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Article

Structure of Iso-Symmetric Operators

Bhagwati Prashad Duggal [1,*] and In-Hyoun Kim [2]

[1] Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia
[2] Department of Mathematics, Incheon National University, Incheon 22012, Korea; ihkim@inu.ac.kr
* Correspondence: bpduggal@yahoo.co.uk

Abstract: For a Hilbert space operator $T \in B(\mathcal{H})$, let L_T and $R_T \in B(B(\mathcal{H}))$ denote, respectively, the operators of left multiplication and right multiplication by T. For positive integers m and n, let $\triangle^m_{T^*,T}(I) = (L_{T^*}R_T - I)^m(I)$ and $\delta^n_{T^*,T}(I) = (L_{T^*} - R_T)^m(I)$. The operator T is said to be (m,n)-isosymmetric if $\triangle^m_{T^*,T}\left(\delta^n_{T^*,T}(I)\right) = 0$. Power bounded (m,n)-isosymmetric operators $T \in B(\mathcal{H})$ have an upper triangular matrix representation $T = \begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix} \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ such that $T_1 \in B(\mathcal{H}_1)$ is a $C_{0.}$-operator which satisfies $\delta^n_{T_1^*,T_1}(I|_{\mathcal{H}_1}) = 0$ and $T_2 \in B(\mathcal{H}_2)$ is a $C_{1.}$-operator which satisfies $AT_2 = (V_u \oplus V_b)|_{\mathcal{H}_2}A$, $A = \lim_{t\to\infty} T_2^{*t}T_2^t$, V_u is a unitary and V_b is a bilateral shift. If, in particular, T is cohyponormal, then T is the direct sum of a unitary with a C_{00}-contraction.

Keywords: Hilbert space; left/right multiplication operator; (m,n)-symmetric operator; hyponormal operator; C_{00}-operator; unitary operator

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1. Introduction

Let $B(\mathcal{H})$ denote the algebra of operators, i.e., bounded linear transformations, on an infinite dimensional complex Hilbert space $\mathcal{H}$ into itself. Let $\mathbb{C}$ denote the complex plane and $\bar{z}$ the conjugate of $z \in \mathbb{C}$. For a given polynomial $f(z) = \sum_{i,j} c_{ij}\bar{z}^i z^j$ on $\mathbb{C}$ and an operator $T \in B(\mathcal{H})$, define $f(T)$ by $f(T) = \sum_{i,j} c_{ij}T^{*i}T^j$. Then T is said to be a (hereditary) root of f if $f(T) = 0$. An operator $T \in B(\mathcal{H})$ is n-selfadjoint for some positive integer n if T is a root of the polynomial $f(z) = (\bar{z} - z)^n$, equivalently, if

$$\delta^n_{T^*,T}(I) = \sum_{j=0}^{n}(-1)^j \begin{pmatrix} n \\ j \end{pmatrix} T^{*(n-j)}T^j = 0,$$

and T is m-isometric for some positive integer m if it is a root of the polynomial $f(z) = (\bar{z}z - 1)^m$, equivalently, if

$$\triangle^m_{T^*,T}(I) = \sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} T^{*(m-j)}T^{m-j} = 0.$$

The classes consisting of n-selfadjoint and m-isometric operators have been studied extensively by a large number of authors in the recent past (see list of references for further references).

The development of the theory of m-selfadjoint operators in infinite dimensional Hilbert spaces was motivated by the seminal work of Helton [1], who observed an unexpected, intimate connection with differential equations, in particular conjugate point theory and disconjugacy. McCullough and Rodman [2] in their consideration of algebraic and spectral properties of n-symmetric operators remark [2] (p. 419), that the authors of [1,3,4] were certainly aware of the fact that every 2-symmetric operator is 1-symmetric, even though they do not explicitly state so. More generally, McCullough and Rodman [2] (Theorem 3.1) state that the techniques of Helton [1] lead to a possible proof of the more general result that "$2n$-symmetric operators are $(2n-1)$-symmetric". The class of m-symmetric operators was introduced by Agler [3] and studied in a series of papers by Agler and Stankus [5–7]; properties of m-isometric operators, amongst them the spectral picture, strict m-isometries, perturbation by commuting nilpotents and the product of m-isometries, have since been studied by a large number of authors, amongst them Bayart [8], Bermudez et al. [9–11], Botelho and Jamison [12], Duggal et al. [13–15], and Gu et al. [16–18]. The (hereditary) roots of the polynomial $(\bar{z}z-1)^m(\bar{z}-z)^n = 0$ have been called (m,n)-isosymmetric operators; thus T is (m,n)-isosymmetric if and only if

$$\begin{aligned}
\triangle^m_{T^*,T}(\delta^n_{T^*,T}(I)) &= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}T^{*(m-j)}\left(\sum_{k=0}^{n}(-1)^k\begin{pmatrix} n \\ k \end{pmatrix}T^{*(n-k)}T^k\right)T^{m-j} \\
&= \delta^n_{T^*,T}(\triangle^m_{T^*,T}(I)) \\
&= \sum_{k=0}^{n}(-1)^k\begin{pmatrix} n \\ k \end{pmatrix}T^{*(n-k)}\left(\sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}T^{*(m-j)}T^{m-j}\right)T^k \\
&= 0.
\end{aligned}$$

Examples of (m,n)-isosymmetric operators occur naturally. Thus, every isometric operator $T \in B(\mathcal{H})$ is $(1,1)$-isosymmetric. Indeed, if $T \in B(\mathcal{H})$ is m-isometric, or n-symmetric, then T is (m,n)-isosymmetric. A study of this class of operators has been carried out by Stankus [19,20], and Gu and Stankus [18], amongst others.

For an operator $T \in B(\mathcal{H})$, define the operators L_T and $R_T \in B(B(\mathcal{H}))$ of left multiplication and (respectively) right multiplication by T by

$$L_T(X) = TX,\ R_T(X) = XT.$$

Then T is n-symmetric, respectively, m-isometric, if and only if

$$(L_{T^*} - R_T)^n(I) = \delta^n_{T^*,T}(I) = 0,\ \text{, respectively } (L_{T^*}R_T - I)^m(I) = \triangle^m_{T^*,T}(I) = 0$$

and T is (m,n)-isosymmetric if and only if

$$(L_{T^*}R_T - I)^m((L_{T^*} - R_T)^n(I)) = \triangle^m_{T^*,T}(\delta^n_{T^*,T}(I)) = 0.$$

Trivially, $\delta^n_{T^*,T}(I) = 0$ if and only if $\delta^n_{(T-\lambda)^*,T-\lambda}(I) = 0$ for all $\lambda \in \mathbb{C}$, and if $\lambda \in \mathbb{R}$ is such that $\lambda \notin \sigma(T)$, then

$$\begin{aligned}
\delta^n_{T^*,T}(I) = 0 &\iff \delta^n_{(T-\lambda)^*,T-\lambda}(I) = 0 \\
&\iff R^{-n}_{T-\lambda}\delta^n_{(T-\lambda)^*,T-\lambda}(I) = 0 \\
&\iff \triangle^n_{T^*-\lambda,(T-\lambda)^{-1}}(I) = 0 \\
&\iff L^{-n}_{T^*-\lambda}\delta^n_{(T-\lambda)^*,T-\lambda}(I) = 0 \\
&\iff \triangle^n_{(T^*-\lambda)^{-1},T-\lambda}(I) = 0.
\end{aligned}$$

In this note, we exploit relationships of this type, using little more than some basic properties of elementary operators, to give a formal, simple proof of the result that $2n$-symmetric operators are $(2n-1)$-symmetric. The case $n = 1$ of this result is of some interest, more so for the reason that 2-symmetric operators are cohyponormal. Cohyponormal (m,n)-isosymmetric operators have a particularly simple structure: they are the direct sum of a unitary operator and a C_{00}-contraction (where either of the components may be absent). The cohyponormality condition is redundant in the case in which ($n = 2$ and) $\delta^2_{T^*,T}(I) \geq 0$; if also $m = 2$, then $\triangle^2_{T^*,T}(\delta^2_{T^*,T}(I)) \geq 0$ is sufficient to guarantee T is the direct sum of a unitary operator and a C_{00}-contraction. For hyponormal, more generally normaloid,

(m,n)-isosymmetric T, T is a contraction, hence power bounded. Power bounded (m,n)-isosymmetric operators T have an upper triangular matrix representation $T \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ such that the $(1,1)$-entry is a $C_{0.}$-operator T_1 satisfying $\delta^n_{T_1^*,T_1}(I|_{\mathcal{H}_1}) = 0$ and the $(2,2)$-entry T_2 satisfies $AT_2 = (V_u \oplus V_b)|_{\mathcal{H}_2} A$ for an injective positive operator $A \in B(\mathcal{H}_2)$ (defined by $A = \lim_{t \longrightarrow \infty} T_2^{*t} T_2^t$), unitary V_u and a bilateral shift V_b.

We introduce our notation/terminology, along with some complementary results, in the following section, Section 3 is devoted to considering $2n$-symmetric and related operators, and our Section 4 considers the structure of cohyponormal and power bounded (m,n)-isosymmetric operators.

2. Some Complementary Results

In the following, $\langle .,. \rangle$ will denote the inner product on $\mathcal{H}$. We shall denote the approximate point spectrum and the spectrum of an operator $T \in B(\mathcal{H})$ by $\sigma_a(T)$ and $\sigma(T)$, respectively. We shall denote the open unit disc in the complex plane $\mathbb{C}$ by $\mathbb{D}$ and the boundary of the unit disc in $\mathbb{C}$ by $\partial\mathbb{D}$. The operator T is power bounded if there exists a scalar $M > 0$ such that

$$\sup_{n \in \mathbb{N}} \|T^n\| \leq M.$$

It is clear from the definition that if $T \in B(\mathcal{H})$ is power bounded, then T^* is power bounded, the spectral radius

$$r(T) = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}} \leq 1$$

and the spectrum $\sigma(T)$ of T satisfies $\sigma(T) \subseteq \overline{\mathbb{D}}\,(= \{\lambda \in \mathbb{C} : |\lambda| \leq 1\})$. The operator T is a $C_{0.}$, respectively, $C_{1.}$, operator if

$$\lim_{n \to \infty} \|T^n x\| = 0 \ \text{ for all } \ x \in \mathcal{H},$$

$$\text{respectively, } \inf_{n \in \mathbb{N}} \|T^n x\| > 0 \ \text{ for all } \ 0 \neq x \in \mathcal{H};$$

$T \in C_{.0}$ (resp., $T \in C_{.1}$) if $T^* \in C_{0.}$ (resp., $T^* \in C_{1.}$) and $T \in C_{\alpha\beta}$ if $T \in C_{\alpha.} \cap C_{.\beta}$ $(\alpha, \beta = 0, 1)$. It is well known [21] that every power bounded operator $T \in B(\mathcal{H})$ has an upper triangular matrix representation

$$T = \begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix} \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$$

for some decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of $\mathcal{H}$ such that $T_1 \in C_{0.}$ and $T_2 \in C_{1.}$. Recall that every isometry $V \in B(\mathcal{H})$ has a direct sum decomposition

$$V = V_c \oplus V_u \in B(\mathcal{H}_c \oplus \mathcal{H}_u), V_c \in C_{10} \ \text{ and } \ V_u \in C_{11}$$

into its completely non-unitary (i.e., unilateral shift) and unitary parts [22]. Hyponormal contractions T, i.e., contractions $T \in B(\mathcal{H})$ such that $TT^* \leq T^*T$, are known to have $C_{.0}$ cnu (=completely non-unitary) parts [23].

The following result from [24] will be used in some of our argument below.

Theorem 1. *If $A, B \in B(\mathcal{H})$, then the following statements are pairwise equivalent.*

(i) $\mathrm{ran(A)} \subseteq \mathrm{ran(B)}$.
(ii) *There is a $\mu \geq 0$ such that $AA^* \leq \mu^2 BB^*$.*
(iii) *There is an operator $C \in B(\mathcal{H})$ such that $A = BC$.*

Furthermore, if these conditions hold, then the operator C may be chosen so that (a) $\|C\|^2 = \inf\{\lambda : AA^* \leq \lambda BB^*\}$; (b) $\ker(A) = \ker(C)$; (c) $\mathrm{ran}(C) \subseteq \ker(B)^{\perp}$.

A pair of operators $A, B \in B(\mathcal{H})$ satisfies the Putnam–Fuglede (commutativity) property if $\delta_{A,B}^{-1}(0) \subseteq \delta_{A^*,B^*}^{-1}(0)$. It is easily seen that if A, B satisfy the Putnam–Fuglede property and $\delta_{A,B}(X) = 0$, then $\overline{X(\mathcal{H})}$ reduces A, $\ker^{\perp}(X)$ reduces B, and $A|_{\overline{X(\mathcal{H})}}$ and $B|_{\ker^{\perp}(X)}$ are unitarily equivalent normal operators. Normal operators satisfy the Putnam–Fuglede property [25]. Indeed, more is true. An asymmetric version of the Putnam–Fuglede property holds for a variety of classes of Hilbert space operators [26], amongst them hyponormal pairs A and $B^* \in B(\mathcal{H})$: if A, B^* are hyponormal operators, then $\delta_{A,B^*}^{-1}(0) \subseteq \delta_{A^*,B}^{-1}(0)$. Even more interestingly:

Theorem 2 ([26])**.** *If $A, B^* \in B(\mathcal{H})$ are hyponormal operators and n is some positive integer, then*

$$\delta_{A,B^*}^{n^{-1}}(0) = \delta_{A,B^*}^{-1}(0) \subseteq \delta_{A^*,B}^{-1}(0).$$

3. *n*-Symmetric Operators for *n* Even

We start by proving that n-symmetric operators for n even are $(n-1)$-symmetric. This property of n-symmetric operators is stated in [2] (Theorem 3.4) without a proof (but with the remark that a proof can be given using the techniques of [1]). Our proof below uses little more than some well understood properties of elementary operators of left and right multiplication.

Theorem 3. *If $T \in B(\mathcal{H})$ is n-symmetric for some positive even integer n, then T is $(n-1)$-symmetric.*

Proof. A straightforward argument shows that $\sigma_a(T) \subset \mathbb{R}$ for n-symmetric operator T. Hence $\sigma(T) \subset \mathbb{R}$, and there exists a non-zero real number $\lambda \notin \sigma(T)$. Since

$$\delta_{T^*,T}^{n}(I) = 0 \Longleftrightarrow \delta_{T^*-\mu,T-\mu}^{n}(I) = 0$$

for all real μ, we have

$$\delta_{T^*,T}^{n}(I) = 0 \Longleftrightarrow R_{T-\lambda}^{-n}\delta_{T^*-\lambda,T-\lambda}^{n}(I) = 0 \Longleftrightarrow \triangle_{T^*-\lambda,(T-\lambda)^{-1}}^{n}(I) = 0.$$

It is easily seen (use an induction argument) that

$$\begin{aligned} \triangle_{A,B}^{n}(I) = (L_A R_B - I)^n(I) = 0 \quad &\Longleftrightarrow \quad (L_A R_B)^n(I) - \sum_{j=0}^{n-1} \binom{n}{j} \triangle_{A,B}^{j}(I) = 0 \\ &\Longleftrightarrow \quad (L_A R_B)^n(I) = \sum_{j=0}^{n-1} \binom{n}{j} \triangle_{A,B}^{j}(I) \end{aligned}$$

for all operators $A, B \in B(\mathcal{H})$. Hence, given $\triangle^n_{A,B}(I) = 0$,

$$\begin{aligned}
(L_A R_B)^{n+1}(I) &= \sum_{j=0}^{n-1} \binom{n}{j} L_A R_B \triangle^j_{A,B}(I) \\
&= \sum_{j=0}^{n-1} \binom{n}{j} \triangle^{j+1}_{A,B}(I) + \sum_{j=0}^{n-1} \binom{n}{j} \triangle^j_{A,B}(I) \\
&= \binom{n}{n-1} \triangle^n_{A,B}(I) + \sum_{j=0}^{n-1} \binom{n+1}{j} \triangle^j_{A,B}(I) \\
&= \sum_{j=0}^{n-1} \binom{n+1}{j} \triangle^j_{A,B}(I) \\
&= \binom{n+1}{n-1} \triangle^{n-1}_{A,B}(I) + \sum_{j=0}^{n-2} \binom{n+1}{j} \triangle^j_{A,B}(I),
\end{aligned}$$

and by an induction argument that

$$(L_A R_B)^t(I) = \binom{t}{n-1} \triangle^{n-1}_{A,B}(I) + \sum_{j=0}^{n-2} \binom{t}{j} \triangle^j_{A,B}(I) \tag{1}$$

for all $A, B \in B(\mathcal{H})$ and integers $t \geq n$. Translating to the operator $\delta^n_{T^*,T}(I) = \triangle^n_{T^*-\lambda,(T-\lambda)^{-1}}$ $(I) = 0$, we have

$$(L_{T^*-\lambda} R^{-1}_{T-\lambda})^t(I) = \binom{t}{n-1} \triangle^{n-1}_{T^*-\lambda,(T-\lambda)^{-1}}(I) + \sum_{j=0}^{n-2} \binom{t}{j} \triangle^j_{T^*-\lambda,(T-\lambda)^{-1}}(I)$$

for all $t \geq n$ and real $\lambda \notin \sigma(T)$. Trivially,

$$\begin{aligned}
\triangle^n_{A,B}(I) = L_A R_B \triangle^{n-1}_{A,B}(I) - \triangle^{n-1}_{A,B}(I) = 0 &\implies L_A R_B \triangle^{n-1}_{A,B}(I) = \triangle^{n-1}_{A,B}(I) \\
&\implies \cdots \\
&\implies (L_A R_B)^t \triangle^{n-1}_{A,B}(I) = \triangle^{n-1}_{A,B}(I)
\end{aligned}$$

for all $A, B \in B(\mathcal{H})$ and integers $t \geq 1$. Hence

$$\begin{aligned}
I &= \binom{t}{n-1} \triangle^{n-1}_{T^*-\lambda,(T-\lambda)^{-1}}(I) + \sum_{j=0}^{n-2} \binom{t}{j} (L^{-1}_{T^*-\lambda} R_{T-\lambda})^t \triangle^j_{T^*-\lambda,(T-\lambda)^{-1}}(I) \\
\implies \quad 0 \leq \|x\|^2 &= \binom{t}{n-1} \left\langle \triangle^{n-1}_{T^*-\lambda,(T-\lambda)^{-1}}(I)x, x \right\rangle \\
&\quad + \sum_{j=0}^{n-2} \binom{t}{j} \left\langle \triangle^j_{T^*-\lambda,(T-\lambda)^{-1}}(I)(T-\lambda)^t x, (T-\lambda)^{-t} x \right\rangle
\end{aligned}$$

for all $x \in \mathcal{H}$ and integers $t \geq 1$. Letting $t \to \infty$, and observing that $\binom{t}{n-1}$ is of the order of t^{n-1} and $\binom{t}{j}$, $0 \leq j \leq n-2$, is of the order of t^{n-2} as $t \to \infty$,

$$0 \leq \left\langle \triangle^{n-1}_{T^*-\lambda,(T-\lambda)^{-1}}(I)x, x \right\rangle$$

for all $x \in \mathcal{H}$. Conclusion:

$$\triangle^{n-1}_{T^*-\lambda,(T-\lambda)^{-1}}(I) \geq 0.$$

Equivalently,

$$\begin{aligned} 0 &\leq \triangle^{n-1}_{T^*-\lambda,(T-\lambda)^{-1}}(I) \\ &= R^{-n+1}_{T-\lambda}\delta^{n-1}_{T^*-\lambda,T-\lambda}(I) \\ &= (\triangle^{n-1}_{T^*-\lambda,(T-\lambda)^{-1}}(I))^* \\ &= \triangle^{n-1}_{(T^*-\lambda)^{-1},T-\lambda}(I) \\ &= (-1)^{n-1}L^{-n+1}_{T^*-\lambda}\delta^{n-1}_{T^*-\lambda,T-\lambda}(I). \end{aligned}$$

Thus

$$L^{n-1}_{T^*-\lambda}R^{-n+1}_{T-\lambda}\delta^{n-1}_{T^*-\lambda,T-\lambda}(I) = (-1)^{n-1}\delta^{n-1}_{T^*-\lambda,T-\lambda}(I).$$

Since $\delta^n_{T^*-\lambda,T-\lambda}(I) = 0$ implies $(L_{T^*-\lambda}R^{-1}_{T-\lambda})^{n-1}\delta^{n-1}_{T^*-\lambda,T-\lambda}(I) = \delta^{n-1}_{T^*-\lambda,T-\lambda}(I)$, and the integer n is even,

$$\delta^{n-1}_{T^*-\lambda,T-\lambda}(I) = -\delta^{n-1}_{T^*-\lambda,T-\lambda}(I) \Longleftrightarrow \delta^{n-1}_{T^*-\lambda,T-\lambda}(I) = \delta^{n-1}_{T^*,T}(I) = 0.$$

This completes the proof. □

It is immediate from Theorem 3 that 2-symmetric $B(\mathcal{H})$ operators are symmetric. A proof of this of a different flavour and (in some respects) of interest in itself may be given as follows.

Corollary 1 ([2])**.** *A 2-symmetric $B(\mathcal{H})$ operator is self-adjoint.*

Proof. For operators $T \in B(\mathcal{H})$,

$$0 \leq (\delta_{T^*,T}(I))^*(\delta_{T^*,T}(I)) = T^*T + TT^* - T^2 - T^{*2}.$$

If also T is 2-symmetric, then

$$\delta^2_{T^*,T}(I) = T^{*2} - 2T^*T + T^2 = 0.$$

Hence

$$\delta^2_{T^*,T}(I) = 0 \leq (\delta_{T^*,T}(I))^*(\delta_{T^*,T}(I)) \Longrightarrow T^*T \leq TT^*,$$

i.e., T^* is hyponormal. Set $\delta_{T^*,T}(I) = X$; then T is 2-symmetric if and only if

$$\delta_{T^*,T}(X) = T^*X - XT = 0.$$

Applying the Putnam–Fuglede commutativity theorem for hyponormal operators, we have

$$T^*X - XT = 0 \Longrightarrow TX - XT^* = 0 \Longleftrightarrow T^{*2} - 2TT^* + T^2 = 0.$$

Already $T^{*2} - 2T^*T + T^2 = 0$. Hence $T^*T = TT^*$, i.e., T is normal. However, then

$$\delta^2_{T^*,T}(I) = 0 \Longleftrightarrow \delta_{T^*,T}(I) = 0$$

(see Theorem 2). Hence $T^* = T$. □

The argument of the proof of Corollary 1 is suggestive of an interesting proof of a well known result on invertible 2-isometries [8].

Corollary 2. *Invertible 2-isometric $B(\mathcal{H})$ operators are unitary.*

Proof. The operator $\triangle_{T^*,T}(I) \in B(\mathcal{H})$ being self-adjoint,

$$(\triangle_{T*,T}(I))^2 = (T^*T)^2 - 2T^*T + I \geq 0.$$

Since $\triangle^2_{T^*,T}(I) = T^{*2}T^2 - 2T^*T + I = 0$ and T is invertible, we have

$$T^{*2}T^2 \leq (T^*T)^2 \Longleftrightarrow T^*T \leq TT^*,$$

i.e., T^* is invertible hyponormal (with a hyponormal inverse T^{*-1}). We have

$$\triangle^2_{T^*,T}(I) = 0 \Longleftrightarrow \delta^2_{T^*,T^{-1}}(I) = 0.$$

Putnam–Fuglede commutativity theorem for hyponormal operators applies and we conclude that

$$\delta^2_{T^*,T^{-1}}(I) = 0 \Longleftrightarrow \delta_{T^*,T^{-1}}(I) = 0 \Longleftrightarrow T^*T = TT^* = I,$$

i.e., T is unitary. $\square$

A generalised version of Corollary 2 is known to hold: if $\triangle^m_{T^*,T}(I) = 0$ for an invertible $T \in B(\mathcal{H})$ and an even positive integer m, then $\triangle^{m-1}_{T^*,T}(I) = 0$ [8] (Proposition 2.4). Here the pair (T^*, T) may be replaced by the pair (T^*, T^{-1}).

Corollary 3. *If $\triangle^m_{T^*,T^{-1}}(I) = 0$ for an invertible $T \in B(\mathcal{H})$ and even positive integer m, then $\triangle^{m-1}_{T^*,T^{-1}}(I) = 0$.*

Proof. The proof is an application of Theorem 3. The hypothesis $\triangle^m_{T^*,T}(I) = 0$ implies

$$\begin{aligned} L^{-m}_{T^*}\triangle^m_{T^*,T^{-1}}(I) = (-1)^m\delta^m_{T^{*-1},T^{-1}}(I) = 0 &\Longleftrightarrow \delta^m_{T^{*-1},T^{-1}}(I) = 0 \\ &\Longrightarrow \delta^{m-1}_{T^{*-1},T^{-1}}(I) = 0 \\ &\Longleftrightarrow L^{m-1}_{T^*}\delta^{m-1}_{T^{*-1},T^{-1}}(I) = 0 \\ &\Longleftrightarrow \triangle^{m-1}_{T^*,T^{-1}}(I) = 0. \end{aligned}$$

This completes the proof. $\square$

Yet another generalisation of Corollary 2 is obtained upon considering operators $T \in B(\mathcal{H})$ such that $T \in (m, X)$-isometric, i.e., operators $T \in B(\mathcal{H})$ satisfying $\triangle^m_{T^*,T}(X) = \sum_{j=0}^m (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} T^{*^{m-j}}XT^{m-j} = 0$, for some positive operator $X \in B(\mathcal{H})$. For such operators T, it is clear from the argument leading to equality (1) that

$$0 \leq (L_{T^*}R_T)^t(X) = \begin{pmatrix} t \\ m-1 \end{pmatrix}\triangle^{m-1}_{T^*,T}(X) + \sum_{j=0}^{m-2}\begin{pmatrix} t \\ j \end{pmatrix}\triangle^j_{T^*,T}(X)$$

for all integers $t \geq m$. Letting $t \to \infty$, one obtains

$$0 \leq \triangle^{m-1}_{T^*,T}(X). \tag{2}$$

Proposition 1. *If $T \in B(\mathcal{H})$ is an invertible (m, X)-isometric operator for some positive operator $X \in B(\mathcal{H})$, then $T \in (m-1, X)$-isometric.*

Proof. T being invertible

$$\triangle^m_{T^*,T}(X) = (L_{T^*}R_T - I)^m(X) = (-1)^m(L_{T^*}R_T)^m\Big((L_{T^*}R_T)^{-1} - I\Big)^m(X)$$

and this since $T \in (m, X)$-isometric implies $\triangle^m_{T^{*-1},T^{-1}}(X) = 0$. Arguing as above, we have

$$0 \leq \triangle^{m-1}_{T^{*-1},T^{-1}}(X) = (-1)^{m-1}(L_{T^*}R_T)^{-m+1}\triangle^{m-1}_{T^*,T}(X) \Longrightarrow \triangle^{m-1}_{T^*,T}(X) \leq 0.$$

Combining with inequality (2), we obtain the required equality. □

Remark 1. (i) In the presence of the hyponormality hypothesis on T (or T^*), the hypothesis that T is 2 -symmetric is not necessary. Indeed, hyponormal n-symmetric operators T are self-adjoint. This is seen as follows. A straightforward argument shows $\sigma_a(T) \subset \mathbb{R}$; hence $\sigma(T) \subset \mathbb{R}$. Since hyponormal operators with spectrum in $\mathbb{R}$ are self-adjoint [27], T is self-adjoint.

(ii) It is known that hyponormal m-isometric operators are isometric [28]. The following argument shows that a cohyponormal m-isometric operator is unitary. If T is m-isometric, then $\sigma_a(T)$ is a subset of the boundary of the unit disc in $\mathbb{C}$. Hence T is a contraction and therefore isometric [28] (Proposition 2.6). The proof now follows, since a cohyponormal isometry is necessarily unitary.

4. Structure of (m,n)-Isosymmetric Operators

In this section, we consider the structure of power bounded (m,n)-isosymmetric operators. We start, however, by considering cohyponormal (m,n)-isosymmetric operators. It is seen that such operators T have a particulary simple structure: T is the direct sum of a unitary operator with a C_{00}-contraction satisfying $T \in (1,1)$-isosymmetric.

By the definition of the approximate point spectrum of an operator, if a $\lambda \in \sigma_a(T)$, then there exists a sequence of unit vectors $\{x_t\} \subseteq \mathcal{H}$ such that $\lim_{t\to\infty} \|(T-\lambda)x_t\| = 0$. Hence, if $T \in (m,n)$-isosymmetric and $\lambda \in \sigma_a(T)$, then

$$\begin{aligned}
0 &= \lim_{t\to\infty} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \langle T^{m+j-k}x_t, T^{m+n-j-k}x_t\rangle \\
&= \sum_{j=0}^{n} (-1)^j \binom{n}{j} \overline{\lambda}^{(n-j)} \lambda^j \sum_{k=0}^{m} (-1)^k \binom{m}{k} |\lambda|^{2(m-k)} \\
&= (\overline{\lambda}-\lambda)^n (1-|\lambda|^2)^m \\
\Longrightarrow \quad & \sigma_a(T) \subseteq \partial\mathbb{D} \cup \mathbb{R} \text{ and } \sigma(T) \subseteq \overline{\mathbb{D}} \cup \mathbb{R}.
\end{aligned}$$

Recall that an operator $T \in B(\mathcal{H})$ is normaloid if $\|T\|$ equals the spectral radius $r(T) = \lim_{t\to\infty} \|T^t\|^{\frac{1}{t}}$ of T. Hyponormal operators are normaloid.

Theorem 4. ***(a)*** *If $T \in B(\mathcal{H})$ is cohyponormal, then the following statements are mutually equivalent.*

(i) $\triangle^m_{T^*,T}\left(\delta^n_{T^*,T}(I)\right) = 0$ *for some positive integers m,n.*

(ii) $\triangle_{T^*,T}(\delta_{T^*,T}(I)) = 0$.

(iii) *T is the direct sum of a unitary with a selfadjoint C_{00}-contraction.*

(b) *If $T \in B(\mathcal{H})$ is an invertible operator and m is a positive even integer such that* $\triangle^m_{T^*,T}\left(\delta^n_{T^*,T}(I)\right) \geq 0$ *and* $\delta^n_{T^*,T}(I) \geq 0$, *or,* $\triangle^m_{T^*,T}\left(\delta^n_{T^*,T}(I)\right) \leq 0$ *and* $\delta^n_{T^*,T}(I) \leq 0$, *for some positive integer n, then* $\triangle^m_{T^*,T}\left(\delta^n_{T^*,T}(I)\right) = 0$.

Proof. (**a**) $(iii) \Longrightarrow (ii) \Longrightarrow (i)$. If we let $T = T_u \oplus T_c \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$, T_u unitary and T_c a C_{00}-contraction such that $T_c^* = T_c$, then

$$\begin{aligned}
\triangle_{T^*,T}(\delta_{T^*,T}(I)) &= \triangle_{T^*,T}((T_u^*-T_u)\oplus 0) \\
&= \left(0 \oplus \triangle_{T_c^*,T_c}\right)((T_u^*-T_u)\oplus 0) \\
&= 0
\end{aligned}$$

and

$$\triangle^m_{T^*,T}(\delta^n_{T^*,T}(I)) = \triangle^{m-1}_{T^*,T}\left[\delta^{n-1}_{T^*,T}(\triangle_{T^*,T}(\delta_{T^*,T}(I)))\right] = 0.$$

$(i) \Longrightarrow (iii)$. In view of our observation on the spectrum of operators $T \in B(\mathcal{H})$ satisfying the equality of (i), the hypothesis T^* is hyponormal implies T^*, hence T, is a contraction. Decompose T into its normal and pure (i.e., completely non-normal) parts by $T = T_1 \oplus T_2 \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$. Then T_2 is a cnu (= completely non-unitary) $C_{0.}$-contraction. The hypothesis

$$\triangle^m_{T^*,T}(\delta^n_{T^*,T}(I)) = 0 \Longleftrightarrow \oplus_{i=1}^2 \triangle^m_{T_i^*,T_i}\left(\delta^n_{T_i^*,T_i}(I_i)\right) = 0,$$

where I_i is the identity of $B(\mathcal{H}_i)$. Since

$$\triangle^m_{T_i^*,T_i}\left(\delta^n_{T_i^*,T_i}(I_i)\right) = 0 \Longleftrightarrow \delta^n_{T_i^*,T_i}\left(\triangle^m_{T_i^*,T_i}(I)\right) = 0,$$

if we let $\triangle^m_{T_i^*,T_i}(I_i) = X_i$ and apply Theorem 2 to $\delta^n_{T_i^*,T_i}(X_i) = 0$, then

$$\delta_{T_i^*,T_i}(X_i) = \delta_{T_i^*,T_i}\left(\triangle^m_{T_i^*,T_i}(I_i)\right) = \triangle^m_{T_i^*,T_i}\left(\delta_{T_i^*,T_i}(I_i)\right) = 0.$$

Choose $i = 2$. Set $\triangle^{m-1}_{T_2^*,T_2}\left(\delta_{T_2^*,T_2}(I_2)\right) = Y_{m-1}$ and consider $\triangle_{T_2^*,T_2}(Y_{m-1})$. Since

$$\triangle_{T_2^*,T_2}(Y_{m-1}) = 0 \Longrightarrow T_2^* Y_{m-1} T_2 = Y_{m-1} \Longrightarrow \cdots \Longrightarrow T_2^{*t} Y_{m-1} T_2^t = Y_{m-1}$$

for all positive integers t,

$$|\langle Y_{m-1}x, x\rangle| = \left|\langle Y_{m-1}T_2^t x, T_2^t x\rangle\right| \leq \|Y_{m-1}\| \left\|T_2^t x\right\|^2$$

for all $x \in \mathcal{H}_2$. Since T_2 is a $C_{0.}$-contraction, letting $t \to \infty$, we have

$$|\langle Y_{m-1}x, x\rangle| = 0 \text{ for all } x \in \mathcal{H}_2.$$

Hence $Y_{m-1} = \triangle^{m-1}_{T_2^*,T_2}\left(\delta_{T^*_, T_2}(I_2)\right) = 0$. Repeating the argument, considering $\triangle_{T_2^*,T_2}(Y_{m-2})$ and $\triangle_{T_2^*,T_2}(Y_{m-3})$ etc., it follows that

$$Y_1 = \triangle_{T_2^*,T_2}\left(\delta_{T_2^*,T_2}(I_2)\right) = 0 \Longrightarrow Y_0 = \delta_{T_2^*,T_2}(I_2) = 0.$$

Thus, $T_2 \in C_{00}$ is a selfadjoint contraction.

Considering next the case $i = 1$, the normal contraction T_1 is the direct sum of a unitary and a cnu contraction. Let

$$T_1 = T_{11} \oplus T_{12} \in B(\mathcal{H}_{11} \oplus \mathcal{H}_{12}),\ T_{11} \text{ unitary and } T_{12} \text{ cnu}.$$

Then

$$\triangle^n_{T_1^*,T_1}\left(\delta_{T_1^*,T_1}(I)\right) = \oplus_{j=1}^2 \triangle^n_{T_{1j}^*,T_{1j}}\left(\delta_{T_{1j}^*,T_{1j}}(I_{1j})\right) = 0,$$

where I_{1j} is the identity of $B(\mathcal{H}_{1j})$. Since T_{11} is unitary,

$$\begin{aligned}
\triangle^n_{T_{11}^*,T_{11}}\left(\delta_{T_{11}^*,T_{11}}(I_{11})\right) = 0 \quad &\Longleftrightarrow \quad \delta^n_{T_{11}^*,T_{11}^{-1}}\left(\delta_{T_{11}^*,T_{11}}(I_{11})\right) = 0 \\
&\Longleftrightarrow \quad \delta_{T_{11}^*,T_{11}^{-1}}\left(\delta_{T_{11}^*,T_{11}}(I_{11})\right) = 0 \\
&\Longleftrightarrow \quad \triangle_{T_{11}^*,T_{11}}\left(\delta_{T_{11}^*,T_{11}}(I_{11})\right) = 0.
\end{aligned}$$

The operator T_{12} being a normal cnu-contraction is a C_{00}-contraction. Arguing as above, this implies

$$\triangle^n_{T^*_{12},T_{12}}\left(\delta_{T^*_{12},T_{12}}(I_{12})\right) = 0 \Longrightarrow \delta_{T^*_{12},T_{12}}(I_{12}) = 0,$$

i.e., $T_{12} \in C_{00}$-contraction is selfadjoint. To complete the proof, define T_u and T_c by $T_u = T_{11}$ and $T_c = T_{12} \oplus T_2$.

(**b**) We prove that either of the hypotheses implies equality (i) of part **a**. The proof in both the cases being almost the same, simply substitute $-X$ for X in the argument below, we consider the case $\triangle^m_{T^*,T}(\delta^n_{T^*,T}(I)) \geq 0$ and $\delta^n_{T^*,T}(I) \geq 0$. Let $\delta^n_{T^*,T}(I) = X$; then

$$0 \leq \triangle^m_{T^*,T}(X) \Longleftrightarrow 0 \leq (L_{T^*}R_T)^m(X) - \sum_{j=0}^{m-1} \begin{pmatrix} m \\ j \end{pmatrix} \triangle^j_{T^*,T}(X)$$

implies

$$\begin{aligned} 0 &\leq (L_{T^*}R_T)^{m+1}(X) - \sum_{j=0}^{m-1} \begin{pmatrix} m \\ j \end{pmatrix} L_{T^*}R_T \triangle^j_{T^*,T}(X) \\ &= (L_{T^*}R_T)^{m+1}(X) - \begin{pmatrix} m+1 \\ m-1 \end{pmatrix} \triangle^{m-1}_{T^*,T}(X) - \sum_{j=0}^{m-2} \begin{pmatrix} m+1 \\ j \end{pmatrix} \triangle^j_{T^*,T}(X) \end{aligned}$$

and this (using an induction argument as in the proof of (1)) implies

$$0 \leq (L_{T^*}R_T)^t(X) - \begin{pmatrix} t \\ m-1 \end{pmatrix} \triangle^{m-1}_{T^*,T}(X) + \sum_{j=0}^{m-2} \begin{pmatrix} t \\ j \end{pmatrix} \triangle^j_{T^*,T}(X) \tag{3}$$

for all integers $t \geq m$. Thus

$$\begin{aligned} \left\langle \triangle^{m-1}_{T^*,T}(X)x, x \right\rangle &\leq \frac{1}{\begin{pmatrix} t \\ m-1 \end{pmatrix}} \left[\left\langle \left\{ (L_{T^*}R_T)^t(X) + \sum_{j=0}^{m-2} \begin{pmatrix} t \\ j \end{pmatrix} \triangle^j_{T^*,T}(X) \right\} x, x \right\rangle \right] \\ &= \frac{1}{\begin{pmatrix} t \\ m-1 \end{pmatrix}} \left[\left\| X^{\frac{1}{2}} T^t x \right\|^2 + \left\langle \sum_{j=0}^{m-2} \begin{pmatrix} t \\ j \end{pmatrix} \triangle^j_{T^*,T}(X)x, x \right\rangle \right] \end{aligned}$$

for all $x \in \mathcal{H}$. Since $\begin{pmatrix} t \\ m-1 \end{pmatrix}$ is of the order of t^{m-1} and $\begin{pmatrix} t \\ j \end{pmatrix}$ is of the order of t^{m-2} (for $0 \leq j \leq m-2$) as $t \to \infty$, letting $t \to \infty$ we have

$$\left\langle \triangle^{m-1}_{T^*,T}(X)x, x \right\rangle \leq 0 \text{ for all } x \in \mathcal{H} \Longrightarrow \triangle^{m-1}_{T^*,T}(X) \leq 0.$$

The invertibility of T implies

$$\triangle^m_{T^*,T}(X) = (-1)^m (L_{T^*}R_T)^m \triangle^m_{T^{*-1},T^{-1}}(X)$$

and hence since m is even

$$(L_{T^*}R_T)^m \triangle^m_{T^{*-1},T^{-1}}(X) = 0 \Longleftrightarrow \triangle^m_{T^{*-1},T^{-1}}(X) = 0.$$

Arguing as above, we conclude

$$0 \le \triangle^{m-1}_{T^{*-1},T^{-1}}(X) = (L^{-1}_{T^*}R^{-1}_T)^{m-1}(-1)^{m-1}\triangle^{m-1}_{T^*,T}(X) \quad \Longrightarrow \quad 0 \le (-1)^{m-1}\triangle^{m-1}_{T^*,T}(X)$$
$$\Longleftrightarrow \quad 0 \ge \triangle^{m-1}_{T^*,T}(X).$$

Hence

$$\triangle^{m-1}_{T^*,T}(X) = 0 \Longrightarrow \triangle^{m}_{T^*,T}(X) = 0 \Longleftrightarrow \triangle^{m}_{T^*,T}(\delta^{n}_{T^*,T}(I)) = 0$$

and the proof is complete. □

The hypothesis T^* is hyponormal is redundant in the case in which $n = 2$ and $\delta^2_{T^*,T}(I) \ge 0$. (For then $\delta^2_{T^*,T}(I) \ge 0$ and $(\delta_{T^*,T}(I)^*(\delta_{T^*,T}(I)) \ge 0$ imply $TT^* \ge T^*T$.) Furthermore, if also $m = 2$, then the hypothesis T is invertible may be dispensed with in Theorem 4(b).

Theorem 5. *If $\delta^2_{T^*,T}(I)$ and $\triangle^2_{T^*,T}\left(\delta^2_{T^*,T}(I)\right)$ are both greater than or equal to 0, then $\triangle_{T^*,T}(\delta_{T^*,T}(I)) = 0$ and T is the direct sum of a unitary with a C_{00}-contraction.*

Proof. The cohyponormality of T implies T is a contraction, hence has a direct sum decomposition

$$T = T_u \oplus T_c \in B(\mathcal{H}_1 \oplus \mathcal{H}_2),\ T_u = T|_{\mathcal{H}_1} \text{ unitary and } T_c = T|_{\mathcal{H}_2} \text{ a cnu } C_{0.} - \text{contraction.}$$

If we let

$$X = \delta^2_{T^*,T}(I) = \delta^2_{T_u^*,T_u}(I_1) \oplus \delta^2_{T_c^*,T_c}(I_2) = X_1 \oplus X_2 \in B(\mathcal{H}_1 \oplus \mathcal{H}_2),\ I_i = I|_{\mathcal{H}_i},\ i = 1,2,$$

then $X_i \ge 0$ for $i = 1,2$ and

$$\triangle^2_{T^*,T}(X) = \triangle^2_{T_u^*,T_u}(X_1) \oplus \triangle^2_{T_c^*,T_c}(X_2) \ge 0 \Longleftrightarrow \triangle^2_{T_u^*,T_u}(X_1) \ge 0,\ \triangle^2_{T_c^*,T_c}(X_2) \ge 0.$$

The operator T_u being unitary, Theorem 4(b) implies

$$\triangle^2_{T_u^*,T_u}(X_1) \ge 0 \Longleftrightarrow \triangle_{T_u^*,T_u}\left(\delta_{T_u^*,T_u}(I_1)\right) = 0.$$

Consider now the operator $\triangle^2_{T_c^*,T_c}(X_2) = \triangle_{T_c^*,T_c}(X_{21}) \ge 0$; $X_{21} = \triangle_{T_c^*,T_c}(X_2)$. We have

$$\triangle_{T_c^*,T_c}(X_{21}) \ge 0 \Longrightarrow T_c^* X_{21} T_c \ge X_{21} \Longrightarrow T_c^{*2} X_{21} T_c^2 \ge X_{21} \Longrightarrow \cdots \Longrightarrow T_c^{*t} X_{21} T_c^t \ge X_{21}$$

for all positive integers t. Hence

$$\langle X_{21}x, x\rangle \le \langle T_c^{*t} X_{21} T_c^t x, x\rangle \le \|X_{21}\| \left\|T_c^t x\right\|^2$$

for all $x \in \mathcal{H}_2$. Letting $t \to \infty$, this implies

$$\langle X_{21}x, x\rangle \le \lim_{t\to\infty} \|X_{21}\| \left\|T_c^t x\right\|^2 = 0$$

for all $x \in \mathcal{H}_2$. Hence

$$X_{21} = \triangle_{T_c^*,T_c}\left(\delta^2_{T_c^*,T_c}(I_2)\right) = \delta^2_{T_c^*,T_c}\left(\triangle_{T_c^*,T_c}(I_2)\right) = 0.$$

The operator T_C^* being hyponormal, it follows from an application of Theorem 2 that

$$\delta_{T_c^*,T_c}\left(\triangle_{T_c^*,T_c}(I_2)\right) = \triangle_{T_c^*,T_c}\left(\delta_{T_c^*,T_c}(I_2)\right) = 0.$$

This completes the proof. $\square$

A result similar to that of Theorem 4 does not hold for hyponormal T. For example, if T is the forward unilateral shift $T(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, \cdots)$, then $\triangle^m_{T^*,T}\left(\delta^n_{T^*,T}(I)\right) = 0$ for all positive integers m, n. However, hyponormal T is neither unitary nor self-adjoint nor a direct sum of the two. If T is hyponormal and satisfies $\triangle^m_{T^*,T}\left(\delta^n_{T^*,T}(I)\right) = 0$, then T is a contraction, hence power bounded. For power bounded operators $T \in B(\mathcal{H})$ satisfying $\triangle^m_{T^*,T}\left(\delta^n_{T^*,T}(I)\right) = 0$, Theorem 4 has the following analogue.

Theorem 6. *If a power bounded operator $T \in B(\mathcal{H})$ satisfies $\triangle^m_{T^*,T}\left(\delta^n_{T^*,T}(I)\right) = 0$ for some positive integers m and n, then:*

(i) $\triangle_{T^*,T}\left(\delta^n_{T^*,T}(I)\right) = 0$;

(ii) *there exist decompositions $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}_1 \oplus (\mathcal{H}_{21} \oplus \mathcal{H}_{22})$, a Hilbert space $\mathcal{K} = \mathcal{H}_{22} \oplus (\mathcal{K} \ominus \mathcal{H}_{22})$ and operators $T_1 \in B(\mathcal{H}_1)$, $T_2 \in B(\mathcal{H}_2)$, $T_3 \in B(\mathcal{H}_2, \mathcal{H}_1)$, $V_u \in B(\mathcal{H}_{21})$, $V_c \in B(H_{22})$, $V_b = \begin{pmatrix} V_c & Z \\ 0 & Y \end{pmatrix} \in B(\mathcal{K})$ (for some operators $Z \in B(\mathcal{K} \ominus \mathcal{H}_{22}, \mathcal{H}_{22})$, $Y \in B(\mathcal{K} \ominus \mathcal{H}_{22})$) such that $T = \begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix} \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$, $T_1 \in C_{0.}$ satisfies $\delta^n_{T_1^*,T_1}(I|_{\mathcal{H}_1}) = 0$, V_u is unitary, V_c is a unilateral shift, V_b is a bilateral shift, the positive operator $\lim_{t\to\infty} T_2^{*t}T_2^t = A$ is injective and $AT_2 = (V_u \oplus V_b)|_{\mathcal{H}_2}A$.*

We remark here that either of the components in Theorem 6, as also in Theorem 4, may be missing.

Proof. If we set $\delta^n_{T^*,T}(I) = X$, then $\triangle^m_{T^*,T}(X) = 0$ and

$$(L_{T^*}R_T)^t(X) = \begin{pmatrix} t \\ m-1 \end{pmatrix} \triangle^{m-1}_{T^*,T}(X) + \sum_{j=0}^{m-2} \begin{pmatrix} t \\ j \end{pmatrix} \triangle^j_{T^*,T}(X)$$

for all integers $t \geq m$ (see the proof of Theorem 4(b) above). The operator T being power bounded, there exists a real number $M > 0$ such that $||T^t|| \leq M$ for all integers $t > 0$. We have

$$\left\|\triangle^{m-1}_{T^*,T}(X)\right\| \leq \lim_{t\to\infty} \frac{1}{\begin{pmatrix} t \\ m-1 \end{pmatrix}} \left[\|(L_{T^*}R_T)^t(X)\| + \left\|\sum_{j=0}^{m-2} \begin{pmatrix} t \\ j \end{pmatrix} \triangle^j_{T^*,T}(X)\right\|\right] = 0.$$

Hence

$$\triangle^{m-1}_{T^*,T}(X) = 0.$$

Repeating the argument a finite number of time, we conclude

$$\triangle_{T^*,T}(X) = 0 \; (\iff \triangle_{T^*,T}(\delta^n_{T^*,T}(I)) = \delta^n_{T^*,T}(\triangle_{T^*,T}(I)) = 0).$$

Recall [21], that the power bounded operator T has an upper triangular matrix representation

$$T = \begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix} \in B(\mathcal{H}_1 \oplus \mathcal{H}_2), \tag{4}$$

where $T_1 \in C_{0.}$ and $T_2 \in C_{1.}$. Evidently,

$$\triangle_{T^*,T}(X) = 0 \Longrightarrow \triangle_{T_1^*,T_1}\left(\delta^n_{T_1^*,T_1}(I_1)\right) = 0, \; I_1 = I|_{\mathcal{H}_1}.$$

Set $\delta^n_{T_1^*,T_1}(I_1) = X_1$. Then

$$\triangle_{T_1^*,T_1}(X_1) = 0 \Longleftrightarrow T_1^* X_1 T_1 = X_1 \Longrightarrow T_1^{*2} X_1 T_1^2 = X_1 \Longrightarrow \cdots \Longrightarrow T_1^{*t} X_1 T_1^t = X_1$$

for all integers $t \geq 0$. Since $T_1 \in C_{0.}$, for every $x \in \mathcal{H}_1$,

$$|\langle X_1 x, x\rangle| = \lim_{t\to\infty} \left|\langle T_1^{*t} X_1 T_1^t x, x\rangle\right| \leq \lim_{t\to\infty} \|X_1\| \left\|T_1^t x\right\|^2 = 0.$$

Hence

$$X_1 = \delta^n_{T_1^*,T_1}(I_1) = 0.$$

Consider now the power bounded operator T_2. Since $T_2 \in C_{1.}$, T_2 is injective and

$$\lim_{t\to\infty} T_2^{*t} T_2^t = A$$

exists and is a positive injective operator which satisfies

$$T_2^* A T_2 = A$$

Ref. [29] (Theorem 5.1). An application of Theorem 1 implies the existence of an isometry $V \in B(\mathcal{H}_2)$ satisfying

$$A^{\frac{1}{2}} T_2 = V A^{\frac{1}{2}}.$$

Since every isometry is part of a unitary, there exists a decomposition $\mathcal{H}_2 = \mathcal{H}_{21} \oplus \mathcal{H}_{22}$, a Hilbert space $\mathcal{K} = \mathcal{H}_{22} \oplus (\mathcal{K} \ominus \mathcal{H}_{22})$ and a unitary

$$W = \begin{pmatrix} Vu & 0 & 0 \\ 0 & V_c & Y \\ 0 & 0 & Z \end{pmatrix} \in B(\mathcal{H}_{21} \oplus \mathcal{H}_{22} \oplus (\mathcal{K} \ominus \mathcal{H}_{22})),$$

$Y \in B(\mathcal{K} \ominus \mathcal{H}_{22}, \mathcal{H}_{22})$ and $Z \in B(\mathcal{K} \ominus \mathcal{H}_{22})$ some operators, such that V_u is unitary, V_c is a unilateral shift, $\begin{pmatrix} V_c & Y \\ 0 & Z \end{pmatrix}$ is a bilateral shift and $V = W|_{\mathcal{H}_2}$ [22] (Lemma 5.7, Page 82). Evidently $A^{\frac{1}{2}} T_2 = W|_{\mathcal{H}_2} A^{\frac{1}{2}}$. $\square$

If $n = 2$ in the preceding theorem, then $\delta_{T_1^*,T_1}(I_1) = 0$ and the operator T_1 is a selfadjoint C_{00}-operator. Furthermore, if the normal parts of the operator T reduce T, then $T_3 = 0$.

An operator $S \in B(\mathcal{H})$ is paranormal if $\|Sx\|^2 \leq \|S^2 x\|$ for all unit vectors $x \in \mathcal{H}$. Hyponormal operators are paranormal, paranormal operators are normaloid, the restriction of a paranormal operator to an invaraint subspace is again paranormal [25] and $\delta^{-1}_{S,V^*}(0) \subseteq \delta^{-1}_{S^*,V}(0)$ for paranormal S and isometric $V \in B(\mathcal{H})$ [30] (p. 316). Hence if the operator T^* of Theorem 4 is paranormal, then $\delta_{V,T_2}(A^{\frac{1}{2}}) = 0$ implies $\delta_{V^*,T_2^*}(A^{\frac{1}{2}}) = 0$. Consequently, T_2 is unitary and (since T^* is necessarily a contraction and the unitary parts of a contraction reduce the contraction) $T_3 = 0$ in representation (4) of T. Thus, $T = T_1 \oplus T_2$, $\delta^n_{T_1^*,T_1}(I_1) = 0$ and T_2 is unitary. If we now assume $n = 2$ in Theorem 4, then we have the following generalisation of a result of Stankus [19] (Proposition 5.22).

Corollary 4. *If $\triangle^m_{T^*,T}\left(\delta^2_{T^*,T}(I)\right) = 0$ for some paranormal operator $T^* \in B(\mathcal{H})$ and integer $m \geq 1$, then T is the direct sum of a selfadjoint operator with a unitary.*

Proof. As seen above $T = T_1 \oplus T_2$, where T_2 is unitary and $\delta^2_{T_1^*,T_1}(I_1) = 0$. Since $\delta^2_{T_1^*,T_1}(I_1) = 0$ if and only if $\delta_{T_1^*,T_1}(I_1) = 0$, the proof follows. $\square$

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