi(\sigma_n)) \end{aligned}$$

from (7), $\sigma^\dagger = \Psi(\sigma^\dagger)$. Thus, the sequence $\{\sigma_n\}$ strongly converges to a point in $F(\Psi)$. $\square$

Remark 3. Theorem 2 is an immediate generalization of [25] (Theorem 5) from the setting of Banach spaces to hyperbolic spaces.

Theorem 3. Let (Y, Γ, W) be a complete UCW-hyperbolic space. Let $\mathcal{C}$, Ψ , and $\{\sigma_n\}$ be the same as in Lemma 4. If Ψ is a compact mapping, then the sequence $\{\sigma_n\}$ strongly converges to a point in $F(\Psi)$.

Proof. In view of Lemma 4, the sequence $\{\sigma_n\}$ is bounded. From Lemma 4,

$$\lim_{n \rightarrow +\infty} \Gamma(\sigma_n, \Psi(\sigma_n)) = 0. \quad (9)$$

From the definition of compact mapping, the range of $\mathcal{C}$ under Ψ is contained in a compact set. Therefore, there is a subsequence $\{\Psi(\sigma_{n_j})\}$ of $\{\Psi(\sigma_n)\}$ that strongly converges to $\sigma^\dagger \in \mathcal{C}$. In view of (9), this implies that the subsequence $\{\sigma_{n_j}\}$ strongly converges to $\sigma^\dagger$. Mapping Ψ is nonexpansive, and by the triangle inequality,

$$\begin{aligned} \Gamma(\sigma_{n_j}, \Psi(\sigma^\dagger)) &\leq \Gamma(\sigma_{n_j}, \Psi(\sigma_{n_j})) + \Gamma(\Psi(\sigma_{n_j}), \Psi(\sigma^\dagger)) \\ &\leq \Gamma(\sigma_{n_j}, \Psi(\sigma_{n_j})) + \Gamma(\sigma_{n_j}, \sigma^\dagger). \end{aligned}$$

Therefore, subsequence $\{\sigma_{n_j}\}$ strongly converges to $\Psi(\sigma^\dagger)$, which implies that $\Psi(\sigma^\dagger) = \sigma^\dagger$. Since $\lim_{n \rightarrow +\infty} \Gamma(\sigma_n, \sigma^\dagger)$ exists, the sequence $\{\sigma_n\}$ strongly converges to a point in $F(\Psi)$. $\square$

4. Solution of a Constrained Minimization Problem

Let (Y, Γ) be a complete CAT(0) space and $\zeta : Y \rightarrow (-\infty, +\infty]$ a proper lower semi-continuous and convex function. We present a theorem to find the minimizers of ζ , which is the solution of the following minimization problem

$$\min_{\sigma \in Y} \zeta(\sigma). \quad (10)$$

We take

$$\operatorname{argmin}_{\zeta \in Y} \zeta(\sigma) = \{\sigma \in Y : \zeta(\sigma) \leq \zeta(\zeta), \text{ for all } \zeta \in Y\}$$

as the set of minimizers of ζ .

Proposition 2 ([18]). Let $r > 0$ and J_r be a resolvent associated with ζ . Then, $F(J_r) = \operatorname{argmin}_{\zeta \in Y} \zeta(\sigma)$.

Theorem 4. Suppose that the function ζ has a minimizer. Then, for all $r > 0$, given $\sigma_1 \in \mathcal{C}$, k is a fixed natural number and $\alpha_n \in [\alpha, \beta]$ with $\alpha, \beta \in (0, 1)$, and the sequence $\{\sigma_n\}$ is defined by

$$\sigma_{n+1} = J_r^k \{(1 - \alpha_n)\sigma_n \oplus \alpha_n J_r(\sigma_n)\}, \quad n \in \mathbb{N}.$$

Then, $\{\sigma_n\}$ Δ -converges to some point in Y , which is a minimizer of ζ .

Proof. It can be easily seen that J_r (a resolvent associated with ζ) is a nonexpansive mapping. Therefore, the conclusion directly follows from Theorem 1. $\square$

5. A Zero of a Monotone Operator

Let (Y, Γ) be a complete CAT(0) space having dual space Y^* . Let $A : Y \rightarrow 2^{Y^*}$ be an operator with domain $D(A) := \{\sigma \in Y : A(\sigma) \neq \emptyset\}$, it is monotone if and only if

$$\langle \sigma^* - \zeta^*, \overrightarrow{\sigma\zeta} \rangle \geq 0, \text{ for all } \sigma, \zeta \in D(A), \sigma \neq \zeta, \sigma^* \in A(\sigma), \zeta^* \in A(\zeta).$$

The monotone operator A is maximal if there exists no monotone operator B such that $\text{gra}(B)$ properly contains $\text{gra}(A)$. Finding the solution of the following problem is pivotal in monotone operator theory.

$$\text{Find } \sigma \in D(A), \text{ such that } 0 \in A(\sigma). \quad (11)$$

The solution of the above problem is a solution of an equilibrium point of an evolution equation. Moreover, the solution of (11) is equivalent to the solution of variational inequality associated to the monotone operator, see [4,7]. Let $\lambda > 0$, the resolvent of operator A of order λ is the set-valued mapping $J_\lambda : Y \rightarrow 2^Y$ defined by $J_\lambda(\sigma) := \{z \in Y \mid \left[\frac{1}{\lambda}\overrightarrow{\sigma z}\right] \in A(z)\}$, see [6]. A monotone operator $A : Y \rightarrow 2^{Y^*}$ on a complete CAT(0) space satisfies the range condition if for every $\lambda > 0$, $D(J_\lambda) = Y$.

Lemma 5 ([6]). Let X be a CAT(0) space and J_λ be the resolvent of the operator A of order λ . We have the following:

- (i) If A is monotone with $\lambda \leq \mu$, then $\Gamma(\sigma, J_\lambda(\sigma)) \leq 2\Gamma(\sigma, J_\mu(\sigma))$.
- (ii) For any $\lambda > 0$, $F(J_\lambda) = A^{-1}(0)$.

Now, we present the following result:

Theorem 5. Let Y be a complete CAT(0) space with dual Y^* and $A : Y \rightarrow 2^{Y^*}$ be a monotone operator that satisfies the range condition and $A^{-1}(0) \neq \emptyset$, where $0 \in Y^*$. Let $\{c_n\}$ be a sequence of positive real numbers such that $0 < c \leq c_n$, for all $n \in \mathbb{N}$, and $\{\alpha_n\}$ is a sequence in $[\alpha, \beta]$ with $\alpha, \beta \in (0, 1)$. For a fixed $k \in \mathbb{N}$, given $\sigma_1 \in Y$, the sequence $\{\sigma_n\}$ is defined as

$$\sigma_{n+1} = J_{c_n}^k \{(1 - \alpha_n)\sigma_n \oplus \alpha_n J_{c_n}(\sigma_n)\}, \quad n \in \mathbb{N}.$$

Then, the sequence Δ -converges to a point $\sigma^+ \in A^{-1}(0)$.

Proof. Let $\sigma^+ \in A^{-1}(0)$. From (W1), we have

$$\begin{aligned} \Gamma(\sigma_{n+1}, \sigma^+) &= \Gamma(J_{c_n}^k \{(1 - \alpha_n)\sigma_n \oplus \alpha_n J_{c_n}(\sigma_n)\}, \sigma^+) \\ &\leq \Gamma((1 - \alpha_n)\sigma_n \oplus \alpha_n J_{c_n}(\sigma_n), \sigma^+) \\ &\leq (1 - \alpha_n)\Gamma(\sigma_n, \sigma^+) + \alpha_n\Gamma(J_{c_n}(\sigma_n), \sigma^+) \\ &\leq (1 - \alpha_n)\Gamma(\sigma_n, \sigma^+) + \alpha_n\Gamma(\sigma_n, \sigma^+) \\ &= \Gamma(\sigma_n, \sigma^+). \end{aligned}$$

Thus, $\{\sigma_n\}$ is bounded, and $\lim_{n \rightarrow +\infty} \Gamma(\sigma_n, \sigma^\dagger)$ exists. Let

$$\lim_{n \rightarrow +\infty} \Gamma(\sigma_n, \sigma^\dagger) = r > 0. \quad (12)$$

By the nonexpansiveness of J_{c_n} ,

$$\lim_{n \rightarrow +\infty} \Gamma(J_{c_n}(\sigma_n), \sigma^\dagger) \leq r. \quad (13)$$

From (12),

$$\begin{aligned} r &= \lim_{n \rightarrow +\infty} \Gamma(\sigma_{n+1}, \sigma^\dagger) = \limsup_{n \rightarrow +\infty} \Gamma(J_{c_n}^k \{(1 - \alpha_n)\sigma_n \oplus \alpha_n J_{c_n}(\sigma_n)\}, \sigma^\dagger) \\ &\leq \limsup_{n \rightarrow +\infty} \Gamma((1 - \alpha_n)\sigma_n \oplus \alpha_n J_{c_n}(\sigma_n), \sigma^\dagger) \\ &\leq \lim_{n \rightarrow +\infty} \Gamma(\sigma_n, \sigma^\dagger) = r. \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow +\infty} \Gamma((1 - \alpha_n)\sigma_n \oplus \alpha_n J_{c_n}(\sigma_n), \sigma^\dagger) = r. \quad (14)$$

From (12)–(14) and Lemma 3,

$$\lim_{n \rightarrow +\infty} \Gamma(\sigma_n, J_{c_n}(\sigma_n)) = 0.$$

By Lemma 5,

$$\Gamma(\sigma_n, J_c(\sigma_n)) \leq 2\Gamma(\sigma_n, J_{c_n}(\sigma_n)). \quad (15)$$

The sequence $\{\Gamma(\sigma_n, \sigma^\dagger)\}$ is monotone nonincreasing, and the sequence $\{\sigma_n\}$ is Fejér monotone with respect to $A^{-1}(0)$. From Proposition 1, the sequence $\{\sigma_n\}$ has a unique asymptotic center $w^\dagger$ with respect to $A^{-1}(0)$. Suppose $\{\rho_n\}$ is a subsequence of $\{\sigma_n\}$. Then, $\{\rho_n\}$ has a unique asymptotic center $\rho^\dagger$ with respect to $A^{-1}(0)$. Now,

$$\begin{aligned} \Gamma(\rho_n, J_c(\rho^\dagger)) &\leq \Gamma(J_c(\rho_n), J_c(\rho^\dagger)) + \Gamma(J_c(\rho_n), \rho_n) \\ &\leq \Gamma(\rho_n, \rho^\dagger) + \Gamma(J_c(\rho_n), \rho_n). \end{aligned}$$

From (15) and Lemma 1, it follows that $J_c(\rho^\dagger) = \rho^\dagger$. From Lemma 2, the sequence $\{\sigma_n\}$ Δ -converges to a point in $A^{-1}(0)$. $\square$

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