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Review

Ideals, Nonnegative Summability Matrices and Corresponding Convergence Notions: A Short Survey of Recent Advancements

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Abstract: In this survey article, we look into some recent results concerning summability matrices, both regular as well as those which are not regular (called semi-regular) and generated matrix ideals as the overall view of the inter relationship between the notions of ideal convergence and summability methods by regular summability matrices.

Keywords: ideal; ideal convergence; matrix summability; summability method; regular matrix; semiregular matrix; matrix density; matrix ideal

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1. Introduction

This is intended to be a short survey article in continuation of the survey articles on ideal convergence [1,2] which appeared in the years 2016 and 2013, respectively, and had covered several facets of this line of research up to the year 2014, but this time we do not intend to cover everything (meaning all significant developments regarding ideal convergence done during the interim period). We would rather concentrate mainly on a few research articles where matrix summability methods have also played pivotal roles alongside the notion of ideals and ideal convergence. As this issue is seemingly devoted to "recent advances in operator theory" research, I would like to comment that though from outside this article looks every bit of misplaced, indeed there are some connections, at least with the word "operator". We can start with the result of Mazur ([3], pp. 44–45) which presents a representation of continuous linear functionals defined on separable subspaces of ℓ^∞ which actually tells that every continuous linear functional on a separable subspace of ℓ^∞ is equal to some matrix summability method on that subspace.

Theorem 1 (Mazur). *Let $V \subseteq \ell^\infty$ be a separable linear subspace of ℓ^∞ with sup norm. For every continuous linear functional $\phi : V \to \mathbb{R}$ there is an infinite matrix $A = (a_{i,k})$, such that:*

(1) For every i, $a_{i,k} = 0$ for all but finitely many k;
(2) $\Sigma_{k=1}^{\infty}|a_{i,k}| \leq ||\phi||$ for every i;
(3) $\lim_{i\to\infty} \Sigma_{k=1}^{\infty}|a_{i,k}| = ||\phi||$;
(4) for every $x \in V$, $\phi(x) = \lim_{i\to\infty} \Sigma_{k=1}^{\infty} a_{i,k}x_k$.

This shows that how summability matrices had come into picture long ago in the realms of functional analysis very naturally.

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ which contains an infinite set. Let $c^{\mathcal{I}}$ be the set of all $\mathcal{I}$-convergent sequences. Reproducing from Lemma 2.1 [4] one can show that $V = c^{\mathcal{I}} \cap \ell^\infty$ is a non-separable subspace of ℓ^∞ (with sup norm). Take an infinite set $C \in \mathcal{I}$ and note that balls $B(1_B; \frac{1}{2})$ of the radius $\frac{1}{2}$ and the center 1_B are pairwise disjoint for distinct $B \subseteq C$. As there are many uncountable distinct subsets of C, and for each $B \subseteq C$ we have $B(1_B; \frac{1}{2}) \cap V \neq \emptyset$. Now one can define a continuous linear functional $\lim^{\mathcal{I}} : V \to \mathbb{R}$ mapping a sequence x to the $\mathcal{I}$-limit of x.

So both matrix summability methods as well as ideal convergence generate continuous linear functionals on suitable subspaces of the space ℓ^∞. With the "facile" relationship of the two main notions, namely, ideal convergence and matrix summability methods with "operator theory" done, we can now safely move into the details of this survey article.

One of the main focuses of this article is the "ideals" themselves and the first section is devoted to all the basic and necessary information about ideals. In the last survey article [1], there was almost no focus on these set theoretic objects though over the years it has become more and more clear how different set theoretic properties of ideals strongly influence several aspects of ideal convergence and in fact it has become very apparent that one has to consider suitable set theoretic properties of ideals to obtain deep and interesting results regarding ideal convergence. On the other hand non-negative matrices themselves generate classes of ideals which are now called "matrix ideals". Section 3 mainly contains very briefly the basic ideas of ideal convergence and is in some sense a short reproduction from the survey article [1] to make this article self contained followed by a little detailed discussion of the characterization of the set of all $\mathcal{I}$-limit points in order to showcase the significance of topological nature of ideals in deriving surprisingly interesting results. Section 4 primarily deals with non-negative regular summability matrices and some exciting questions and obviously their answers established in the last few years, specifically in two genuinely indigenous articles by Filipow and Tryba [5,6]. The final section is specifically devoted to "ideals", especially matrix ideals, where both regular matrices, as well as matrices which are not regular (in our terminology "semiregular") are used to generate these ideals and in the process the complete solution is provided for a folklore problem in summability theory which is referred to as "Connor's conjecture", as the same was indeed conveyed and discussed to me by Prof. Jeff Connor himself way back in two international conferences held in Turkey in the years 2011 and 2016.

Unlike [1], where more often detailed proofs and examples were provided, here we refrain from doing that. So primarily just definitions and results have been presented from the articles cited and interested readers are advised to consult those papers for the concerned details.

2. Ideals

We start by recalling the basic notions of ideals and filters. A family $\mathcal{I} \subset 2^Y$ of subsets of a non-empty set Y is said to be an ideal in Y if

(i) $A,\ B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(ii) $A \in \mathcal{I},\ B \subset A$ imply $B \in \mathcal{I}$.

Further, an admissible ideal $\mathcal{I}$ of Y satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$. Such ideals are also called free ideals. If $\mathcal{I}$ is a proper non-trivial ideal in Y (i.e., $Y \notin \mathcal{I},\ \mathcal{I} \neq \{\phi\}$), then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset Y : M^c \in \mathcal{I}\}$ is a filter in Y called the filter associated with the ideal $\mathcal{I}$ while $\mathcal{I}^+ = \{M \subset Y : M \notin \mathcal{I}\}$ is called the co-ideal. The ideal of all finite subsets of an infinite set Y is denoted by $Fin(Y)$ (or just Fin if Y is clear from the context).

Definition 1. *Two Ideals $\mathcal{I}$ and $\mathcal{J}$ on X and Y, respectively, are isomorphic (in short $\mathcal{I} \approx \mathcal{J}$) if there exists a bijection $f : X \to Y$, such that $A \in \mathcal{I} \iff f[A] \in \mathcal{J}$ for every $A \subseteq X$.*

In this article, we are primarily interested with ideals on $\mathbb{N}$ where $\mathbb{N}$ stands for the set of all natural numbers, and so, henceforth, by ideals, we would mean admissible ideals on $\mathbb{N}$, unless otherwise mentioned.

Following are some very useful ideals and methods of generating new ideals from given ones.

Definition 2. *For ideals $\mathcal{I}, \mathcal{J}$ and $A \notin \mathcal{I}$ we define the following new ideals:*

(1) $\mathcal{I} \restriction A = \{B \subseteq A : B \in \mathcal{I}\} = \{B \cap A : B \in \mathcal{I}\}$;
(2) $\mathcal{I} \oplus \mathcal{J} = \{A \subset \mathbb{N} \times \{0,1\} : \{n : (n,0) \in A\} \in \mathcal{I} \wedge \{n : (n,1) \in A\} \in \mathcal{J}\}$;
(3) $\mathcal{I} \oplus \mathcal{P}(\mathbb{N}) = \{A \subset \mathbb{N} \times \{0,1\} : \{n : (n,0) \in A\} \in \mathcal{I}\}$;
(4) $\mathcal{I} \otimes \mathcal{J} = \{A \subset \mathbb{N} \times \mathbb{N} : \forall n \in \mathbb{N}\{n : \{k : (n,k) \in A\} \notin \mathcal{J}\} \in \mathcal{I}\}$;
(5) $\emptyset \otimes \mathcal{J} = \{A \subset \mathbb{N} \times \mathbb{N} : \{k : (n,k) \in A\} \in \mathcal{J}\}$ *(special case of (4) taking $\mathcal{I} = \{\emptyset\}$);*

(6) $\mathcal{I} \otimes \emptyset = \{A \subset \mathbb{N} \times \mathbb{N} : \{n : \{k : (n,k) \in A\} \neq \emptyset\} \in \mathcal{I}\}$ *(special case of (4) taking* $\mathcal{J} = \{\emptyset\}$*).*

We now look into a particular property of ideals which have again and again found several remarkable applications in the theory of ideal convergence from the beginning.

Definition 3. *An admissible ideal $\mathcal{I}$ is said to satisfy the condition (AP) (or is generally called a P-ideal or sometimes AP-ideal) if for every countable family of mutually disjoint sets $(A_1, A_2, ...)$ from $\mathcal{I}$ there exists a countable family of sets $(B_1, B_2, ...)$ such that $A_j \triangle B_j$ is finite for each $j \in \mathbb{N}$ and $\bigcup_{k=1}^{\infty} B_k \in \mathcal{I}$. Equivalently $\mathcal{I}$ is a P-ideal if for every countable family $\mathcal{F} \subseteq \mathcal{I}$, there is $A \in \mathcal{I}$, such that $F \setminus A$ is finite for every $F \in \mathcal{F}$.*

Recall that after identifying the power set $\mathcal{P}(\mathbb{N})$ of $\mathbb{N}$ with the Cantor space $C = \{0,1\}^{\mathbb{N}}$ in a standard manner we may consider an ideal as a subset of C. An ideal is called an analytic or Borel (in particular F_σ or $F_{\sigma\delta}$) ideal if it corresponds to an analytic or Borel (in particular F_σ or $F_{\sigma\delta}$) subset of C. This topological aspects of ideals were not that much considered or used for the first five six years of the development of the theory of ideal convergence but since then have been found to be remarkable useful in obtaining several very deep and interesting results.

Let S be a set. We say that a map $\varphi : \mathcal{P}(S) \to [0, \infty]$ is a submeasure on S if it satisfies the following conditions:

- $\varphi(\phi) = 0$ and $\varphi(\{s\}) < \infty$ for every $s \in S$;
- φ is monotone: if $A \subset B \subset S$, then $\varphi(A) \leq \varphi(B)$;
- φ is subadditive: if $A, B \subset S$, then $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$.

A submeasure φ on $\mathbb{N}$ is lower semi-continuous if for every $A \subset \mathbb{N}$ we have $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap [1,n])$.

Note that a submeasure on $\mathbb{N}$ is lower semi-continuous if, and only if, it is lower semi-continuous as a function from $\mathcal{P}(\mathbb{N})$ to $[0, \infty]$.

Definition 4. *A submeasure φ is called non-pathological if*

$$\varphi(A) = \sup\{\mu(A) : \mu \leq \varphi;\ \mu \ \textit{is a measure on } \mathcal{P}(\mathbb{N})\}$$

for each $A \subseteq \mathbb{N}$.

Mazur [7] (see also [8]) had shown that $\mathcal{I}$ is an F_σ ideal if, and only if, $\mathcal{I} = Fin(\varphi) = \{A \subseteq \mathbb{N} : \varphi(\mathbb{N}) < \infty\}$ for some lower semi-continuous submeasure φ on $\mathbb{N}$, such that $\varphi(\mathbb{N}) = 1$.

For a submeasure φ we define $||A||_\varphi = \lim_{n\to\infty} \varphi(A \setminus \{1,2,\ldots,n\})$ and $Exh(\varphi) = \{A \subset \mathbb{N} : \lim_{n\to\infty} \varphi(A \setminus [1,n]) = 0\}$. If $||\mathbb{N}||_\varphi \neq 0$, then $Exh(\varphi)$ is an ideal (see e.g., [8]).

$\mathcal{I}$ is an F_σ P-ideal (see [8]) if, and only if, $\mathcal{I} = Fin(\varphi) = Exh(\varphi)$ for some lower semi-continuous submeasure φ on $\mathbb{N}$, such that $\varphi(\mathbb{N}) = 1$ and $||\mathbb{N}||_\varphi \neq 0$. As for analytic P-ideals, Solecki in [9] proved that the following conditions are equivalent.

(1) $\mathcal{I}$ is an analytic P-ideal;
(2) $\mathcal{I}$ is an $F_{\sigma\delta}$ P-ideal;
(3) $\mathcal{I} = Exh(\varphi)$ for some lower semi-continuous submeasure φ on $\mathbb{N}$, such that $\varphi(\mathbb{N}) = 1$ and $||\mathbb{N}||_\varphi \neq 0$.

Definition 5. *An ideal $\mathcal{I}$ is tall if for every infinite $B \subseteq \mathbb{N}$ there is an infinite $C \in \mathcal{I}$ such that $C \subseteq B$. It is easy to see that $\mathcal{I}$ is tall $\Longleftrightarrow \forall\, B \in [\mathbb{N}]_0^{\aleph} (\mathcal{I} \restriction B \neq Fin)$.*

The name "tall ideal" was introduced by Mathias [10], but later Todorčević as also Farah [8] used the name "dense ideal" as a tall ideal is dense in the poset $([\mathbb{N}]_0^{\aleph}, \subseteq)$ and now both the names are used in the literature.

Definition 6 (see [5])**.** *We say that an ideal $\mathcal{I}$ is nowhere tall if $\mathcal{I} \restriction A$ is not tall for every $A \notin \mathcal{I}$.*

It is obvious that Fin is nowhere tall and it is not difficult to see that $\emptyset \otimes Fin$ and $Fin \oplus \mathcal{P}(\mathbb{N})$ are nowhere tall as well. In [5], references to these ideals and other nomenclatures used can be found. Further, in ([5], Proposition 2.27) it was shown that if $\mathcal{I}$ and $\mathcal{J}$ are nowhere tall ideals, then $\emptyset \otimes \mathcal{J}, \mathcal{I} \otimes \emptyset, \mathcal{I} \oplus \mathcal{J}$ and $\mathcal{I} \oplus \mathcal{P}(\mathbb{N})$ are nowhere tall as well.

Theorem 2 (see e.g., [8], Corollary 1.2.11)**.** *Let $\mathcal{I}$ be an analytic P-ideal. Then $\mathcal{I}$ is nowhere tall $\Longleftrightarrow$ $\mathcal{I}$ is isomorphic to one of the three ideals: Fin, $Fin \oplus P(\mathbb{N})$ or $\emptyset \otimes Fin$.*

Finally, we recall the notion of the following special kind of ideal which will be needed in the final section.

Definition 7. *For every $f : \mathbb{N} \to [0, \infty)$, such that $\sum_{n=1}^{\infty} f(n) = \infty$ we define a* summable ideal generated by a function f *by $\mathcal{I}_f = \{B \subset \mathbb{N} : \sum_{n \in B} f(n) < \infty\}$. In particular, for $f(n) = 1/n$ we obtain the ideal $\mathcal{I}_{1/n} = \{B \subset \mathbb{N} : \sum_{n \in B} \frac{1}{n} < \infty\}$ (which we will actually use). It is known that summable ideals are F_σ P-ideals.*

3. Basic Facts of Ideal Convergence, Role of Nice Ideals

Throughout ℓ^∞ will denote the set of all bounded real sequences endowed with the sup norm while by m we will denote just the set of all bounded real sequences, while c will, as usual, denote the set of all convergent real sequences.

The usual notion of convergence does not always capture in fine details the properties of vast class of sequences that are not convergent. Additionally, many times in different investigations in Mathematics we come across sequences that are not convergent but almost all of its terms (in some sense) have the properties of a convergent sequence. So it always seems better to include more sequences under purview, while discussing convergence. One way of including more sequences under purview is to consider those sequences that are convergent when restricted to some 'big' set of natural numbers which is a big set in certain prevalent sense. This is perhaps the motivation behind the introduction of the notion of statistical convergence or more generally ideal convergence which we will define below.

For $K \subset \mathbb{N}$, $K(m, n)$ denote the cardinality of the set $K \cap [m, n]$. The upper and lower natural density of the subset K [11] is defined by

$$\overline{d}(K) = \limsup_{n\to\infty} \frac{K(1, n)}{n} \text{ and } \underline{d}(K) = \liminf_{n\to\infty} \frac{K(1, n)}{n}.$$

If $\overline{d}(K) = \underline{d}(K)$ then we say that the natural density of K exists and it is denoted simply by $d(K)$. Clearly $d(K) = \lim_{n\to\infty} \frac{K(1, n)}{n}$. The notion of natural density was first introduced to define a more general notion of convergence by Fast [12] and independently by Steinhaus [13] in 1951. After the works of Šalăt [14] and particularly of Fridy and Connor [15–20] it became one of the major thirst areas of summability theory and since then a lot of work has been done on statistical convergence and its further generalizations.

Most importantly the idea of statistical convergence have been extended to two types of convergence, namely, $\mathcal{I}$ and $\mathcal{I}^*$ convergence by Kostyrko et. al. in 2000 [21] with the help of ideals. This approach is much more general as most of the known convergence methods become special cases of ideal convergence taking appropriate ideals. However one should know that the notion of the ideal convergence was already there in the literature much before and is in fact dual (equivalent) to the notion of the filter convergence introduced by Cartan in 1937 [22]. Even in 1990, Connor had presented the idea of convergence with respect to a two valued measure [17] which is again nothing but ideal convergence which somehow went unnoticed. However, there is no denying of the fact that this notion of convergence came into prominence only after the article [21], maybe this approach has been easy to understand and the overall importance of "ideals" in set theory.

Definition 8 ([21])**.** *A sequence (x_n) is said to be $\mathcal{I}$-convergent to $\xi \in \mathbb{R}$ $(\xi = \mathcal{I} - \lim_{n\to\infty} x_n)$ if, and only if, for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - \xi| \geq \varepsilon\} \in \mathcal{I}$. The element ξ is called the $\mathcal{I}$-limit of the sequence (x_n).*

Some of the examples of ideals and corresponding convergence notions are described below (see [21]).

Example 1. *(a) If $\mathcal{I}$ is the class Fin then $\mathcal{I}$-convergence coincides with the usual convergence of real sequences;*

(b) If $\mathcal{I}_d$ is the class of all $A \subset \mathbb{N}$ with $d(A) = 0$, then $\mathcal{I}_d$ is a non-trivial admissible ideal and $\mathcal{I}_d$-convergence coincides with the statistical convergence;

(c) The uniform density of a set $A \subset \mathbb{N}$ is defined as follows: For integers $t \geq 0$ and $s \geq 1$ let $A(t+1, t+s) = card\{n \in A : t+1 \leq n \leq t+s\}$. Put

$$\beta_s = \lim_{t\to\infty} inf\, A(t+1, t+s),\ \beta^s = \lim_{t\to\infty} sup\, A(t+1, t+s).$$

It can be shown that the following limits exist :

$$\underline{u}(A) = \lim_{s\to\infty} \frac{\beta_s}{s},\ \overline{u}(A) = \lim_{s\to\infty} \frac{\beta^s}{s}.$$

If $\underline{u}(A) = \overline{u}(A)$, then $u(A) = \underline{u}(A)$ is called the uniform density of the set A.
Put $\mathcal{I}_u = \{A \subset \mathbb{N} : u(A) = 0\}$. Then $\mathcal{I}_u$ is a non-trivial ideal and $\mathcal{I}_u$-convergence is said to be the uniform statistical convergence;

(d) A wide class of $\mathcal{I}$-convergence can be obtained as follows.

For $E \subset \mathbb{N}$ and a non-negative regular matrix $A = (a_{i,k})$ (see Definition 11), we put

$$d_A^{(n)}(E) = \sum_{k=1}^{\infty} a_{n,k}\, \chi_E(k)$$

for $n \in \mathbb{N}$. If $\lim_{n\to\infty} d_A^{(n)}(E) = d_A(E)$ exists, then $d_A(E)$ is called A-density of E [23]. From the regularity of A it follows that $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1$ and from this we see that $d_A(E) \in [0,1]$ (if it exists). Consequently $\mathcal{I}(A) = \{E \subset \mathbb{N} : d_A(E) = 0\}$ is a non-trivial ideal which we call the matrix ideal generated by A. Note that $\mathcal{I}_d$-convergence can be obtained from $\mathcal{I}(A)$-convergence by choosing $a_{n,k} = \frac{1}{n}$ for $k \leq n$ and $a_{n,k} = 0$ for $k > n$. On the other hand if $a_{n,k} = \frac{1}{s_n}$ for $k \leq n$ and $a_{n,k} = 0$ for $k > n$ where $s_n = \sum_{j=1}^{n} \frac{1}{j}$ for $n \in \mathbb{N}$, then we obtain the notion of $\mathcal{I}_\delta$-convergence (logarithmic convergence). Finally choosing $a_{n,k} = \frac{\phi(k)}{n}$ for $k \leq n, k|n$ and $a_{n,k} = 0$ for $k \leq n, k$ does not divide n and $a_{n,k} = 0$ for $k > n$ we obtain ϕ-convergence of Schoenberg (see [24]), where ϕ is the Euler function.

Recall the following result from the theory of statistical convergence. A sequence (x_n) of real numbers is statistically convergent to ξ if, and only if, there exist a set $M = \{m_1 < m_2 < ...\} \subset \mathbb{N}$, such that $d(M) = 1$ and $\lim_{k\to\infty} x_{m_k} = \xi$ (See [14]).

This result influenced the introduction of the following concept of convergence, namely, $\mathcal{I}^*$-convergence.

Definition 9 ([21])**.** *A sequence $x = (x_n)$ is said to be $\mathcal{I}^*$-convergent to $\xi \in \mathbb{R}$ if, and only if, there exists a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m_1 < m_2 < \ldots\}$, such that $\lim\limits_{k\to\infty} |x_{m_k} - \xi| = 0$.*

The similar results such as that of the following were originally proved in [21] in the more general settings of a metric space.

Theorem 3. *Let $\mathcal{I}$ be an admissible ideal. If $\mathcal{I}^* - \lim x_n = \xi$ then $\mathcal{I} - \lim x_n = \xi$.*

The converse implication between $\mathcal{I}$- and $\mathcal{I}^*$-convergence depends essentially on the structure of the metric space (X, ρ) and in [21] it was shown that if X has no accumulation point then $\mathcal{I}$- and $\mathcal{I}^*$- convergence coincide for each admissible ideal $\mathcal{I}$. Otherwise we can have a result like following.

Theorem 4 ([21])**.** *There exist an admissible ideal $\mathcal{I}$ and a sequence (y_n) of real numbers, such that $\mathcal{I} - \lim y_n = \xi$ but $\mathcal{I}^* - \lim y_n$ does not exist.*

Throughout $c^{\mathcal{I}}$ and $c^{\mathcal{I}^*}$ will stand for the sets of all $\mathcal{I}$ and $\mathcal{I}^*$-convergent real sequences. It is known that $c^{\mathcal{I}} \cap m$ is a closed subset of ℓ^∞ whereas $c^{\mathcal{I}^*} \cap m$ is dense in $c^{\mathcal{I}} \cap m$ (for the reference as also detailed proof see [1]). Further it is known that $c^{\mathcal{I}} \cap m = m$ if $\mathcal{I}$ is a maximal ideal.

One can naturally ask as to when the two notions of convergence coincide and in [21] it was proved that they actually coincide when the concerned ideal is a P-ideal and moreover P-ideals are in fact characterized by this property.

Theorem 5 ([21])**.** *Let $\mathcal{I}$ be an admissible ideal.*

(i) *If $\mathcal{I}$ is a P-ideal then for any arbitrary sequence (x_n) of real numbers, $\mathcal{I} - \lim x_n = \xi$ implies $\mathcal{I}^* - \lim x_n = \xi$;*

(ii) *If for every arbitrary sequence (x_n) of real numbers, $\mathcal{I} - \lim x_n = \xi$ implies $\mathcal{I}^* - \lim x_n = \xi$, then $\mathcal{I}$ is a P-ideal.*

There are many more instances where P ideals come into picture, which interested readers can see from several papers on ideal convergence and the survey article [1]. However for certain investigations, one need to look further, typically into the topological aspects of ideals. Following is a classic case of such application.

Definition 10 (cf. [21])**.** *Let $x = (x_n)$ be a sequence of real numbers.*

(i) *An element $\xi \in \mathbb{R}$ is said to be an $\mathcal{I}$-limit point of x provided that there is a set $M = (m_1 < m_2 < \ldots) \subset \mathbb{N}$, such that $M \notin \mathcal{I}$ and $\lim\limits_{k\to\infty} x_{m_k} = \xi$;*

(ii) *An element $\xi \in \mathbb{R}$ is said to be an $\mathcal{I}$-cluster point of x if, and only if, for each $\varepsilon > 0$ we have $\{n \in \mathbb{N} : |x_n - \xi| < \varepsilon\} \notin \mathcal{I}$.*

Denote by $\mathcal{I}(L_x)$ and $\mathcal{I}(C_x)$ the sets of all $\mathcal{I}$-limit and $\mathcal{I}$-cluster points of x, respectively. The similar results like that of the following results may be found in [25,26].

Theorem 6. *Let $\mathcal{I}$ be an admissible ideal. Then, for each sequence $x = (x_n)$ of real numbers we have $\mathcal{I}(L_x) \subset \mathcal{I}(C_x)$.*

It was also observed in [25] in a topological space X that if $x = (x_n)$ and $y = (y_n)$ are two sequences in X such that $\{n \in \mathbb{N} : x_n \neq y_n\} \notin \mathcal{I}$ then $\mathcal{I}(C_x) = \mathcal{I}(C_y)$, $\mathcal{I}(L_x) = \mathcal{I}(L_y)$.

Theorem 7 ([26])**.** *Let $\mathcal{I}$ be an admissible ideal.*

(i) *The set $\mathcal{I}(C_x)$ is closed for each sequence $x = (x_n)$ of real numbers;*

(ii) *For each closed set $F \subset \mathbb{R}$ there exists a sequence $x = (x_n)$ of real numbers, such that $F = \mathcal{I}(C_x)$.*

Theorem 8. *For any sequence (x_n) of real numbers, the set $\mathcal{I}(L_x)$ is a F_σ-set provided $\mathcal{I}$ is an analytic P-ideal.*

The statistical versions of the result appeared in Theorem 1.1 [27] for real sequences, in Theorem 2.6 [28] in topological spaces and then in Theorem 2 [25] from where the statement has been reproduced. Later with a different line of proof it appeared in Theorem 3.1 [29] for metric spaces and in Theorem 2.2 [30] (topological spaces). The converse of the above result was established in Theorem 3 [25] but there was a gap in the argument which was later modified and is given a little later.

The above mentioned results have since been investigated in more detail and the following results from [30] show how one can obtain nice results when ideals have some specific topological properties.

Theorem 9 ([30])**.** *Let $x = (x_n)$ be a sequence taking values in a first countable space X and let $\mathcal{I}$ be an F_σ-ideal. Then, $\mathcal{I}(L_x) = \mathcal{I}(C_x)$. In particular, $\mathcal{I}(L_x)$ is closed.*

Corollary 1 ([30])**.** *Let x be a real sequence and let $\mathcal{I}$ be a summable ideal. Then, $\mathcal{I}(L_x)$ is closed.*

In a really remarkable observation it turns out that, within the class of analytic P-ideals, the property that the set of $\mathcal{I}$-limit points is always closed characterizes the subclass of F_σ-ideals.

Theorem 10 ([30])**.** *Let X be a first countable space which admits a non-trivial convergent sequence. Let also $\mathcal{I}$ be an analytic P-ideal where $\mathcal{I} = Exh(\varphi)$. Then the following are equivalent:*

(i) $\mathcal{I} = Fin(\varphi)$*, i.e., it is also an F_σ-ideal;*
(ii) $\mathcal{I}(L_x) = \mathcal{I}(C_x)$ *for all sequences x;*
(iii) $\mathcal{I}(L_x)$ *is closed for all sequences x;*
(iv) *there does not exist a partition $\{A_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ such that $||A_n||_\varphi > 0$ for all n and* $\lim_n || \bigcup_{k>n} A_k ||_\varphi = 0$.

Note that, if X is a first countable space which admits a non-trivial convergent sequence and $\mathcal{I} = Exh(\varphi)$ is an analytic P-ideal which is not F_σ, then there exists a sequence x such that $\mathcal{I}(L_x)$ is a non-closed F_σ-set. In this case, indeed, all the F_σ-sets can be obtained.

Theorem 11 ([30])**.** *Let X be a first countable space where all closed sets are separable and assume that there exists a non-trivial convergent sequence. Fix also an analytic P-ideal $\mathcal{I} = Exh(\varphi)$ which is not F_σ and let $B \subseteq X$ be a non-empty F_σ-set. Then, there exists a sequence x such that $\mathcal{I}(L_x) = B$.*

There are several other instances where analytic P-ideals (or more precisely, the property of being "analytical") are found to be extremely helpful. If we assume that $\mathcal{I}$ is an analytic P-ideal then ideal limits of continuous functions behave like ordinary limits. In [21], it is shown that $\mathcal{I}_d$-limits of sequences of continuous functions are of the first Baire class. Finally, it was generalized to all analytic P-ideals and all Baire classes in [31]. In [32], it was shown that if $\mathcal{I}$ is an analytic P-ideal then for any finite measure space (X, M, μ), real valued measurable functions $f_n, f (n \in \mathbb{N})$ defined almost everywhere on X such that (f_n) is pointwise $\mathcal{I}$-convergent to f almost everywhere on X and every $\varepsilon > 0$ there is an $A \in M$, such that $\mu(X \setminus A) < \varepsilon$ and f_n equally ideally converges to f on A. There are several other instances which can be found from the literature (see the survey article [2] in particular). As this is not our main goal in this survey article we stop here.

4. Regular Summability Matrices, Summability Methods, and Relation with Ideal Convergence

For an infinite matrix of reals $A = (a_{i,k})$, a sequence of reals $x = (x_k)$ and $n \in \mathbb{N}$ we write $A_n(x) = \sum_{k=1}^{\infty} a_{n,k} x_k$ whenever the infinite series is well defined.

Let $A = (a_{i,k})$ be an infinite matrix of reals. We say that a sequence $x = (x_k)$ is A-summable (see [33,34] for details about this summability method and associated research) if:

- The series $A_n(x)$ is convergent for all but finitely many $n \in \mathbb{N}$;
- The sequence $(A_n(x))$ is convergent.

The real $\lim_{n\to\infty} A_n(x)$ is called the A-limit of the sequence x and is denoted by $\lim^A x$. The set of all A summable sequences will be denoted by c^A.

The famous Silverman–Toeplitz theorem [Silverman, 1913 [35] and independently Toeplitz, 1913 [36] says that a matrix A is regular if, and only if, $\lim_{n\to\infty} A_n(x) = \lim_{n\to\infty} x_n$ for every ordinary convergent sequence x (i.e., when $x \in c$) which is equivalent to the following conditions which we now write as the definition of a non-negative regular matrix.

Definition 11. *A non-negative matrix $A = (a_{i,k})$ is called regular if*

(i) $\sup_j \left(\sum_{k=1}^{\infty} |a_{j,k}| \right) < +\infty$;

(ii) $\lim_j a_{j,k} = 0$ *for each* $k \in \mathbb{N}$;

(iii) $\lim_j \left(\sum_{k=1}^{\infty} a_{j,k} \right) = 1.$

Throughout REG will stand for the family of all non-negative regular summability matrices (i.e., non-negative matrices satisfying all the three conditions of Definition 11).

Lemma 1 (see [5] (Folklore))**.** *If A is a regular matrix, then there is a regular matrix B, such that:*

(1) B has only finitely many non-zero elements in each row;

(2) Each row of B sums to 1;

(3) $\lim^A x = \lim^B x$ for every bounded real sequence x;

(4) $\mathcal{I}_A = \mathcal{I}_B$.

In general, Lemma 1 (3) cannot be extended for unbounded sequences x as can be seen from [5]. For the ideal $\mathcal{I}_d$ the question whether the functional $\lim^{\mathcal{I}_d}$ has a representation in terms of matrix summability methods was posed by Mazur as Problem 5 in "The Scottish Book" ([7], Problem 5, p. 55 or [37], Problem 5, p. 69). In straight words the problem is "is the notion of statistical convergence of bounded sequences equivalent to some matrix summability method?" No clear answer to that problem is given in the book, but Mazur wrote down in the book two claims, and from the second it follows that a matrix method summing all bounded statistically convergent sequences must also sum other bounded sequences. That corollary would mean that the answer to that problem is negative. Khan and Orhan, seemingly unaware of "The Scottish Book" problems, have shown in ([38], Theorem 2.2) that for every non-negative regular matrix summability method A there exists a non-negative regular matrix method B, such that A-statistical convergence and B-summability are equivalent over all bounded sequences. Since statistical convergence is A-statistical convergence when A is the Cesáro matrix, that theorem gives us a positive answer to Problem 5 of "The Scottish Book". We would also like to mention here another related observation, namely, Corollary 2.4 [39] which states that for any regular matrix A, there exists a $\{0,1\}$-valued sequence x, such that Ax is not statistically convergent.

In the following, we say that "ideal convergence with respect to an ideal $\mathcal{I}$ is contained in some matrix summability with respect to a regular matrix A" if every $\mathcal{I}$-convergent sequence x is A-summable with $\lim^{\mathcal{I}} x = \lim^A x$.

Very recently the full scope of the relationship between ideal convergence and matrix summability in the realm of bounded and unbounded sequences were investigated by Filipow and Tryba [5,6] where several very interesting observations were presented, which we discuss below.

Theorem 12 ([5])**.** *The ideal convergence generated by an ideal $\mathcal{I}$ is contained in some matrix summability if and only if $\mathcal{I}$ is not dense.*

Since the ideal $\mathcal{I}_d$ is dense, so this means that statistical convergence is not contained in any matrix summability (already proved by Fridy [19], Theorem 2).

Theorem 13 ([5])**.** *The ideal convergence generated by an ideal $\mathcal{I}$ is equal to some matrix summability if, and only if, $\mathcal{I} = Fin$ or $\mathcal{I} \approx Fin \oplus \mathcal{P}(\mathbb{N})$.*

Lemma 2 ([5])**.** *The ideal convergence generated by an ideal $\mathcal{I}$ is equal to the matrix summability generated by a non-negative regular matrix in the realm of bounded sequences if, and only if, $\mathcal{I}$ is a matrix ideal generated by a non-negative regular matrix.*

Lemma 3 ([5])**.** *The ideal convergence generated by an ideal $\mathcal{I}$ is contained in the matrix summability generated by a non-negative regular matrix in the realm of bounded sequences if, and only if, the ideal $\mathcal{I}$ can be extended to the matrix summability ideal generated by a non-negative regular matrix.*

Proposition 1 ([5])**.** *Let A be a non-negative regular matrix.*

(1) The ideal convergence generated by the ideal $\mathcal{I}(A)$ is contained in the matrix summability generated by the matrix A in the realm of bounded sequences;
(2) If there exists $B \subseteq \mathbb{N}$ such that $d_A(B)$ exists and is not equal to 0 nor 1 then there is a bounded sequence which is A-summable but is not $\mathcal{I}(A)$-convergent.

However, one can provide some necessary conditions for ideal convergence to be equal to (or contained in) some matrix summability in the realm of bounded sequences in a easier way (i.e., to say that it is easier to check this than showing that an ideal is not equal to (or contained in) a matrix ideal).

Proposition 2 ([5])**.** *Let $\mathcal{I}$ be an ideal.*

(1) If there is a non-negative regular matrix A with $\lim^{\mathcal{I}} = \lim^{A}$ on m, then $\mathcal{I}$ is an $F_{\sigma\delta}P$-ideal;
(2) If there is a non-negative regular matrix A, such that $\lim^{\mathcal{I}}$ is a restriction of $\lim^{A}$, then $\mathcal{I}$ is contained in an $F_{\sigma\delta}P$-ideal.

In ([31], Lemma 11) Laczkovich and Reclaw had shown that the ideal convergence generated by the ideal $Exh(\varphi)$ with a non-pathological submeasure φ is always weaker than the matrix summability generated by some non-negative regular matrix in the realm of bounded sequences. In [5] it has recently been shown that there are analytic P-ideals generated by pathological submeasures that are not contained in any matrix ideal.

Theorem 14 ([5])**.** *There is an $F_{\sigma}P$-ideal which is not contained in any matrix ideal.*

Definition 12 ([5])**.** *An ideal $\mathcal{I}$ is said to have the property (M) if for every regular non-negative matrix A, such that $\lim^{A} \restriction m \subseteq \lim^{\mathcal{I}} \restriction m$ there is $F \in \mathcal{F}(\mathcal{I})$, such that for every $x \in m \cap c^{A}$ the subsequence $(x)_F$ is ordinarily convergent.*

In "The Scottish Book" ([7], p. 56) Mazur had claimed that the ideal $\mathcal{I}_d$ has the property (M). In [5] it was shown that Mazur's claim about the ideal $\mathcal{I}_d$ was incorrect. Actually, the problem of an ideal $\mathcal{I}$ having or not having the property (M) was studied in much more details there from where we now present some results.

Proposition 3 ([5]). *Let $\mathcal{I}$ be an ideal with the property (M). If $\mathcal{J}$ is isomorphic to $\mathcal{I}$, then $\mathcal{J}$ has the property (M) as well.*

Proposition 4 ([5]). *Fin and $Fin \oplus \mathcal{P}(\mathbb{N})$ have the property (M).*

Proposition 5 ([5]). *An ideal $\mathcal{I}$ has the property (M) if and only if for every regular non-negative matrix A, such that $\lim^A \upharpoonright m \subseteq \lim^{\mathcal{I}} \upharpoonright m$ there is $F \in \mathcal{F}(\mathcal{I})$, such that $\lim^A \upharpoonright m \subseteq \lim^B \upharpoonright m$, where the matrix $B = (b_{i,k})$ is given by $b_{i,e_F(i)} = 1$ for $i \in \mathbb{N}$ and $b_{i,k} = 0$ otherwise.*

Theorem 15 ([5]). *An ideal $\mathcal{I}$ has the property (M) if, and only if, the ideal $\mathcal{I} \upharpoonright C$ has the property (M) for every $C \notin \mathcal{I}$.*

Proposition 6 ([5]). *Let $\mathcal{I}$ be an ideal. If there are $C \subseteq \mathbb{N}$ and a regular non-negative matrix A, such that the ideal $\mathcal{I}(A) \upharpoonright C$ is dense and $\mathcal{I}(A) \upharpoonright C \subseteq \mathcal{I} \upharpoonright C$, then $\mathcal{I}$ does not have the property (M).*

In [5] naturally the following open problems were raised which, to my knowledge, still remain open.

Question 1. Is the converse of Proposition 4.6 true?

Question 2. Does there exist an ideal with the property (M) which is not isomorphic to *Fin* nor $Fin \oplus P(\mathbb{N})$?

Next, let us look into another very interesting line of investigations involving ideal convergence and matrix summability methods. We start with the observation of Fridy and Miller ([40], Theorem 4) who had shown that the ideal limit function generated by a matrix ideal $\mathcal{I}(A)$ is equal to an intersection of some matrix summability methods in the realm of all bounded sequences, more specifically they showed that there is a family of matrices $\mathcal{M}$, such that

$$\forall x \in \ell^\infty \, (\lim^{\mathcal{I}} x = L \iff \forall A \in \mathcal{M}(\lim^A x = L)).$$

Later, Gogola, Macaj and Visnyai ([41], Theorem 4.4) established a similar result for another family of ideals and had asked ([41], Problem 4.6) whether the same holds for every ideal $\mathcal{I}$. A negative answer was given in [5] where the existence of an F_σ ideal $\mathcal{I}$ was established which does not fulfil this property ([5], Proposition 6.8). As a natural consequence the question arises as to for which ideals the above mention property holds and precisely this problem has been investigated in the very recent article [6].

Definition 13. *For an ideal $\mathcal{I}$ on $\mathbb{N}$ we define*

$$\mathcal{M}(\mathcal{I}) = \{A \in REG : \mathcal{I} \subseteq \mathcal{I}(A)\}.$$

Proposition 7 ([6]). *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and $\mathcal{M} \subseteq REG$. If $\lim^{\mathcal{I}} \upharpoonright \ell^\infty = \bigcap\{\lim^A \upharpoonright \ell^\infty : A \in \mathcal{M}\}$ or $\lim^{\mathcal{I}^*} \upharpoonright \ell^\infty = \bigcap\{\lim^A \upharpoonright \ell^\infty : A \in \mathcal{M}\}$ then,*

(1) $\mathcal{M} \subseteq \mathcal{M}(\mathcal{I})$;

(2) $\mathcal{I} = \bigcap\{\mathcal{I}(A) : A \in \mathcal{M}\}$.

Proposition 8 ([6]). *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and $\mathcal{M} \subseteq REG$. If $\lim^{\mathcal{I}^*} \upharpoonright \ell^\infty = \bigcap\{\lim^A \upharpoonright \ell^\infty : A \in \mathcal{M}\}$ then $\mathcal{I}$ is a P-ideal.*

Theorem 16 ([6]). *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following conditions are equivalent.*

(1) $\exists \mathcal{M} \subseteq REG \, (\lim^{\mathcal{I}} = \bigcap\{\lim^A : A \in \mathcal{M}\})$;

(2) *$\mathcal{I}$ is nowhere tall.*

Theorem 17 ([6]). *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following conditions are equivalent.*

(1) $\exists \mathcal{M} \subseteq REG\ (\lim^{\mathcal{I}^*} = \bigcap\{\lim^A : A \in \mathcal{M}\})$;
(2) $\mathcal{I}$ *is a nowhere tall P-ideal.*

Theorem 18 ([6])**.** *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following conditions are equivalent.*

(1) $\exists \mathcal{M} \subseteq REG(\lim^{\mathcal{I}} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}\})$;
(2) $\mathcal{I} = \bigcap\{\mathcal{I}(A) : A \in \mathcal{M}(\mathcal{I})\}$;
(3) $\lim^{\mathcal{I}} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\}$.

Theorem 19 ([6])**.** *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following conditions are equivalent.*

(1) $\exists \mathcal{M} \subseteq REG(\lim^{\mathcal{I}^*} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}\})$;
(2) $\mathcal{I}$ *is a P-ideal and* $\mathcal{I} = \bigcap\{\mathcal{I}(A) : A \in \mathcal{M}(\mathcal{I})\}$;
(3) $\lim^{\mathcal{I}^*} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\}$.

Instead of general ideals if one considers ideals defined with the aid of submeasures, namely, analytic P-ideals and F_σ ideals which have already been discussed in Section 2, then one gets into certain specific situations regarding the representation of ideal limit functions and we present below results from [6] where these problems have been investigated. In the case of $\mathcal{I}$-limits in the realm of all sequences, one only has to check if $\mathcal{I}$ is nowhere tall ideal. So one essentially has only 3 analytic P-ideals for which $\mathcal{I}$-limit can be represented as an intersection of matrix summability methods.

In Propositions 6.8 and 6.15 [5] it was shown that there is a pathological submeasure giving an ideal for which $\mathcal{I}$-limit in the realm of bounded sequences is not representable as an intersection of matrix summability methods. We skip here the details of that example as it is quite complicated. However for non-pathological case one has the following result.

Theorem 20 ([6])**.** *If φ is a non-pathological lower semi-continuous submeasure and $\mathcal{I} = Fin(\varphi)$, then $\lim^{\mathcal{I}} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\}$. Moreover, $\lim^{\mathcal{I}^*} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\} \iff \mathcal{I}$ is a P-ideal.*

Theorem 21 ([6])**.** *If φ is a non-pathological lower semi-continuous submeasure and $\mathcal{I} = Exh(\varphi)$, then $\lim^{\mathcal{I}} \restriction \ell^\infty = \lim^{\mathcal{I}^*} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\}$.*

The following open problem was subsequently posed in [6] which remains open.

Question 3. Does there exist a pathological lower semi-continuous submeasure φ, such that

$$\overset{\mathcal{I}}{\lim} \restriction \ell^\infty = \bigcap\{\overset{A}{\lim} \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\}$$

where $\mathcal{I} = Fin(\varphi)$ or $\mathcal{I} = Exh(\varphi)$?

We end this discussion with the following observation for the ideal $\mathcal{I}_u$ defined before for which again a positive result holds but it does not follow from the results mentioned above as $\mathcal{I}_u$ is not a P-ideal.

Theorem 22 ([6])**.** $\lim^{\mathcal{I}_u} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\}$.

We end this section with a very recent extension of the idea of regular matrices by Connor and Leonetti [42]. Given sequence spaces X, Y, we let (X, Y) be the set of infinite real matrices A, such that Ax is well defined and belongs to Y for all $x \in X$. Accordingly, a matrix A is regular if $A \in (c, c)$ and preserves the (ordinary) limits.

Definition 14 ([42])**.** *Given ideals $\mathcal{I}, \mathcal{J}$ on $\mathbb{N}$, a matrix A is said to be $(\mathcal{I}, \mathcal{J})$-regular if $A \in (c^{\mathcal{I}} \cap \ell^\infty, c^{\mathcal{I}} \cap \ell^\infty)$ and*

$$\forall x \in c^{\mathcal{I}} \cap \ell^\infty,\ \mathcal{I} - \lim x = \mathcal{J} - \lim Ax.$$

Theorem 23 ([42]). *Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. Then, a matrix A is $(\mathcal{I}, \mathcal{J})$-regular provided that*

(i) $\sup_j \left(\sum_{k=1}^{\infty} |a_{j,k}|\right) < +\infty$;

(ii) $\mathcal{J} - \lim_{j \in E} a_{j,k} = 0$ *for each* $E \in \mathcal{I}$;

(iii) $\mathcal{J} - \lim_j \left(\sum_{k=1}^{\infty} a_{j,k}\right) = 1.$

The converse holds if $A \geq 0$ or $\mathcal{I} = Fin$ or $\mathcal{J} = Fin$.

5. How Many Distinct Analytic *P*-Ideals Are There? Some Remarks

We have already seen in Section 3 that the class of all analytic P-ideals are the most important class of "so called nice ideals" which helps to obtain several deep and interesting results in the theory of ideal convergence. This class itself have been topics of research in the field of Set Theory which we would not dwell upon much here. It had all started in the 1950s. Points of $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ (i.e., the remainder in the Stone–Cech compactification of the space of natural numbers with discrete topology) are identified with free ultrafilters on the set $\mathbb{N}$. Recall that ultrafilters are those filters which are not properly contained in any other filter. P-points are precisely those ultrafilters whose dual ideals are P-ideals. In 1956, Rudin showed that the space $\mathbb{N}^*$ has P-points if the continuum hypothesis is assumed. There has since been a lot of work on analytic P-ideals and ideals in general which can be seen from the excellent survey article [43] where all the important references up to the year 2010 can be found.

In view of all these, we can ask a very natural question as to how many distinct such ideals are there. In this section, we present certain answers where concrete examples could be constructed. One can obtain the inference from other credible sources from articles in set theory also, but we would not dwell upon them. Instead we would use definitions and results where certain specific types of non-negative summabilty matrices (not necessarily regular) have been used.

The notion of natural density can be generalized as follows. Let $g : \mathbb{N} \to [0, \infty)$ be a function with $\lim_{n\to\infty} g(n) = \infty$ and $n/g(n) \nrightarrow 0$. The upper simple density (or density of weight g) [44] was defined by the formula

$$\overline{d}_g(A) = \limsup_{n\to\infty} \frac{|A \cap [1, n]|}{g(n)}$$

for $A \subset \mathbb{N}$. This definition extended the notion of natural density of order α [45].

Theorem 24 ([44]). *If $g \colon \mathbb{N} \to [0, \infty)$ is such that $g(n) \to \infty$ and $n/g(n) \nrightarrow 0$, then the ideal $\mathcal{I}_g$ is equal to $Exh(\varphi)$ where*

$$\varphi(E) = \sup_{n\in\mathbb{N}} \frac{card(E \cap n)}{g(n)} \quad \textit{for } E \subset \mathbb{N},$$

and φ is a lower semi-continuous submeasure on $\mathbb{N}$. Consequently, $\mathcal{I}_g$ is an $F_{\sigma\delta}$ P-ideal on $\mathbb{N}$.

The collection of all weight functions $g \colon \mathbb{N} \to [0, \infty)$ with the properties that $g(n) \to \infty$ and $n/g(n) \nrightarrow 0$ will be denoted by G. These ideals were later renamed as simple density ideals in [46,47]. Now the following result provides the first basic answer to our question and we can conclude that there are uncountably many distinct analytic P-ideals.

Theorem 25 ([44]). *There exists a family $G_0 \subset G$ of cardinality $\mathfrak{c}$, such that $\mathcal{I}_f$ is incomparable with $\mathcal{I}_d$ for every $f \in G$, and $\mathcal{I}_f$ and $\mathcal{I}_g$ are incomparable for any distinct $f, g \in G_0$.*

Actually, one can say a lot more and we now look into some stronger results from [47].

Proposition 9 ([47])**.** *$\bigcap_{g\in G}\mathcal{I}_g = Fin$ and $\bigcap_{g\in G}\mathcal{I}_g = \mathcal{I}_l$ where $\mathcal{I}_l = \{A \subseteq \mathbb{N} : \underline{d}(A) = 0\}$.*

Instead of the basic inclusion relation on the set of ideals, one can consider a more stronger notion of pre-order. We say that an ideal $\mathcal{I}$ is below an ideal $\mathcal{J}$ in the Katětov order ($\mathcal{I} \leq_K \mathcal{J}$) if there is such a function (not necessarily a bijection) $f : \mathbb{N} \to \mathbb{N}$, such that $A \in \mathcal{I}$ implies $f^{-1}(A) \in \mathcal{J}$ for all $A \subseteq \mathbb{N}$.

Theorem 26 ([47])**.** *Among ideals of the form $\mathcal{I}_g$, for $g \in G$, there exists an antichain in the sense of $\leq_K$ of size $\mathfrak{c}$.*

Corollary 2 ([47])**.** *There are $\mathfrak{c}$ many non-isomorphic ideals of the form $\mathcal{I}_g$, for $g \in G$.*

The last result ensures the existence of $\mathfrak{c}$ many non-isomorphic analytic P-ideals. Observe that we have actually considered ideals of the form $\mathcal{I}(A)$ where the non-negative matrix $A = (a_{i,k})$ is defined by choosing $a_{n,k} = \frac{1}{g(n)}$ for $k \leq n$ and $a_{n,k} = 0$ for $k > n$ where $g \in G$. In particular taking $g = \frac{1}{n^\alpha}$, $0 < \alpha < 1$ one obtains non-negative matrices A which are not necessarily regular as they may fail to satisfy the condition (iii) in the Definition 11.

Interested readers can consult the articles [46,47], in particular, for several interesting facets of these simple density ideals.

6. Semi-Regular Matrices and Ideals

We have already seen that non-negative regular matrices also generate ideals. Since the class of these matrices is also huge (at least uncountable) and many of the known nice ideals such as $\mathcal{I}_d$ are nothing but the ideals generated by suitable matrices, so Connor conjectured that the class of all analytic P-ideals coincide with the class of all ideals generated by non-negative regular matrices. As has been mentioned in the introduction, this problem remained largely "folklore" without any specific mention of it in any research article. In the rest of this article we state results from the very recent paper [48] which shows that the conjecture is false.

In order to understand how things work, we first break down the definition of these matrices and introduce notions of certain matrices which are not necessarily regular.

Definition 15. *We say that a matrix $A = (a_{i,k})$ is:*

- Non-negative *if $a_{i,k} \geq 0$ for every $i, k \in \mathbb{N}$;*
- Admissible *if $\lim_{i\to\infty} a_{i,k} = 0$ for every $k \in \mathbb{N}$.*

Definition 16. *A matrix $A = (a_{i,k})$ is semi-regular if:*

- *A is admissible;*
- *$\lim_{i\to\infty} \sum_{k\in\omega} a_{i,k} = \infty$.*

A semi-regular matrix $A = (a_{i,k})$ is of:

- type 1 *if $\sum_{k\in\mathbb{N}} |a_{i,k}| < \infty$ for all but finitely many i;*
- type 2 *if $\sum_{k\in\mathbb{N}} |a_{i,k}| = \infty$ for infinitely many i.*

For $E \subset \mathbb{N}$ and a semi-regular matrix $A = (a_{i,k})$ of type 1 we put

$$d^{(n)}_{ST1}(E) = \sum_{k=1}^{\infty} a_{n,k}\, \chi_E(k)$$

for $n \in \mathbb{N}$. If $\lim\limits_{n\to\infty} d^{(n)}_{ST1}(E) = d_{ST1}(E)$ exists, then it can be verified that the functions d_{ST1} for semi-regular matrices of type 1, and d_{ST2} which can be similarly defined for semi-regular matrices of type 2, both satisfy basic properties of density functions. Here one must understand that just for technical reason we are denoting any such density function generated by a semi-regular matrix of type 1 by a common notation d_{ST1} (of course for different such matrices one may obtain different density functions, per say) because in this section we are more interested in the whole class rather than individual functions. Subsequently one can generate the corresponding ideals

$$\mathcal{I}(ST1) = \{E \subset \mathbb{N} : d_{ST1}(E) = 0\}$$

and

$$\mathcal{I}(ST2) = \{E \subset \mathbb{N} : d_{ST2}(E) = 0\}.$$

The following result obtained in [49] showed that indeed all non-negative regular matrices generate analytic P-ideals.

Theorem 27 ([49])**.** *Every ideal belonging to $\mathcal{I}(REG)$ is a $F_{\sigma\delta}P$-ideal where $\mathcal{I}(REG) = \{\mathcal{I}_A : A$ is a regular matrix$\}$ and $\mathcal{I}(A) = \{E \subset \mathbb{N} : d_A(E) = 0\}$ (as defined in Section 3).*

Finally for semi-regular matrices we have the following results.

Proposition 10 ([48])**.** *If A is a non-negative, admissible matrix, such that $d_A(\mathbb{N}) \neq 0$, then $\mathcal{I}(A) = \{B \subseteq \mathbb{N} : d_A(B) = 0\}$ is an ideal on $\mathbb{N}$. We call it the matrix ideal generated by A.*

Theorem 28 ([48])**.** *Every member of $\mathcal{I}(ST1)$ is also a $F_{\sigma\delta}P$-ideal whereas members of $\mathcal{I}(ST2)$ are $F_{\sigma\delta}$ but not necessarily P-ideal.*

Theorem 29 ([48])**.** *(1) $\mathcal{I}(REG) \cup \mathcal{I}(ST1) \cup \mathcal{I}(ST2) \subseteq \mathcal{I}(F_{\sigma\delta})$ where $\mathcal{I}(F_{\sigma\delta})$ stands for all $F_{\sigma\delta}$ ideals;*
(2) $\mathcal{I}(REG) \cup \mathcal{I}(ST1) \subseteq \mathcal{I}(P)$. where $\mathcal{I}(P)$ stands for the set of all P-ideals.

Now coming back to the folklore Connor's conjecture, it is not actually true. In [48] the following examples were considered to present an overall picture when all the ideals generated by non-negative regular and semi-regular matrices come into picture.

- The most common ideal $\mathcal{I}_d$ (w.r.t natural density) is in $\mathcal{I}(REG)$ which can not be generated by any semi-regular matrix;
- $\mathcal{I}_{1/n}$ is a dense $F_{\sigma\delta}P$-ideal belonging to $\mathcal{I}(ST2)$ which can not be generated by any regular matrix or semi-regular matrix of type 1;
- $\mathcal{I}_{1/n} \oplus Fin$ is a non-dense $F_{\sigma\delta}P$-ideal belonging to $\mathcal{I}(ST2)$ which can not be generated by any regular matrix or semi-regular matrix of type 1;
- $Fin \otimes \emptyset$ is a $F_{\sigma\delta}$ non P-ideal generated by a semi-regular matrix of type 2;
- $\mathcal{I}_u \oplus Fin$ is an ideal which can not be generated by any matrix, regular or semi-regular.

However there are several interesting observations regarding ideals generated by regular, as well as semi-regular matrices which give a clearer picture as to the relationships, as well as behaviors of the generated ideals. When we consider the summable ideals, we have the following observations.

Proposition 11 ([5])**.** *$\mathcal{I}(SUM) \cap \mathcal{I}(DENSE) \cap \mathcal{I}(REG) = \emptyset$.*

Proposition 12 ([48])**.** *$\mathcal{I}(SUM) \subseteq \mathcal{I}(ST2)$*

Proposition 13 ([48])**.** *(i) $\mathcal{I}(ST1) \cup I(ST2) \subseteq \mathcal{I}(SUM - EXT)$;*
(ii) $\mathcal{I}(REG) \cap \mathcal{I}(SUM - EXT) \subseteq \mathcal{I}(ST2)$.
where $\mathcal{I}(SUM - EXT)$ stands for the class of all ideals which are extendable to summable ideals.

Proposition 14 ([48])**.** *$\mathcal{I}(ST1) \subseteq \mathcal{I}(ST2)$ and, in fact, $\mathcal{I}(REG) \cap \mathcal{I}(ST2) = \mathcal{I}(ST1)$.*

Proposition 15 ([48])**.** *(1) If $\mathcal{I}, \mathcal{J} \in \mathcal{I}(REG)$, then $\mathcal{I} \oplus \mathcal{J} \in \mathcal{I}(REG)$;*

(2) *If* $\mathcal{I} \in \mathcal{I}(ST1)$ *and* $\mathcal{J} \in \mathcal{I}(REG) \cup \mathcal{I}(ST1)$*, then* $\mathcal{I} \oplus \mathcal{I} \in \mathcal{I}(ST1)$*;*
(3) *If* $\mathcal{I} \in \mathcal{I}(ST2)$ *and* $\mathcal{J} \in \mathcal{I}(REG) \cup \mathcal{I}(ST1) \cup \mathcal{I}(ST2)$*, then* $\mathcal{I} \oplus \mathcal{I} \in \mathcal{I}(ST1)$*. I(ST2).*

Several other interesting observations can be seen from [48], in particular when some other important classes of ideals come into picture which we skip here.

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