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Degenerated and Competing Dirichlet Problems with Weights and Convection

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Abstract: This paper focuses on two Dirichlet boundary value problems whose differential operators in the principal part exhibit a lack of ellipticity and contain a convection term (depending on the solution and its gradient). They are driven by a degenerated (p, q)-Laplacian with weights and a competing (p, q)-Laplacian with weights, respectively. The notion of competing (p, q)-Laplacians with weights is considered for the first time. We present existence and approximation results that hold under the same set of hypotheses on the convection term for both problems. The proofs are based on weighted Sobolev spaces, Nemytskij operators, a fixed point argument and finite dimensional approximation. A detailed example illustrates the effective applicability of our results.

Keywords: degenerated (p, q)-Laplacian; competing (p, q)-Laplacian; weighted Sobolev space; convection; finite dimensional approximation; weak solution; generalized solution

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1. Introduction

Consider a bounded domain Ω in $\mathbb{R}^N$ $(N \geq 1)$ with a Lipschitz boundary $\partial\Omega$, numbers $1 < q < p < \infty$, functions $a, b \in L^1(\Omega)$ with $a(x), b(x) > 0$ for a.e. $x \in \Omega$ and a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ (i.e., $f(\cdot, t, \xi)$ is measurable on Ω for each $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $f(x, \cdot, \cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$ for a.e. $x \in \Omega$). The aim of this paper is to investigate the quasilinear Dirichlet problems

$$\begin{cases} -\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u + b(x)|\nabla u|^{q-2}\nabla u) = f(x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

and

$$\begin{cases} -\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u - b(x)|\nabla u|^{q-2}\nabla u) = f(x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Notice that problem (1) is driven by a sum of weighted p-Laplacians, whereas problem (2) by a difference of weighted p-Laplacians. The weights $a \in L^1(\Omega)$ and $b \in L^1(\Omega)$ are strongly related to the ellipticity property, but act in a fundamentally different way in these problems. The celebrated p-Laplacian and q-Laplacian are used instead of more general operators in the above formulations just to highlight the main ideas.

The differential operator in the principal part of Equation (1) is the sum

$$u \mapsto \mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u) + \mathrm{div}(b(x)|\nabla u|^{p-2}\nabla u)$$

of the degenerated p-Laplacian with weight $a \in L^1(\Omega)$ and the degenerated q-Laplacian with weight $b \in L^1(\Omega)$ that should be consistent. This operator was introduced in [1] where it was called the degenerated (p, q)-Laplacian with weights $a, b \in L^1(\Omega)$. Its construction is reviewed in Section 2. The characteristic property of this operator is the degeneracy, meaning that one cannot guarantee the existence of a constant $k > 0$ to have

$$\langle -\mathrm{div}(a(x)|\nabla u|^{p-2} + b(x)|\nabla u|^{q-2})\nabla u), u\rangle \geq k \int_\Omega (|\nabla u(x)|^p + |\nabla u(x)|^q) dx.$$

Due to this, one cannot apply the classical elliptic theory.

The differential operator in the principal part of Equation (2) is the difference

$$u \mapsto \mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u) - \mathrm{div}(b(x)|\nabla u|^{p-2}\nabla u)$$

of the degenerated p-Laplacian with weight $a \in L^1(\Omega)$ and of the degenerated q-Laplacian with weight $b \in L^1(\Omega)$. Such a nonlinear operator with weights is considered for the first time. We call it the competing (p,q)-Laplacian with weights $a, b \in L^1(\Omega)$. In this case, we go beyond the degeneracy, actually completely dropping the ellipticity because the quantity

$$\langle -\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u - b(x)|\nabla u|^{q-2}\nabla u), u\rangle = \int_\Omega (a(x)|\nabla u(x)|^p - b(x)|\nabla u(x)|^q) dx$$

can have an arbitrary sign (note that $a(x)$ and $b(x)$ are positive). For problem (2), any method of monotone type, including the use of pseudomonotone operators, fails to apply.

The right-hand side $f(x, u, \nabla u)$ of the equations in (1) and (2) is a convection term; that is, it depends on the solution u and on its gradient ∇u. The dependence on the gradient ∇u generally prevents having a variational structure for problems (1) and (2), so the variational methods are not applicable. In order to find the needed estimates, an essential part of our development is devoted to the Nemytskij operator associated with the convection term $f(x, u, \nabla u)$ under an appropriate growth condition for the function $f(x, t, \xi)$ on $\Omega \times \mathbb{R} \times \mathbb{R}^N$. Different results regarding unweighted problems involving (p,q)-Laplacian and convection terms can be found in [2].

The problems (1) and (2) have only recently been regarded in their generality. To the best of our knowledge, there is solely the existence theorem for problem (1), obtained in [1] through the theory of pseudomonotone operators. For the particular case of (1) where the equation is governed by a degenerated p-Laplacian (i.e., $b = 0$ in (1)), existence results based on minimization and degree theoretic methods can be found in [3] and a method to create a sub-supersolution was developed in [4]. Concerning problem (2) driven by competing operators, there is no available result except for the most particular situation where $a(x) = b(x) \equiv 1$ in Ω (i.e., the problem without weights), whose study was initiated in [5] and continued in [6,7].

In the present paper, we overcome the lack of ellipticity, monotonicity and variational structure in problems (1) and (2) by means of a passing to limit process involving approximate solutions generated through fixed point arguments on finite dimensional spaces. This approach was implemented in [6,7] for unweighted problems (i.e., $a(x) = b(x) \equiv 1$ in Ω). Here, the development is substantially modified due to the completely different functional setting under the weights $a \in L^1(\Omega)$ and $b \in L^1(\Omega)$.

For problem (1), we are able to establish the existence of a solution in a weak sense, whereas for problem (2), we prove the existence of a solution in a generalized sense. It is worth noting that in the case of problem (1) any generalized solution is a weak solution. Moreover, our results can be viewed as providing approximations in the sense of strong convergence for solutions to problems (1) and (2) by finite dimensional approximate solutions.

Inspired by [3], a major step in our treatment is a reduction within the framework of classical Sobolev spaces. We impose a suitable growth condition for the convection term $f(x, u, \nabla u)$ to match this reduction. The growth condition is expressed using a positive quantity (p_s in the text) described by the weights $a \in L^1(\Omega)$ and $b \in L^1(\Omega)$, which provide the best integrability rate.

We plan to use the present work for studying evolutionary counterparts for problems (1) and (2).

The rest of the paper is organized as follows. Section 2 presents the degenerated and competing (p,q)-Laplacians with weights. Section 3 sets forth the associated Nemytskij operator. Section 4 contains our main result on the solvability and approximation for problem (1). Section 5 focuses on the solvability of problem (2). Section 6 illustrates by an example the effective applicability of our theorems.

2. Degenerated and Competing (P,Q)-Laplacians with Weights

Throughout the text, we denote by $\to$ the strong convergence and by $\rightharpoonup$ the weak convergence in any normed space X under consideration. The norm on X is denoted by $\|\cdot\|_X$, while the notation $\langle\cdot,\cdot\rangle_X$ stands for the duality pairing between X and its dual X^*. For the rest of the paper, by a bounded map we understand a map between normed spaces that maps bounded sets to bounded sets.

We fix the framework for the underlying weighted Sobolev spaces related to problems (1) and (2). For a systematic study of weighted Sobolev spaces, we refer to [3,8]. The completeness property for such spaces is discussed in [9]. This functional setting was also discussed in [1].

Given a real number $p \in (1,+\infty)$ and a positive function $a \in L^1(\Omega)$, the weighted space

$$W^{1,p}(a,\Omega) := \{u \in L^p(\Omega) : \int_\Omega a(x)|\nabla u(x)|^p dx < \infty\},$$

is endowed with the norm

$$\|u\|_{W^{1,p}(a,\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + \int_\Omega a(x)|\nabla u(x)|^p dx\right)^{\frac{1}{p}}, \quad \forall u \in W^{1,p}(a,\Omega).$$

We note that $C_0^\infty(\Omega) \subset W^{1,p}(a,\Omega)$. The closure of $C_0^\infty(\Omega)$ in $W^{1,p}(a,\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p}(a,\Omega)}$ is the space $W_0^{1,p}(a,\Omega)$. The dual spaces of $W^{1,p}(a,\Omega)$ and $W_0^{1,p}(a,\Omega)$ are denoted by $W^{1,p}(a,\Omega)^*$ and $W_0^{1,p}(a,\Omega)^*$, respectively.

A reduction in the setting of classical Sobolev spaces is based on the following condition from [3] (p. 26):

(H1). $a^{-s} \in L^1(\Omega)$ for some $s \in \left(\max\left\{\frac{N}{p}, \frac{1}{p-1}\right\}, +\infty\right)$.

Proposition 1. *Under condition (H1), there are the continuous embeddings*

$$W^{1,p}(a,\Omega) \hookrightarrow W^{1,p_s}(\Omega) \hookrightarrow L^p(\Omega), \tag{3}$$

where

$$p_s = \frac{ps}{s+1}. \tag{4}$$

In addition, the embedding $W^{1,p_s}(\Omega) \hookrightarrow L^p(\Omega)$ is compact. Furthermore,

$$\|u\|_{W_0^{1,p}(a,\Omega)} := \left(\int_\Omega a(x)|\nabla u(x)|^p dx\right)^{\frac{1}{p}}, \quad \forall u \in W_0^{1,p}(a,\Omega),$$

is an equivalent norm on $W_0^{1,p}(a,\Omega)$ for which $W_0^{1,p}(a,\Omega)$ becomes a uniformly convex Banach space.

Proof. The proof is essentially completed in [3]. For the sake of clarity, we highlight aspects relevant for problems (1) and (2).

It can be seen from (4) that $p_s > 1$ if and only if $s > 1/(p-1)$, which by assumption (H1) is true. In order to prove the first inclusion in (3), let $u \in W^{1,p}(a,\Omega)$. Using Hölder's inequality, hypothesis (H1) and (4) (note $p_s < p$), we infer that

$$\begin{aligned}
&\int_\Omega |\nabla u(x)|^{p_s} dx = \int_\Omega (a(x)^{\frac{p_s}{p}} |\nabla u(x)|^{p_s}) a(x)^{-\frac{p_s}{p}} dx \\
&\leq \left(\int_\Omega a(x) |\nabla u(x)|^p dx \right)^{\frac{p_s}{p}} \left(\int_\Omega a(x)^{-\frac{p_s}{p-p_s}} dx \right)^{\frac{p-p_s}{p}} \\
&\leq \|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{s+1}} \|u\|_{W^{1,p}(a,\Omega)}^{p_s}, \quad \forall u \in W_0^{1,p}(a,\Omega).
\end{aligned}$$

The continuous inclusion $W^{1,p}(a,\Omega) \hookrightarrow W^{1,p_s}(\Omega)$ is proven.

The Rellich–Kondrachov embedding theorem ensures the compact embedding $W^{1,p_s}(\Omega) \hookrightarrow L^r(\Omega)$, with $1 \leq r < p_s^*$, where p_s^* is the critical exponent corresponding to p_s, that is,

$$p_s^* := \begin{cases} \frac{Np_s}{N-p_s} & \text{if } N > p_s (\Leftrightarrow ps < N(s+1)) \\ +\infty & \text{if } N \leq p_s (\Leftrightarrow ps \geq N(s+1)). \end{cases}$$

We have that $p_s^* > p$ if and only if $s > N/p$. Since the latter holds by assumption (H1), the compactness of the second inclusion in (3) follows.

The desired equivalence of norms is a consequence of (3) and the Poincaré inequality on $W_0^{1,p_s}(\Omega)$ because with a positive constant C,

$$\|u\|_{L^p(\Omega)} \leq C \|u\|_{W^{1,p}(a,\Omega)}, \quad \forall u \in W_0^{1,p}(a,\Omega).$$

It remains to show that $W_0^{1,p}(a,\Omega)$ is a uniformly convex Banach space. It suffices to have $a^{-\frac{1}{p-1}} \in L^1(\Omega)$ (see [3] Theorem 1.3). From hypothesis (H1), it is known that $a^{-s} \in L^1(\Omega)$ with $s > 1/(p-1)$, which results in

$$\begin{aligned}
&\int_\Omega a(x)^{-\frac{1}{p-1}} dx = \int_{\{a(x)<1\}} a(x)^{-\frac{1}{p-1}} dx + \int_{\{a(x)\geq 1\}} a(x)^{-\frac{1}{p-1}} dx \\
&\leq \int_\Omega a(x)^{-s} dx + \operatorname{meas}(\Omega) < \infty,
\end{aligned}$$

thus completing the proof. □

The degenerated p-Laplacian with the weight $a \in L^1(\Omega)$ is defined as the map $\Delta_p^a : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ given by $\Delta_p^a(u) = \operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u)$ for all $u \in W_0^{1,p}(a,\Omega)$, i.e.,

$$\langle -\Delta_p^a(u), v \rangle_{W_0^{1,p}(a,\Omega)} = \int_\Omega a(x) |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) dx, \quad \forall u, v \in W_0^{1,p}(a,\Omega).$$

The definition makes sense as can be seen through Hölder's inequality

$$\begin{aligned}
&\left| \int_\Omega a(x) |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) dx \right| \\
&\leq \int_\Omega (a(x)^{\frac{p-1}{p}} |\nabla u(x)|^{p-1}) (a(x)|^{\frac{1}{p}} |\nabla v(x)|) dx \\
&\leq \left(\int_\Omega a(x) |\nabla u(x)|^p dx \right)^{\frac{p-1}{p}} \left(\int_\Omega a(x) |\nabla v(x)|^p dx \right)^{\frac{1}{p}} < \infty, \quad \forall u, v \in W_0^{1,p}(a,\Omega).
\end{aligned}$$

The ordinary p-Laplacian is recovered when $a(x) \equiv 1$ in Ω.

The degenerated p-Laplacian $\Delta_p^a : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ is continuous and bounded. We denote by λ_1 the first eigenvalue of $-\Delta_p^a : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ corresponding to the weight $a \in L^1(\Omega)$ with $a^{-\frac{1}{p-1}} \in L^1(\Omega)$. Specifically, λ_1 is the least $\lambda > 0$ for which the problem

$$\begin{cases} -\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

possesses a nontrivial solution. It can be variationally characterized as

$$\lambda_1 = \inf_{u \in W_0^{1,p}(a,\Omega)\setminus\{0\}} \frac{\int_\Omega a(x)|\nabla u(x)|^p dx}{\|u\|_{L^p(\Omega)}^p}. \tag{5}$$

More details on the degenerated p-Laplacian with weight can be seen in [3].

For the positive weights $a \in L^1(\Omega)$ and $b \in L^1(\Omega)$ entering problems (1) and (2), we have the degenerated p-Laplacian $\Delta_p^a : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ with weight $a \in L^1(\Omega)$ and the degenerated q-Laplacian $\Delta_q^b : W_0^{1,2}(b,\Omega) \to W_0^{1,2}(b,\Omega)^*$ with weight $b \in L^1(\Omega)$. The two operators need to be consistent, which is achieved under the following compatibility condition for the weights:

(H2). $1 < q < p < +\infty$ and $a^{-\frac{q}{p-q}} b^{\frac{p}{p-q}} \in L^1(\Omega)$.

Proposition 2. *Assume that condition (H2) holds. Then, one has the continuous embedding* $W_0^{1,p}(a,\Omega) \hookrightarrow W_0^{1,q}(b,\Omega)$.

Proof. By hypothesis (H2) and Hölder's inequality, we infer that

$$\begin{aligned} &\int_\Omega b(x)|\nabla u(x)|^q dx = \int_\Omega (a(x)^{-\frac{q}{p}} b(x))(a(x)^{\frac{q}{p}}|\nabla u(x)|^q) dx \\ &\le \left(\int_\Omega a(x)^{-\frac{q}{p-q}} b(x)^{\frac{p}{p-q}} dx\right)^{\frac{p-q}{p}} \left(\int_\Omega a(x)|\nabla u(x)|^p dx\right)^{\frac{q}{p}} \\ &\le \|a^{-\frac{q}{p-q}} b^{\frac{p}{p-q}}\|_{L^1(\Omega)}^{\frac{p-q}{p}} \|u\|_{W^{1,p}(a,\Omega)}^q, \quad \forall u,v \in W_0^{1,p}(a,\Omega), \end{aligned}$$

which proves the result. □

Under condition (H2), on the basis of Proposition 2, the map $\Delta_p^a + \Delta_q^b : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ called the degenerated (p,q)-Laplacian with weights $a, b \in L^1(\Omega)$ is well-defined. It is given by

$$\begin{aligned} &\langle -(\Delta_p^a + \Delta_q^b)u, v\rangle_{W_0^{1,p}(a,\Omega)} \\ &= \int_\Omega (a(x)|\nabla u(x)|^{p-2}\nabla u(x) + b(x)|\nabla u(x)|^{q-2}\nabla u(x))\nabla v(x) dx, \quad \forall u,v \in W_0^{1,p}(a,\Omega). \end{aligned} \tag{6}$$

The degenerated (p,q)-Laplacian with weights $a, b \in L^1(\Omega)$ was introduced in [1].

Again on the basis of Proposition 2, the map $\Delta_p^a - \Delta_q^b : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ given by

$$\begin{aligned} &\langle -(\Delta_p^a - \Delta_q^b)u, v\rangle_{W_0^{1,p}(a,\Omega)} \\ &= \int_\Omega (a(x)|\nabla u(x)|^{p-2}\nabla u(x) + b(x)|\nabla u(x)|^{q-2}\nabla u(x))\nabla v(x) dx, \quad \forall u,v \in W_0^{1,p}(a,\Omega), \end{aligned} \tag{7}$$

is well-defined provided condition (H2) is satisfied. We call it the competing (p,q)-Laplacian with weights $a, b \in L^1(\Omega)$ and is introduced here for the first time.

Proposition 3. *Under assumption (H2), the maps* $\Delta_p^a + \Delta_q^b : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ *and* $\Delta_p^a - \Delta_q^b : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ *are continuous and bounded. In addition, under (H1) and (H2), the* $(S)_+$ *property holds for the map* $-(\Delta_p^a + \Delta_q^b) : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$*; that is, any sequence* $\{u_n\} \subset W_0^{1,p}(a,\Omega)$ *satisfying* $u_n \rightharpoonup u$ *in* $W_0^{1,p}(a,\Omega)$ *and*

$$\limsup_{n\to\infty} \langle -(\Delta_p^a + \Delta_q^b)u_n, u_n - u\rangle_{W_0^{1,p}(a,\Omega)} \le 0 \tag{8}$$

is strongly convergent. Thus, $u_n \to u$ *in* $W_0^{1,p}(a,\Omega)$.

Proof. Due to the continuous embedding $W^{1,p}(a,\Omega) \hookrightarrow W^{1,q}(b,\Omega)$ in Proposition 2, $\Delta_p^a + \Delta_q^b$ and $\Delta_p^a - \Delta_q^b$ inherit the continuity and boundedness from Δ_p^a and Δ_q^b.

For the second part of the statement, let a sequence $\{u_n\} \subset W_0^{1,p}(a,\Omega)$ with the required properties. By (6), the monotonicity of $-\Delta_q^b$ and Hölder's inequality, we obtain

$$\begin{aligned}
&\langle -(\Delta_p^a + \Delta_q^b)(u_n) + (\Delta_p^a + \Delta_q^b)(u), u_n - u\rangle_{W_0^{1,p}(a,\Omega)} \\
&\ge \langle -\Delta_p^a(u_n) + \Delta_p^a(u), u_n - u\rangle_{W_0^{1,p}(a,\Omega)} \\
&\ge (\|u_n\|_{W_0^{1,p}(a,\Omega)} - \|u\|_{W_0^{1,p}(a,\Omega)})(\|u_n\|^{p-1}_{W_0^{1,p}(a,\Omega)} - \|u\|^{p-1}_{W_0^{1,p}(a,\Omega)}) \ge 0.
\end{aligned}$$

It follows from the above estimate, (8) and $u_n \rightharpoonup u$ in $W_0^{1,p}(a,\Omega)$ that there holds $\lim_{n\to+\infty} \|u_n\|_{W_0^{1,p}(a,\Omega)} = \|u\|_{W_0^{1,p}(a,\Omega)}$. From Proposition 1, we know that the space $W_0^{1,p}(a,\Omega)$ is uniformly convex. Therefore, we can conclude that $u_n \to u$ in $W_0^{1,p}(a,\Omega)$. □

3. An Associated Nemytskij Operator

In this section we focus on the right-hand side of the equations in (1) and (2), i.e., the convection term $f(x,u,\nabla u)$. Our goal is to identify the growth condition for the function $f(x,t,\xi)$ to match the reduction in Proposition 1 to the unweighted Sobolev space $W_0^{1,p_s}(\Omega)$. The appropriate growth for $f(x,t,\xi)$ is the one used in [1].

In order to simplify the presentation, for any real number $r > 1$, we denote $r' := r/(r-1)$ (the Hölder conjugate of r). This convention will be preserved for the rest of the paper.

Lemma 1. *Assume (H1) and (H2) and in addition that the Carathéodory function* $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *satisfies the growth condition:*

(***H3***).

$$|f(x,t,\xi)| \le \sigma(x) + c_1|t|^\alpha + c_2|\xi|^\beta \text{ for a.e } x \in \Omega,\ \ \forall (t,\xi) \in \mathbb{R}\times\mathbb{R}^N, \tag{9}$$

with $\sigma \in L^{\gamma'}(\Omega)$ *for* $\gamma \in (1, p_s^*)$ *and constants* $c_1 > 0$, $c_2 > 0$, $\alpha \in [0, p_s^* - 1)$, $\beta \in [0, \frac{p_s}{(p_s^*)'})$.

Set

$$\theta := \min\left\{\gamma', \frac{p_s^*}{\alpha}, \frac{p_s}{\beta}\right\}. \tag{10}$$

Then, the Nemytskij operator $\mathcal{N}_f : L^{p_s^*}(\Omega) \times (L^{p_s}(\Omega))^N \to L^\theta(\Omega)$ *associated with the function* f *which is given by*

$$\mathcal{N}_f(v,z) = f(\cdot, v(\cdot), z(\cdot)), \quad \forall v,z \in L^{p_s^*}(\Omega)\times(L^{p_s}(\Omega))^N \tag{11}$$

is well-defined, continuous and bounded.

Proof. The requirements in (H3) postulate $\gamma' > 1$ (note $\gamma > 1$),

$$\frac{p_s^*}{\alpha} > \frac{p_s^*}{p_s^* - 1} = (p_s^*)' > 1, \; \frac{p_s}{\beta} > (p_s^*)' > 1$$

(note that $p_s^* > p > 1$). Then, a consequence of (10) is that $\theta > 1$.

We observe that (9) yields

$$|f(x,t,\xi)| \le \tilde{\sigma}(x) + c_1|t|^{\frac{p_s^*}{\theta}} + c_2|\xi|^{\frac{p_s}{\theta}} \text{ for a.e } x \in \Omega, \; \forall (t,\xi) \in \mathbb{R} \times \mathbb{R}^N, \tag{12}$$

with $\tilde{\sigma} \in L^\theta(\Omega)$. Indeed, (10) gives

$$\alpha \le \frac{p_s^*}{\theta} \text{ and } \beta \le \frac{p_s}{\theta}.$$

Hence (12) is derived from (9) with $\tilde{\sigma}(x) = \sigma(x) + c_1 + c_2$ for a.e. $x \in \Omega$ obtaining $\tilde{\sigma} \in L^{\gamma'}(\Omega) \subset L^\theta(\Omega)$.

Using Krasnoselskij's theorem, we infer from (12) that $\mathcal{N}_f$ introduced in (11) has the required properties, thus proving the result. □

Let $N_f : W^{1,p}(a,\Omega) \to W^{1,p}(a,\Omega)^*$ be defined by

$$\langle N_f(u), v\rangle_{W^{1,p}(a,\Omega)} = \int_\Omega f(x,u(x),\nabla u(x))v(x)dx, \quad \forall u,v \in W_0^{1,p}(a,\Omega).$$

Due to the first inclusion in (3), it holds that $(u,\nabla u) \in L^{p_s^*}(\Omega) \times (L^{p_s}(\Omega))^N$ whenever $u \in W_0^{1,p}(a,\Omega)$. It turns out

$$\langle N_f(u), v\rangle_{W^{1,p}(a,\Omega)} = \langle \mathcal{N}_f(u,\nabla u), v\rangle_{L^\theta(\Omega)}, \quad \forall u,v \in W_0^{1,p}(a,\Omega). \tag{13}$$

The assertion below provides a key tool for investigating problems (1) and (2).

Proposition 4. *Assume (H1)–(H3). If $u_n \rightharpoonup u$ in $W_0^{1,p}(a,\Omega)$, it holds that*

$$\lim_{n\to\infty} \langle N_f(u_n), u_n - u\rangle_{W_0^{1,p}(a,\Omega)} = 0. \tag{14}$$

Proof. Recalling the convention made in the beginning of this section, (10) entails

$$\theta' = \max\left\{\gamma, \left(\frac{p_s^*}{\alpha}\right)', \left(\frac{p_s}{\beta}\right)'\right\}. \tag{15}$$

By (H3) we have $\gamma < p_s^*$,

$$\left(\frac{p_s^*}{\alpha}\right)' = \frac{\frac{p_s^*}{\alpha}}{\frac{p_s^*}{\alpha} - 1} = \frac{p_s^*}{p_s^* - \alpha} < p_s^*,$$

$$\left(\frac{p_s}{\beta}\right)' = \frac{\frac{p_s}{\beta}}{\frac{p_s}{\beta} - 1} = \frac{p_s}{p_s - \beta} < \frac{p_s}{p_s - \frac{p_s}{(p_s^*)'}} = \frac{(p_s^*)'}{(p_s^*)' - 1} = p_s^*.$$

Hence, (15) yields $1 < \theta' < p_s^*$ and we can apply the Rellich–Kondrachov compact embedding theorem to deduce that the embedding $W^{1,p_s}(\Omega) \hookrightarrow L^{\theta'}(\Omega)$ is compact, which results in $u_n \to u$ in $L^{\theta'}(\Omega)$.

Since (13) implies that

$$|\langle N_f(u_n), u_n - u\rangle_{W^{1,p}(a,\Omega)}| \le \|\mathcal{N}_f(u_n,\nabla u_n)\|_{L^\theta(\Omega)} \|u_n - u\|_{L^{\theta'}(\Omega)}. \tag{16}$$

and Lemma 1 ensures that $\{\mathcal{N}_f(u_n, \nabla u_n)\}$ is bounded in $L^{\theta}(\Omega)$, from (16) we arrive at (14), as desired. □

4. Solvability and Approximation for the Degenerate Elliptic Problem (1)

The object of this section is to develop an approach based on finite dimensional approximations for problem (1).

Since the Banach space $W_0^{1,p}(a,\Omega)$ is separable (see Section 2), there exists a Galerkin basis for it. This amounts to saying that there is a sequence $\{X_n\}$ of vector subspaces of $W_0^{1,p}(a,\Omega)$ such that

(i) $\dim(X_n) < \infty, \quad \forall n;$
(ii) $X_n \subset X_{n+1}, \quad \forall n;$
(iii)

$$\overline{\bigcup_n X_n} = W_0^{1,p}(a,\Omega).$$

We fix such a sequence of subspaces $\{X_n\}$. Each approximate problem on X_n will be resolved by means of a consequence of Brouwer's fixed point theorem.

Proposition 5. *Assume the conditions (H1)–(H3) and in addition*

(H4). *there exists $\rho \in L^1(\Omega)$ and constants $d_1 > 0$ and $d_2 > 0$ provided $\lambda_1^{-1}d_1 + d_2 < 1$, where λ_1 denotes the first eigenvalue of $-\Delta_p^a$ on $W_0^{1,p}(a,\Omega)$, such that*

$$f(x,t,\xi)t \leq \rho(x) + d_1|t|^p + d_2 a(x)|\xi|^p \tag{17}$$

for a.e $x \in \Omega$ and all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

Then for each n there exists $u_n \in X_n$ such that

$$\langle -(\Delta_p^a + \Delta_q^b)(u_n), v\rangle_{W_0^{1,p}(a,\Omega)} = \int_\Omega f(x,u_n(x),\nabla u_n(x))v(x)dx, \quad \forall v \in X_n. \tag{18}$$

Proof. For each n, consider the continuous map $A_n : X_n \to X_n^*$ defined by

$$\langle A_n(u), v\rangle_{X_n} = \langle -(\Delta_p^a + \Delta_q^b)(u), v\rangle_{W_0^{1,p}(a,\Omega)} - \int_\Omega f(x,u(x),\nabla u(x))v(x)dx, \quad \forall v \in X_n.$$

The definition of the operator A_n, (17) and (5) lead to

$$\begin{aligned}
\langle A_n(v), v\rangle_{X_n} &= \int_\Omega (a(x)|\nabla v|^p + b(x)|\nabla v|^q - f(x,v,\nabla v)v)dx \\
&\geq \|v\|^p_{W_0^{1,p}(a,\Omega)} - \|\rho\|_{L^1(\Omega)} - d_1\|v\|^p_{L^p(\Omega)} - d_2\|v\|^p_{W_0^{1,p}(a,\Omega)} \\
&\geq (1 - d_1\lambda_1^{-1} - d_2)\|v\|^p_{W_0^{1,p}(a,\Omega)} - \|\rho\|_{L^1(\Omega)}, \quad \forall v \in X_n.
\end{aligned}$$

Thanks to the assumption $1 - d_1\lambda_1^{-1} - d_2 > 0$ in (H4), it follows that

$$\langle A_n(v), v\rangle_{X_n} \geq 0 \text{ whenever } v \in X_n \text{ with } \|v\|_{W_0^{1,p}(a,\Omega)} = R$$

provided $R = R(n) > 0$ is sufficiently large. In view of the fact that X_n is a finite dimensional space, by a well-known consequence of Brouwer's fixed point theorem (see, e.g., [10] (p. 37)) there exists $u_n \in X_n$ solving the equation $A_n(u_n) = 0$. This means exactly that $u_n \in X_n$ is a solution for problem (18), which completes the proof. □

We are in a position to state our main result on problem (1).

Theorem 1. *Assume that the conditions (H1)–(H4) are fulfilled. Then, the sequence* $\{u_n\}$*, with* $u_n \in X_n$ *constructed in Proposition 5, contains a subsequence which is strongly convergent in* $W_0^{1,p}(a,\Omega)$ *to a weak solution of problem (1) meaning that*

$$\int_\Omega (a(x)|\nabla u|^{p-2}\nabla u + b(x)|\nabla u|^{q-2}\nabla u)\nabla v dx = \int_\Omega f(x,u,\nabla u)v dx \tag{19}$$

for all $v \in W_0^{1,p}(a,\Omega)$.

Proof. We claim that the sequence $\{u_n\}$ built in Proposition 5 is bounded in $W_0^{1,p}(a,\Omega)$. Acting with $v = u_n$ in (18) gives

$$\|u_n\|^p_{W_0^{1,p}(a,\Omega)} + \|u_n\|^q_{W_0^{1,q}(a,\Omega)} = \int_\Omega f(x,u_n,\nabla u_n)u_n dx.$$

Then, through (17) and (5) we obtain

$$\begin{aligned}\|u_n\|^p_{W_0^{1,p}(a,\Omega)} &\le \|\rho\|_{L^1(\Omega)} + d_1\|u\|^p_{L^p(\Omega)} + d_2\|u\|^p_{W_0^{1,p}(a,\Omega)}\\ &\le \|\rho\|_{L^1(\Omega)} + (d_1\lambda_1^{-1} + d_2)\|u\|^p_{W_0^{1,p}(a,\Omega)}.\end{aligned}$$

Thanks to $\lambda_1^{-1}d_1 + d_2 < 1$, as known from hypothesis (H4), the claim is verified.

Recall from Proposition 1 that $W_0^{1,p}(a,\Omega)$ is a uniformly convex Banach space, so it is reflexive. Hence, the bounded sequence $\{u_n\}$ possesses a subsequence denoted again $\{u_n\}$ such that for some $u \in W_0^{1,p}(a,\Omega)$ it holds $u_n \rightharpoonup u$ in $W_0^{1,p}(a,\Omega)$.

Proposition 3 and Lemma 1 ensure that the operators $\Delta_p^a + \Delta_q^b : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ and $N_f : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ are bounded. Then, in view of the reflexivity of $W_0^{1,p}(a,\Omega)$ along a relabeled subsequence, one has

$$-(\Delta_p^a + \Delta_q^b)(u_n) - N_f(u_n) \rightharpoonup \eta \text{ in } W_0^{1,p}(a,\Omega)^* \tag{20}$$

for some $\eta \in W_0^{1,p}(a,\Omega)^*$.

Let us prove that $\eta = 0$. For $v \in \bigcup_n X_n$ choose m with $v \in X_m$. According to Proposition 5 and property (ii) in the definition of Galerkin basis, we may apply (18) for all $n \ge m$, which reads as

$$\langle -(\Delta_p^a + \Delta_q^b)(u_n) - N_f(u_n), v\rangle_{W_0^{1,p}(a,\Omega)} = 0 \text{ for all } n \ge m.$$

Letting $n \to \infty$ enables us to derive from (20) that

$$\langle \eta, v\rangle_{W_0^{1,p}(a,\Omega)} = 0, \quad \forall v \in \bigcup_n X_n.$$

The property (iii) in the definition of Galerkin basis $\{X_n\}$ highlights the density of the set $\bigcup_n X_n$ in $W_0^{1,p}(a,\Omega)$. As η vanishes on $\bigcup_n X_n$, it follows that $\eta = 0$.

Therefore, (20) becomes

$$-(\Delta_p^a + \Delta_q^b)(u_n) - N_f(u_n) \rightharpoonup 0 \text{ in } W_0^{1,p}(a,\Omega)^*. \tag{21}$$

In particular, we have

$$\lim_{n\to\infty} \langle -(\Delta_p^a + \Delta_q^b)(u_n) - N_f(u_n), u\rangle_{W_0^{1,p}(a,\Omega)} = 0. \tag{22}$$

Now, we return to (18) and insert $v = u_n$, obtaining

$$\langle -(\Delta_p^a + \Delta_q^b)(u_n) - N_f(u_n), u_n \rangle_{W_0^{1,p}(a,\Omega)} = 0, \quad \forall n,$$

which in conjunction with (22) yields

$$\lim_{n\to\infty} \langle -(\Delta_p^a + \Delta_q^b)(u_n) - N_f(u_n), u_n - u \rangle_{W_0^{1,p}(a,\Omega)} = 0.$$

Taking into account Proposition 4, this amounts to saying that

$$\lim_{n\to\infty} \langle -(\Delta_p^a + \Delta_q^b)(u_n), u_n - u \rangle_{W_0^{1,p}(a,\Omega)} = 0.$$

Consequently, the sequence $\{u_n\}$ satisfies (8). We are thus allowed to apply Proposition 3 which provides the strong convergence $u_n \to u$ in $W_0^{1,p}(a,\Omega)$.

Using the continuity of the nonlinear operators $\Delta_p^a + \Delta_q^b : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ and $N_f : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ as known by Proposition 2 and Lemma 1, we infer from the strong convergence $u_n \to u$ in $W_0^{1,p}(a,\Omega)$ that

$$-(\Delta_p^a + \Delta_q^b)(u_n) - N_f(u_n) \to -(\Delta_p^a + \Delta_q^b)(u) - N_f(u) \text{ in } W_0^{1,p}(a,\Omega)^*.$$

A simple comparison with (21) confirms that

$$-(\Delta_p^a + \Delta_q^b)(u) - N_f(u) = 0,$$

which is just (19). The proof is complete. □

5. Resolving the Non-Elliptic Problem (2)

Due to the total lack of ellipticity of the competing (p,q)-Laplacian $\Delta_p^a - \Delta_q^b$ with weights $a \in L^1(\Omega)$ and $b \in L^1(\Omega)$ as introduced in (7), i.e., the differential operator $\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u - b(x)|\nabla u|^{q-2}\nabla u)$, when the weights $a(x)$ and $b(x)$ are positive, we are not able to prove the existence of a weak solution for problem (2) in the weak sense. For this reason, we seek a solution in the following generalized sense.

Definition 1. *An element $u \in W_0^{1,p}(a,\Omega)$ is called a generalized solution to problem* (2) *if there exists a sequence $\{u_n\} \subset W_0^{1,p}(a,\Omega)$ such that*

(j) $u_n \rightharpoonup u$ *in* $W_0^{1,p}(\Omega)$;

(jj) *for every $v \in W_0^{1,p}(a,\Omega)$, it holds that*

$$\lim_{n\to\infty} \int_\Omega ((a(x)|\nabla u_n|^{p-2}\nabla u_n - b(x)|\nabla u_n|^{q-2}\nabla u_n)\nabla v - f(x,u_n,\nabla u_n)v)dx = 0;$$

(jjj)

$$\lim_{n\to\infty} \int_\Omega ((a(x)|\nabla u_n|^{p-2}\nabla u_n - b(x)|\nabla u_n|^{q-2}\nabla u_n)\nabla(u_n - u)dx = 0.$$

Our result for the non-elliptic problem (2) is as follows.

Theorem 2. *Assume for $1 < q < p < +\infty$ that the positive weights $a \in L^1(\Omega)$ and $b \in L^1(\Omega)$ and the Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ that the conditions (H1)–(H4) hold. Then, there exists at least a generalized solution $u \in W_0^{1,p}(a,\Omega)$ of problem* (2) *in the sense of Definition 1.*

Proof. The proof is carried over along the pattern of Theorem 1. Fix a Galerkin basis $\{X_n\}$ of $W_0^{1,p}(a,\Omega)$, i.e., a sequence of finite dimensional vector subspaces of $W_0^{1,p}(a,\Omega)$ such that the properties (i)–(iii) in Section 4 hold.

We claim that for each n there exists $u_n \in X_n$ such that

$$\int_\Omega ((a(x)|\nabla u_n|^{p-2}\nabla u_n - b(x)|\nabla u_n|^{q-2}\nabla u_n)\nabla v - f(x,u_n,\nabla u_n)v)dx = 0, \quad \forall v \in X_n. \tag{23}$$

To this end, define the continuous map $B_n : X_n \to X_n^*$ by

$$\langle B_n(u), v\rangle_{X_n} = \int_\Omega ((a(x)|\nabla u|^{p-2}\nabla u - b(x)|\nabla u|^{q-2}\nabla u)\nabla v - f(x,u,\nabla u)v)dx$$

for all $u, v \in X_n$. The continuous embedding $W^{1,p}(a,\Omega) \hookrightarrow W^{1,q}(b,\Omega)$ (see Proposition 2), (17) and (5) imply that

$$\langle B_n(v), v\rangle_{X_n} \geq (1 - d_1\lambda_1^{-1} - d_2)\|v\|^p_{W_0^{1,p}(a,\Omega)} - C\|v\|^q_{W_0^{1,p}(a,\Omega)} - \|\rho\|_{L^1(\Omega)}, \quad \forall v \in X_n.$$

with a constant $C > 0$. By the assumption $1 - d_1\lambda_1^{-1} - d_2 > 0$ in (H4) and the fact that $p > q$, it turns out

$$\langle B_n(v), v\rangle_{X_n} \geq 0 \text{ for all } v \in X_n \text{ with } \|v\|_{W_0^{1,p}(a,\Omega)} = R$$

if $R = R(n) > 0$ is sufficiently large. According to condition (i) in the definition of Galerkin basis, the space X_n is finite dimensional. This enables us to apply a well-known consequence of Brouwer's fixed point theorem (see, e.g., [10] (p. 37)) obtaining a $u_n \in X_n$ with $B_n(u_n) = 0$. Therefore, we obtain (23), thus proving the claim.

Next, we show that the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(a,\Omega)$. Since $u_n \in X_n$, we can take $v = u_n$ as a test function in (23), where

$$\|u_n\|^p_{W_0^{1,p}(a,\Omega)} = \|u_n\|^q_{W_0^{1,q}(a,\Omega)} + \int_\Omega f(x,u_n,\nabla u_n)u_n dx. \tag{24}$$

The continuous embedding $W^{1,p}(a,\Omega) \hookrightarrow W^{1,q}(b,\Omega)$ in Proposition 2, in conjunction with (17) and (5), ensures the estimate

$$\begin{aligned}\|u_n\|^p_{W_0^{1,p}(a,\Omega)} &\leq C\|u_n\|^q_{W_0^{1,p}(a,\Omega)} + \|\rho\|_{L^1(\Omega)} + d_1\|u\|^p_{L^p(\Omega)} + d_2\|u\|^p_{W_0^{1,p}(a,\Omega)} \\ &\leq C\|u_n\|^q_{W_0^{1,p}(a,\Omega)} + \|\rho\|_{L^1(\Omega)} + (d_1\lambda_1^{-1} + d_2)\|u\|^p_{W_0^{1,p}(a,\Omega)},\end{aligned}$$

with a constant $C > 0$. On account of $p > q$ and assumption $\lambda_1^{-1}d_1 + d_2 < 1$ in (H4), we conclude that the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(a,\Omega)$.

Proposition 1 guarantees the reflexivity of the space $W_0^{1,p}(a,\Omega)$. We are thus allowed to extract a subsequence still denoted as $\{u_n\}$ such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ for some $u \in W_0^{1,p}(a,\Omega)$. The requirement (j) in Definition 1 is fulfilled.

Equality (23) expresses that

$$\langle -\Delta_p^a(u_n) + \Delta_q^b(u_n) - N_f(u_n), v\rangle_{W_0^{1,p}(a,\Omega)} = 0, \quad \forall v \in X_n. \tag{25}$$

Inserting $v = u_n$ in (25) leads to

$$\langle -\Delta_p^a(u_n) + \Delta_q^b(u_n) - N_f(u_n), u_n\rangle_{W_0^{1,p}(a,\Omega)} = 0, \quad \forall n. \tag{26}$$

The sequence $\{(-\Delta_p^a + \Delta_q^b - N_f)(u_n)\}$ is bounded in $W_0^{1,p}(a,\Omega)^*$ because the nonlinear operators $\Delta_p^a, \Delta_q^b, N_f : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ are bounded. Due to the reflexivity of the space $W_0^{1,p}(\Omega)^*$, we can pass to a relabeled subsequence such that for a $\zeta \in W_0^{1,p}(\Omega)^*$ it holds that

$$(-\Delta_p^a + \Delta_q^b - N_f)(u_n) \rightharpoonup \zeta \text{ in } W_0^{1,p}(\Omega)^*. \tag{27}$$

Let $v \in X_m$ for some m. Assertion (ii) in the definition of Galerkin basis renders $v \in X_n$ for every $n \geq m$. Then, (25) and (27) imply

$$\langle \zeta, v \rangle_{W_0^{1,p}(a,\Omega)} = 0.$$

By (iii) in the definition of Galerkin basis $\{X_n\}$, the set $\bigcup_n X_n$ is dense in $W_0^{1,p}(a,\Omega)$. Therefore, $\zeta = 0$, so that (27) becomes

$$(-\Delta_p^a + \Delta_q^b - N_f)(u_n) \rightharpoonup 0 \text{ in } W_0^{1,p}(\Omega)^*,$$

which establishes property (jj) in Definition 1.

Setting $v = u$ in (jj) provides

$$\lim_{n\to\infty} \langle -\Delta_p^a(u_n) + \Delta_q^b(u_n) - N_f(u_n), u \rangle_{W_0^{1,p}(a,\Omega)} = 0,$$

which, with (24), produces

$$\lim_{n\to\infty} \langle -\Delta_p^a(u_n) + \Delta_q^b(u_n) - N_f(u_n), u_n - u \rangle_{W_0^{1,p}(a,\Omega)} = 0. \tag{28}$$

Proposition 4 and (28) ensure that

$$\lim_{n\to\infty} \langle -\Delta_p^a(u_n) + \Delta_q^b(u_n), u_n - u \rangle_{W_0^{1,p}(a,\Omega)} = 0$$

which shows the validity of part (jjj) in Definition 1. Summarizing, $u \in W_0^{1,p}(a,\Omega)$ is a generalized solution to problem (2) in the sense of Definition 1. □

Remark 1. *The notion of a generalized solution can be introduced for problem* (2)*, too. Precisely, $u \in W_0^{1,p}(a,\Omega)$ is called a generalized solution to problem* (1) *if there exists a sequence $\{u_n\} \subset W_0^{1,p}(a,\Omega)$ such that (j) in Definition 1 holds with*

(jj)′ for every $v \in W_0^{1,p}(a,\Omega)$ one has

$$\lim_{n\to\infty} \int_\Omega ((a(x)|\nabla u_n|^{p-2}\nabla u_n + b(x)|\nabla u_n|^{q-2}\nabla u_n)\nabla v - f(x,u_n,\nabla u_n)v)dx = 0;$$

(jjj)″with $u \in W_0^{1,p}(a,\Omega)$ in (j),

$$\lim_{n\to\infty} \int_\Omega ((a(x)|\nabla u_n|^{p-2}\nabla u_n + b(x)|\nabla u_n|^{q-2}\nabla u_n)\nabla (u_n - u)dx = 0.$$

In the case of problem (1)*, $u \in W_0^{1,p}(a,\Omega)$ is a generalized solution if and only if it is a weak solution in the sense of* (19)*. Indeed, if $u \in W_0^{1,p}(a,\Omega)$ is a weak solution to problem* (1)*, then the constant sequence $\{u_n = u\} \subset W_0^{1,p}(a,\Omega)$ verifies (j), (jj)′, (jjj)″; thus, u is a generalized solution. Conversely, let $u \in W_0^{1,p}(a,\Omega)$ be a generalized solution for* (1) *with the sequence $\{u_n\} \subset W_0^{1,p}(a,\Omega)$ satisfying (j), (jj)′, (jjj)″. Condition (jjj)″ reads as*

$$\lim_{n\to\infty} \langle -(\Delta_p^a + \Delta_q^b)(u_n), u_n - u \rangle_{W_0^{1,p}(a,\Omega)} = 0.$$

By (j) and Proposition 3 we deduce that $u_n \to u$ in $W_0^{1,p}(\Omega)$. Then from (jj)′ we get (19), where u is a weak solution of problem (1).

6. An Application

The goal of this section is to illustrate the effective applicability of our results. For the sake of simplicity, we focus on problems of types (1) and (2) on the unit open ball $B = \{x \in \mathbb{R}^3 : |x| < 1\}$ in $\mathbb{R}^3$ and for degenerated and competing $(3,2)$-Laplacians with weights.

Consider the Dirichlet problems

$$\begin{cases} -\mathrm{div}(|x|^r|\nabla u|\nabla u + (1-|x|^2)^h\nabla u) = g(x,u) + k_0\frac{|x|^r u}{1+u^2}|\nabla u|^\mu - k_1 u|\nabla u|^\nu & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases} \tag{29}$$

and

$$\begin{cases} -\mathrm{div}(|x|^r|\nabla u|\nabla u - (1-|x|^2)^h\nabla u) = g(x,u) + k_0\frac{|x|^r u}{1+u^2}|\nabla u|^\mu - k_1 u|\nabla u|^\nu & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases} \tag{30}$$

on B, with constants $r \in (0, \frac{3}{2})$, $h \geq 0$, $k_0 \in [0,1)$, $k_1 \geq 0$, $\mu \in [0, \frac{5}{3})$, $\nu \in [0, \frac{5}{6})$, and a Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying

$$|g(x,t)| \leq a_0 t^2 + b_0 \text{ for a.e } x \in \Omega, \ \ \forall t \in \mathbb{R},$$

with constants $a_0 \geq 0$ and $b_0 \geq 0$ provided $(a_0 + b_0)\lambda_1^{-1} + k_0 < 1$, where λ_1 represents the first eigenvalue of $-\Delta_3^a$ on $W_0^{1,3}(a, B)$ with $a(x) = |x|^r$. Notice that (29) and (30) are particular cases of problems (1) and (2), respectively, with $N = 3$, $p = 3$, $q = 2$, $\Omega = B$, $a(x) = |x|^r$, $b(x) = (1-|x|^2)^h$ and

$$f(x,t,\xi) = g(x,t) + k_0\frac{|x|^r t}{1+t^2}|\xi|^\mu - k_1 t|\xi|^\nu.$$

Let us check the conditions (H1)–(H4). Condition (H1) requires having $a^{-s} = |x|^{-rs} \in L^1(B)$ for some $s \in \left(\max\left\{\frac{N}{p}, \frac{1}{p-1}\right\}, +\infty\right) = (1, +\infty)$, which amounts to choosing $1 < s < \frac{3}{r}$. Taking into account that $r \in (0, \frac{3}{2})$, condition (H1) is fulfilled for instance with $s = 2$, a choice that we keep in the sequel.

Since $2r < 3$, we have

$$a^{-\frac{q}{p-q}} b^{\frac{p}{p-q}} = |x|^{-2r}(1-|x|^2)^{3h} \in L^1(B),$$

therefore, assumption (H2) is verified. For $s = 2$, it holds $p_s = ps/(s+1) = 2$, so we are in the situation of $N = 3 > p_s = 2$, where $p_s^* = \frac{Np_s}{N-p_s} = 6$, so $(p_s^*)' = \frac{6}{5}$ and $\frac{p_s}{(p_s^*)'} = \frac{5}{3}$.

We note that

$$\begin{aligned} |f(x,t,\xi)| &\leq a_0 t^2 + b_0 + k_0|x|^r|\xi|^\mu + k_1|t||\xi|^\nu \\ &\leq a_0 t^2 + b_0 + k_0|\xi|^\mu + \frac{k_1}{2}(t^2 + |\xi|^{2\nu}) \\ &\leq b_0 + k_0 + \frac{k_1}{2} + (a_0 + \frac{k_1}{2})t^2 + (k_0 + \frac{k_1}{2})|\xi|^\gamma \end{aligned}$$

for a.e $x \in B$ and all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^3$, where

$$\gamma := \max\{\mu, 2\nu\} < \frac{5}{3} = \frac{p_s}{(p_s^*)'}.$$

Therefore assumption (H3) is satisfied with $\sigma(x) = b_0 + k_0 + \frac{k_1}{2}$, $c_1 = a_0 + \frac{k_1}{2}$, $c_2 = k_0 + \frac{k_1}{2}$, $\alpha = 2$ and $\beta = \gamma$. We also derive

$$
\begin{aligned}
&f(x,t,\xi)t = g(x,t)t + k_0\frac{|x|^r t^2}{1+t^2}||^{\mu} - k_1 t^2|\xi|^{\nu} \\
&\leq a_0|t|^3 + b_0|t| + k_0|x|^r|\xi|^{\mu} \\
&\leq b_0 + k_0 + (a_0 + b_0)|t|^3 + k_0|x|^r|\xi|^3
\end{aligned}
$$

for a.e $x \in B$ and all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^3$. Assumption (H4) is verified with $\rho(x) = b_0 + k_0$, $d_1 = a_0 + b_0$ and $d_2 = k_0$ having been supposed that $\lambda_1^{-1}(a_0 + b_0) + k_0 < 1$.

Since the assumptions (H1)-(H4) are satisfied, Theorems 1 and 2 can be applied to ensure the existence of a weak solution to problem (29) and of a generalized solution to problem (30). The weak solution to problem (29) can be approximated as described in Theorem 1.

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