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Sequential Riemann–Liouville and Hadamard–Caputo Fractional Differential Systems with Nonlocal Coupled Fractional Integral Boundary Conditions

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Abstract: In this paper, we initiate the study of existence of solutions for a fractional differential system which contains mixed Riemann–Liouville and Hadamard–Caputo fractional derivatives, complemented with nonlocal coupled fractional integral boundary conditions. We derive necessary conditions for the existence and uniqueness of solutions of the considered system, by using standard fixed point theorems, such as Banach contraction mapping principle and Leray–Schauder alternative. Numerical examples illustrating the obtained results are also presented.

Keywords: coupled systems; Riemann–Liouville fractional derivative; Hadamard–Caputo fractional derivative; nonlocal boundary conditions; existence; fixed point

1. Introduction

Fractional differential equations have played a very important role in almost all branches of applied sciences because they are considered a valuable tool to model many real world problems. For details and applications, we refer the reader to monographs [1–11]. The study of coupled systems of fractional differential equations is important as such systems appear in various problems in applied sciences, see [12–16].

On the other hand, multi-term fractional differential equations also gained considerable importance in view of their occurrence in the mathematical models of certain real world problems, such as behavior of real materials [17], continuum and statistical mechanics [18], an inextensible pendulum with fractional damping terms [19], etc.

Fractional differential equations have several kinds of fractional derivatives, such as Riemann–Liouville fractional derivative, Caputo fractional derivative, Hadamard fractional derivative, and so on. In the literature, there are many papers studying existence and uniqueness results for boundary value problems and coupled systems of fractional differential equations and used mixed types of fractional derivatives, see [20–29]. In [23], the following boundary value problem is considered:

$$\begin{cases} {}^{RL}D^q[{}^{C}D^r x(t) - g(t,x(t))] = f(t,x(t)), \;\; 0<t<T, \\ x(\eta) = \phi(x), \quad I^p x(T) = h(x), \end{cases} \tag{1}$$

where ${}^{RL}D^q$, ${}^{C}D^r$ are Riemann–Liouville and Caputo fractional derivatives of orders $q, r \in (0,1)$, respectively, I^p is the Riemann–Liouville fractional integral of order $p > 0$, $f, g : J \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\phi, h : C(J,\mathbb{R}) \to \mathbb{R}$ are two given functionals.

In [24], the authors initiated the study of a coupled system of sequential mixed Caputo and Hadamard fractional differential equations supplemented with coupled separated boundary conditions. To be more precisely, in [24], existence and uniqueness results are established for the following couple system:

$$\begin{cases} {}^{C}D^{p_1}\,{}^{H}D^{q_1}x(t) = f(t, x(t), y(t)), & t \in [a,b], \\ {}^{H}D^{q_2}\,{}^{C}D^{p_2}y(t) = g(t, x(t), y(t)), & t \in [a,b], \\ \alpha_1 x(a) + \alpha_2\,{}^{C}D^{p_2}y(a) = 0, \quad \beta_1 x(b) + \beta_2\,{}^{C}D^{p_2}y(b) = 0, \\ \alpha_3 y(a) + \alpha_4\,{}^{H}D^{q_1}x(a) = 0, \quad \beta_3 y(b) + \beta_4\,{}^{H}D^{q_1}x(b) = 0, \end{cases} \tag{2}$$

where ${}^{C}D^{p_i}$ and ${}^{H}D^{q_i}$ are notations of the Caputo and Hadamard fractional derivatives of orders p_i and q_i, respectively, $0 < p_i, q_i \le 1$, $i = 1,2$, $f, g : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are nonlinear continuous functions, $a > 0$, $\alpha_i \in \mathbb{R} \setminus \{0\}$, $\beta_i \in \mathbb{R}$, $i = 1, \ldots, 4$.

In [25], the existence and uniqueness of solutions for neutral fractional order coupled systems containing mixed Caputo and Riemann–Liouville sequential fractional derivatives were studied, complemented with nonlocal multi-point and Riemann–Stieltjes integral multi-strip conditions of the form:

$$\begin{cases} {}^{c}D^{q}({}^{RL}D^{p}x(t) + f(t, x(t)) = g(t, x(t), y(t)), \ t \in (0,1), \\ {}^{c}D^{q_1}({}^{RL}D^{p_1}y(t) + f_1(t, y(t)) = g_1(t, x(t), y(t)), \ t \in (0,1), \\ x(0) = 0, \quad bx(1) = a\displaystyle\int_0^1 y(s)dH(s) + \sum_{i=1}^{n}\alpha_i\int_{\xi_i}^{\eta_i} y(s)ds, \\ y(0) = 0, \quad b_1 y(1) = a_1\displaystyle\int_0^1 x(s)dH(s) + \sum_{j=1}^{m}\beta_j\int_{\theta_j}^{\zeta_j} x(s)ds, \end{cases} \tag{3}$$

where ${}^{RL}D^p, {}^{RL}D^{p_1}$, and ${}^{c}D^q, {}^{c}D^{q_1}$ denote the Riemann–Liouville and Caputo fractional derivatives of order p, p_1 and q, q_1, respectively, $0 < p, p_1, q, q_1 \le 1$, with $1 < p + q \le 2$, $1 < p_1 + q_1 \le 2$, f, f_1 and g, g_1 are given continuous functions, $0 < \xi_i < \eta_i < 1$, $0 < \theta_j < \zeta_j < 1$, $\alpha_i, \beta_j \in \mathbb{R}, i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m, a, a_1, b, b_1 \in \mathbb{R}$, and $H(\cdot)$ is a function of bounded variation.

To the best of the authors' knowledge, there are some papers dealing with sequential mixed type fractional derivatives, but we not find in the literature papers dealing with coupled systems with sequential Riemann–Liouville and Hadamard–Caputo fractional differential equations. Motivated by this fact, and to fill this gap, in the present paper, we investigate the existence and uniqueness of solutions for the following coupled system of sequential Riemann–Liouville and Hadamard–Caputo fractional differential equations supplemented with nonlocal coupled fractional integral boundary conditions

$$\begin{cases} {}^{RL}D^{p_1}\left({}^{HC}D^{q_1}x\right)(t) = f(t, x(t), y(t)), & t \in [0,T], \\ {}^{RL}D^{p_2}\left({}^{HC}D^{q_2}y\right)(t) = g(t, x(t), y(t)), & t \in [0,T], \\ {}^{HC}D^{q_1}x(0) = 0, \quad x(T) = \displaystyle\sum_{i=1}^{m}\alpha_i\,{}^{RL}I^{\beta_i}y(\xi_i), \\ {}^{HC}D^{q_2}y(0) = 0, \quad y(T) = \displaystyle\sum_{j=1}^{k}\lambda_j\,{}^{RL}I^{\delta_j}x(\eta_j), \end{cases} \tag{4}$$

where ${}^{RL}D^{p_r}$ and ${}^{HC}D^{q_r}$ are the Riemann–Liouville and Hadamard–Caputo fractional derivatives of orders p_r and q_r, respectively, $0 < p_r, q_r < 1$, $r = 1, 2$, the nonlinear continuous functions $f, g : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$, ${}^{RL}I^{\phi}$ is the Riemann–Liouville fractional integral of orders $\phi > 0$, $\phi \in \{\beta_i, \delta_j\}$ and given constants $\alpha_i, \lambda_j \in \mathbb{R}$, $\xi_i, \eta_j \in (0, T)$, $i = 1, \ldots, m$, $j = 1, \ldots, k$.

Let us compare the coupled system (4) with the coupled system (2) studied in [24].

(i) In (2), we studied a coupled system consisting by mixed Caputo and Hadamard fractional derivatives, while, in (4), we consider mixed Riemann–Liouville and Hadamard–Caputo fractional derivatives.
(ii) In (2), the coupled system was subjected to coupled separated boundary conditions, while, in (4), the coupled system is subjected to nonlocal coupled fractional integral boundary conditions.
(iii) In both problems (4) and (2), the same method is used to establish the existence and uniqueness results, and based on standard fixed point theorems, but their presentation in the framework of mixed coupled Caputo and Hadamard and Riemann–Liouville and Hadamard–Caputo fractional derivatives is new.

We also notice that the conditions ${}^{HC}D^{q_1}x(0) = 0$ and ${}^{HC}D^{q_2}y(0) = 0$ are necessary for the well-posedness of the problem.

By using standard tools from fixed point theory in the present study, we establish existence and uniqueness results for the coupled system (4). The Banach contraction mapping principle is used to obtain the existence and uniqueness result, while an existence result is derived via the Leray–Schauder alternative.

The rest of the paper is organized as follows. In Section 2, some basic definitions and lemmas from fractional calculus are recalled. In addition, an auxiliary lemma, concerning a linear variant of (4), which plays a key role in obtaining the main results, is proved. The main results are presented in Section 3, which also include examples illustrating the basic results. We emphasize that our results are new and significantly enhance the existing literature on the topic, and, as far as we know, they are the first results concerning a coupled system with sequential mixed Riemann–Liouville and Hadamard–Caputo fractional derivatives.

2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [2,30] and present preliminary results needed in our proofs later.

Definition 1. *The Riemann–Liouville fractional derivative of order $p > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by*

$$ {}^{RL}D^{p} f(t) = \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-p-1} f(s) ds, \quad n - 1 < p < n, $$

where $n = [p] + 1$, $[p]$ denotes the integer part of a real number p and Γ is the Gamma function defined by $\Gamma(p) = \int_0^{\infty} e^{-s} s^{p-1} ds$.

Definition 2. *The Riemann–Liouville fractional integral of order p of a function $f : (0, \infty) \to \mathbb{R}$, is defined as*

$$ {}^{RL}I^{p} f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{p-1} f(s) ds, \;\; p > 0, $$

provided the right side is pointwise defined on $\mathbb{R}_+$.

Definition 3. *For an at least n-times differentiable function $g : (0,\infty) \to \mathbb{R}$, the Hadamard–Caputo derivative of fractional order $q > 0$ is defined as*

$$ {}^{HC}D^{q}g(t) = \frac{1}{\Gamma(n-q)} \int_0^t \left(\log \frac{t}{s}\right)^{n-q-1} \delta^n g(s)\frac{ds}{s}, \;\; n-1 < q < n, \;\; n = [q]+1, $$

where $\delta = t\frac{d}{dt}$ and $\log(\cdot) = \log_e(\cdot)$.

Definition 4. *The Hadamard fractional integral of order $q > 0$ is defined as*

$$ {}^{H}I^{q}g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\log \frac{t}{s}\right)^{q-1} g(s)\frac{ds}{s}, $$

provided the integral exists.

Lemma 1 (see [2])**.** *Let $p > 0$. Then, for $y \in C(0,T) \cap L(0,T)$, it holds that*

$$ {}^{RL}I^{p}\left({}^{RL}D^{p}y\right)(t) = y(t) + c_1 t^{p-1} + c_2 t^{p-2} + \cdots + c_n t^{p-n}, $$

where $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$ and $n-1 < p < n$.

Lemma 2 ([30])**.** *Let $u \in AC^n_\delta[0,T]$ or $C^n_\delta[0,T]$ and $q \in \mathbb{C}$, where $X^n_\delta[0,T] = \{g : [0,T] \to \mathbb{C} : \delta^{n-1}g(t) \in X[0,T]\}$. Then, we have*

$$ {}^{H}I^{q}({}^{HC}D^{q})u(t) = u(t) + c_0 + c_1\log t + c_2(\log t)^2 + \cdots + c_{n-1}(\log t)^{n-1}, $$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$ $(n = [q]+1)$.

Lemma 3 ([2], p. 113)**.** *Let $q > 0$ and $\beta > 0$ be given constants. Then, the following formula*

$$ {}^{H}I^{q}t^{\beta} = \beta^{-q}t^{\beta}, $$

holds.

Next, the integral equations are obtained by transformation of a linear variant of problem (4). For convenience in computation, we set some constants

$$ \Omega_1 = \sum_{i=1}^{m} \frac{\alpha_i \xi_i^{\beta_i}}{\Gamma(\beta_i+1)}, \quad \Omega_2 = \sum_{j=1}^{k} \frac{\lambda_j \eta_j^{\delta_j}}{\Gamma(\delta_j+1)} $$

and $\Lambda = \Omega_1\Omega_2 - 1 \neq 0$.

Lemma 4. *Let $f^*, g^* \in C([a,b],\mathbb{R})$ be two given functions. Then, the linear system equivalent to problem (4) of sequential Riemann–Liouville and Hadamard–Caputo fractional differential equations*

$$ \begin{cases} {}^{RL}D^{p_1}\left({}^{HC}D^{q_1}x\right)(t) = f^*(t), & t \in [0,T], \\ {}^{RL}D^{p_2}\left({}^{HC}D^{q_2}y\right)(t) = g^*(t), & t \in [0,T], \\ {}^{HC}D^{q_1}x(0) = 0, \quad x(T) = \sum_{i=1}^{m} \alpha_i {}^{RL}I^{\beta_i}y(\xi_i), \\ {}^{HC}D^{q_2}y(0) = 0, \quad y(T) = \sum_{j=1}^{k} \lambda_j {}^{RL}I^{\delta_j}x(\eta_j), \end{cases} \tag{5} $$

can be written into integral equations as

$$
\begin{aligned}
x(t) \quad = \quad & -\frac{1}{\Lambda}\sum_{i=1}^{m}\alpha_i\,{}^{RL}I^{\beta_i}\Big({}^{H}I^{q_2}\Big({}^{RL}I^{p_2}g^*\Big)\Big)(\xi_i)+\frac{1}{\Lambda}{}^{H}I^{q_1}\Big({}^{RL}I^{p_1}f^*\Big)(T)\\
& +\frac{\Omega_1}{\Lambda}{}^{H}I^{q_2}\Big({}^{RL}I^{p_2}g^*\Big)(T)-\frac{\Omega_1}{\Lambda}\sum_{j=1}^{k}\lambda_j\,{}^{RL}I^{\delta_j}\Big({}^{H}I^{q_1}\Big({}^{RL}I^{p_1}f^*\Big)\Big)(\eta_j)\\
& +{}^{H}I^{q_1}\Big({}^{RL}I^{p_1}f^*\Big)(t),
\end{aligned}\tag{6}
$$

and

$$
\begin{aligned}
y(t) \quad = \quad & -\frac{\Omega_2}{\Lambda}\sum_{i=1}^{m}\alpha_i\,{}^{RL}I^{\beta_i}\Big({}^{H}I^{q_2}\Big({}^{RL}I^{p_2}g^*\Big)\Big)(\xi_i)+\frac{\Omega_2}{\Lambda}{}^{H}I^{q_1}\Big({}^{RL}I^{p_1}f^*\Big)(T)\\
& +\frac{1}{\Lambda}{}^{H}I^{q_2}\Big({}^{RL}I^{p_2}g^*\Big)(T)-\frac{1}{\Lambda}\sum_{j=1}^{k}\lambda_j\,{}^{RL}I^{\delta_j}\Big({}^{H}I^{q_1}\Big({}^{RL}I^{p_1}f^*\Big)\Big)(\eta_j)\\
& +{}^{H}I^{q_2}\Big({}^{RL}I^{p_2}g^*\Big)(t).
\end{aligned}\tag{7}
$$

Proof. For $t \in [0,T]$ and by taking the Riemann–Liouville fractional integral of order p_1 to the first equation of (5), we obtain

$$
{}^{HC}D^{q_1}x(t)=c_1t^{p_1-1}+{}^{RL}I^{p_1}f^*(t),\quad c_1\in\mathbb{R}.\tag{8}
$$

Similarly, for the second equation of (5), we have

$$
{}^{HC}D^{q_2}y(t)=d_1t^{p_2-1}+{}^{RL}I^{p_2}g^*(t),\quad d_1\in\mathbb{R}.\tag{9}
$$

Since $0<p_r<1$, $r=1,2$, the conditions ${}^{HC}D^{q_1}x(0)=0$ and ${}^{HC}D^{q_2}y(0)=0$ imply $c_1=0$ and $d_1=0$, respectively. Applying the Hadamard fractional integral of orders q_1 and q_2 to (8) and (9), respectively, and substituting the values of c_1, d_1, we get

$$
x(t)=c_0+{}^{H}I^{q_1}\Big({}^{RL}I^{p_1}f^*\Big)(t),\tag{10}
$$

and

$$
y(t)=d_0+{}^{H}I^{q_2}\Big({}^{RL}I^{p_2}g^*\Big)(t).\tag{11}
$$

Now, we consider the terms

$$
\sum_{i=1}^{m}\alpha_i\,{}^{RL}I^{\beta_i}y(\xi_i)=d_0\sum_{i=1}^{m}\frac{\alpha_i\xi_i^{\beta_i}}{\Gamma(\beta_i+1)}+\sum_{i=1}^{m}\alpha_i\,{}^{RL}I^{\beta_i}\Big({}^{H}I^{q_2}\Big({}^{RL}I^{p_2}g^*\Big)\Big)(\xi_i)\tag{12}
$$

and

$$
\sum_{j=1}^{k}\lambda_j\,{}^{RL}I^{\delta_j}x(\eta_j)=c_0\sum_{j=1}^{k}\frac{\lambda_j\eta_j^{\delta_j}}{\Gamma(\delta_j+1)}+\sum_{j=1}^{k}\lambda_j\,{}^{RL}I^{\delta_j}\Big({}^{H}I^{q_1}\Big({}^{RL}I^{p_1}f^*\Big)\Big)(\eta_j).\tag{13}
$$

Consequently, by (10)–(13) and boundary fractional integral conditions in (5), it follows that

$$
\begin{aligned}
c_0 \quad = \quad & -\frac{1}{\Lambda}\sum_{i=1}^{m}\alpha_i\,{}^{RL}I^{\beta_i}\Big({}^{H}I^{q_2}\Big({}^{RL}I^{p_2}g^*\Big)\Big)(\xi_i)+\frac{1}{\Lambda}{}^{H}I^{q_1}\Big({}^{RL}I^{p_1}f^*\Big)(T)\\
& +\frac{\Omega_1}{\Lambda}{}^{H}I^{q_2}\Big({}^{RL}I^{p_2}g^*\Big)(T)-\frac{\Omega_1}{\Lambda}\sum_{j=1}^{k}\lambda_j\,{}^{RL}I^{\delta_j}\Big({}^{H}I^{q_1}\Big({}^{RL}I^{p_1}f^*\Big)\Big)(\eta_j),
\end{aligned}
$$

and

$$\begin{aligned} d_0 &= -\frac{\Omega_2}{\Lambda}\sum_{i=1}^{m}\alpha_i\,{}^{RL}I^{\beta_i}\left({}^{H}I^{q_2}\left({}^{RL}I^{p_2}g^*\right)\right)(\xi_i)+\frac{\Omega_2}{\Lambda}{}^{H}I^{q_1}\left({}^{RL}I^{p_1}f^*\right)(T) \\ &\quad+\frac{1}{\Lambda}{}^{H}I^{q_2}\left({}^{RL}I^{p_2}g^*\right)(T)-\frac{1}{\Lambda}\sum_{j=1}^{k}\lambda_j\,{}^{RL}I^{\delta_j}\left({}^{H}I^{q_1}\left({}^{RL}I^{p_1}f^*\right)\right)(\eta_j), \end{aligned}$$

Substituting the values of c_0 and d_0 in (10) and (11), we obtain integral equations in (6) and (7), respectively, as desired.

The converse follows by direct computation. This completes the proof. □

Next, we establish formulas for multiple fractional integrals of Riemann–Liouville and Hadamard types.

Lemma 5. *Let $a, b, c > 0$ be constants. Then, we have*

(i)

$$ {}^{H}I^{b}\left({}^{RL}I^{a}(1)\right)(t)=\frac{a^{-b}t^{a}}{\Gamma(a+1)}. $$

(ii)

$$ {}^{RL}I^{c}\left({}^{H}I^{b}\left({}^{RL}I^{a}(1)\right)\right)(t)=\frac{a^{-b}}{\Gamma(a+c+1)}t^{a+c}. $$

Proof. Since ${}^{RL}I^{a}(1)=\dfrac{t^{a}}{\Gamma(a+1)}$, we have

$$ {}^{H}I^{b}\left({}^{RL}I^{a}(1)\right)(t)=\frac{1}{\Gamma(a+1)}{}^{H}I^{b}t^{a}=\frac{a^{-b}t^{a}}{\Gamma(a+1)}, \tag{14} $$

by using Lemma 3, and (i) is proved. To prove (ii), taking the Riemann–Liouville fractional integral of order $c > 0$ in (14), we have

$$ {}^{RL}I^{c}\left({}^{H}I^{b}\left({}^{RL}I^{a}(1)\right)\right)(t)=\frac{a^{-b}}{\Gamma(a+1)}{}^{RL}I^{c}t^{a}=\frac{a^{-b}}{\Gamma(a+c+1)}t^{a+c}, $$

from ${}^{RL}I^{c}t^{a}=\dfrac{\Gamma(a+1)}{\Gamma(a+c+1)}t^{a+c}$. The proof is completed. □

Corollary 1. *Let constants p_r, q_r, $r = 1, 2$, $\beta_i, \xi_i, \delta_j, \eta_j$ be defined in problem (4). Then, from Lemma 5, we have*

$$\begin{aligned} {}^{H}I^{q_1}\left({}^{RL}I^{p_1}1\right)(T) &= \frac{p_1^{-q_1}T^{p_1}}{\Gamma(p_1+1)}, \\ {}^{H}I^{q_2}\left({}^{RL}I^{p_2}1\right)(T) &= \frac{p_2^{-q_2}T^{p_2}}{\Gamma(p_2+1)}, \\ {}^{RL}I^{\beta_i}\left({}^{H}I^{q_2}\left({}^{RL}I^{p_2}1\right)\right)(\xi_i) &= \frac{p_2^{-q_2}}{\Gamma(p_2+\beta_i+1)}\xi_i^{p_2+\beta_i}, \\ {}^{RL}I^{\delta_j}\left({}^{H}I^{q_1}\left({}^{RL}I^{p_1}1\right)\right)(\eta_j) &= \frac{p_1^{-q_1}}{\Gamma(p_1+\delta_j+1)}\eta_j^{p_1+\delta_j}, \end{aligned}$$

which will be used in the next section.

3. Main Results

Let $\mathcal{C} = C([0, T], \mathbb{R})$ be the Banach space of all continuous functions from $[0, T]$ to $\mathbb{R}$. Let $X = \{x(t) : x(t) \in C^2([0, T], \mathbb{R})\}$ be the space endowed with the norm $\|x\| =$

$\sup\{|x(t)|, t \in [0,T]\}$. Obviously, $(X, \|\cdot\|)$ is a Banach space. Next, we set $Y = \{y(t) : y(t) \in C^2([0,T],\mathbb{R})\}$ with the norm $\|y\| = \sup\{|y(t)|, t \in [0,T]\}$. The product space $(X \times Y, \|(x,y)\|)$ is Banach space with the norm $\|(x,y)\| = \|x\| + \|y\|$.

In the following, for brevity, we use the subscript notation

$$h_{x,y}(t) = h(t, x(t), y(t)), \;\; h \in \{f, g\}, \tag{15}$$

in fractional integral as

$${}^{RL}I^p h_{x,y}(\phi) = \frac{1}{\Gamma(p)} \int_a^{\phi} (\phi - s)^{p-1} h(s, x(s), y(s))\, ds, \tag{16}$$

where $\phi \in \{t, T, \xi_i, \eta_j\}$. In addition, we use it in multiple fractional integrations.

In view of Lemma 4, we define the operator $\mathcal{P} : X \times Y \to X \times Y$ by

$$\mathcal{P}(x,y)(t) = \begin{pmatrix} \mathcal{P}_1(x,y)(t) \\ \mathcal{P}_2(x,y)(t) \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{P}_1(x,y)(t) \;=\; & -\frac{1}{\Lambda}\sum_{i=1}^{m} \alpha_i \, {}^{RL}I^{\beta_i}\Big({}^{H}I^{q_2}\Big({}^{RL}I^{p_2} g_{x,y}\Big)\Big)(\xi_i) + \frac{1}{\Lambda}\, {}^{H}I^{q_1}\Big({}^{RL}I^{p_1} f_{x,y}\Big)(T) \\ & + \frac{\Omega_1}{\Lambda}\, {}^{H}I^{q_2}\Big({}^{RL}I^{p_2} g_{x,y}\Big)(T) - \frac{\Omega_1}{\Lambda}\sum_{j=1}^{k} \lambda_j \, {}^{RL}I^{\delta_j}\Big({}^{H}I^{q_1}\Big({}^{RL}I^{p_1} f_{x,y}\Big)\Big)(\eta_j) \\ & + {}^{H}I^{q_1}\Big({}^{RL}I^{p_1} f_{x,y}\Big)(t) \end{aligned}$$

and

$$\begin{aligned} \mathcal{P}_2(x,y)(t) \;=\; & -\frac{\Omega_2}{\Lambda}\sum_{i=1}^{m} \alpha_i \, {}^{RL}I^{\beta_i}\Big({}^{H}I^{q_2}\Big({}^{RL}I^{p_2} g_{x,y}\Big)\Big)(\xi_i) + \frac{\Omega_2}{\Lambda}\, {}^{H}I^{q_1}\Big({}^{RL}I^{p_1} f_{x,y}\Big)(T) \\ & + \frac{1}{\Lambda}\, {}^{H}I^{q_2}\Big({}^{RL}I^{p_2} g_{x,y}\Big)(T) - \frac{1}{\Lambda}\sum_{j=1}^{k} \lambda_j \, {}^{RL}I^{\delta_j}\Big({}^{H}I^{q_1}\Big({}^{RL}I^{p_1} f_{x,y}\Big)\Big)(\eta_j) \\ & + {}^{H}I^{q_2}\Big({}^{RL}I^{p_2} g_{x,y}\Big)(t). \end{aligned}$$

For computational convenience, we set

$$\begin{aligned} M_1 &= \left(\frac{1+|\Lambda|}{|\Lambda|}\right)\left(\frac{p_1^{-q_1} T^{p_1}}{\Gamma(p_1+1)}\right) + \frac{|\Omega_1|}{|\Lambda|}\left(p_1^{-q_1}\sum_{j=1}^{k} \frac{|\lambda_j|\eta_j^{p_1+\delta_j}}{\Gamma(p_1+\delta_j+1)}\right), \\ M_2 &= \frac{|\Omega_1|}{|\Lambda|}\left(\frac{p_2^{-q_2} T^{p_2}}{\Gamma(p_2+1)}\right) + \frac{1}{|\Lambda|}\left(p_2^{-q_2}\sum_{i=1}^{m} \frac{|\alpha_i|\xi_i^{p_2+\beta_i}}{\Gamma(p_2+\beta_i+1)}\right), \\ M_3 &= \frac{|\Omega_2|}{|\Lambda|}\left(\frac{p_1^{-q_1} T^{p_1}}{\Gamma(p_1+1)}\right) + \frac{1}{|\Lambda|}\left(p_1^{-q_1}\sum_{j=1}^{k} \frac{|\lambda_j|\eta_j^{p_1+\delta_j}}{\Gamma(p_1+\delta_j+1)}\right), \\ M_4 &= \left(\frac{1+|\Lambda|}{|\Lambda|}\right)\left(\frac{p_2^{-q_2} T^{p_2}}{\Gamma(p_2+1)}\right) + \frac{|\Omega_2|}{|\Lambda|}\left(p_2^{-q_2}\sum_{i=1}^{m} \frac{|\alpha_i|\xi_i^{p_2+\beta_i}}{\Gamma(p_2+\beta_i+1)}\right). \end{aligned}$$

In the first result, Banach's contraction mapping principle is used to prove existence and uniqueness of solutions of system (4).

Theorem 1. *Suppose that $f, g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions. In addition, we assume that f, g satisfies the Lipchitz condition:*

(H_1) *there exist constants m_i, n_i, $i = 1, 2$*

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq m_1|u_1 - u_2| + m_2|v_1 - v_2|$$

and

$$|g(t,u_1,v_1)-g(t,u_2,v_2)|\le n_1|u_1-u_2|+n_2|v_1-v_2|,$$

for all $t\in[0,T]$ and $u_i,v_i\in\mathbb{R}$, $i=1,2$. Then, the system (4) has a unique solution on $[0,T]$, if

$$(M_1+M_3)(m_1+m_2)+(M_2+M_4)(n_1+n_2)<1. \tag{17}$$

Proof. Let us define $\sup_{t\in[0,T]}f(t,0,0)=N_1<\infty$ and $\sup_{t\in[0,T]}g(t,0,0)=N_2<\infty$. Choose a constant $r>0$ satisfying

$$r>\frac{(M_1+M_3)N_1+(M_2+M_4)N_2}{1-[(M_1+M_3)(m_1+m_2)+(M_2+M_4)(n_1+n_2)]}.$$

At first, we shall show that the set $\mathcal{P}B_r\subset B_r$, where a ball $B_r=\{(x,y)\in X\times Y:\|(x,y)\|\le r\}$. For $(x,y)\in B_r$, and using

$$|f_{x,y}|\le|f_{x,y}-f_{0,0}|+|f_{0,0}|\le m_1\|x\|+m_2\|y\|+N_1,$$

and

$$|g_{x,y}|\le|g_{x,y}-g_{0,0}|+|g_{0,0}|\le n_1\|x\|+n_2\|y\|+N_2,$$

we get relations

$$\begin{aligned}
&|\mathcal{P}_1(x,y)(t)|\\
\le\;& \frac{1}{|\Lambda|}\sum_{i=1}^{m}|\alpha_i|\,{}^{RL}I^{\beta_i}\left({}^{H}I^{q_2}\left({}^{RL}I^{p_2}|g_{x,y}|\right)\right)(\xi_i)+\frac{1}{|\Lambda|}{}^{H}I^{q_1}\left({}^{RL}I^{p_1}|f_{x,y}|\right)(T)\\
&+\frac{|\Omega_1|}{|\Lambda|}{}^{H}I^{q_2}\left({}^{RL}I^{p_2}|g_{x,y}|\right)(T)+\frac{|\Omega_1|}{|\Lambda|}\sum_{j=1}^{k}|\lambda_j|\,{}^{RL}I^{\delta_j}\left({}^{H}I^{q_1}\left({}^{RL}I^{p_1}|f_{x,y}|\right)\right)(\eta_j)\\
&+{}^{H}I^{q_1}\left({}^{RL}I^{p_1}|f_{x,y}|\right)(T)\\
\le\;& \frac{1}{|\Lambda|}\sum_{i=1}^{m}|\alpha_i|\,{}^{RL}I^{\beta_i}\left({}^{H}I^{q_2}\left({}^{RL}I^{p_2}1\right)\right)(\xi_i)(n_1\|x\|+n_2\|y\|+N_2)\\
&+\frac{1}{|\Lambda|}{}^{H}I^{q_1}\left({}^{RL}I^{p_1}1\right)(T)(m_1\|x\|+m_2\|y\|+N_1)\\
&+\frac{|\Omega_1|}{|\Lambda|}{}^{H}I^{q_2}\left({}^{RL}I^{p_2}1\right)(T)(n_1\|x\|+n_2\|y\|+N_2)\\
&+\frac{|\Omega_1|}{|\Lambda|}\sum_{j=1}^{k}|\lambda_j|\,{}^{RL}I^{\delta_j}\left({}^{H}I^{q_1}\left({}^{RL}I^{p_1}1\right)\right)(\eta_j)(m_1\|x\|+m_2\|y\|+N_1)\\
&+{}^{H}I^{q_1}\left({}^{RL}I^{p_1}1\right)(T)(m_1\|x\|+m_2\|y\|+N_1)\\
=\;& \frac{1}{|\Lambda|}\left(p_2^{-q_2}\sum_{i=1}^{m}\frac{|\alpha_i|\xi_i^{p_2+\beta_i}}{\Gamma(p_2+\beta_i+1)}\right)(n_1\|x\|+n_2\|y\|+N_2)\\
&+\frac{1}{|\Lambda|}\left(\frac{p_1^{-q_1}T^{p_1}}{\Gamma(p_1+1)}\right)(m_1\|x\|+m_2\|y\|+N_1)\\
&+\frac{|\Omega_1|}{|\Lambda|}\left(\frac{p_2^{-q_2}T^{p_2}}{\Gamma(p_2+1)}\right)(n_1\|x\|+n_2\|y\|+N_2)\\
&+\frac{|\Omega_1|}{|\Lambda|}\left(p_1^{-q_1}\sum_{j=1}^{k}\frac{|\lambda_j|\eta_j^{p_1+\delta_j}}{\Gamma(p_1+\delta_j+1)}\right)(m_1\|x\|+m_2\|y\|+N_1)\\
&+\left(\frac{p_1^{-q_1}T^{p_1}}{\Gamma(p_1+1)}\right)(m_1\|x\|+m_2\|y\|+N_1)\\
=\;& M_1(m_1\|x\|+m_2\|y\|+N_1)+M_2(n_1\|x\|+n_2\|y\|+N_2)\\
=\;& (M_1m_1+M_2n_1)\|x\|+(M_1m_2+M_2n_2)\|y\|+M_1N_1+M_2N_2\\
\le\;& [M_1(m_1+m_2)+M_2(n_1+n_2)]r+M_1N_1+M_2N_2.
\end{aligned}$$

Therefore, we deduce that

$$\|\mathcal{P}_1(x,y)\| \leq [M_1(m_1+m_2)+M_2(n_1+n_2)]r + M_1N_1 + M_2N_2.$$

In a similar way of computation, we get

$$\begin{aligned}
|\mathcal{P}_2(x,y)(t)| &\leq \frac{|\Omega_2|}{|\Lambda|}\sum_{i=1}^{m}|\alpha_i|\,{}^{RL}I^{\beta_i}\left({}^{H}I^{q_2}\left({}^{RL}I^{p_2}1\right)\right)(\xi_i)(n_1\|x\|+n_2\|y\|+N_2) \\
&\quad + \frac{|\Omega_2|}{|\Lambda|}{}^{H}I^{q_1}\left({}^{RL}I^{p_1}1\right)(T)(m_1\|x\|+m_2\|y\|+N_1) \\
&\quad + \frac{1}{|\Lambda|}{}^{H}I^{q_2}\left({}^{RL}I^{p_2}1\right)(T)(n_1\|x\|+n_2\|y\|+N_2) \\
&\quad + \frac{1}{|\Lambda|}\sum_{j=1}^{k}|\lambda_j|\,{}^{RL}I^{\delta_j}\left({}^{H}I^{q_1}\left({}^{RL}I^{p_1}1\right)\right)(\eta_j)(m_1\|x\|+m_2\|y\|+N_1) \\
&\quad + {}^{H}I^{q_2}\left({}^{RL}I^{p_2}1\right)(T)(n_1\|x\|+n_2\|y\|+N_2) \\
&= \frac{|\Omega_2|}{|\Lambda|}\left(p_2^{-q_2}\sum_{i=1}^{m}\frac{|\alpha_i|\xi_i^{p_2+\beta_i}}{\Gamma(p_2+\beta_i+1)}\right)(n_1\|x\|+n_2\|y\|+N_2) \\
&\quad + \frac{|\Omega_2|}{|\Lambda|}\left(\frac{p_1^{-q_1}T^{p_1}}{\Gamma(p_1+1)}\right)(m_1\|x\|+m_2\|y\|+N_1) \\
&\quad + \frac{1}{|\Lambda|}\left(\frac{p_2^{-q_2}T^{p_2}}{\Gamma(p_2+1)}\right)(n_1\|x\|+n_2\|y\|+N_2) \\
&\quad + \frac{1}{|\Lambda|}\left(p_1^{-q_1}\sum_{j=1}^{k}\frac{|\lambda_j|\eta_j^{p_1+\delta_j}}{\Gamma(p_1+\delta_j+1)}\right)(m_1\|x\|+m_2\|y\|+N_1) \\
&\quad + \left(\frac{p_2^{-q_2}T^{p_2}}{\Gamma(p_2+1)}\right)(n_1\|x\|+n_2\|y\|+N_2) \\
&= M_3(m_1\|x\|+m_2\|y\|+N_1) + M_4(n_1\|x\|+n_2\|y\|+N_2),
\end{aligned}$$

which yields

$$\|\mathcal{P}_2(x,y)\| \leq [M_3(m_1+m_2)+M_4(n_1+n_2)]r + M_3N_1 + M_4N_2.$$

Then, we conclude that

$$\begin{aligned}
\|\mathcal{P}(x,y)\| &\leq [M_1(m_1+m_2)+M_2(n_1+n_2)]r + M_1N_1 + M_2N_2 \\
&\quad + [M_3(m_1+m_2)+M_4(n_1+n_2)]r + M_3N_1 + M_4N_2 \leq r,
\end{aligned}$$

which leads to $\mathcal{P}B_r \subset B_r$.

In the next step, we will show that the $\mathcal{P}$ is a contraction operator. For any (x_1,y_1), $(x_2,y_2) \in X \times Y$, we have

$$
\begin{aligned}
& |\mathcal{P}_1(x_1,y_1)(t)-\mathcal{P}_1(x_2,y_2)(t)| \\
\leq\ & \frac{1}{|\Lambda|}\sum_{i=1}^{m}|\alpha_i|\,{}^{RL}I^{\beta_i}\left({}^{H}I^{q_2}\left({}^{RL}I^{p_2}|g_{x_1,y_1}-g_{x_2,y_2}|\right)\right)(\xi_i) \\
& +\frac{1}{|\Lambda|}{}^{H}I^{q_1}\left({}^{RL}I^{p_1}|f_{x_1,y_1}-f_{x_2,y_2}|\right)(T)+\frac{|\Omega_1|}{|\Lambda|}{}^{H}I^{q_2}\left({}^{RL}I^{p_2}|g_{x_1,y_1}-g_{x_2,y_2}|\right)(T) \\
& +\frac{|\Omega_1|}{|\Lambda|}\sum_{j=1}^{k}|\lambda_j|\,{}^{RL}I^{\delta_j}\left({}^{H}I^{q_1}\left({}^{RL}I^{p_1}|f_{x_1,y_1}-f_{x_2,y_2}|\right)\right)(\eta_j) \\
& +{}^{H}I^{q_1}\left({}^{RL}I^{p_1}|f_{x_1,y_1}-f_{x_2,y_2}|\right)(T) \\
\leq\ & \frac{1}{|\Lambda|}\sum_{i=1}^{m}|\alpha_i|\,{}^{RL}I^{\beta_i}\left({}^{H}I^{q_2}\left({}^{RL}I^{p_2}1\right)\right)(\xi_i)(n_1\|x_1-x_2\|+n_2\|y_1-y_2\|) \\
& +\frac{1}{|\Lambda|}{}^{H}I^{q_1}\left({}^{RL}I^{p_1}1\right)(T)(m_1\|x_1-x_2\|+m_2\|y_1-y_2\|) \\
& +\frac{|\Omega_1|}{|\Lambda|}{}^{H}I^{q_2}\left({}^{RL}I^{p_2}1\right)(T)(n_1\|x_1-x_2\|+n_2\|y_1-y_2\|) \\
& +\frac{|\Omega_1|}{|\Lambda|}\sum_{j=1}^{k}|\lambda_j|\,{}^{RL}I^{\delta_j}\left({}^{H}I^{q_1}\left({}^{RL}I^{p_1}1\right)\right)(\eta_j)(m_1\|x_1-x_2\|+m_2\|y_1-y_2\|) \\
& +{}^{H}I^{q_1}\left({}^{RL}I^{p_1}1\right)(T)(m_1\|x_1-x_2\|+m_2\|y_1-y_2\|) \\
=\ & M_1(m_1\|x_1-x_2\|+m_2\|y_1-y_2\|)+M_2(n_1\|x_1-x_2\|+n_2\|y_1-y_2\|) \\
=\ & (M_1m_1+M_2n_1)\|x_1-x_2\|+(M_1m_2+M_2n_2)\|y_1-y_2\|.
\end{aligned}
$$

Then, we get the result that

$$
\|\mathcal{P}_1(x_1,y_1)-\mathcal{P}_1(x_2,y_2)\|\leq M_1(m_1+m_2)+M_2(n_1+n_2)(\|x_1-x_2\|+\|y_1-y_2\|). \tag{18}
$$

In addition, we have

$$
\begin{aligned}
& |\mathcal{P}_2(x_1,y_1)(t)-\mathcal{P}_2(x_2,y_2)(t)| \\
\leq\ & \frac{|\Omega_2|}{|\Lambda|}\sum_{i=1}^{m}|\alpha_i|\,{}^{RL}I^{\beta_i}\left({}^{H}I^{q_2}\left({}^{RL}I^{p_2}1\right)\right)(n_1\|x_1-x_2\|+n_2\|y_1-y_2\|) \\
& +\frac{|\Omega_2|}{|\Lambda|}{}^{H}I^{q_1}\left({}^{RL}I^{p_1}1\right)(T)(m_1\|x_1-x_2\|+m_2\|y_1-y_2\|) \\
& +\frac{1}{|\Lambda|}{}^{H}I^{q_2}\left({}^{RL}I^{p_2}1\right)(T)(n_1\|x_1-x_2\|+n_2\|y_1-y_2\|) \\
& +\frac{1}{|\Lambda|}\sum_{j=1}^{k}|\lambda_j|\,{}^{RL}I^{\delta_j}\left({}^{H}I^{q_1}\left({}^{RL}I^{p_1}1\right)\right)(\eta_j)(m_1\|x_1-x_2\|+m_2\|y_1-y_2\|) \\
& +{}^{H}I^{q_2}\left({}^{RL}I^{p_2}1\right)(T)(n_1\|x_1-x_2\|+n_2\|y_1-y_2\|) \\
=\ & M_3(m_1\|x_1-x_2\|+m_2\|y_1-y_2\|)+M_4(n_1\|x_1-x_2\|+n_2\|y_1-y_2\|) \\
=\ & (M_3m_1+M_4n_1)\|x_1-x_2\|+(M_3m_2+M_4n_2)\|y_1-y_2\|,
\end{aligned}
$$

which yields

$$
\|\mathcal{P}_2(x_1,y_1)-\mathcal{P}_2(x_2,y_2)\|\leq M_3(m_1+m_2)+M_4(n_1+n_2)(\|x_1-x_2\|+\|y_1-y_2\|). \tag{19}
$$

The above results in (18) and (19) imply

$$
\begin{aligned}
\|\mathcal{P}(x_1,y_1)-\mathcal{P}(x_2,y_2)\| \leq\ & [(M_1+M_3)(m_1+m_2)+(M_2+M_4)(n_1+n_2)] \\
& \times(\|x_1-x_2\|+\|y_1-y_2\|).
\end{aligned}
$$

Since $(M_1+M_3)(m_1+m_2)+(M_2+M_4)(n_1+n_2)<1$, then the operator $\mathcal{P}$ is a contraction. From the benefits of Banach's fixed point theorem, the operator $\mathcal{P}$ has a unique fixed point, which is the unique solution of (4) on $[0,T]$. The proof is completed. □

The Leray–Schauder alternative is applied to our second existence result.

Lemma 6. *(Leray–Schauder alternative) [31]. Let $Q : U \to U$ be a completely continuous operator. Let*

$$\mu(Q)=\{x\in U : x=\theta Q(x)\ for\ some\ 0<\theta<1\}.$$

Then, either the set $\mu(Q)$ is unbounded, or Q has at least one fixed point.

Theorem 2. *Suppose that there exist constants $a_r, b_r \geq 0$ for $r=1,2$ and $a_0, b_0>0$. In addition, for any $u,v\in\mathbb{R}$, we assume that*

$$\begin{aligned} |f(t,u,v)| &\leq a_0+a_1|u|+a_2|v|,\\ |g(t,u,v)| &\leq b_0+b_1|u|+b_2|v|. \end{aligned}$$

If $(M_1+M_3)a_1+(M_2+M_4)b_1<1$ and $(M_1+M_3)a_2+(M_2+M_4)b_2<1$, then (4) has at least one solution on $[0,T]$.

Proof. The first task of the proof is to show that the operator $\mathcal{P} : X\times Y \to X\times Y$ is completely continuous. The continuity of the functions f,g on $[0,T]\times\mathbb{R}\times\mathbb{R}$ can be used to claim that the operator $\mathcal{P}$ is continuous. Now, we let Φ be the bounded subset of $X\times Y$. Then, there exist positive constants G_1 and G_2 such that

$$|f(t,x,y)|\leq G_1,\ |g(t,x,y)|\leq G_2,\ \forall (x,y)\in\Phi.$$

For any $(x,y)\in\Phi$, we have

$$\begin{aligned} |\mathcal{P}_1(x,y)(t)| &\leq \frac{1}{|\Lambda|}\sum_{i=1}^{m}|\alpha_i|\,{}^{RL}I^{\beta_i}\left({}^{H}I^{q_2}\left({}^{RL}I^{p_2}|g_{x,y}|\right)\right)(\xi_i)+\frac{1}{|\Lambda|}\,{}^{H}I^{q_1}\left({}^{RL}I^{p_1}|f_{x,y}|\right)(T)\\ &\quad+\frac{|\Omega_1|}{|\Lambda|}\,{}^{H}I^{q_2}\left({}^{RL}I^{p_2}|g_{x,y}|\right)(T)+\frac{|\Omega_1|}{|\Lambda|}\sum_{j=1}^{k}|\lambda_j|\,{}^{RL}I^{\delta_j}\left({}^{H}I^{q_1}\left({}^{RL}I^{p_1}|f_{x,y}|\right)\right)(\eta_j)\\ &\quad+{}^{H}I^{q_1}\left({}^{RL}I^{p_1}|f_{x,y}|\right)(T)\\ &\leq \frac{1}{|\Lambda|}\left(p_2^{-q_2}\sum_{i=1}^{m}\frac{|\alpha_i|\xi_i^{p_2+\beta_i}}{\Gamma(p_2+\beta_i+1)}\right)G_2+\frac{1}{|\Lambda|}\left(\frac{p_1^{-q_1}T^{p_1}}{\Gamma(p_1+1)}\right)G_1\\ &\quad+\frac{|\Omega_1|}{|\Lambda|}\left(\frac{p_2^{-q_2}T^{p_2}}{\Gamma(p_2+1)}\right)G_2+\frac{|\Omega_1|}{|\Lambda|}\left(p_1^{-q_1}\sum_{j=1}^{k}\frac{|\lambda_j|\eta_j^{p_1+\delta_j}}{\Gamma(p_1+\delta_j+1)}\right)G_1\\ &\quad+\left(\frac{p_1^{-q_1}T^{p_1}}{\Gamma(p_1+1)}\right)G_1, \end{aligned}$$

which leads to

$$\|\mathcal{P}_1(x,y)\|\leq G_1M_1+G_2M_2.$$

Furthermore, we get

$$
\begin{aligned}
\|\mathcal{P}_2(x,y)\| \quad &\leq \quad \frac{|\Omega_2|}{|\Lambda|}\sum_{i=1}^{m}|\alpha_i|\,{}^{RL}I^{\beta_i}\left({}^{H}I^{q_2}\left({}^{RL}I^{p_2}1\right)\right)G_2+\frac{|\Omega_2|}{|\Lambda|}{}^{H}I^{q_1}\left({}^{RL}I^{p_1}1\right)(T)G_1\\
&\quad +\frac{1}{|\Lambda|}{}^{H}I^{q_2}\left({}^{RL}I^{p_2}1\right)(T)G_2+\frac{1}{|\Lambda|}\sum_{j=1}^{k}|\lambda_j|\,{}^{RL}I^{\delta_j}\left({}^{H}I^{q_1}\left({}^{RL}I^{p_1}1\right)\right)(\eta_j)G_1\\
&\quad +{}^{H}I^{q_2}\left({}^{RL}I^{p_2}1\right)(T)G_2\\
&= \quad G_1M_3+G_2M_4.
\end{aligned}
$$

Therefore, from above two results, we deduce that the set $\mathcal{P}\Phi$ is uniformly bounded. The next is to prove that the set $\mathcal{P}\Phi$ is equicontinuous. Choosing two points $\tau_1,\tau_2\in[0,T]$ such that $\tau_1<\tau_2$, we have, for any $(x,y)\in\Phi$, that

$$
\begin{aligned}
|\mathcal{P}_1(x,y)(\tau_2)-\mathcal{P}_1(x,y)(\tau_1)| \quad &= \quad \left|{}^{H}I^{q_1}\left({}^{RL}I^{p_1}f_{x,y}\right)(\tau_2)-{}^{H}I^{q_1}\left({}^{RL}I^{p_1}f_{x,y}\right)(\tau_1)\right|\\
&\leq \quad G_1\left|{}^{H}I^{q_1}\left({}^{RL}I^{p_1}1\right)(\tau_2)-{}^{H}I^{q_1}\left({}^{RL}I^{p_1}1\right)(\tau_1)\right|\\
&= \quad G_1\frac{p_1^{-q_1}}{\Gamma(p_1+1)}\left|\tau_2^{p_1}-\tau_1^{p_1}\right|,
\end{aligned}
$$

which implies

$$
|\mathcal{P}_1(x,y)(\tau_2)-\mathcal{P}_1(x,y)(\tau_1)|\to 0,\quad \text{as}\ \ \tau_1\to\tau_2.
$$

In addition, we obtain

$$
\begin{aligned}
|\mathcal{P}_2(x,y)(\tau_2)-\mathcal{P}_2(x,y)(\tau_1)| \quad &= \quad \left|{}^{H}I^{q_2}\left({}^{RL}I^{p_2}g_{x,y}\right)(\tau_2)-{}^{H}I^{q_2}\left({}^{RL}I^{p_2}g_{x,y}\right)(\tau_1)\right|\\
&\leq \quad G_2\left|{}^{H}I^{q_2}\left({}^{RL}I^{p_2}1\right)(\tau_2)-{}^{H}I^{q_2}\left({}^{RL}I^{p_2}1\right)(\tau_1)\right|\\
&= \quad G_2\frac{p_2^{-q_2}}{\Gamma(p_2+1)}\left|\tau_2^{p_2}-\tau_1^{p_2}\right|.
\end{aligned}
$$

Then,

$$
|\mathcal{P}_2(x,y)(\tau_2)-\mathcal{P}_2(x,y)(\tau_1)|\to 0,\quad \text{as}\ \ \tau_1\to\tau_2.
$$

Thus, the set $\mathcal{P}\Phi$ is equicontinuous. By taking into account the Arzelá-Ascoli theorem, the set $\mathcal{P}\Phi$ is relatively compact. Then, operator $\mathcal{P}$ is completely continuous.

Finally, we will claim that the set $\mu=\{(x,y)\in X\times Y:(x,y)=\theta\mathcal{P}(x,y),0\leq\theta\leq 1\}$ is bounded. For any $(x,y)\in\mu$, then $(x,y)=\theta\mathcal{P}(x,y)$. Hence, for $t\in[a,b]$, we have

$$
x(t)=\theta\mathcal{P}_1(x,y)(t)\quad \text{and}\quad y(t)=\theta\mathcal{P}_2(x,y)(t).
$$

Therefore, we obtain

$$
\begin{aligned}
\|x\| \quad &\leq \quad (a_0+a_1\|x\|+a_2\|y\|)M_1+(b_0+b_1\|x\|+b_2\|y\|)M_2,\\
\|y\| \quad &\leq \quad (a_0+a_1\|x\|+a_2\|y\|)M_3+(b_0+b_1\|x\|+b_2\|y\|)M_4,
\end{aligned}
$$

which lead to

$$
\begin{aligned}
\|x\|+\|y\| \quad &\leq \quad (M_1+M_3)a_0+(M_2+M_4)b_0+[(M_1+M_3)a_1+(M_2+M_4)b_1]\|x\|\\
&\quad +[(M_1+M_3)a_2+(M_2+M_4)b_2]\|y\|.
\end{aligned}
$$

Thus, the following inequality holds:

$$
\|(x,y)\|\leq\frac{(M_1+M_3)a_0+(M_2+M_4)b_0}{M^*}, \tag{20}
$$

where $M^* = \min\{1-(M_1+M_3)a_1-(M_2+M_4)b_1, 1-(M_1+M_3)a_2-(M_2+M_4)b_2\}$. Hence, the set μ is a bounded set. Then, by using Lemma 6, the operator $\mathcal{P}$ has at least one fixed point. Therefore, we conclude that problem (4) has at least one solution on $[0,T]$. The proof is complete. $\square$

If $a_r, b_r = 0$, $r = 1,2$, in Theorem 2, we have following corollary.

Corollary 2. *Assume that $|f(t,x,y)| \le a_0$ and $|g(t,x,y)| \le b_0$, where $a_0, b_0 > 0$, $\forall (t,x,y) \in [0,T] \times \mathbb{R}^2$. Then, problem (4) has at least one solution on $[0,T]$.*

Next, we present examples to illustrate our results.

Example 1. *Consider the following sequential Riemann–Liouville and Hadamard–Caputo fractional differential system with coupled fractional integral boundary conditions of the form*

$$\begin{cases} {}^{RL}D^{\frac{1}{5}}\left({}^{HC}D^{\frac{4}{5}}x\right)(t) = f(t,x(t),y(t)), \qquad t \in [0,7/4], \\ {}^{RL}D^{\frac{2}{5}}\left({}^{HC}D^{\frac{3}{5}}y\right)(t) = g(t,x(t),y(t)), \qquad t \in [0,7/4], \\ {}^{HC}D^{\frac{4}{5}}x(0) = 0, \quad x\left(\frac{7}{4}\right) = \frac{1}{3}{}^{RL}I^{\frac{3}{4}}y\left(\frac{1}{2}\right) + \frac{2}{7}{}^{RL}I^{\frac{5}{4}}y\left(\frac{5}{4}\right), \\ {}^{HC}D^{\frac{3}{5}}y(0) = 0, \; y\left(\frac{7}{4}\right) = \frac{3}{11}{}^{RL}I^{\frac{1}{2}}x\left(\frac{1}{4}\right) + \frac{4}{17}{}^{RL}I^{\frac{7}{8}}x\left(\frac{3}{4}\right) \\ \qquad\qquad\qquad + \frac{5}{19}{}^{RL}I^{\frac{11}{8}}x\left(\frac{3}{2}\right). \end{cases} \tag{21}$$

Here, $p_1 = 1/5$, $p_2 = 2/5$, $q_1 = 4/5$, $q_2 = 3/5$, $T = 7/4$, $m = 2$, $\alpha_1 = 1/3$, $\alpha_2 = 2/7$, $\beta_1 = 3/4$, $\beta_2 = 5/4$, $\xi_1 = 1/2$, $\xi_2 = 5/4$, $k = 3$, $\lambda_1 = 3/11$, $\lambda_2 = 4/17$, $\lambda_3 = 5/19$, $\delta_1 = 1/2$, $\delta_2 = 7/8$, $\delta_3 = 11/8$, $\eta_1 = 1/4$, $\eta_2 = 3/4$, $\eta_3 = 3/2$. Form all constants, we find that $\Omega_1 \approx 0.5489581728$, $\Omega_2 \approx 0.7217268652$, $|\Lambda| \approx 0.6038021388$, $M_1 \approx 13.82028787$, $M_2 \approx 3.420721316$, $M_3 \approx 9.093047627$, $M_4 \approx 7.354860071$.

Let the two nonlinear Lipschitzian functions $f, g : [0,7/4] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$f(t,x,y) = \frac{1}{12(t+12)}\left(\frac{x^2+2|x|}{1+|x|}\right) + \frac{e^{-t}\sin y}{15(3t+5)} + \frac{1}{2}, \tag{22}$$

$$g(t,x,y) = \frac{\cos \pi t}{6(2t+9)}\tan^{-1} x + \frac{1}{36(4t+7)}\left(\frac{3y^2+4|y|}{1+|y|}\right) + \frac{3}{4}. \tag{23}$$

From (22)–(23), we see that

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \le \frac{1}{72}|x_1-x_2| + \frac{1}{75}|y_1-y_2|$$

and

$$|g(t,x_1,y_1) - g(t,x_2,y_2)| \le \frac{1}{54}|x_1-x_2| + \frac{1}{63}|y_1-y_2|,$$

for all $x_r, y_r \in \mathbb{R}$, $r = 1,2$, we obtain $(M_1+M_3)(1/72+1/75) + (M_2+M_4)(1/54+1/63) \approx 0.9943406888 < 1$. From the benefits of Theorem 1, the problem of a sequential Riemann–Liouville and Hadamard–Caputo fractional differential system with coupled fractional integral boundary conditions (21) with f and g given by (22)–(23), respectively, has a unique solution on $[0,7/4]$.

Example 2. *Consider the sequential Riemann–Liouville and Hadamard–Caputo fractional differential system with coupled fractional integral boundary conditions of the Example 1, where the nonlinear functions* $f, g : [0, 7/4] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ *are defined by*

$$f(t,x,y) = \frac{2e^{-t}}{13} + \frac{1}{2(5t+23)}\left(\frac{x^{16}}{1+|x|^{15}}\right) + \frac{\cos \pi t}{3(2t+15)} y \sin^2 x, \quad (24)$$

$$g(t,x,y) = \frac{4t}{3} + \frac{xe^{-y^2}}{2(4t+11)} + \frac{|y|^{19}\cos^4 x}{3(3t+8)(1+y^{18})}. \quad (25)$$

It is easy to obtain that $|f(t,x,y)| \leq (2/13) + (1/46)|x| + (1/45)|y|$ *and* $|g(t,x,y)| \leq (7/3) + (1/22)|x| + (1/24)|y|$. *By setting* $a_0 = 2/13$, $a_1 = 1/46$, $a_2 = 1/45$, $b_0 = 7/3$, $b_1 = 1/22$ *and* $b_2 = 1/24$, *we can find that* $(M_1 + M_3)a_1 + (M_2 + M_4)b_1 \approx 0.9879151432 < 1$ *and* $(M_1 + M_3)a_2 + (M_2 + M_4)b_2 \approx 0.9581677912 < 1$. *The conclusion of Theorem 2 can be implied that system (21) with* f *and* g *given by (24)–(25), respectively, has at least one solution on* $[0, 7/4]$.

Example 3. *Consider the sequential Riemann–Liouville and Hadamard–Caputo fractional differential system with coupled fractional integral boundary conditions of the Example 1, where the nonlinear functions* $f, g : [0, 7/4] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ *are given by*

$$f(t,x,y) = \frac{1}{2}(1+\cos^2 t) + \frac{|x|e^{-t}}{(1+|x|)} + \frac{2}{\pi}\tan^{-1} y, \quad (26)$$

$$g(t,x,y) = \frac{1}{4}(3+\sin^2 \pi t) + e^{-x^4} + \frac{3y^{22}}{1+y^{22}}. \quad (27)$$

We can check that $|f(t,x,y)| \leq 3$, $|g(t,x,y)| \leq 5$ *for all* $x, y \in \mathbb{R}$. *Using the Corollary 2, the problem (21) with* f *and* g *given by (26) and (27), respectively, has at least one solution on* $[0, 7/4]$.

4. Conclusions

In this paper, we studied a new system of sequential fractional differential equations which consists of mixed fractional derivatives of Riemann–Liouville and Hadamard–Caputo types, supplemented with nonlocal coupled fractional integral boundary conditions. To the best of our knowledge, this is the first system of this type that appeared in the literature. After proving a basic lemma, helping us to transform the considered system into a fixed point problem, we use the standard tools from functional analysis to establish existence and uniqueness results. We use a Banach contraction mapping principle to derive the uniqueness result and Leray–Schauder alternative to obtain an existence result. The obtained results are well illustrated by numerical examples. The obtained results enrich the existing literature on sequential systems of fractional differential equations. Other cases of fractional systems with other types of mixed fractional derivatives or other types of boundary conditions can be studied using the methodology of this paper.

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