



# Article The Existence of Nontrivial Solution for a Class of Kirchhoff-Type Equation of General Convolution Nonlinearity without Any Growth Conditions

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**Abstract:** In this paper, we consider the following Kirchhoff-type equation:  $\begin{cases} -(a+b\int_{\mathbb{R}^N}|\nabla u|^2 dx)\Delta u+V(x)u=(I_{\alpha}*F(u))f(u)+\lambda g(u), \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$  where  $a > 0, b \ge 0, \lambda > 0, \alpha \in (N-2, N), N \ge 3, V : \mathbb{R}^N \to \mathbb{R}$  is a potential function and  $I_{\alpha}$  is a Riesz potential of order  $\alpha \in (N-2, N)$ . Under certain assumptions on V(x), f(u) and g(u), we prove that the equation has at least one nontrivial solution by variational methods.

Keywords: Kirchhoff equation; no growth conditions; cutoff function

MSC: 35J60; 35J35; 35R11; 35A15

## 1. Introduction

In this article, we study the following Kirchhoff-type equation:

$$\begin{cases} -(a+b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x)\Delta u+V(x)u=(I_{\alpha}*F(u))f(u)+\lambda g(u), \text{ in } \mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N), \end{cases}$$
(1)

where  $a > 0, b \ge 0, \lambda > 0, \alpha \in (N-2, N), N \ge 3, F(t) = \int_0^t f(s) ds$  and  $I_{\alpha}$  is a Riesz potential for which its order is  $\alpha \in (N-2, N)$ . Here,  $I_{\alpha}$  is defined by  $I_{\alpha} = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^{\alpha}|x|^{N-\alpha}}$ .

Moreover,  $V(x) : \mathbb{R}^N \to \mathbb{R}$  is a potential function satisfying the following. (V)  $\inf_{x \in \mathbb{R}^N} V(x) = V_0 > 0$  and for any M > 0, there exist r > 0 such that:

$$\lim_{y|\to+\infty} meas\{x\in\mathbb{R}^N: |x-y|\leq r, V(x)\leq M\}=0.$$

Additionally, we suppose that the function  $f \in C^1(\mathbb{R}, \mathbb{R})$  verifies: (*f*1)  $f(t) = o(t^{\frac{\alpha}{N}})$  as  $t \to 0$ ;

$$f2) \lim_{\substack{|t|\to+\infty \\ t \to 2}} \frac{f(t)}{t} = 0;$$

$$f3) \frac{f(t)}{t} \text{ is increasing on } (0, +\infty) \text{ and decreasing on } (-\infty, 0);$$

$$f4) f(t) \text{ is increasing on } \mathbb{R}.$$
Furthermore, we assume that the function  $g \in C(\mathbb{R}, \mathbb{R})$  satisfies the following.
$$g1) g(t) = o(t) \text{ as } t \to 0;$$

$$g2) \lim_{|t|\to+\infty} \frac{g(t)}{t} = +\infty.$$

It is worth mentioning that, here, *g* maybe critical or supercritical.

In the past decades, many scholars have studied the existence of nontrivial solutions for the Kirchhoff-type problem:



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$$\begin{cases} -(a+b\int_{\mathbb{R}^3} |\nabla u|^2 dx) \Delta u + V(x)u = g(x,u), \text{ in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases}$$
(2)

where a > 0,  $b \ge 0$ ,  $V : \mathbb{R}^3 \to \mathbb{R}$  is a potential function and  $g \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ . Problem (2) is a nonlocal problem due to the presence of the term  $b \int_{\mathbb{R}^3} |\nabla u|^2 dx$ , which causes some mathematical difficulties but, at the same time, renders the research problem particular interesting. This problem has a profound and interesting physical context. Indeed, if we set V(x) = 0 and replace  $\mathbb{R}^3$  by a bounded domain  $\Omega \subset \mathbb{R}^3$  in (2), then we obtain the following Kirchhoff Dirichlet problem.

$$\begin{cases} -(a+b\int_{\mathbb{R}^3} |\nabla u|^2 dx) \Delta u = g(x,u), & x \in \Omega, \\ u = 0 & x \in \partial \Omega \end{cases}$$

It is related to the stationary analogue of the equation, as shown as follows:

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left|\frac{\partial u}{\partial x}\right| dx\right) \frac{\partial^2 u}{\partial x^2} = 0$$

which was proposed by G.Kirchhoff as an extension of classical D'Alembert's wave equations for the free vibration of elastic strings. Kirchhoff's model takes into account the changes in the length of the string produced by transverse vibrations. J. L. Lions soon completed the pioneer work. He introduced a functional analysis approach. Since then, Kirchhoff equations have attracted the attention of many researchers. The works include Readers can see [1–11] and the references therein. Unfortunately, most of the studies assume that the growth condition and the Ambrosetti–Rabinowtiz condition are satisfied. On the bright side, in [4], Guo studied the following Kirchhoff-type problem.

$$\begin{cases} -(a+b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x)\Delta u + V(x)u = f(u), \text{ in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases}$$
(3)

He proved the existence of a positive ground state solution to (2) without any (A-R) type condition. Furthermore, in [10], the authors obtained the existence of a nontrivial solution for the following Kirchhoff-type equation.

$$\begin{cases} -(a+b\int_{\mathbb{R}^3} |\nabla u|^2 dx) \Delta u + V(x)u = |u|^{p-2}u + \lambda f(u), \text{ in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases}$$
(4)

In [10], there is no Ambrosetti–Rabinowtiz and no growth condition. Moreover, their conclusion holds for general supercritical nonlinearity.

On the other hand, when a = 1, b = 0 and f = 0, the Equation (1) reduces to:

$$-\Delta u + V(x)u = (I_{\alpha} * |u|^{p})|u|^{p-2}u,$$
(5)

which is called nonlinear Choquard type equation. Its physical background can be found in [12] and the references therein. Furthermore, readers can investigate [6,13–20] for recent achievements.

Motivated by the works mentioned above, especially by [10,21,22], we consider the combination of the two types of Equations (4) and (5) and extend to the general convolution case in  $\mathbb{R}^N$ . In our paper, we obtain the nontrivial solution of Equation (1).

The main outcome of our investigation is as follows.

**Theorem 1.** *If* (*V*), (f1)-(f4) and (g1), (g2) hold, then problem (1) has at least one nontrivial solution for  $\lambda$  small.

For the convenience of expression, hereafter, we use the following notations:

- $X := \{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) u^2 dx < \infty \}$  is equipped with an equivalent norm  $\|u\| = [\int_{\mathbb{R}^N} (a|\nabla u|^2 + V(x) u^2) dx]^{\frac{1}{2}};$
- L<sup>s</sup>(ℝ<sup>N</sup>)(1 ≤ s ≤ ∞) denotes the Lebesgue space with the norm |u|<sub>s</sub> = (∫<sub>ℝ<sup>N</sup></sub> |u|<sup>s</sup>dx)<sup>1/s</sup>;
  For any u ∈ H<sup>1</sup>(ℝ<sup>N</sup>) \ {0}, u<sub>t</sub> is denoted as follows.

$$u_t = \begin{cases} 0, & t = 0, \\ \sqrt{t}u(\frac{x}{t}), & t > 0, \end{cases}$$

- For any  $x \in \mathbb{R}^N$  and  $r > 0, B_r(x) := \{y \in \mathbb{R}^N : |y x| < r\};$
- $C, C_1, C_2, \ldots$  represent positive constants that are possibly different in different lines.

**Remark 1.** According to the condition (V) and [23],  $X \hookrightarrow L^r(\mathbb{R}^N)$  is compact,  $r \in [2, 2^*)$ , where  $2^* = \frac{2N}{N-2}$  if  $N \ge 3$  and  $2^* = \infty$  if N = 1 or 2.

## 2. Preliminaries

In this section, we will provide the revised functional and some lemmas. Notice that there is no growth condition and no Ambrosetti–Rabinowitz condition and so we require the cutoff function.

According to (*g*2), we have g(M) > 0 as M > 0 large and so we define the cutoff function as follows.

$$h_M(t) = \begin{cases} g(t), & 0 < t \le M, \\ C_M t^{p-1}, & t > M, \\ 0, & t \le 0. \end{cases}$$

Here  $C_M = \frac{g(M)}{M^{p-1}}$ ,  $2 . Since <math>g \in C(\mathbb{R}, \mathbb{R})$ ,  $h_M$  is also continuous. Moreover, by (g1),  $h_M$  satisfies the following.

- (h1)  $h_M(t) = o(t)$  as  $t \to 0$ ;
  - (h2)  $\lim_{t \to +\infty} \frac{H_M(t)}{t^4} = +\infty$ , where  $H_M(t) = \int_0^t h_M(s) ds$ ;
  - (h3)  $|h_M(t)| \le C'_M |t| + C_M |t|^{p-1}$ , where  $C'_M = \max_{t \in [0,M]} \frac{|g(t)|}{t}$ ;
- (*h*4) there exist  $\theta = \theta(M) > 0$  such that  $th_M(t) 4H_M(t) \ge -\theta t^2$ ,  $t \ge 0$ . Next, we first consider the following revised problem.

$$-(a+b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x)\Delta u+V(x)u=(I_\alpha*F(u))f(u)+\lambda h_M(u),\ x\ \mathrm{in}\ \mathbb{R}^N,\qquad(6)$$

Problem (6) has a variational structure, i.e., the critical points of the functional  $I_{\lambda}^{M}$ :  $X \to \mathbb{R}$  is defined as follows:

$$I_{\lambda}^{M}(u) = \frac{1}{2} \int_{\mathbb{R}^{N}} [a|\nabla u|^{2} + V(x)u^{2}] dx + \frac{b}{4} \left( \int_{\mathbb{R}^{N}} |\nabla u|^{2} dx \right)^{2} - \frac{1}{2} \int_{\mathbb{R}^{N}} (I_{\alpha} * F(u))F(u) dx - \lambda \int_{\mathbb{R}^{N}} H_{M}(u) dx, \ u \in X$$

$$(7)$$

which are weak solutions of problem (6).

It is obvious that  $I_{\lambda}^{M}$  is of class  $C^{1}$  and the following is the case.

$$\langle (I_{\lambda}^{M})'(u), v \rangle = \int_{\mathbb{R}^{N}} [a \nabla u \nabla v + V(x) uv] dx + b \int_{\mathbb{R}^{N}} |\nabla u|^{2} dx \int_{\mathbb{R}^{N}} \nabla u \nabla v dx - \int_{\mathbb{R}^{N}} (I_{\alpha} * F(u)) f(u) v dx - \lambda \int_{\mathbb{R}^{N}} h_{M}(u) v dx.$$

$$(8)$$

**Lemma 1.** *Assume* (*f*1)–(*f*4) *are fulfilled, then we have:* 

- (1) for all  $\varepsilon > 0$ , there is a  $C_{\varepsilon} > 0$  such that  $|f(t)| \leq \varepsilon |t|^{\frac{\alpha}{N}} + C_{\varepsilon} |t|^{\frac{\alpha+2}{N-2}}$  and  $|F(t)| \leq \varepsilon |t|^{\frac{\alpha}{N-2}}$  $\varepsilon |t|^{\frac{N+\alpha}{N}} + C_{\varepsilon} |t|^{\frac{N+\alpha}{N-2}};$
- (2) for all  $\varepsilon > 0$ , there is a  $C_{\varepsilon} > 0$  such that for every  $p \in (2, 2^*)$ ,  $|F(t)| \le \varepsilon(|t|^{\frac{N+\alpha}{N}} + \varepsilon)$ (3)  $|t|^{\frac{N+\alpha}{N-2}}) + C_{\varepsilon}|t|^{\frac{p(N+\alpha)}{2N}}$ , and  $|F(t)|^{\frac{2N}{N+\alpha}} \le \varepsilon(|t|^2 + |t|^{\frac{2N}{N-2}}) + C_{\varepsilon}|t|^p$ ; for any  $s \ne 0$ , sf(s) > 2F(s) and F(s) > 0.

**Proof.** One can easily obtain the results by elementary calculation.  $\Box$ 

**Lemma 2.** (*Hardy–Littlewood–Sobolev inequality* [24]). Let  $0 < \alpha < N$ , p, q > 1 and  $1 \le r < 1$  $s < \infty$  be such that

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{\alpha}{N}, \ \frac{1}{r} - \frac{1}{s} = \frac{\alpha}{N}$$

(1) For any  $f \in L^p(\mathbb{R}^N)$  and  $g \in L^q(\mathbb{R}^N)$ , one has

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)g(y)}{|x-y|^{N-\alpha}}dxdy\right| \leq C(N,\alpha,p)\|f\|_{L^p(\mathbb{R}^N)}\|g\|_{L^q(\mathbb{R}^N)}$$

(2) For any  $f \in L^r(\mathbb{R}^N)$  one has

$$\left\|\frac{1}{|\cdot|^{N-\alpha}}*f\right\|_{L^{s}(\mathbb{R}^{N})}\leq C(N,\alpha,r)\|f\|_{L^{r}(\mathbb{R}^{N})}.$$

**Remark 2.** By Lemma 1 (1), Lemma 2 (1) and the Sobolev imbedding theorem, we can obtain the following.

$$\left| \int_{\mathbb{R}^{N}} \left( I_{\alpha} * F(u) \right) F(u) dx \right| \leq C |F(u)|_{\frac{2N}{N+\alpha}}^{2}$$

$$\leq C \left[ \int_{\mathbb{R}^{N}} \left( |u|^{\frac{N+\alpha}{N}} + |u|^{\frac{N+\alpha}{N-2}} \right)^{\frac{(2N)}{N+\alpha}} dx \right]^{\frac{N+\alpha}{N}}$$

$$\leq C \left[ \int_{\mathbb{R}^{N}} \left( |u|^{2} + |u|^{\frac{2N}{N-2}} \right) dx \right]^{\frac{N+\alpha}{N}}$$

$$\leq C (||u||^{\frac{2N+2\alpha}{N}} + ||u||^{\frac{2N+2\alpha}{N-2}})$$
(9)

#### 3. Variational Formulation

In this section, we will prove the following results.

**Lemma 3.**  $I_{\lambda}^{M}(u)$  satisfies (*PS*) condition.

**Proof.** Let  $\{u_n\}$  be  $(PS)_c$  sequence of  $I_{\lambda}^M(u)$ . Then by (h4) we have the following.

$$\begin{aligned} c + o(1) \|u_n\| &\geq 4I_{\lambda}^{M}(u_n) - \langle (I_{\lambda}^{M})'(u_n), u_n \rangle \\ &= \int_{\mathbb{R}^N} [a|\nabla u_n|^2 + V(x)|u_n|^2] dx + \int_{\mathbb{R}^N} (I_{\alpha} * F(u_n))[f(u_n)u_n - 2F(u_n)] dx + \lambda \int_{\mathbb{R}^N} [h_M(u_n)u_n - 4H_M(u_n)] dx \\ &\geq \int_{\mathbb{R}^N} [a|\nabla u_n|^2 + V(x)|u_n|^2] dx + \lambda \int_{\mathbb{R}^N} [h_M(u_n)u_n - 4H_M(u_n)] dx \\ &\geq \int_{\mathbb{R}^N} [a|\nabla u_n|^2 + V(x)|u_n|^2] dx - \lambda \int_{\mathbb{R}^N} \theta |u_n^+|^2 dx. \end{aligned}$$
(10)

Here  $u_n^+ = \max\{u_n, 0\}, u_n^- = \min\{u_n, 0\}$  and  $u_n = u_n^+ + u_n^-$ . If  $\{u_n\}$  is unbounded, i.e.,  $||u_n|| \to \infty$ . Let  $v_n = \frac{u_n}{||u_n||}$ , then  $v_n = v_n^+ + v_n^-$  and  $||v_n|| = 1$ . Thus, we can obtain the fact that there exist a  $v \in X$  such that:

and

$$\begin{cases} v_n^+ \rightharpoonup v^+ \text{ in } X, \\ v_n^+ \rightarrow v^+ \text{ in } L^s(\mathbb{R}^N), \ \forall \ s \in [2, 2^*) \\ v_n^+ \rightarrow v^+ \ a.e. \text{ on } \mathbb{R}^N. \end{cases}$$

By (10) we have the following.

$$o(1) \geq \|v_n\|^2 - \lambda \theta \int_{\mathbb{R}^N} |v_n^+|^2 \mathrm{d}x = 1 - \lambda \theta \int_{\mathbb{R}^N} |v^+|^2 \mathrm{d}x + o(1).$$

Therefore  $v^+ \neq 0$ . By (10) and  $(I^M_{\lambda})'(u_n) \to 0$ , we have the following.

$$o(1) = \frac{\langle (I_{\lambda}^{M})'(u_{n}), u_{n} \rangle}{\|u_{n}\|^{4}}$$
  
=  $o(1) + b \left( \int_{\mathbb{R}^{N}} |\nabla v_{n}|^{2} dx \right)^{2} - \|u_{n}\|^{-4} \int_{\mathbb{R}^{N}} (I_{\alpha} * F(u_{n})) f(u_{n}) u_{n} dx - \lambda \int_{\mathbb{R}^{N}} \frac{h_{M}(u_{n}^{+})u_{n}^{+}}{|u_{n}^{+}|^{4}} |v_{n}^{+}|^{4}$  (11)  
 $\leq o(1) + b \left( \int_{\mathbb{R}^{N}} |\nabla v_{n}|^{2} dx \right)^{2} - \lambda \int_{\mathbb{R}^{N}} \frac{h_{M}(u_{n}^{+})u_{n}^{+}}{|u_{n}^{+}|^{4}} |v_{n}^{+}|^{4}.$ 

It is obvious that  $u_n^+ = v_n^+ ||u_n|| \to +\infty$  a.e. $x \in \{x \in \mathbb{R}^N : v_n^+(x) \neq 0\}$ . Together with (11) and (*h*2), we have  $0 \le -\infty$ , which is a contradiction. Therefore,  $\{u_n\}$  is bounded in *X*. Then, by standard methods we can obtain the convergence of  $\{u_n\}$ .

**Lemma 4.** The functional  $I_{\lambda}^{M}$  possesses the mountain-pass geometry, i.e.:

- (1)
- There exist  $\rho, \delta > 0$  such that  $I_{\lambda}^{M} \ge \delta$  for all  $||u|| = \rho$ ; There exist  $e \in H^{1}(\mathbb{R}^{3})$  such that  $||e|| > \rho$  and  $I_{\lambda}^{M}(e) < 0$ . (2)

**Proof.** (1) By (h3) and Lemma 1, we have the following.

$$I_{\lambda}^{M}(u) \geq C_{1} \|u\|^{2} - C_{2}(\|u\|^{\frac{2N+2\alpha}{N}} + \|u\|^{\frac{2N+2\alpha}{N-2}}) - C_{\varepsilon} \|u\|^{p}.$$

Thus, there exist  $\rho$ ,  $\delta > 0$  such that  $I_{\lambda}^{M} \ge \delta$  for all  $||u|| = \rho > 0$  is small enough. (2) We freely choose  $u \in C_{0}^{\infty}(\mathbb{R}^{3})$ , then we can obtain:

$$I_{\lambda}^{M}(u_{t}) = \frac{at^{N-1}}{2} \int_{\mathbb{R}^{N}} |\nabla u|^{2} dx + \frac{t^{N+1}}{2} \int_{\mathbb{R}^{N}} V(x)u^{2} dx + \frac{bt^{2N-2}}{4} \left( \int_{\mathbb{R}^{N}} |\nabla u|^{2} dx \right)^{2} - \frac{t^{N+\alpha}}{2} \int_{\mathbb{R}^{N}} (I_{\alpha} * F(\sqrt{t}u))F(\sqrt{t}u) dx - \lambda \int_{\mathbb{R}^{N}} H_{M}(\sqrt{t}u) dx \to -\infty,$$
(12)

as  $t \to +\infty$ .

Note the following.

$$||u_t||^2 = at^{N-1} \int_{\mathbb{R}^N} |\nabla u|^2 dx + t^{N+1} \int_{\mathbb{R}^N} V_{\infty} u^2 dx.$$

Thus, in taking  $e = t_0 u$  with  $t_0 > 0$  large, we have  $||e|| > \rho$  and  $I_{\lambda}^M(e) < 0$ .  $\Box$ 

**Remark 3.** Now we can define the mountain-pass level of  $I_{\lambda}^{M}$ :

$$c_{\lambda}^{M} = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_{\lambda}^{M}(\gamma(t)) > 0,$$

where:  $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, I_{\lambda}^{M}(\gamma(1)) < 0\}$ . Then, according to [25] and Lemma 3,  $I_{\lambda}^{M}$  has a critical point  $u_{\lambda}$  with  $I_{\lambda}^{M}(u_{\lambda}) = c_{\lambda}^{M}$ .

## 4. Solution for Equation (1)

In this section, we prove the main theorem. By the similar Moser iteration Lemma in [21,22], we only need to prove the following lemma.

**Lemma 5.** There exist two constants B, D > 0 independent on m such that  $|u_0|_{\infty} \leq B(1+\lambda)^D$ .

**Proof.** Similar to (10), we can obtain the following:

$$\begin{aligned} 4c_{\lambda}^{M} &= 4I_{\lambda}^{M}(u_{0}) - \langle (I_{\lambda}^{M})'(u_{0}), u_{0} \rangle \\ &= \int_{\mathbb{R}^{N}} [a|\nabla u_{0}|^{2} + V(x)|u_{0}|^{2}] dx + \int_{\mathbb{R}^{N}} (I_{\alpha} * F(u_{0}))[f(u_{0})u_{0} - 2F(u_{0})] dx + \lambda \int_{\mathbb{R}^{N}} [h_{M}(u_{0})u_{n} - 4H_{M}(u_{0})] dx \\ &\geq \int_{\mathbb{R}^{N}} [a|\nabla u_{0}|^{2} + V(x)|u_{0}|^{2}] dx + \lambda \int_{\mathbb{R}^{N}} [h_{M}(u_{0})u_{0} - 4H_{M}(u_{0})] dx \\ &\geq \int_{\mathbb{R}^{N}} [a|\nabla u_{0}|^{2} + V(x)|u_{0}|^{2}] dx - \lambda \int_{\mathbb{R}^{N}} \theta |u_{0}^{+}|^{2} dx. \end{aligned}$$
(13)

then, by similar argument as the proof of Lemma 3, we can know that  $||u_0||$  is bounded and that there exists Q > 0 such that  $||u_0|| \le Q$ .

Next, set T > 2, r > 0 and  $\tilde{u}_0^T := b(u_0)$ , where  $b : \mathbb{R} \to \mathbb{R}$  is a smooth function satisfying b(s) = s for  $|s| \le T - 1$ , b(-s) = -b(s); b'(s) = 0 for  $s \ge T$  and b'(s) is decreasing in [T - 1, T]. This implies the following:

$$\begin{cases} \tilde{u}_0^T = u_0, & \text{for } |u_0| \le T - 1, \\ |\tilde{u}_0^T| = |b(u_0)| \le |u_0|, & \text{for } T - 1 \le |u_0| \le T, \\ |\tilde{u}_0^T| = C_T > 0, & \text{for } |u_0| \ge T, \end{cases}$$

where  $T - 1 \le C_T \le T$ . Moreover, one can easily obtain the following.

$$0 \leq rac{sb'(s)}{b(s)} \leq 1, \; orall s 
eq 0.$$

Let  $\psi = u_0 |\tilde{u}_0^T|^{2r}$ . Then  $\psi \in X$ , hence by taking  $\psi$  as the test function, one obtains the following.

$$\int_{\mathbb{R}^{N}} (I_{\alpha} * F(u_{0})) f(u_{0}) \psi dx + \lambda \int_{\mathbb{R}^{N}} h_{M}(u_{0}) \psi dx$$

$$= a \int_{\mathbb{R}^{N}} \nabla u_{0} \nabla \psi dx + b \int_{\mathbb{R}^{N}} |\nabla u_{0}|^{2} dx \int_{\mathbb{R}^{N}} \nabla u_{0} \nabla \psi dx + \int_{\mathbb{R}^{N}} V(x) u_{0} \psi dx.$$
(14)

Note that the following obtains.

$$\begin{split} &\int_{\mathbb{R}^{N}} \nabla u_{0} \nabla \psi \mathrm{d}x \\ &\geq \int_{|u_{0}| \leq T-1} (1+r) |\tilde{u}_{0}^{T}|^{2r} |\nabla u_{0}|^{2} \mathrm{d}x + \int_{|u_{0}| \geq T} |\tilde{u}_{0}^{T}|^{2r} |\nabla u_{0}|^{2} \mathrm{d}x \\ &+ \int_{T-1 < |u_{0}| < T} [[\tilde{u}_{0}^{T}]^{2r} + 2ru_{0}b(u_{0})b'(u_{0})]\tilde{u}_{0}^{T}]^{2r-2}] |\nabla u_{0}|^{2} \mathrm{d}x \\ &\geq \int_{|u_{0}| \leq T-1} |\tilde{u}_{0}^{T}|^{2r} |\nabla u_{0}|^{2} \mathrm{d}x + \int_{|u_{0}| \geq T} |\tilde{u}_{0}^{T}|^{2r-2}] |\nabla u_{0}|^{2} \mathrm{d}x \\ &+ \int_{T-1 < |u_{0}| < T} [[\tilde{u}_{0}^{T}]^{2r} + 2ru_{0}b(u_{0})b'(u_{0})]\tilde{u}_{0}^{T}]^{2r-2}] |\nabla u_{0}|^{2} \mathrm{d}x \\ &\geq \frac{1}{(1+r)^{2}} \int_{|u_{0}| \leq T-1} |\nabla [u_{0}(\tilde{u}_{0}^{T})^{r}]|^{2} \mathrm{d}x + \int_{|u_{0}| \geq T} |\nabla [u_{0}(\tilde{u}_{0}^{T})^{r}]|^{2} \mathrm{d}x \\ &+ \int_{T-1 < |u_{0}| < T} [[\tilde{u}_{0}^{T}]^{2r} + 2ru_{0}^{2}(b'(u_{0}))^{2}]\tilde{u}_{0}^{T}]^{2r-2}] |\nabla u_{0}|^{2} \mathrm{d}x \\ &\geq \frac{1}{(1+r)^{2}} \int_{|u_{0}| \leq T-1} |\nabla [u_{0}(\tilde{u}_{0}^{T})^{r}]|^{2} \mathrm{d}x + \int_{|u_{0}| \geq T} |\nabla [u_{0}(\tilde{u}_{0}^{T})^{r}]|^{2} \mathrm{d}x \\ &+ \int_{T-1 < |u_{0}| < T} \left[\frac{1}{(1+r)^{2}} |\tilde{u}_{0}^{T}|^{2r} + \frac{r}{(1+r)^{2}} 2ru_{0}^{2}(b'(u_{0}))^{2} |\tilde{u}_{0}^{T}|^{2r-2}\right] |\nabla u_{0}|^{2} \mathrm{d}x \\ &= \frac{1}{(1+r)^{2}} \int_{|u_{0}| \leq T-1} |\nabla [u_{0}(\tilde{u}_{0}^{T})^{r}]|^{2} \mathrm{d}x + \int_{|u_{0}| \geq T} |\nabla [u_{0}(\tilde{u}_{0}^{T})^{r}]|^{2} \mathrm{d}x \\ &+ \int_{T-1 < |u_{0}| < T} \left[\frac{1}{(1+r)^{2}} |\tilde{u}_{0}^{T}|^{2r} + \frac{r}{(1+r)^{2}} 2ru_{0}^{2} (b'(u_{0}))^{2} |\tilde{u}_{0}^{T}|^{2r-2}\right] |\nabla u_{0}|^{2} \mathrm{d}x \\ &= \frac{1}{(1+r)^{2}} \int_{|u_{0}| \leq T-1} |\nabla [u_{0}(\tilde{u}_{0}^{T})^{r}]|^{2} \mathrm{d}x + \int_{|u_{0}| \geq T} |\nabla [u_{0}(\tilde{u}_{0}^{T})^{r}]|^{2} \mathrm{d}x \\ &+ \int_{T-1 < |u_{0}| < T} \left[\frac{1}{(1+r)^{2}} b^{2r}(u_{0}) |\nabla u_{0}|^{2} + \frac{2}{(1+r)^{2}} u_{0}^{2} |\nabla b^{r}(u_{0})|^{2}\right] \mathrm{d}x \\ &\geq \frac{1}{(1+r)^{2}} \int_{|u_{0}| \leq T-1} |\nabla [u_{0}(\tilde{u}_{0}^{T})^{r}]|^{2} \mathrm{d}x + \int_{|u_{0}| \geq T} |\nabla [u_{0}(\tilde{u}_{0}^{T})^{r}]|^{2} \mathrm{d}x \\ &+ \frac{2C_{1}}{(1+r)^{2}} \int_{|u_{0}| \leq T-1} |\nabla [u_{0}(\tilde{u}_{0}^{T})^{r}]|^{2} \mathrm{d}x + \int_{|u_{0}| \geq T} |\nabla [u_{0}(\tilde{u}_{0}^{T})^{r}]|^{2} \mathrm{d}x \\ &\geq \frac{1}{(1+r)^{2}} \int_{|u_{0}| \leq T-1} |\nabla [u_{0}(\tilde{u}_{0}^{T})^{r}]|^{2} \mathrm{d}x + \int_{|u_{0}| \geq T} |\nabla [u_{0}(\tilde{u}_{0}^{T})^{r}]|^{2} \mathrm{d}x \\ &$$

Hence, by (14), we obtain the following.

$$\begin{split} \int_{\mathbb{R}^N} (I_{\alpha} * |u_0|^p) |u_0|^p |\tilde{u}_0^T|^{2r} \mathrm{d}x + \lambda \int_{\mathbb{R}^N} h_M(u_0) u_0 |\tilde{u}_0^T|^{2r} \mathrm{d}x \\ &\geq \frac{C_1}{(1+r)^2} \int_{\mathbb{R}^N} |\nabla [u_0(\tilde{u}_0^T)^r]|^2 \mathrm{d}x + \int_{\mathbb{R}^N} V(x) |u_0|^2 |\tilde{u}_0^T|^{2r} \mathrm{d}x. \end{split}$$

For any  $\varepsilon > 0$ , by properties of  $\tilde{u}_m^T$  and  $h_M$ , there exists  $C_{\varepsilon} > 0$  such that:

$$\int_{\mathbb{R}^{N}} (I_{\alpha} * F(u_{0})) f(u_{0}) u_{0} |\tilde{u}_{0}^{T}|^{2r} \mathrm{d}x \leq T_{1} \int_{\mathbb{R}^{N}} (I_{\alpha} * F(u_{0})) f(u_{0}) u_{0} \mathrm{d}x \leq T_{2} (\|u\|^{\frac{2N+2\alpha}{N}} + \|u\|^{\frac{2N+2\alpha}{N-2}}) \leq M_{0}.$$

where  $T_1$ ,  $T_2$  and  $M_0$  are positive constants and the following applies.

$$|h_M(t)| \le C'_M |t| + C_M |t|^{2^* - 1}$$

For all  $t \in \mathbb{R}$ . Therefore, for fixed  $\lambda > 0$  and small  $\varepsilon > 0$ , we can deduce the following:

$$\begin{split} \frac{C_1}{(1+r)^2} \int_{\mathbb{R}^N} |\nabla[u_0(\tilde{u}_0^T)^r]|^2 \mathrm{d}x \\ &\leq \int_{\mathbb{R}^N} (I_\alpha * F(u_0)) f(u_0) u_0 |\tilde{u}_0^T|^{2r} \mathrm{d}x + \lambda \int_{\mathbb{R}^N} h_M(u_0) u_0 |\tilde{u}_0^T|^{2r} \mathrm{d}x - \int_{\mathbb{R}^N} V(x) |u_0|^2 |\tilde{u}_0^T|^{2r} \mathrm{d}x \\ &\leq M + \int_{\mathbb{R}^N} V_0 |u_0|^2 |\tilde{u}_0^T|^{2r} \mathrm{d}x + \lambda C \int_{\mathbb{R}^N} u_0^p |\tilde{u}_0^T|^{2r} \mathrm{d}x - \int_{\mathbb{R}^N} V_0 |u_0|^2 |\tilde{u}_0^T|^{2r} \mathrm{d}x \\ &\leq (1+\lambda) C \int_{\mathbb{R}^N} u_0^p |\tilde{u}_0^T|^{2r} \mathrm{d}x. \end{split}$$

Notice that the following is the case.

$$\frac{C_2}{(1+r)^2} \left[ \int_{\mathbb{R}^N} |u_0|^{2^*} |\tilde{u}_0^T|^{2r \cdot \frac{2^*}{2}} dx \right]^{\frac{2}{2^*}} \le \frac{C_1}{(1+r)^2} \int_{\mathbb{R}^N} |\nabla [u_0(\tilde{u}_0^T)^r]|^2 dx.$$

Consequently, the following is obtained.

$$\left[\int_{\mathbb{R}^N} |u_0|^{2^*} |\tilde{u}_0^T|^{2r \cdot \frac{2^*}{2}} dx\right]^{\frac{2}{2^*}} \le (1+\lambda)C(r+1)^2 \int_{\mathbb{R}^N} u_0^{2^*} |\tilde{u}_0^T|^{2r} dx.$$

Take  $r_0 > 0$  and  $r_k = r_0(\frac{2^*}{2})^k = r_{k-1} \cdot \frac{2^*}{2}$ . Then,

$$\begin{split} \left[ \int_{\mathbb{R}^{N}} |u_{0}|^{2^{*}} |\tilde{u}_{0}^{T}|^{2r_{k}} dx \right]^{\frac{1}{2r_{k}}} \\ &\leq \left[ \sqrt{1+\lambda} \sqrt{C}(r_{k-1}+1) \right]^{\frac{1}{r_{k-1}}} \left[ \int_{\mathbb{R}^{N}} |u_{0}|^{2^{*}} |\tilde{u}_{0}^{T}|^{2r_{k-1}} dx \right]^{\frac{1}{2r_{k-1}}} \\ &\leq \prod_{i=0}^{k-1} \left[ \sqrt{1+\lambda} \sqrt{C}(r_{i}+1) \right]^{\frac{1}{r_{i}}} \left[ \int_{\mathbb{R}^{N}} |u_{0}|^{2^{*}} |\tilde{u}_{0}^{T}|^{2r_{0}} dx \right]^{\frac{1}{2r_{0}}} \\ &= \prod_{i=0}^{k-1} (1+\lambda)^{\frac{1}{2r_{i}}} \prod_{i=0}^{k-1} \left[ \sqrt{C}(r_{i}+1) \right]^{\frac{1}{r_{i}}} \left[ \int_{\mathbb{R}^{N}} |u_{0}|^{2^{*}} |\tilde{u}_{0}^{T}|^{2r_{0}} dx \right]^{\frac{1}{2r_{0}}} \\ &= \prod_{i=0}^{k-1} (1+\lambda)^{\frac{1}{2r_{i}}} exp\{ \sum_{i=0}^{k-1} \frac{1}{r_{i}} ln [\sqrt{C}(r_{i}+1)] \} \left[ \int_{\mathbb{R}^{N}} |u_{0}|^{2^{*}} |\tilde{u}_{0}^{T}|^{2r_{0}} dx \right]^{\frac{1}{2r_{0}}}. \end{split}$$

Notice that the following is the case:

$$\begin{split} \left[ \int_{\mathbb{R}^N} |u_0|^{2^*} |\tilde{u}_0^T|^{2r_0 \cdot \frac{N}{N-2}} dx \right]^{\frac{N-2}{N}} \\ &\leq C(r_0+1)^2 \int_{\mathbb{R}^N} |u_0|^{2^*} |\tilde{u}_0^T|^{2r_0} dx \\ &\leq C(r_0+1)^2 \int_{|u_0(x)| < \rho} |u_0|^{2^*} |\tilde{u}_0^T|^{2r_0} dx + \\ &C(r_0+1)^2 \bigg( \int_{|u_0(x)| \ge \rho} |u_0|^{2^*} dx \bigg)^{\frac{2}{N}} \bigg( \int_{\mathbb{R}^N} |u_0|^{2^*} |\tilde{u}_0^T|^{2r_0 \cdot \frac{N}{N-2}} dx \bigg)^{\frac{N-2}{N}}. \end{split}$$

Take  $\rho > 0$  to be such that:

$$C(r_0+1)^2 \left(\int_{|u_0(x)|\geq \rho} |u_0|^{2^*} \mathrm{d}x\right)^{\frac{2}{N}} < \frac{1}{2}.$$

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Then the following obtains:

$$\left[\int_{\mathbb{R}^N} |u_0|^{2^*} |\tilde{u}_0^T|^{2r_0 \cdot \frac{N}{N-2}} \mathrm{d}x\right]^{\frac{N-2}{N}} \le C(r_0+1)^2 \int_{|u_0(x)| < \rho} |u_0|^{2^*} |\tilde{u}_0^T|^{2r_0} \mathrm{d}x \le C.$$

Set the following:

$$d_k = \prod_{i=0}^{k-1} \left[ \sqrt{C}(r_i+1) \right]^{\frac{1}{r_i}} = exp\left\{ \sum_{i=0}^{k-1} \frac{1}{r_i} ln[\sqrt{C}(r_i+1)] \right\}$$

and:

$$e_k = \prod_{i=0}^{k-1} (1+\lambda)^{\frac{1}{2r_i}} = (1+\lambda)^{\frac{2^*}{(2^*-2)2r_0}[1-(\frac{2}{2^*})^k]}.$$

Then  $d_k \to d_\infty$  as  $k \to \infty$  and  $e_k \to e_\infty = (1 + \lambda)^{\frac{2^*}{(2^* - 2)2r_0}}$  as  $k \to \infty$ . By (15), we know that the following is the case.

$$\begin{split} \left[ \int_{\mathbb{R}^{N}} |u_{0}|^{2^{*}} |\tilde{u}_{0}^{T}|^{2r_{k}} dx \right]^{\frac{1}{2r_{k}}} \\ &\leq d_{k} e_{k} \left[ \int_{\mathbb{R}^{N}} |u_{0}|^{2^{*}} |\tilde{u}_{0}^{T}|^{2r_{0}} dx \right]^{\frac{1}{2r_{0}}} \\ &\leq d_{k} e_{k} \left[ \left( \int_{\mathbb{R}^{N}} |u_{0}|^{2^{*}} dx \right)^{\frac{2}{N}} \left( \int_{\mathbb{R}^{N}} |u_{0}|^{2^{*}} |\tilde{u}_{0}^{T}|^{2r_{0}} \cdot \frac{N}{N-2} dx \right)^{\frac{N-2}{N}} \right]^{\frac{1}{2r_{0}}} \\ &\leq C d_{k} e_{k} \left( \int_{\mathbb{R}^{N}} |u_{0}|^{2^{*}} dx \right)^{\frac{1}{Nr_{0}}} \leq C d_{k} e_{k}. \end{split}$$
(16)

From (16), by Fatou Lemma with  $T \to +\infty$ , one has the following.

$$|u_0|_{2^*+2r_k}^{\frac{2^*+2r_k}{2r_k}} \le Cd_k e_k$$

Consequently, let  $k \to \infty$  and we obtain the following:

$$|u_0|_{\infty} \leq Cd_{\infty}e_{\infty} = Cd_{\infty}(1+\lambda)^{\frac{2^*}{(2^*-2)2r_0}} := B(1+\lambda)^D,$$

where B > 0 and D > 0. Thus we complete the proof.  $\Box$ 

**Proof of Theorem 1.** By Lemma 5, for large M > 0, we can choose small  $\lambda_0 > 0$  such that  $|u_0|_{\infty} \leq B(1+\lambda)^D \leq M$  for all  $\lambda \in (0, \lambda_0]$ . The consequence  $u_0$  is also a nontrivial solution of Equation (1) with  $\lambda \in (0, \lambda_0]$ . Thus, we complete the proof.  $\Box$ 

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