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An Affine Model of a Riemann Surface Associated to a Schwarz–Christoffel Mapping

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Abstract: In this paper, we construct an affine model of a Riemann surface with a flat Riemannian metric associated to a Schwarz–Christoffel mapping of the upper half plane onto a rational triangle. We explain the relation between the geodesics on this Riemann surface and billiard motions in a regular stellated n-gon in the complex plane.

Keywords: Schwarz–Christofell; Rieman surface; discrete subgroup

MSC: 30C30; 30F10

Citation: Cushman, R. An Affine Model of a Riemann Surface Associated to a Schwarz–Christoffel Mapping. *Axioms* **2021**, *10*, 49. https://doi.org/10.3390/axioms10020049

Academic Editor: Anna Maria Fino

Received: 19 January 2021
Accepted: 23 March 2021
Published: 1 April 2021

1. Introduction

We give a section by section summary of the contents of this paper.

In §1 we define the Schwarz–Christoffel conformal map F_Q (2) of the complex plane less $\{0,1\}$ onto a quadrilateral Q, which is formed by reflecting a rational triangle $T_{n_0 n_1 n_\infty}$ in the real axis.

In §2, following Aurell and Itzykson [1] we associate to the map F_Q the affine Riemann surface $\mathcal{S}$ in $\mathbb{C}^2$ defined by $\eta^n = \xi^{n-n_0}(1-\xi)^{n-n_1}$, where $\mathbb{C}^2$ has coordinates (ξ,η) and $n = n_0 + n_1 + n_\infty$. Thinking of $\mathcal{S}$ as a branched covering

$$\pi : \mathcal{S} \to \mathbb{C} \setminus \{0,1\} : (\xi,\eta) \mapsto \xi$$

with branch points at $(0,0)$, $(1,0)$ and ∞ corresponding to the branch values 0, 1, and ∞, respectively, we show that $\mathcal{S}$ has genus $\frac{1}{2}\left(n+2-(d_0+d_1+d_\infty)\right)$, where $d_j = \gcd(n,n_j)$ for $j = 0,1,\infty$. Let $\mathcal{S}_{\mathrm{reg}}$ be the set of nonsingular points of $\mathcal{S}$. The map $\widehat{\pi} = \pi_{|\mathcal{S}_{\mathrm{reg}}} : \mathcal{S}_{\mathrm{reg}} \to \mathbb{C}\setminus\{0,1\}$ is a holomorphic n-fold covering map with covering group the cyclic group generated by

$$\mathcal{R} : \mathcal{S}_{\mathrm{reg}} \subseteq \mathbb{C}^2 \to \mathcal{S}_{\mathrm{reg}} \subseteq \mathbb{C}^2 : (\xi,\eta) \mapsto (\xi, \mathrm{e}^{2\pi i/n}\eta).$$

In §3 we build a model $\widetilde{\mathcal{S}}_{\mathrm{reg}}$ of the affine Riemann surface $\mathcal{S}_{\mathrm{reg}}$. The quadrilateral Q is holomorphically diffeomorphic to a fundamental domain $\mathcal{D}$ of the action of the covering group on $\mathcal{S}_{\mathrm{reg}}$. Rotating Q by

$$R : \mathbb{C} \to \mathbb{C} : z \mapsto \mathrm{e}^{2\pi i/n} z$$

gives a regular stellated n-gon K^*, which is invariant under the dihedral group G generated by the mappings R and $U : \mathbb{C} \to \mathbb{C} : z \mapsto \overline{z}$. We study the group theoretic properties of K^*. We show that K^* is invariant under the reflection $S^{(j)} = R^{n_j}U$ in the ray $\{t\,\mathrm{e}^{2\pi i n_n/n} \in \mathbb{C} \,\big|\, t \ge 0\}$ for $j = 0,1,\infty$. To construct the model $\widetilde{\mathcal{S}}_{\mathrm{reg}}$ of the affine Riemann surface $\mathcal{S}_{\mathrm{reg}}$ from the regular stellated n-gon K^* we follow Richens and Berry [2]. We identify two nonadjacent closed edges of $\mathrm{cl}(K^*)$, the closure of K^*, if one edge is obtained from the other by a reflection $S^{(j)}_k = R^k S^{(j)} R^{-k}$ for some $j = 0,1,\infty$. The identification space $(\mathrm{cl}(K^*)\setminus \mathrm{O})^{\sim}$, where O is the center of K^*, is a complex manifold except at points corresponding to O or a vertex of $\mathrm{cl}(K^*)$, where it has a conical singularity. The action of G on $K^* \setminus \mathrm{O}$ induces a free and proper action on the identification space $(K^* \setminus \mathrm{O})^{\sim}$, whose orbit space $\widetilde{\mathcal{S}}_{\mathrm{reg}}$ is a complex manifold with compact closure in $\mathbb{CP}^2$, with genus $\frac{1}{2}\big(n+2-(d_0+d_1+d_\infty)\big)$. Moreover $\widetilde{\mathcal{S}}_{\mathrm{reg}}$ is holomorphically diffeomorphic to the affine Riemann surface $\mathcal{S}_{\mathrm{reg}}$.

In §4, we construct an affine model $\widetilde{S}_{\mathrm{reg}}$ of the Riemann surface $\mathcal{S}_{\mathrm{reg}}$. In other words, we find a discrete subgroup $\mathfrak{G}$ of the 2-dimensional Euclidean group $\mathrm{E}(2)$, which acts freely and properly on $\mathbb{C} \setminus \mathbb{V}^+$ such that after forming an identification space $(\mathbb{C} \setminus \mathbb{V}^+)^{\sim}$ the $\mathfrak{G}$ orbit space $(\mathbb{C} \setminus \mathbb{V}^+)^{\sim} \big/ \mathfrak{G}$ is holomorphically diffeomorphic to $\mathcal{S}_{\mathrm{reg}}$. We now describe the group $\mathfrak{G}$. Reflect the regular stellated n-gon K^* in its edges, and then in the edges of the reflected regular stellated n-gons, et cetera. We obtain a group $\mathcal{T}$ generated by $2n$ translations τ_k so that $\tau_1^{\ell_1} \circ \cdots \circ \tau_{2n}^{\ell_{2n}}$ sends the center O of K^* to the center of a repeatedly reflected reflected n-gon. The set $\mathbb{V}^+$ is the union of the image under $\tau_1^{\ell_1} \circ \cdots \circ \tau_{2n}^{\ell_{2n}}$ of a vertex of $\mathrm{cl}(K^*)$ and its center O for every $(\ell_1, \ldots, \ell_{2n}) \in (\mathbb{Z}_{\ge 0})^{2n}$. Let $\mathfrak{G}$ be the semi-direct product $G \ltimes \mathcal{T}$. The fundamental domain of the $\mathfrak{G}$ action on $\mathbb{C} \setminus \mathbb{V}^+$ is $\mathrm{cl}(K^*)$ less its vertices and center. Identifying equivalent open edges of $K^* \setminus \mathrm{O}$ and then taking G orbits, it follows that the affine model $\widetilde{S}_{\mathrm{reg}}$ of the affine Riemann surface $\mathcal{S}_{\mathrm{reg}}$ is the $\mathfrak{G}$ orbit space $(C \setminus \mathbb{V}^+)^{\sim} \big/ \mathfrak{G}$.

In §5 we show that the mapping

$$\delta_Q : \mathcal{D} \subseteq \mathcal{S}_{\mathrm{reg}} \subseteq \mathbb{C}^2 \to Q \subseteq \mathbb{C} : (\xi, \eta) \mapsto (F_Q \circ \widehat{\pi})(\xi, \eta) = z$$

straightens the nowhere vanishing holomorphic vector field X (11) on $\mathcal{S}_{\mathrm{reg}}$, that is, $T_{(\xi,\eta)}\delta_Q\, X(\xi,\eta) = \frac{\partial}{\partial z}\Big|_{z=\delta_Q(\xi,\eta)}$ for every $(\xi,\eta) \in \mathcal{D}$. We pull back the flat metric $\gamma = \mathrm{d}z \odot \mathrm{d}\overline{z}$ on $\mathbb{C}$ by δ_Q to the metric Γ on $\mathcal{S}_{\mathrm{reg}}$. So δ_Q is a local developing map. Since $\frac{\partial}{\partial z}$ is the geodesic vector field on $(Q, \gamma|_Q)$, it follows that X is a holomorphic geodesic vector field on $(\mathcal{S}_{\mathrm{reg}}, \Gamma)$.

In §6 we study the geometry of the developing map δ_Q. The dihedral group $\mathcal{G}$ generated by $\mathcal{R}$ and $\mathcal{U} : \mathcal{S}_{\mathrm{reg}} \to \mathcal{S}_{\mathrm{reg}} : (\xi, \eta) \mapsto (\overline{\xi}, \overline{\eta})$ is a group of isometries of $(\mathcal{S}_{\mathrm{reg}}, \Gamma)$. The group G generated by R and $U : \mathbb{C} \to \mathbb{C} : z \mapsto \overline{z}$ is a group of isometries of $(Q, \gamma|_Q)$. Extend the holomorphic map δ_Q to a holomorphic map map $\delta_{K^*} : \mathcal{S}_{\mathrm{reg}} \to K^*$ by requiring that $R^j \circ \delta_{K^*} = \delta_Q \circ \mathcal{R}^j$ on $\mathcal{R}^{-j}(\mathcal{D})$. This works since $\mathcal{D}$ is a fundamental domain of the action of the covering group on $\mathcal{S}_{\mathrm{reg}}$, which implies $\mathcal{S}_{\mathrm{reg}} = \amalg_{0 \le j \le n} \mathcal{R}^j(\mathcal{D})$. Thus, the local holomorphic diffeomorphism δ_{K^*} intertwines the $\mathcal{G}$ action on $(\mathcal{S}_{\mathrm{reg}}, \Gamma)$ with the G action on $(K^*, \gamma|_{K^*})$ and intertwines the local geodesic flow of the holomorphic geodesic vector field X with the local geodesic flow of the holomorphic vector field $\frac{\partial}{\partial z}$.

Following Richens and Berry [2] we impose the condition: when a geodesic, starting at a point in $\mathrm{int}(\mathrm{cl}(K^*) \setminus \mathrm{O})$, meets ∂K^* it undergoes a reflection in the edge of K^* that it meets. Such geodesics never meet a vertex of $\mathrm{cl}(K^*)$. Thus, this type of geodesic becomes a billiard motion in $K^* \setminus \mathrm{O}$, which is defined for all time. Billiard motions in polygons have been extensively studied. For a nice overview see Berger ([3], chpt. XI) and references therein. An argument shows that $\widehat{\mathcal{G}}$ invariant geodesics on $(\mathcal{S}_{\mathrm{reg}}, \Gamma)$ correspond under the map $\delta_{K^* \setminus \mathrm{O}}$ to billiard motions on $(K^* \setminus \mathrm{O}, \gamma|_{(K^* \setminus \mathrm{O})})$.

Repeatedly reflecting a billiard motion in an edge of $K^* \setminus \mathrm{O}$ and suitable edges of suitable $\mathcal{T}$ translations of $K^* \setminus \mathrm{O}$ gives a straight line motion λ on $\mathbb{C} \setminus \mathbb{V}^+$. The image of the segment of a billiard motion, where λ intersects $K^* \setminus \mathrm{O}$, in the orbit space $(\mathbb{C} \setminus \mathbb{V}^+)^{\sim} \big/ \mathfrak{G} = \widetilde{S}_{\mathrm{reg}}$, is a geodesic. Here we use the flat Riemannian metric $\widehat{\gamma}$ on $\widetilde{S}_{\mathrm{reg}}$, which is induced by the $\mathfrak{G}$ invariant Euclidean metric γ on $\mathbb{C} \setminus \mathbb{V}^+$ restricted to $K^* \setminus \mathrm{O}$. Consequently, $(\widetilde{S}_{\mathrm{reg}}, \widehat{\gamma})$ is an affine analogue of the affine Riemann surface $\mathcal{S}_{\mathrm{reg}}$ thought of as the orbit space of a discrete subgroup of $\mathrm{PGl}(2, \mathbb{C})$ acting on $\mathbb{C}$ with the Poincaré metric, see Weyl [4].

2. A Schwarz–Christoffel Mapping

Consider the conformal Schwarz–Christoffel mapping

$$F_T : \mathbb{C}^+ = \{\xi \in \mathbb{C} \big| \operatorname{Im} \xi \geq 0\} \to T = T_{n_0 n_1 n_\infty} \subseteq \mathbb{C} : \xi \mapsto \int_0^\xi \frac{\mathrm{d}w}{w^{1-\frac{n_0}{n}}(1-w)^{1-\frac{n_1}{n}}} = z \quad (1)$$

of the upper half plane $\mathbb{C}^+$ to the rational triangle $T = T_{n_0 n_1 n_\infty}$ with interior angles $\frac{n_0}{n}\pi$, $\frac{n_1}{n}\pi$, and $\frac{n_\infty}{n}\pi$, see Figure 1. Here $n_0 + n_1 + n_\infty = n$ and $n_j \in \mathbb{Z}_{\geq 1}$ for $j = 0, 1$ and ∞ with $1 \leq n_0 \leq n_1 \leq n_\infty$. Because n_∞ is greater than or equal to either n_0 or n_1, it follows that the corresponding side OC is the longest side of the triangle $T = \triangle OCD$.

In the integrand of (1) we use the following choice of complex nth root. Suppose that $w \in \mathbb{C} \setminus \{0, 1\}$. Let $w = r_0 \mathrm{e}^{i\theta_0}$ and $1 - w = r_1 \mathrm{e}^{i\theta_1}$ where $r_0, r_1 \in \mathbb{R}_{>0}$ and $\theta_0, \theta_1 \in [0, 2\pi)$. For $w \in (0, 1)$ on the real axis we have $\theta_0 = \theta_1 = 0$, $w = r_0 > 0$, and $1 - w = r_1 > 0$. So $\big(w^{n-n_0}(1-w)^{n-n_1}\big)^{1/n} = (r_0^{n-n_0} r_1^{n-n_1})^{1/n}$. In general for $w \in \mathbb{C} \setminus \{0, 1\}$, we have

$$\big(w^{n-n_0}(1-w)^{n-n_1}\big)^{1/n} = (r_0^{n-n_0} r_1^{n-n_1})^{1/n} \mathrm{e}^{i((n-n_0)\theta_0 + (n-n_1)\theta_1)/n}.$$

From (1) we get

$$F_T(0) = 0, \ F_T(1) = C, \text{ and } F_T(\infty) = D,$$

where $C = \int_0^1 \frac{\mathrm{d}w}{w^{1-\frac{n_0}{n}}(1-w)^{1-\frac{n_1}{n}}}$ and $D = \mathrm{e}^{\frac{n_0}{n}\pi i}\big(\frac{\sin \frac{n_1}{n}\pi}{\sin \frac{n_\infty}{n}\pi}\big)C$. Consequently, the bijective holomorphic mapping F_T sends $\operatorname{int}(\mathbb{C}^+ \setminus \{0, 1\})$, the interior of the upper half plane less 0 and 1, onto $\operatorname{int} T$, the interior of the rational triangle $T = T_{n_0 n_1 n_\infty}$, and sends the boundary of $\mathbb{C}^+ \setminus \{0, 1\}$ to the edges of ∂T less their end points O, C and D, see Figure 1. Thus, the image of $\mathbb{C}^+ \setminus \{0, 1\}$ under F_T is $\operatorname{cl}(T) \setminus \{O, C, D\}$. Here $\operatorname{cl}(T)$ is the closure of T in $\mathbb{C}$.

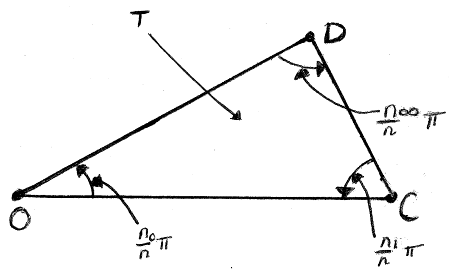

Figure 1. The rational triangle $T = T_{n_0 n_1 n_\infty}$.

Because $F_T|_{[0,1]}$ is real valued, we may use the Schwarz reflection principle to extend F_T to the holomorphic diffeomorphism

$$F_Q : \mathbb{C} \setminus \{0, 1\} \to Q = T \cup \overline{T} \subseteq \mathbb{C} : \xi \mapsto z = \begin{cases} F_T(\xi), & \text{if } \xi \in \mathbb{C}^+ \setminus \{0, 1\} \\ \overline{F_T(\overline{\xi})}, & \text{if } \xi \in \overline{\mathbb{C}^+ \setminus \{0, 1\}}. \end{cases} \quad (2)$$

Here $Q = Q_{n_0 n_1 n_\infty}$ is a quadrilateral with internal angles $2\pi\frac{n_0}{n}$, $\pi\frac{n_\infty}{n}$, $2\pi\frac{n_1}{n}$, and $\pi\frac{n_\infty}{n}$ and vertices at O, D, C, and $\overline{D}$, see Figure 2. The conformal mapping F_Q sends $\mathbb{C} \setminus \{0, 1\}$ onto $\operatorname{cl}(Q) \setminus \{O, D, C, \overline{D}\}$.

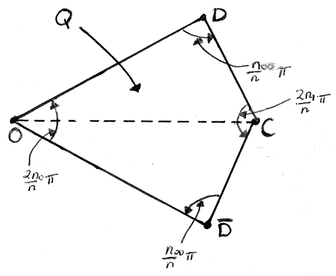

Figure 2. The rational quadrilateral Q.

3. The Geometry of an Affine Riemann Surface

Let ξ and η be coordinate functions on $\mathbb{C}^2$. Consider the affine Riemann surface $\mathcal{S} = \mathcal{S}_{n_0,n_1,n_\infty}$ in $\mathbb{C}^2$, associated to the holomorphic mapping F_Q, defined by

$$g(\xi,\eta) = \eta^n - \xi^{n-n_0}(1-\xi)^{n-n_1} = 0, \tag{3}$$

see Aurell and Itzykson [1]. We determine the singular points of $\mathcal{S}$ by solving

$$\begin{aligned} 0 &= \mathrm{d}g(\xi,\eta) \\ &= -(n-n_0)\xi^{n-n_0-1}(1-\xi)^{n-n_1-1}(1 - \tfrac{2n-n_0-n_1}{n-n_0}\xi)\,\mathrm{d}\xi + n\eta^{n-1}\,\mathrm{d}\eta \end{aligned} \tag{4}$$

For $(\xi,\eta) \in \mathcal{S}$, we have $\mathrm{d}g(\xi,\eta) = 0$ if and only if $(\xi,\eta) = (0,0)$ or $(1,0)$. Thus, the set $\mathcal{S}_{\mathrm{sing}}$ of singular points of $\mathcal{S}$ is $\{(0,0),(1,0)\}$. So the affine Riemann surface $\mathcal{S}_{\mathrm{reg}} = \mathcal{S} \setminus \mathcal{S}_{\mathrm{sing}}$ is a complex submanifold of $\mathbb{C}^2$. Actually, $\mathcal{S}_{\mathrm{reg}} \subseteq \mathbb{C}^2 \setminus \{\eta = 0\}$, for if $(\xi,\eta) \in \mathcal{S}$ and $\eta = 0$, then either $\xi = 0$ or $\xi = 1$.

Lemma 1. *Topologically $\mathcal{S}_{\mathrm{reg}}$ is a compact Riemann surface $\overline{\mathcal{S}} \subseteq \mathbb{CP}^2$ of genus $g = \frac{1}{2}\big(n+2-(d_0+d_1+d_\infty)\big)$ less three points: $[0:0:1]$, $[1:0:1]$, and $[0:1:0]$. Here $d_j = \gcd(n_j,n)$ for $j = 0,1,\infty$.*

Proof. Consider the (projective) Riemann surface $\overline{\mathcal{S}} \subseteq \mathbb{CP}^2$ specified by the condition $[\xi:\eta:\zeta] \in \overline{\mathcal{S}}$ if and only if

$$G(\xi,\eta,\zeta) = \zeta^{n-n_0-n_1}\eta^n - \xi^{n-n_0}(\zeta-\xi)^{n-n_1} = 0. \tag{5}$$

Thinking of G as a polynomial in η with coefficients which are polynomials in ξ and ζ, we may view $\overline{\mathcal{S}}$ as the branched covering

$$\overline{\pi} : \overline{\mathcal{S}} \subseteq \mathbb{CP}^2 \to \mathbb{CP} : [\xi:\eta:\zeta] \mapsto [\xi:\zeta]. \tag{6}$$

When $\zeta = 1$ we get the affine branched covering

$$\pi = \overline{\pi}|\mathcal{S} : \mathcal{S} = \overline{\mathcal{S}} \cap \{\zeta = 1\} \subseteq \mathbb{C}^2 \to \mathbb{C} = \mathbb{CP} \cap \{\zeta = 1\} : (\xi,\eta) \mapsto \xi. \tag{7}$$

From (3) it follows that $\eta = \omega_k(\xi^{n-n_0}(1-\xi)^{n-n_1})^{1/n}$, where ω_k for $k = 0,1,\ldots,n-1$ is an nth root of unity with and $(\ \ )^{1/n}$ is the complex nth root used in the definition of the mapping F_T (1). Thus, the branched covering mapping $\overline{\pi}$ (6) has n "sheets" except at its branch points. Since

$$\eta = \xi^{1-\frac{n_0}{n}}(1-\xi)^{1-\frac{n_1}{n}} = \xi^{1-\frac{n_0}{n}}\big(1-(1-\tfrac{n_1}{n})\xi + \cdots\big) \tag{8a}$$

and

$$\eta = (1-\xi)^{1-\frac{n_1}{n}}\big(1-(1-\xi)\big)^{1-\frac{n_0}{n}}$$

$$= (1-\xi)^{1-\frac{n_1}{n}}\big(1-(1-\tfrac{n_0}{n})(1-\xi)+\cdots\big), \tag{8b}$$

it follows that $\xi = 0$ and $\xi = 1$ are branch points of the mapping $\overline{\pi}$ of degree $\frac{n}{d_0}$ and $\frac{n}{d_1}$, since $d_j = \gcd(n, n_j) = \gcd(n-n_j, n_j)$ for $j = 0, 1$, see McKean and Moll ([5], p. 39). Because

$$\begin{aligned}\eta &= \big(\tfrac{1}{\xi}\big)^{-(1-\frac{n_0}{n})}\big(1-\frac{1}{\frac{1}{\xi}}\big)^{1-\frac{n_1}{n}} = (-1)^{1-\frac{n_1}{n}}\xi^{2-\frac{n_0+n_1}{n}}\big(1-\tfrac{1}{\xi}\big)^{1-\frac{n_1}{n}} \\ &= (-1)^{1-\frac{n_1}{n}}\xi^{1+\frac{n_\infty}{n}}\big(1-(1-\tfrac{n_1}{n})\tfrac{1}{\xi}+\cdots\big),\end{aligned} \tag{8c}$$

∞ is a branch point of the mapping $\overline{\pi}$ of degree $\frac{n}{d_\infty}$, where $d_\infty = \gcd(n, n_\infty)$. Hence the ramification index of 0, 1, ∞ is $d_0(\frac{n}{d_0}-1) = n-d_0$, $n-d_1$, and $n-d_\infty$, respectively. Thus, the map $\overline{\pi}$ has d_0 fewer sheets at 0, d_1 fewer at 1, and d_∞ fewer at ∞ than an n-fold covering of $\mathbb{CP}$. Thus, the total ramification index r of the mapping $\overline{\pi}$ is $r = (n-d_0)+(n-d_1)+(n-d_\infty)$. By the Riemann–Hurwitz formula, the genus g of $\overline{\mathcal{S}}$ is $r = 2n+2g-2$. In other words,

$$g = \tfrac{1}{2}\big(n+2-(d_0+d_1+d_\infty)\big). \tag{9}$$

Consequently, the affine Riemann surface $\mathcal{S}$ is the compact connected surface $\overline{\mathcal{S}}$ less the point at ∞, namely, $\mathcal{S} = \overline{\mathcal{S}} \setminus \{[0:1:0]\}$. So $\mathcal{S}_{\mathrm{reg}}$ is the compact connected surface $\overline{\mathcal{S}}$ less three points: $[0:0:1]$, $[1:0:1]$, and $[0:1:0]$. □

Examples of $\overline{\mathcal{S}} = \overline{\mathcal{S}}_{n_0,n_1,n_\infty}$

1. $n_0 = 1$, $n_0 = 1$, $n_\infty = 4$; $n = 6$. So $d_0 = 1$, $d_1 = 1$, $d_\infty = 2$. Hence $2g = 8-4 = 4$. So $g = 2$.
2. $n_0 = 2$, $n_1 = 2$, $n_\infty = 3$; $n = 7$. So $d_0 = d_1 = d_\infty = 1$. Hence $2g = 9-3 = 6$. So $g = 3$.

Table 1 below lists all the partitions $\{n_1, n_0, n_\infty\}$ of n, which give a low genus Riemann surface $\overline{\mathcal{S}} = \overline{\mathcal{S}}_{n_0,n_1,n_\infty}$

Table 1. Based on the table in Aurell and Itzykson ([1], p. 193).

g	$n_0, n_1, n_\infty; n$	g	$n_0, n_1, n_\infty; n$
1	1, 1, 1; 3	3	2, 2, 3; 7
1	1, 1, 2; 4	3	1, 3, 3; 7
1	1, 2, 3; 6	3	1, 1, 5; 7
2	1, 2, 2; 5	3	2, 3, 3; 8
2	1, 1, 3; 5	3	1, 2, 5; 8
2	1, 1, 4; 6	3	1, 1, 6; 8
2	1, 3, 4; 8	3	2, 3, 4; 9
2	2, 3, 5; 10	3	1, 3, 5; 9
2	1, 4, 5; 10	3	1, 2, 6; 9
		3	3, 4, 5; 12
		3	1, 5, 6; 12
		3	1, 3, 8; 12
		3	2, 5, 7; 14
		3	1, 6, 7; 14

Corollary 1. *If n is an odd prime number and $\{n_1, n_0, n_\infty\}$ is a partition of n into three parts, then the genus of $\overline{\mathcal{S}}$ is $\frac{1}{2}(n-1)$.*

Proof. Because n is prime, we get $d_0 = d_1 = d_\infty = 1$. Using the formula $g = \frac{1}{2}\big(n+2-(d_0+d_1+d_\infty)\big)$ we obtain $g = \frac{1}{2}(n-1)$. □

Corollary 2. *The singular points of the Riemann surface $\overline{\mathcal{S}}$ are $[0:0:1]$, $[1:0:1]$, and if $n_\infty > 1$ then also $[0:1:0]$.*

Proof. A point $[\xi:\eta:\zeta] \in \overline{\mathcal{S}}_{\mathrm{sing}}$ if and only if $[\xi:\eta:\zeta] \in \overline{\mathcal{S}}$, that is,

$$0 = G(\xi,\eta,\zeta) = \zeta^{n-(n_0+n_1)}\eta^n - \xi^{n-n_0}(\zeta-\xi)^{n-n_1} \tag{10a}$$

and

$$\begin{aligned}(0,0,0) &= DG(\xi,\eta,\zeta) \\ &= \big(-\xi^{n-n_0-1}(\zeta-\xi)^{n-n_1-1}\big((n-n_0)(\zeta-\xi)-(n-n_1)\xi\big), \\ &\qquad n\eta^{n-1}\zeta^{n-(n_0+n_1)}, (n-(n_0+n_1))\eta^n\zeta^{n-n_0-n_1-1} \\ &\qquad\qquad -(n-n_1)\xi^{n-n_0}(\zeta-\xi)^{n-n_1-1}\big)\end{aligned} \tag{10b}$$

We need only check the points $[0:0:1]$, $[1:0:1]$ and $[0:1:0]$. Since the first two points are singular points of $\mathcal{S} = \overline{\mathcal{S}} \setminus \{[0:1:0]\}$, they are singular points of $\overline{\mathcal{S}}$. Thus, we need to see if $[0:1:0]$ is a singular point of $\overline{\mathcal{S}}$. Substituting $(0,1,0)$ into the right hand side of (10b) we get $\begin{cases}(0,0,1), & \text{if } n_\infty = n-(n_0+n_1) = 1 \\ (0,0,0), & \text{if } n_\infty > 1.\end{cases}$ Thus, $[0:1:0]$ is a singular point of $\overline{\mathcal{S}}$ only if $n_\infty > 1$. □

Lemma 2. *The mapping*

$$\widehat{\pi} = \pi|\mathcal{S}_{\mathrm{reg}} : \mathcal{S}_{\mathrm{reg}} \subseteq \mathbb{C}^2 \to \mathbb{C}\setminus\{0,1\} : (\xi,\eta) \mapsto \xi \tag{11}$$

is a surjective holomorphic local diffeomorphism.

Proof. Let $(\xi,\eta) \in \mathcal{S}_{\mathrm{reg}}$ and let

$$X(\xi,\eta) = \eta\frac{\partial}{\partial\xi} + \tfrac{n-n_0}{n}\frac{\xi^{n-n_0-1}(1-\xi)^{n-n_1-1}\big(1-\frac{2n-n_0-n_1}{n-n_0}\xi\big)}{\eta^{n-2}}\frac{\partial}{\partial\eta}. \tag{12}$$

By hypothesis $(\xi,\eta) \in \mathcal{S}_{\mathrm{reg}}$ implies that $\eta \neq 0$. The vector $X(\xi,\eta)$ is defined and is nonzero. From $(X \lrcorner\, \mathrm{d}g)(\xi,\eta) = 0$ and $T_{(\xi,\eta)}\mathcal{S}_{\mathrm{reg}} = \ker \mathrm{d}g(\xi,\eta)$, it follows that $X(\xi,\eta) \in T_{(\xi,\eta)}\mathcal{S}_{\mathrm{reg}}$. Using the definition of $X(\xi,\eta)$ (12) and the definition of the mapping π (7), we see that the tangent of the mapping $\widehat{\pi}$ (11) at $(\xi,\eta) \in \mathcal{S}_{\mathrm{reg}}$ is given by

$$T_{(\xi,\eta)}\widehat{\pi} : T_{(\xi,\eta)}\mathcal{S}_{\mathrm{reg}} \to T_\xi(\mathbb{C}\setminus\{0,1\}) = \mathbb{C} : X(\xi,\eta) \mapsto \eta\frac{\partial}{\partial\xi}. \tag{13}$$

Since $X(\xi,\eta)$ and $\eta\frac{\partial}{\partial\xi}$ are nonzero vectors, they form a complex basis for $T_{(\xi,\eta)}\mathcal{S}_{\mathrm{reg}}$ and $T_\xi(\mathbb{C}\setminus\{0,1\})$, respectively. Thus, the complex linear mapping $T_{(\xi,\eta)}\widehat{\pi}$ is an isomorphism. Hence $\widehat{\pi}$ is a local holomorphic diffeomorphism. □

Corollary 3. *$\widehat{\pi}$ (11) is a surjective holomorphic n to 1 covering map.*

Proof. We only need to show that $\widehat{\pi}$ is a covering map. First we note that every fiber of $\widehat{\pi}$ is a finite set with n elements, since for each fixed $\xi \in \mathbb{C}\setminus\{0,1\}$ we have $\widehat{\pi}^{-1}(\xi) = \{(\xi,\eta) \in \mathcal{S}_{\mathrm{reg}} \,\big|\, \eta = \omega_k(\xi^{n-n_0}(1-\xi)^{n-n_1})^{1/n}\}$. Here ω_k for $k = 0,1,\ldots,n-1$, is an n^{th} root of 1 and $(\;)^{1/n}$ is the complex n^{th} root used in the definition of the Schwarz–Christoffel map F_Q (2). Hence the map $\widehat{\pi}$ is a proper surjective holomorphic submersion, because each fiber is compact. Thus, the mapping $\widehat{\pi}$ is a presentation of a locally trivial fiber bundle with fiber consisting of n distinct points. In other words, the map $\widehat{\pi}$ is a n to 1 covering mapping. □

Consider the group $\widehat{\mathcal{G}}$ of linear transformations of $\mathbb{C}^2$ generated by

$$\mathcal{R} : \mathbb{C}^2 \to \mathbb{C}^2 : (\xi, \eta) \mapsto (\xi, \mathrm{e}^{2\pi i/n}\eta).$$

Clearly $\mathcal{R}^n = \mathrm{id}_{\mathbb{C}^2} = e$, the identity element of $\widehat{\mathcal{G}}$ and $\widehat{\mathcal{G}} = \{e, \mathcal{R}, \ldots, \mathcal{R}^{n-1}\}$. For each $(\xi, \eta) \in \mathcal{S}$ we have

$$(\mathrm{e}^{2\pi i/n}\eta)^n - \xi^{n-n_0}(1-\xi)^{n-n_1} = \eta^n - \xi^{n-n_0}(1-\xi)^{n-n_1} = 0.$$

So $\mathcal{R}(\xi, \eta) \in \mathcal{S}$. Thus, we have an action of $\widehat{\mathcal{G}}$ on the affine Riemann surface $\mathcal{S}$ given by

$$\Phi : \widehat{\mathcal{G}} \times \mathcal{S} \to \mathcal{S} : \big(g, (\xi, \eta)\big) \mapsto g(\xi, \eta). \tag{14}$$

Since $\widehat{\mathcal{G}}$ is finite, and hence is compact, the action Φ is proper. For every $g \in \widehat{\mathcal{G}}$ we have $\Phi_g(0,0) = (0,0)$ and $\Phi_g(1,0) = (1,0)$. So Φ_g maps $\mathcal{S}_{\mathrm{reg}}$ into itself. At $(\xi, \eta) \in \mathcal{S}_{\mathrm{reg}}$ the isotropy group $\widehat{\mathcal{G}}_{(\xi,\eta)}$ is $\{e\}$, that is, the $\widehat{\mathcal{G}}$-action Φ on $\mathcal{S}_{\mathrm{reg}}$ is free. Thus, the orbit space $\mathcal{S}_{\mathrm{reg}}/\widehat{\mathcal{G}}$ is a complex manifold.

Corollary 4. *Consider the holomorphic mapping*

$$\rho : \mathcal{S}_{\mathrm{reg}} \subseteq \mathbb{C}^2 \to \mathcal{S}_{\mathrm{reg}}/\widehat{\mathcal{G}} \subseteq \mathbb{C}^2 : (\xi, \eta) \mapsto [(\xi, \eta)],$$

where $[(\xi, \eta)]$ *is the* $\widehat{\mathcal{G}}$*-orbit* $\{\Phi_g(\xi, \eta) \in \mathcal{S}_{\mathrm{reg}} \big| g \in \widehat{\mathcal{G}}\}$ *of* (ξ, η) *in* $\mathcal{S}_{\mathrm{reg}}$*. The* $\widehat{\mathcal{G}}$ *principal bundle presented by the mapping* ρ *is isomorphic to the bundle presented by the mapping* $\widehat{\pi}$ *(11).*

Proof. We use invariant theory to determine the orbit space $\mathcal{S}/\widehat{\mathcal{G}}$. The algebra of polynomials on $\mathbb{C}^2$, which are invariant under the $\widehat{\mathcal{G}}$-action Φ, is generated by $\pi_1 = \xi$ and $\pi_2 = \eta^n$. Since $(\xi, \eta) \in \mathcal{S}$, these polynomials are subject to the relation

$$\pi_2 - \pi_1^{n-n_0}(1-\pi_1)^{n-n_1} = \eta^n - \xi^{n-n_0}(1-\xi)^{n-n_1} = 0. \tag{15}$$

Equation (15) defines the orbit space $\mathcal{S}/\widehat{\mathcal{G}}$ as a complex subvariety of $\mathbb{C}^2$. This subvariety is homeomorphic to $\mathbb{C}$, because it is the graph of the function $\pi_1 \mapsto \pi_1^{n-n_0}(1-\pi_1)^{n-n_1}$. Consequently, the orbit space $\mathcal{S}_{\mathrm{reg}}/\widehat{\mathcal{G}}$ is holomorphically diffeomorphic to $\mathbb{C} \setminus \{0,1\}$.

It remains to show that $\widehat{\mathcal{G}}$ is the group of covering transformations of the bundle presented by the mapping $\widehat{\pi}$ (11). For each $\xi \in \mathbb{C} \setminus \{0,1\}$ look at the fiber $\widehat{\pi}^{-1}(\xi)$. If $(\xi, \eta) \in \widehat{\pi}^{-1}(\xi)$, then $\mathcal{R}^{\pm 1}(\xi, \eta) = (\xi, \mathrm{e}^{\pm 2\pi i/n}\eta) \in \mathcal{S}_{\mathrm{reg}}$, since $(\xi, \mathrm{e}^{\pm 2\pi i/n}\eta) \neq (0,0)$ or $(1,0)$ and $(\xi, \mathrm{e}^{\pm 2\pi i/n}\eta) \in \mathcal{S}$. Thus, $\Phi_{\mathcal{R}^{\pm 1}}\big(\widehat{\pi}^{-1}(\xi)\big) \subseteq \widehat{\pi}^{-1}(\xi)$. So $\widehat{\pi}^{-1}(\xi) \subseteq \Phi_{\mathcal{R}}\big(\widehat{\pi}^{-1}(\xi)\big) \subseteq \widehat{\pi}^{-1}(\xi)$. Hence $\Phi_{\mathcal{R}}\big(\widehat{\pi}^{-1}(\xi)\big) = \widehat{\pi}^{-1}(\xi)$. Thus, $\Phi_{\mathcal{R}}$ is a covering transformation for the bundle presented by the mapping $\widehat{\pi}$. So $\widehat{\mathcal{G}}$ is a subgroup of the group of covering transformations. These groups are equal because $\widehat{\mathcal{G}}$ acts transitively on each fiber of the mapping $\widehat{\pi}$. □

4. Another Model for $\mathcal{S}_{\mathrm{reg}}$

We construct another model $\widetilde{\mathcal{S}}_{\mathrm{reg}}$ for the smooth part $\mathcal{S}_{\mathrm{reg}}$ of the affine Riemann surface $\mathcal{S}$ (3) as follows. Let $\mathcal{D} \subseteq \mathcal{S}_{\mathrm{reg}}$ be a fundamental domain for the $\widehat{\mathcal{G}}$ action Φ (14) on $\mathcal{S}_{\mathrm{reg}}$. So $(\xi, \eta) \in \mathcal{D}$ if and only if for $\xi \in \mathbb{C} \setminus \{0,1\}$ we have $\eta = \big(\xi^{n-n_0}(1-\xi)^{n-n_1}\big)^{1/n}$. Here $(\;\;)^{1/n}$ is the n^{th} root used in the definition of the mapping F_Q (2). The domain $\mathcal{D}$ is a connected subset of $\mathcal{S}_{\mathrm{reg}}$ with nonempty interior. Its image under the map $\widehat{\pi}$ (11) is $\mathbb{C} \setminus \{0,1\}$. Thus, $\mathcal{D}$ is one "sheet" of the covering map $\widehat{\pi}$. So $\widehat{\pi}|_{\mathcal{D}}$ is one to one.

Using the extended Schwarz–Christoffel mapping F_Q (2), we give a more geometric description of the fundamental domain $\mathcal{D}$. Consider the mapping

$$\delta : \mathcal{D} \subseteq \mathcal{S}_{\mathrm{reg}} \to Q \subseteq \mathbb{C} : (\xi, \eta) \mapsto F_Q\big(\widehat{\pi}(\xi, \eta)\big), \tag{16}$$

where the map $\widehat{\pi}$ is given by Equation (11). The map δ is a holomorphic diffeomorphism of $\operatorname{int}\mathcal{D}$ onto $\operatorname{int} Q$, which sends $\partial\mathcal{D}$ homeomorphically onto ∂Q. Look at $\mathrm{cl}(Q)$, which is a closed quadrilateral with vertices O, D, C, and $\overline{D}$. The set $\delta(\mathcal{D})$ contains the open edges OD, DC, and $C\overline{D}$ but *not* the open edge $O\overline{D}$ of $\mathrm{cl}(Q)$, see Figure 3 above.

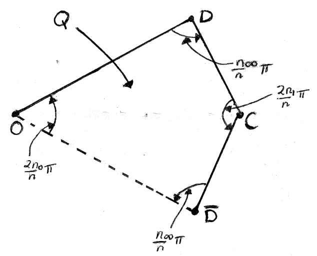

Figure 3. The image Q of the fundamental domain $\mathcal{D}$ under the mapping δ. The open edges OD, $C\overline{D}$, and CD of the quadrilateral are included; while the open edge $O\overline{D}$ is excluded.

Let $K^* = K^*_{n_0,n_1,n_\infty} = \amalg_{0\le j\le n-1} R^j\big(\delta(\mathcal{D})\big)$ be the region in $\mathbb{C}$ formed by repeatedly rotating $Q = \delta(\mathcal{D})$ through an angle $2\pi/n$. Here R is the rotation $\mathbb{C} \to \mathbb{C} : z \mapsto \mathrm{e}^{2\pi i/n} z$. We say that the quadrilateral $Q = Q_{2n_0,n_\infty,2n_1,n_\infty}$ *forms* K^*, see Figure 4 above.

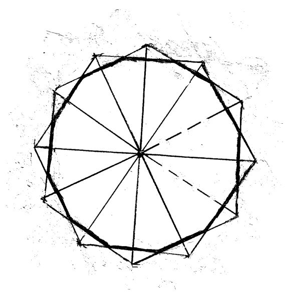

Figure 4. The regular duodecagon K and the stellated regular duodecagon $K^* = K^*_{4,4,4}$ formed by rotating the quadrilateral $Q_{4,4,4}$ through an angle $2\pi/12$ around the origin.

Theorem 1. *The connected set K^* is a regular stellated n-gon with its $2n$ vertices omitted, which is formed from the quadrilateral $Q' = OD'C\overline{D'}$, see Figure 5.*

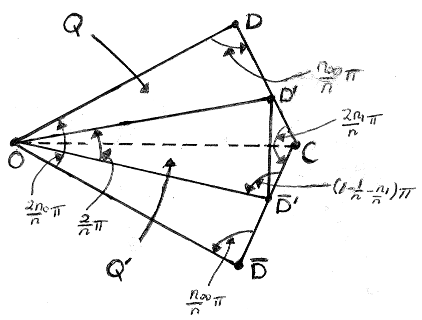

Figure 5. The dart in the figure is the quadrilateral $Q' = OD'C\overline{D'}$, which is the union of the triangles $T = \Delta OD'C$ and the triangle $\overline{T'}$.

Proof. By construction the quadrilateral $Q' = OD'C\overline{D'}$ is contained in the quadrilateral $Q = ODC\overline{D}$. Note that $Q \subseteq \bigcup_{j=[-\frac{n_1+1}{2}]}^{[\frac{n_1+1}{2}]} R^j(Q')$. Thus,

$$K^* = \bigcup_{j=0}^{n} R^j(Q) \subseteq \bigcup_{j=0}^{n} R^j(Q') \subseteq \bigcup_{j=0}^{n} R^j(Q) = K^*.$$

So $K^* = \bigcup_{j=0}^{n} R^j(Q')$. Thus, K^* is the regular stellated n-gon less its vertices, one of whose open sides is the diagonal $D'\overline{D'}$ of Q'. $\square$

We would like to extend the mapping δ (16) to a mapping of $\mathcal{S}_{\mathrm{reg}}$ onto K^*. Let

$$\delta_{\Phi_{\mathcal{R}^j}(\mathcal{D})} : \Phi_{\mathcal{R}^j}(\mathcal{D}) \subseteq \mathcal{S}_{\mathrm{reg}} \to R^j\big(\delta(\mathcal{D})\big) \subseteq K^* : (\xi,\eta) \mapsto R^j\delta\big(\Phi_{\mathcal{R}^{-j}}(\xi,\eta)\big),$$

where Φ is the $\widehat{\mathcal{G}}$ action defined in Equation (14). So we have a mapping

$$\delta_{K^*} : \mathcal{S}_{\mathrm{reg}} \subseteq \mathbb{C}^2 \to K^* \subseteq \mathbb{C} \tag{17}$$

defined by $(\delta_{K^*})|_{\Phi_{\mathcal{R}^j}(\mathcal{D})} = \delta|_{\Phi_{\mathcal{R}^j}(\mathcal{D})}$. The mapping δ_{K^*} is defined on $\mathcal{S}_{\mathrm{reg}}$, because $\mathcal{S}_{\mathrm{reg}} = \amalg_{0\le j\le n-1}\Phi_{\mathcal{R}^j}(\mathcal{D})$, since $\mathcal{D}$ is a fundamental domain for the $\widehat{\mathcal{G}}$-action Φ (14) on $\mathcal{S}_{\mathrm{reg}}$. Because $K^* = \amalg_{0\le j\le n-1}R^j\big(\delta(\mathcal{D})\big)$, the mapping δ_{K^*} is surjective. Hence δ_{K^*} is holomorphic, since it is continuous on $\mathcal{S}_{\mathrm{reg}}$ and is holomorphic on the dense open subset $\amalg_{0\le j\le n-1}\mathcal{R}^j(\mathrm{int}\,\mathcal{D})$ of $\mathcal{S}_{\mathrm{reg}}$. Let $U : \mathbb{C} \to \mathbb{C} : z \mapsto \overline{z}$ and let G be the group generated by the rotation R and the reflection U subject to the relations $R^n = U^2 = e$ and $RU = UR^{-1}$. Shorthand $G = \langle U, R \,|\, U^2 = e = R^n \;\&\; RU = UR^{-1}\rangle$. Then $G = \{e; R^pU^\ell,\ \ell = 0,1 \;\&\; p = 0,1,\ldots,n-1\}$. The group G is the dihedral group D_{2n}. The closure $\mathrm{cl}(K^*)$ of $K^* = \amalg_{0\le j\le n-1}R^j(Q)$ in $\mathbb{C}$ is invariant under $\widehat{G}$, the subgroup of G generated by the rotation R. Because the quadrilateral Q is invariant under the reflection $U : z \mapsto \overline{z}$, and $UR^j = R^{-j}U$, it follows that $\mathrm{cl}(K^*)$ is invariant under the reflection U. So $\mathrm{cl}(K^*)$ is invariant under the group G.

We now look at some group theoretic properties of K^*.

Lemma 3. *If F is a closed edge of the polygon $\mathrm{cl}(K^*)$ and $g|_F = \mathrm{id}|_F$ for some $g \in G$, then $g = e$.*

Proof. Suppose that $g \ne e$. Then $g = R^pU^\ell$ for some $\ell \in \{0,1\}$ and some $p \in \{0,1,\ldots,n-1\}$. Let $g = R^pU$ and suppose that F is an edge of $\mathrm{cl}(K^*)$ such that $\mathrm{int}(F) \cap \mathbb{R} \ne \varnothing$, where $\mathbb{R} = \{\mathrm{Re}\, z \,|\, z \in \mathbb{C}\}$. Then $U(F) = F$, but $U|_F \ne \mathrm{id}_F$. So $g|_F = R^pU|_F \ne \mathrm{id}_F$. Now suppose that $\mathrm{int}(F) \cap \mathbb{R} = \varnothing$. Then $U(F) \ne F$. So $U|_F \ne \mathrm{id}_F$. Hence $g|_F \ne \mathrm{id}_F$. Finally, suppose that $g = R^p$ with $p \ne 0$. Then $g(F) \ne F$. So $g|_F \ne \mathrm{id}|_F$. $\square$

Lemma 4. *For $j = 0,1,\infty$ put $S^{(j)} = R^{n_j}U$. Then $S^{(j)}$ is a reflection in the closed ray $\ell^j = \{te^{i\pi n_j/n} \in \mathbb{C} \,|\, t \in OD\}$. The ray ℓ^0 is the closure of the side OD of the quadrilateral $Q = ODC\overline{D}$ in Figure 5.*

Proof. $S^{(j)}$ fixes every point on the closed ray ℓ^j, because

$$S^{(j)}(\{te^{i\pi n_j/n} \,|\, t \in OD\}) = R^{n_j}(\{te^{-i\pi n_j/n} \,|\, t \in OD\}) = \{te^{i\pi n_j/n} \,|\, t \in OD\}.$$

Since $(S^{(j)})^2 = (R^{n_j}U)(R^{n_j}U) = R^{n_j}(UU)R^{-n_j} = e$, it follows that $S^{(j)}$ is a reflection in the closed ray ℓ^j. $\square$

Corollary 5. *For every $j = 0,1,\infty$ and every $k \in \{0,1,\ldots,n-1\}$ let $S_k^{(j)} = R^kS^{(j)}R^{-k}$. Here $S_n^{(j)} = S_0^{(j)} = S^{(j)}$, because $R^n = e$. Then $S_k^{(j)}$ is a reflection in the closed ray $R^k\ell^j$.*

Proof. This follows because $(S_k^{(j)})^2 = R^k(S^{(j)})^2R^{-k} = e$ and $S_k^{(j)}$ fixes every point on the closed ray $R^k\ell^j$, for

$$\begin{aligned} S_k^{(j)}\big(R^k(\{te^{i\pi n_j/n} \,|\, t \in OD\})\big) &= R^kS^{(j)}(\{te^{i\pi n_j/n} \,|\, t \in OD\}) \\ &= R^k(\{te^{i\pi n_j/n} \,|\, t \in OD\}). \end{aligned}$$

$\square$

Corollary 6. *For every $j = 0, 1, \infty$, every $S_k^{(j)}$ with $k = 0, 1, \ldots, n-1$, and every $g \in G$, we have $gS_k^{(j)}g^{-1} = S_r^{(j)}$ for a unique $r \in \{0, 1, \ldots, n-1\}$.*

Proof. We compute. For every $k = 0, 1, \ldots, n-1$ we have

$$RS_k^{(j)}R^{-1} = R(R^k S^{(j)} R^{-k})R^{-1} = R^{(k+1)}S^{(j)}R^{-(k+1)} = S_{k+1}^{(j)} \tag{18}$$

and

$$\begin{aligned} US_k^{(j)}U^{-1} &= U(R^{(k+n_j)}UR^{-(k+n_j)})U = R^{-(k+n_j)}UR^{(k+n_j)} \\ &= S_{-(k+2n_j)}^{(j)} = S_t^{(j)}, \end{aligned} \tag{19}$$

where $t = -(k + 2n_j) \bmod n$. Since R and U generate the group G, the corollary follows. □

Corollary 7. *For $j = 0, 1, \infty$ let G^j be the group generated by the reflections $S_k^{(j)}$ for $k = 0, 1, \ldots, n-1$. Then G^j is a normal subgroup of G.*

Proof. Clearly G^j is a subgroup of G. From Equations (18) and (19) it follows that $gS_k^{(j)}g^{-1} \in G^j$ for every $g \in G$ and every $k = 0, 1 \ldots, n-1$, since G is generated by R and U. However, G^j is generated by the reflections $S_k^{(j)}$ for $k = 0, 1, \ldots, n-1$, that is, every $g' \in G^j$ may be written as $S_{i_1}^{(j)} \cdots S_{i_p}^{(j)}$, where for $\ell \in \{1, \ldots p\}$ we have $i_\ell \in \{0, 1, \ldots, n-1\}$. So $gg'g^{-1} = g(S_{i_1}^{(j)} \cdots S_{i_p}^{(j)})g^{-1} = (gS_{i_1}^{(j)}g^{-1}) \cdots (gS_{i_p}^{(j)}g^{-1}) \in G^j$ for every $g \in G$, that is, G^j is a normal subgroup of G. □

As a first step in constructing the model $\widetilde{\mathcal{S}}_{\mathrm{reg}}$ of $\mathcal{S}_{\mathrm{reg}}$ from the regular stellated n-gon K^* we look at certain pairs of edges of $\mathrm{cl}(K^*)$. For each $j = 0, 1, \infty$ we say two distinct closed edges E and E' of $\mathrm{cl}(K^*)$ are *adjacent* if and only if they intersect at a vertex of $\mathrm{cl}(K^*)$. For $j = 0, 1, \infty$ let $\mathcal{E}^j$ be the set of unordered pairs of *equivalent* closed edges E and E' of $\mathrm{cl}(K^*)$, that is, the edges E and E' are not adjacent and $E' = S_m^{(j)}(E)$ for some generator $S_m^{(j)}$ of G^j. Recall that for x and y in some set, the unordered pair $[x, y]$ is precisely one of the ordered pairs (x, y) or (y, x). Note that $\bigcup_{j=0,1,\infty} \mathcal{E}^j$ is the set of all unordered pairs of nonadjacent edges of $\mathrm{cl}(K^*)$. Geometrically, two nonadjacent closed edges E' and E of $\mathrm{cl}(K^*)$ are equivalent if and only if E' is obtained from E by reflection in the line $R^m \ell^j$ for some $m \in \{0, 1, \ldots, n-1\}$ and some $j = 0, 1, \infty$. In Figure 6, where $K^*, = K^*_{1,1,4}$, parallel edges of K^*, which are labeled with the same letter, are G^0-equivalent. This is no coincidence.

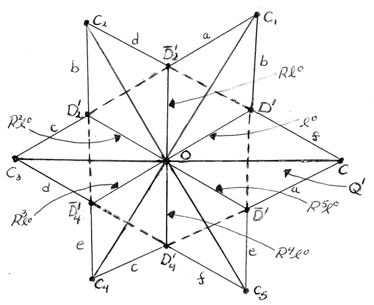

Figure 6. The triangulation $\mathcal{T}_{cl}(K^*)$ of the regular stellated hexagon K^*. The vertices of $\mathrm{cl}(K^*)$ are labeled $X_j = R^j X$ for $X = A, B, C$ and equivalent edges by a, b, c, d, e, f.

Lemma 5. *Let K^* be formed from the quadrilateral $Q = T \cup \overline{T}$, where T is the isosceles rational triangle $T_{n_0 n_0 n_\infty}$ less its vertices. Then nonadjacent edges of $\partial\,\mathrm{cl}(K^*)$ are G^0-equivalent if and only if they are parallel, see Figure 7.*

Proof. In Figure 7, let OAB be the triangle T with $\angle AOB = \alpha$, $\angle OAB = \beta$, and $\angle ABO = \gamma$. Let $OABA''$ be the quadrilateral formed by reflecting the triangle OAB in its edge OB. The quadrilateral $OABA''$ reflected it its edge OA is the quadrilateral $OAB'A'$. Let AC' be perpendicular to $A'B'$ and AC be perpendicular to $A''B$, see Figure 7. Then CAC' is a straight line if and only if $\angle C'AB' + \angle B'AB + \angle BAC = \pi$. By construction $\angle C'AB' = \angle BAC = \pi/2 - 2\gamma$ and $\angle B'AB = 2\pi - 2\beta$. So

$$\begin{aligned}\pi &= 2(\tfrac{\pi}{2} - 2\gamma) + 2(\pi - \beta) = 3\pi - 2(\beta + \gamma) - 2\gamma \\ &= 3\pi - 2(\alpha + \beta + \gamma) + 2(\alpha - \gamma) = \pi + 2(\alpha - \gamma),\end{aligned}$$

if and only if $\alpha = \gamma$. Hence the edges $A''B$ and $A'B'$ are parallel if and only if the triangle OAB is isosceles. $\square$

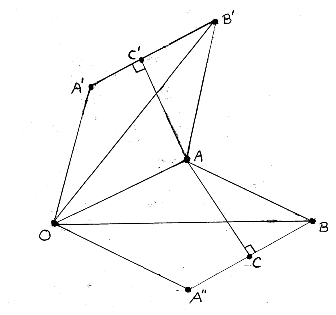

Figure 7. The geometric configuration.

Theorem 2. *Let K^* be the regular stellated n-gon formed from the rational quadrilateral $Q_{n_0 n_1 n_\infty}$ with $d_j = \gcd(n_j, n)$ for $j = 0, 1, \infty$. The G orbit space formed by first identifiying equivalent edges of the regular stellated n-gon K^* formed from Q less O and then acting on the identification space by the group G is $\widetilde{\mathcal{S}}_{\mathrm{reg}}$, which is a smooth 2-sphere with g handles, where $g = \frac{1}{2}\big(n + 2 - (d_0 + d_1 + d_\infty)\big)$, less some points corresponding to the image of the vertices of $\mathrm{cl}(K^*)$.*

Example 1. *Before we begin proving Theorem 2 we consider the following special case. Let $K^* = K^*_{1,1,4}$ be a regular stellated hexagon formed by repeatedly rotating the quadrilateral $Q' = OD'C\overline{D'}$ by R through an angle $2\pi/6$, see Figure 6.*

Let G^0 be the group generated by the reflections $S_k^{(0)} = R^k S^{(0)} R^{-k} = R^{2k+1}U$ for $k = 0, 1, \ldots, 5$. Here $S^{(0)} = RU$ is the reflection which leaves the closed ray $\ell^0 = \{te^{i\rho/6} \,\big|\, t \in OD'\}$ fixed. Define an equivalence relation on $\mathrm{cl}(K^*)$ by saying that two points x and y in $\mathrm{cl}(K^*)$ are *equivalent*, $x \sim y$, if and only if 1) x and y lie on $\partial\,\mathrm{cl}(K^*)$ with x on the closed edge E and $y = S_m^{(0)}(x) \in S_m^{(0)}(E)$ for some reflection $S_m^{(0)} \in G^0$ or 2) if x and y lie in the interior of $\mathrm{cl}(K^*)$ and $x = y$. Let $\mathrm{cl}(K^*)^\sim$ be the space of equivalence classes and let

$$\rho : \mathrm{cl}(K^*) \to \mathrm{cl}(K^*)^\sim : p \mapsto [p] \tag{20}$$

be the identification map which sends a point $p \in \mathrm{cl}(K^*)$ to the equivalence class $[p]$, which contains p. Give $\mathrm{cl}(K^*)$ the topology induced from $\mathbb{C}$. Placing the quotient topology on $\mathrm{cl}(K^*)^\sim$ turns it into a connected topological manifold without boundary, whose closure is compact. Let K^* be $\mathrm{cl}(K^*)$ less its vertices. The identification space $(K^* \setminus \mathrm{O})^\sim = \rho(K^* \setminus \mathrm{O})$ is a connected 2-dimensional smooth manifold without boundary.

Let $G = \langle R, U \,|\, R^6 = e = U^2 \ \& \ RU = UR^{-1} \rangle$. The usual G-action

$$G \times \mathrm{cl}(K^*) \subseteq G \times \mathbb{C} \to \mathrm{cl}(K^*) \subseteq \mathbb{C} : (g, z) \mapsto g(z)$$

preserves equivalent edges of $\mathrm{cl}(K^*)$ and is free on $K^* \setminus \mathrm{O}$. Hence it induces a G action on $(K^* \setminus \mathrm{O})^\sim$, which is free and proper. Thus, its orbit map

$$\sigma : (K^* \setminus \mathrm{O})^\sim \to (K^* \setminus \mathrm{O})^\sim / G = \widetilde{\mathcal{S}}_{\mathrm{reg}} : z \mapsto zG$$

is surjective, smooth, and open. The orbit space $\widetilde{\mathcal{S}}_{\mathrm{reg}} = \sigma((K^* \setminus \mathrm{O})^\sim)$ is a connected 2-dimensional smooth manifold. The identification space $(K^* \setminus \mathrm{O})^\sim$ has the orientation induced from an orientation of $K^* \setminus \mathrm{O}$, which comes from $\mathbb{C}$. So $\widetilde{\mathcal{S}}_{\mathrm{reg}}$ has a complex structure, since each element of G is a conformal mapping of $\mathbb{C}$ into itself.

Our aim is to specify the topology of $\widetilde{\mathcal{S}}_{\mathrm{reg}}$. The regular stellated hexagon $K^* \setminus \mathrm{O}$ less the origin has a triangulation $\mathcal{T}_{K^* \setminus \mathrm{O}}$ made up of 12 open triangles $R^j(\triangle OCD')$ and $R^j(\triangle OC\overline{D}')$ for $j = 0, 1, \ldots, 5$; 24 open edges $R^j(OC)$, $R^j(O\overline{D}')$, $R^j(C\overline{D}')$, and $R^j(CD')$ for $j = 0, 1, \ldots, 5$; and 12 vertices $R^j(D')$ and $R^j(C)$ for $j = 0, 1, \ldots, 5$, see Figure 6.

Consider the set $\mathcal{E}^0$ of unordered pairs of equivalent closed edges of $\mathrm{cl}(K^*)$, that is, $\mathcal{E}^0$ is the set $[E, S_k^{(0)}(E)]$ for $k = 0, 1, \ldots, 5$, where E is a closed edge of $\mathrm{cl}(K^*)$. Table 2 lists the elements of $\mathcal{E}^0$. G acts on $\mathcal{E}^0$, namely, $g \cdot [E, S_k^{(0)}(E)] = [g(E), g S_k^{(0)} g^{-1}(g(E))]$, for $g \in G$. Since G^0 is the group generated by the reflections $S_k^{(0)}$, $k = 0, 1, \ldots, 5$, it is a normal subgroup of G. Hence the action of G on $\mathcal{E}^0$ restricts to an action of G^0 on $\mathcal{E}^0$ and the G action permutes G^0-orbits in $\mathcal{E}^0$. Thus, the set of G^0-orbits in $\mathcal{E}^0$ is G-invariant.

Table 2. The set $\mathcal{E}^0$. Here $D'_k = R^k(D')$ and $\overline{D'_k} = R^k(\overline{D'})$ for $k = 0, 2, 4$ and $C_k = R^k(C)$ for $k = \{0, 1, \ldots, 5\}$, see Figure 6.

$a = [\overline{D'}C, S_0^{(0)}(\overline{D'}C) = \overline{D'_2}C_1]$	$b = [D'C_1, S_1^{(0)}(D'C_1) = D'_2C_2]$
$d = [\overline{D'_2}C_2, S_2^{(0)}(\overline{D'_2}C_2) = \overline{D'_4}C_3]$	$c = [D'_2C_3, S_3^{(0)}(D'_2C_3) = D'_4C_4]$
$e = [\overline{D'_4}C_4, S_4^{(0)}(\overline{D'_4}C_4) = \overline{D'}C_5]$	$f = [D'_4C_5, S_5^{(0)}(D'_4C_5) = D'C]$

We now look at the G^0-orbits on $\mathcal{E}^0$. We compute the G^0-orbit of $d \in \mathcal{E}^0$ as follows. We have

$$\begin{aligned}(UR) \cdot d &= [UR(\overline{D'_2}C_2), UR(S_2^{(0)}(\overline{D'_2}C_2))] = [UR(\overline{D'_2}C_2), UR(\overline{D'_4}C_3))] \\ &= [U(D'_2C_3), U(D'_4C_4)] = [\overline{D'_4}C_5, \overline{D'_2}C_2] = d.\end{aligned}$$

Since

$$\begin{aligned}R^2 \cdot d = R^2 \cdot [\overline{D'_2}C_2, S_2^{(0)}(\overline{D'_2}C_2)] &= [R^2(\overline{D'_2}C_2), R^2 S_2^{(0)} R^{-2}(R^2(\overline{D'_2}C_2))] \\ &= [\overline{D'_4}C_4, S_4^{(0)}(\overline{D'_4}C_4)] = [\overline{D'_4}C_4, \overline{D'}C_5] = e\end{aligned}$$

and

$$\begin{aligned}R^4 \cdot d &= [R^4(\overline{D'_4}C_2), R^4 S_2^{(0)} R^{-4}(R^4(\overline{D'_2}C_2))] \\ &= [\overline{D'}C, S_6^{(0)}(\overline{D'}C)] = [\overline{D'}C, S_0^{(0)}(\overline{D'}C)] = [\overline{D'}C, \overline{D'_2}C_1] = a,\end{aligned}$$

the G^0 orbit $G^0 \cdot d$ of $d \in \mathcal{E}^0$ is $(G^0 / \langle UR \,|\, (UR)^2 = e \rangle) \cdot d = H^0 \cdot d = \{a, d, e\}$. Here $H^0 = \langle V = R^2 \,|\, V^3 = e \rangle$, since $G^0 = \langle V = R^2, UR \,|\, V^3 = e = (UR)^2 \ \& \ V(UR) = (UR)V^{-1} \rangle$. Similarly, the G^0-orbit $G^0 \cdot f$ of $f \in \mathcal{E}^0$ is $H^0 \cdot f = \{b, c, f\}$. Since $G^0 \cdot d \cup G^0 \cdot f = \mathcal{E}^0$, we have found all G^0-orbits on $\mathcal{E}^0$. The G-orbit of OC is $R^j(OC)$ for $j = 0, 1, \ldots, 5$, since

$U(OC) = OC$; while the G-orbit of OD' is $R^j(OD')$, $R^j(O\overline{D'})$ for $j = 0, 1, \ldots, 5$, since $U(OD') = O\overline{D'}$.

Suppose that B is an end point of the closed edge E of $\mathrm{cl}(K^*)$. Then E lies in a unique $[E, S_m^{(0)}(E)]$ of $\mathcal{E}^0$. Let $G^0 \cdot [E, S_m^{(0)}(E)]$ be the G^0-orbit of $[E, S_m^{(0)}(E)]$. Then $g' \cdot B$ is an end point of the closed edge $g'(E)$ of $g' \cdot [E, S_m^{(0)}(E)] \in \mathcal{E}^0$ for every $g' \in G^0$. So $\mathcal{O}(B) = \{g' \cdot B \,|\, g' \in G^0\}$ the G^0-orbit of the vertex B. It follows from the classification of G^0-orbits on $\mathcal{E}^0$ that $\mathcal{O}(D') = \{D', D'_2, D'_4\}$ and $\mathcal{O}(\overline{D'}) = \{\overline{D'}, \overline{D'}_2, \overline{D'}_4\}$ are G^0-orbits of the vertices of $\mathrm{cl}(K^*)$, which are permuted by the action of G on $\mathcal{E}^0$. Furthermore, $\mathcal{O}(C) = \{C, C_1, \ldots, C_5\}$ and $\mathcal{O}(D' \& \overline{D'}) = \{D', \overline{D'}, D'_2, \overline{D'}_2, D'_4, \overline{D'}_4\}$ are G-orbits of vertices of $\mathrm{cl}(K^*)$, which are end points of the G-orbit of the rays OC and OD', respectively.

To determine the topology of the G orbit space $\widetilde{\mathcal{S}}_{\mathrm{reg}}$ we find a triangulation of $\widetilde{\mathcal{S}}_{\mathrm{reg}}$. Note that the triangulation $\mathcal{T}_{K^* \setminus \mathrm{O}}$ of $K^* \setminus \mathrm{O}$, illustrated in Figure 6, is G-invariant. Its image under the identification map ρ is a G-invariant triangulation $\mathcal{T}_{(K^* \setminus \mathrm{O})^\sim}$ of $(K^* \setminus \mathrm{O})^\sim$. After identification of equivalent edges, each vertex $\rho(v)$, each open edge $\rho(E)$, having $\rho(O)$ as an end point, or each open edge $\rho([F, F'])$, where $[F, F']$ is a pair of equivalent edges of $\mathrm{cl}(K^*)$, and each open triangle $\rho(T)$ in $\mathcal{T}_{(K^* \setminus \mathrm{O})^\sim}$ lies in a unique G orbit. It follows that $\sigma(\rho(v))$, $\sigma(\rho(E))$ or $\sigma(\rho([F, F']))$, and $\sigma(\rho(T))$ is a vertex, an open edge, and an open triangle, respectively, of a triangulation $\mathcal{T}_{\widetilde{\mathcal{S}}_{\mathrm{reg}}} = \sigma(\mathcal{T}_{(K^* \setminus \mathrm{O})^\sim})$ of $\widetilde{\mathcal{S}}_{\mathrm{reg}}$. The triangulation $\mathcal{T}_{\widetilde{\mathcal{S}}_{\mathrm{reg}}}$ has 4 vertices, corresponding to the G orbits $\sigma(\rho(\mathcal{O}(D')))$, $\sigma(\rho(\mathcal{O}(\overline{D'})))$, $\sigma(\rho(\mathcal{O}(C)))$, and $\sigma(\rho(\mathcal{O}(D' \& \overline{D'})))$; 18 open edges corresponding to $\sigma(\rho(R^j(OC)))$, $\sigma(\rho(R^j(OD')))$, and $\sigma(\rho(R^j(CD')))$ for $j = 0, 1, \ldots, 5$; and 12 open triangles $\sigma(\rho(R^j(\triangle OCD')))$ and $\sigma(\rho(R^j(\triangle OC\overline{D'})))$ for $j = 0, 1, \ldots, 5$. Thus, the Euler characteristic $\chi(\widetilde{\mathcal{S}}_{\mathrm{reg}})$ of $\widetilde{\mathcal{S}}_{\mathrm{reg}}$ is $4 - 18 + 12 = -2$. Since $\widetilde{\mathcal{S}}_{\mathrm{reg}}$ is a 2-dimensional smooth real manifold, $\chi(\widetilde{\mathcal{S}}_{\mathrm{reg}}) = 2 - 2g$, where g is the genus of $\widetilde{\mathcal{S}}_{\mathrm{reg}}$. Hence $g = 2$. So $\widetilde{\mathcal{S}}_{\mathrm{reg}}$ is a smooth 2-sphere with 2 handles, less a finite number of points, which lies in a compact topological space $\widetilde{S} = \mathrm{cl}(K^*)^\sim / G$, that is its closure, see Figure 8. This completes the example.

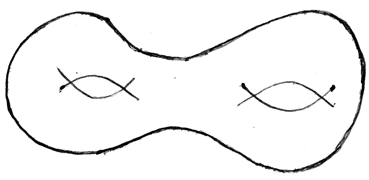

Figure 8. The G-orbit space $\widetilde{\mathcal{S}}_{\mathrm{reg}}$ is 2-sphere with two handles.

Proof of Theorem 2. We now begin the construction of $\widetilde{\mathcal{S}}_{\mathrm{reg}}$ by identifying equivalent edges of $\mathrm{cl}(K^*)$. For each $j = 0, 1, \infty$ let $[E, S_m^{(j)}(E)]$ be an unordered pair of equivalent closed edges of $\mathrm{cl}(K^*)$. We say that x and y in $\mathrm{cl}(K^*)$ are *equivalent*, $x \sim y$, if 1) x and y lie in $\partial\, \mathrm{cl}(K^*)$ with $x \in E$ and $y = S_m^{(j)}(x) \in S_m^{(0)}(E)$ for some $m \in \{0, 1, \ldots, n-1\}$ and some $j = 0, 1, \infty$ or 2) x and y lie in $\mathrm{int}\, \mathrm{cl}(K^*)$ and $x = y$. The relation $\sim$ is an equivalence relation on $\mathrm{cl}(K^*)$. Let $\mathrm{cl}(K^*)^\sim$ be the set of equivalence classes and let

$$\rho : \mathrm{cl}(K^*) \to \mathrm{cl}(K^*)^\sim : p \mapsto [p] \tag{21}$$

be the map which sends p to the equivalence class $[p]$, that contains p. Compare this argument with that of Richens and Berry [2]. Give $\mathrm{cl}(K^*)$ the topology induced from $\mathbb{C}$ and put the quotient topology on $\mathrm{cl}(K^*)^\sim$. $\square$

Theorem 3. *Let K^* be $\mathrm{cl}(K^*)$ less its vertices. Then $(K^* \setminus \mathrm{O})^\sim = \rho(K^* \setminus \mathrm{O})$ is a smooth manifold. Furthermore, $\mathrm{cl}(K^*)^\sim$ is a topological manifold.*

Proof. To show that $(K^* \setminus \mathrm{O})^\sim$ is a smooth manifold, let E_+ be an open edge of K^*. For $p_+ \in E_+$ let D_{p_+} be a disk in $\mathbb{C}$ with center at p_+, which does not contain a vertex of

$\mathrm{cl}(K^*)$. Set $D^+_{p_+} = K^* \cap D_{p_+}$. For each $j = 0, 1, \infty$ let E_- be an open edge of K^*, which is equivalent to E_+ via the reflection $S^{(j)}_m$, that is, $[\mathrm{cl}(E_+), \mathrm{cl}(E_-) = S^{(j)}_m(\mathrm{cl}(E_+))] \in \mathcal{E}^j$ is an unordered pair of $S^{(j)}_m$ equivalent edges. Let $p_- = S^{(j)}_m(p_+)$ and set $D^-_{p_-} = S^{(j)}_m(D^+_{p_+})$. Then $V_{[p]} = \rho(D^+_{p_+} \cup D^-_{p_-})$ is an open neighborhood of $[p] = [p_+] = [p_-]$ in $(K^* \setminus \mathrm{O})^\sim$, which is a smooth 2-disk, since the identification mapping ρ is the identity on $\mathrm{int}\, K^*$. It follows that $(K^* \setminus \mathrm{O})^\sim$ is a smooth 2-dimensional manifold without boundary.

We now handle the vertices of $\mathrm{cl}(K^*)$. Let v_+ be a vertex of $\mathrm{cl}(K^*)$ and set $D_{v_+} = \widetilde{D} \cap \mathrm{cl}(K^*)$, where $\widetilde{D}$ is a disk in $\mathbb{C}$ with center at the vertex $v_+ = r_0 \mathrm{e}^{i\pi\theta_0}$. The map

$$W_{v_+} : D_+ \subseteq \mathbb{C} \to D_{v_+} \subseteq \mathbb{C} : r\mathrm{e}^{i\pi\theta} \mapsto |r - r_0| \mathrm{e}^{i\pi s(\theta-\theta_0)}$$

with $r \geq 0$ and $0 \leq \theta \leq 1$ is a homeomorphism, which sends the wedge with angle π to the wedge with angle πs. The latter wedge is formed by the closed edges E'_+ and E_+ of $\mathrm{cl}(K^*)$, which are adjacent at the vertex v_+ such that $\mathrm{e}^{i\pi s} E'_+ = E_+$ with the edge E'_+ being swept out through $\mathrm{int}\,\mathrm{cl}(K^*)$ during its rotation to the edge E_+. Because $\mathrm{cl}(K^*)$ is a rational regular stellated n-gon, the value of s is a rational number for each vertex of $\mathrm{cl}(K^*)$. For each $j = 0, 1, \infty$ let $E_- = S^{(j)}_m(E_+)$ be an edge of $\mathrm{cl}(K^*)$, which is equivalent to E_+ and set $v_- = S^{(j)}_m(v_+)$. Then v_- is a vertex of $\mathrm{cl}(K^*)$, which is the center of the disk $D_{v_-} = S^{(j)}_m(D_{v_+})$. Set $D_- = \overline{D}_+$. Then $D = D_+ \cup D_-$ is a disk in $\mathbb{C}$. The map $W : D \to \rho(D_{v_+} \cup D_{v_-})$, where $W|_{D_+} = \rho \circ W_{v_+}$ and $W|_{D_-} = \rho \circ S^{(0)}_m \circ W_{v_+} \circ \overline{}$, is a homeomorphism of D into a neighborhood $\rho(D_{v_+} \cup D_{v_-})$ of $[v] = [v_+] = [v_-]$ in $\mathrm{cl}(K^*)^\sim$. Consequently, the identification space $\mathrm{cl}(K^*)^\sim$ is a topological manifold. $\square$

We now describe a triangulation of $K^* \setminus \mathrm{O}$. Let $T' = T_{1,n_1,n-(1+n_1)}$ be the open rational triangle $\triangle OCD'$ with vertex at the origin O, longest side OC on the real axis, and interior angles $\frac{1}{n}\pi$, $\frac{n_1}{n}\pi$, and $\frac{n-1-n_1}{n}\pi$. Let Q' be the quadrilateral $T' \cup \overline{T'}$. Then Q' is a subset of the quadrilateral $Q = ODC\overline{D}$, see Figure 5. Moreover $K^* = \bigcup_{\ell=0}^{n-1} R^\ell(Q')$. The $2n$ triangles $\mathrm{cl}(R^j(T')) \setminus \{O\}$ and $\mathrm{cl}(R^k(\overline{T'})) \setminus \mathrm{O}$ with $k = 0, 1, \ldots, n-1$ form a triangulation $\mathcal{T}_{K^* \setminus \mathrm{O}}$ of $K^* \setminus \mathrm{O}$ with $2n$ vertices $R^k(C)$ and $R^k(D')$ for $k = 0, 1, \ldots, n-1$; $4n$ open edges $R^k(OC)$, $R^k(OD')$, $R^k(CD')$, and $R^k(C\overline{D'})$ for $k = 0, 1, \ldots, n-1$; and $2n$ open triangles $R^k(T')$, $R^k(\overline{T'})$ with $k = 0, 1, \ldots, n-1$. The image of the triangulation $\mathcal{T}_{K^* \setminus \mathrm{O}}$ under the identification map ρ (21) is a triangulation $\mathcal{T}_{(K^* \setminus \mathrm{O})^\sim}$ of the identification space $(K^* \setminus \mathrm{O})^\sim$.

The action of G on $\mathrm{cl}(K^*)$ preserves the set of unordered pairs of $S^{(j)}_m$ equivalent edges of $\mathrm{cl}(K^*)$ for each $j = 0, 1, \infty$. Hence G induces an action on $\mathrm{cl}(K^*)^\sim$, which is proper, since G is finite. The G action is free on $K^* \setminus \mathrm{O}$ and thus on $(K^* \setminus \mathrm{O})^\sim$ by Lemma A2. We have proved

Lemma 6. *The G-orbit space $\widetilde{\mathcal{S}} = \mathrm{cl}(K^*)^\sim / G$ is a compact connected topological manifold with $\widetilde{\mathcal{S}}_{\mathrm{reg}} = (K^* \setminus \mathrm{O})^\sim / G$ being a smooth manifold. Let*

$$\sigma : \mathrm{cl}(K^*)^\sim \to \widetilde{\mathcal{S}} = \mathrm{cl}(K^*)^\sim / G : z \mapsto zG.$$

Then σ is the G orbit map, which is surjective, continuous, and open. The restriction of σ to $K^* \setminus \mathrm{O}$ has image $\widetilde{\mathcal{S}}_{\mathrm{reg}}$ and is a smooth open mapping.

We now determine the topology of the orbit space $\widetilde{\mathcal{S}}_{\mathrm{reg}}$. For each $j = 0, 1, \infty$ and $\ell_j = 0, 1, \ldots, d_j - 1$ let $A^j_{\ell_j}$ be an end point of a closed edge E of $\mathrm{cl}(K^*)$, which lies on the unordered pair $[E, S^{(j)}_{\ell_j}(E)] \in \mathcal{E}^j$. Then $H^j \cdot A^{(j)}_{\ell_j}$ is an end point of the edge $H^j \cdot E$ of the unordered pair $H^j \cdot [E, S^{(j)}_{\ell_j}(E)]$ of $\mathcal{E}^j$. See Appendix A for the definition of the group H_j. The sets $\mathcal{O}(A^{(j)}_\ell) = \{H^j \cdot A^{(j)}_{\ell_j}\}$ with $\ell_j = 0, 1, \ldots, d_j - 1$ are permuted by G. The action of G on $K^* \setminus \mathrm{O}$ preserves the set of open edges of the triangulation $\mathcal{T}_{K^* \setminus \mathrm{O}}$. There are

$3n$-orbits: $R^k(OC)$; $R^k(O\overline{D'})$, since $OD' = R(O\overline{D'})$; and $R^k(CD)$, since $C\overline{D'} = U(CD)$ for $k = 0, 1, \ldots, n-1$. So the image of the triangulation $\mathcal{T}_{K^* \setminus O}$ under the continuous open map

$$\mu = \sigma \circ \pi|_{K^* \setminus O} : K^* \setminus O \rightarrow \widetilde{\mathcal{S}}_{\mathrm{reg}} \tag{22}$$

is a triangulation $\mathcal{T}_{\widetilde{\mathcal{S}}_{\mathrm{reg}}}$ of the G-orbit space $\widetilde{\mathcal{S}}_{\mathrm{reg}}$ with $d_0 + d_1 + d_\infty$ vertices $\mu(\mathcal{O}(A^{(j)}_{\ell_j}))$, where $j = 0, 1, \infty$ and $\ell_j = 0, 1, \ldots, d_j - 1$; $3n$ open edges $\mu(R^k(OC))$, $\mu(R^j(O\overline{D'}))$, and $\mu(R^k(CD))$ for $k = 0, 1, \ldots, n-1$; and $2n$ open triangles $\mu(R^k(T'))$ and $\mu(R^k(\overline{T'}))$ for $k = 0, 1, \ldots n-1$. Thus, the Euler characteristic $\chi(\widetilde{\mathcal{S}}_{\mathrm{reg}})$ of $\widetilde{\mathcal{S}}_{\mathrm{reg}}$ is $d_0 + d_1 + d_\infty - 3n + 2n = d_0 + d_1 + d_\infty - n$. However, $\widetilde{\mathcal{S}}_{\mathrm{reg}}$ is a smooth manifold. So $\chi(\widetilde{\mathcal{S}}_{\mathrm{reg}}) = 2 - 2g$, where g is the genus of $\widetilde{\mathcal{S}}_{\mathrm{reg}}$. Hence $g = \frac{1}{2}\big(n + 2 - (d_0 + d_1 + d_\infty)\big)$. Compare this argument with that of Weyl ([4], p. 174). This proves Theorem 2.

Since the quadrilateral Q is a fundamental domain for the action of G on K^*, the G orbit map $\overline{\mu} = \sigma \circ \pi : K^* \subseteq \mathbb{C} \rightarrow \widetilde{\mathcal{S}}$ restricted to Q is a bijective continuous open mapping. However, $\delta_Q : \mathcal{D} \subseteq \mathcal{S} \rightarrow Q \subseteq \mathbb{C}$ is a bijective continous open mapping of the fundamental domain $\mathcal{D}$ of the $\mathcal{G}$ action on $\mathcal{S}$. Consequently, the $\mathcal{G}$ orbit space is homeomorphic to the G orbit space $\widetilde{\mathcal{S}}$. The mapping $\overline{\mu}$ is holomorphic except possibly at 0 and the vertices of $\mathrm{cl}(K^*)$. So the mapping $\overline{\mu} \circ \delta_{K^*} : \mathcal{S}_{\mathrm{reg}} \rightarrow \widetilde{\mathcal{S}}_{\mathrm{reg}}$ is a holomorphic diffeomorphism.

5. An Affine Model of $\mathcal{S}_{\mathrm{reg}}$

We construct an affine model of the affine Riemann surface $\mathcal{S}_{\mathrm{reg}}$ as follows. Return to the regular stellated n-gon $K^* = K^*_{n_0 n_1 n_\infty}$, which is formed from the quadrilateral $Q = Q_{n_0 n_1 n_\infty}$ less its vertices. Repeatedly reflecting in the edges of K^* and then in the edges of the resulting reflections of K^* et cetera, we obtain a covering of $\mathbb{C} \setminus \mathbb{V}^+$ by certain translations of K^*. Here $\mathbb{V}^+$ is the union of the translates of the vertices of $\mathrm{cl}(K^*)$ and its center O. Let $\mathfrak{T}$ be the group generated by these translations. The semidirect product $\mathfrak{G} = G \ltimes \mathfrak{T}$ acts freely, properly and transitively on $\mathbb{C} \setminus \mathbb{V}^+$. It preserves equivalent edges of $\mathbb{C} \setminus \mathbb{V}^+$ and it acts freely and properly on $(\mathbb{C} \setminus \mathbb{V}^+)^\sim$, the space formed by identifying equivalent edges in $\mathbb{C} \setminus \mathbb{V}^+$. The orbit space $(\mathbb{C} \setminus \mathbb{V}^+)^\sim / \mathfrak{G}$ is holomorphically diffeomorphic to $\widetilde{\mathcal{S}}_{\mathrm{reg}}$ and is the desired affine model of $\mathcal{S}_{\mathrm{reg}}$. We now justify these assertions.

First we determine the group $\mathcal{T}$ of translations.

Lemma 7. *Each of the $2n$ sides of the regular stellated n-gon K^* is perpendicular to exactly one of the directions*

$$\mathrm{e}^{[\frac{1}{2} - \frac{n_1}{n} + 2k\frac{1}{n}]\pi i} \quad \text{or} \quad \mathrm{e}^{[-\frac{1}{2} - \frac{1}{n} + \frac{n_1}{n} + (2k+1)\frac{1}{n}]\pi i}, \tag{23}$$

for $k = 0, 1, \ldots, n-1$.

Proof. From Figure 9 we have $\angle D'CO = \frac{n_1}{n}\pi$. So $\angle COH = \frac{1}{2}\pi - \frac{n_1}{n}\pi$. Hence the line ℓ_0, containing the edge CD' of K^*, is perpendicular to the direction $\mathrm{e}^{[\frac{1}{2} - \frac{n_1}{n}]\pi}$. Since $\triangle CO\overline{D'}$ is the reflection of $\triangle COD'$ in the line segment OC, the line ℓ_1, containing the edge $C\overline{D'}$ of K^*, is perpendicular to the direction $\mathrm{e}^{[-\frac{1}{2} + \frac{n_1}{n}]\pi}$. Because the regular stellated n-gon K^* is formed by repeatedly rotating the quadrilateral $Q' = OD'C\overline{D'}$ through an angle $\frac{2\pi}{n}$, we find that Equation (23) holds. □

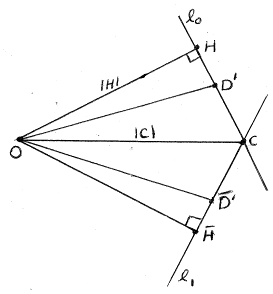

Figure 9. The regular stellated n-gon K^* two of whose sides are CD' and $C\overline{D'}$.

Since $\angle COH = \frac{1}{2}\pi - \frac{n_1}{n}\pi$, it follows that $|H| = |C| \sin \pi \frac{n_1}{n}$ is the distance from the center O of K^* to the line ℓ_0 containing the side CD', or to the line ℓ_1 containing the side $C\overline{D'}$. So $u_0 = (|C| \sin \pi \frac{n_1}{n}) e^{[\frac{1}{2} - \frac{n_1}{n}]\pi i}$ is the closest point H on ℓ_0 to O and $u_1 = (|C| \sin \pi \frac{n_1}{n}) e^{[-\frac{1}{2} + \frac{n_1}{n}]\pi i}$ is the closest point $\overline{H}$ on ℓ_1 to O. Since the regular stellated n-gon K^* is formed by repeatedly rotating the quadrilateral $Q' = OD'C\overline{D'}$ through an angle $\frac{2\pi}{n}$, the point

$$u_{2k} = R^k u_0 = (|C| \sin \pi \frac{n_1}{n}) e^{[\frac{1}{2} - \frac{n_1}{n} + 2k\frac{1}{n}]\pi i} \tag{24}$$

lies on the line $\ell_{2k} = R^k \ell_0$, which contains the edge $R^k(CD')$ of K^*; while

$$u_{2k+1} = R^k u_1 = (|C| \sin \pi \frac{n_1}{n}) e^{[-\frac{1}{2} + \frac{n_1}{n} - \frac{1}{n} + (2k+1)\frac{1}{n}]\pi i} \tag{25}$$

lies on the line $\ell_{2k+1} = R^k \ell_1$, which contains the edge $R^k(C\overline{D'})$ of K^* for every $k \in \{0, 1, \ldots, n-1\}$. Furthermore, the line segments Ou_{2k} and Ou_{2k+1} are perpendicular to the line ℓ_{2k} and ℓ_{2k+1}, respectively, for $k \in \{0, 1, \ldots, n-1\}$.

Corollary 8. *For $k = 0, 1, \ldots, n-1$ we have*

$$\overline{u_{2k}} = u_{2(n-k)+1} \text{ and } \overline{u_{2k+1}} = u_{2(n-k)}. \tag{26}$$

Proof. We compute. From (24) it follows that

$$\begin{aligned} \overline{u_{2k}} &= U(u_{2k}) = UR^k(u_0) = R^{-k}(U(u_0)) \\ &= R^{-k}(u_1) = R^{n-k}(u_1) = u_{2(n-k)+1}, \quad \text{using (25);} \end{aligned}$$

while from (25) we get

$$\overline{u_{2k+1}} = U(u_{2k+1}) = UR^k(u_1) = R^{-k}(U(u_1)) = R^{n-k}(u_0) = u_{2(n-k)}.$$

□

Corollary 9. *For $k, \ell \in \{0, 1, \ldots, 2n-1\}$ we have*

$$u_{(k+2\ell) \bmod 2n} = R^\ell u_k. \tag{27}$$

Proof. If $k = 2i$, then $u_k = R^i u_0$, by definition. So

$$R^\ell u_k = R^{\ell+i} u_0 = u_{(2i+2\ell) \bmod 2n} = u_{(k+2\ell) \bmod 2n}.$$

If $k = 2i + 1$, then $u_\ell = R^i u_1$, by definition. So

$$R^\ell u_k = R^{\ell+i} u_1 = u_{(2(i+\ell)+1) \bmod 2n} = u_{(k+2\ell) \bmod 2n}.$$

□

For $k = 0, 1, \ldots, 2n-1$ let τ_k be the translation

$$\tau_k : \mathbb{C} \to \mathbb{C} : z \mapsto z + 2u_k. \tag{28}$$

Corollary 10. *For $k, \ell \in \{0, 1, \ldots, 2n-1\}$ we have*

$$\tau_{(k+2\ell) \bmod 2n} \circ R^\ell = R^\ell \circ \tau_k. \tag{29}$$

Proof. For every $z \in \mathbb{C}$, we have

$$\begin{aligned} \tau_{(k+2\ell) \bmod 2n}(z) &= z + 2u_{(k+2\ell) \bmod 2n}, \quad \text{using (28)} \\ &= z + 2R^\ell u_k \quad \text{by (27)} \\ &= R^\ell(R^{-\ell} z + 2u_k) = R^\ell \circ \tau_k(R^{-\ell} z). \end{aligned}$$

□

Reflecting the regular stellated n-gon K^* in its edge CD' contained in ℓ_0 gives a congruent regular stellated n-gon K_0^* with the center O of K^* becoming the center $2u_0$ of K_0^*.

Lemma 8. *The collection of all the centers of the regular stellated n-gons, formed by reflecting K^* in its edges and then reflecting in the edges of the reflected regular stellated n-gons et cetera, is*

$$\begin{aligned} \{\tau_0^{\ell_0} \circ \cdots \circ \tau_{2n-1}^{\ell_{2n-1}}(0) \in \mathbb{C} \big| (\ell_0, \ldots, \ell_{2n-1}) \in (\mathbb{Z}_{\geq 0})^{2n}\} = \\ = \{2 \sum_{\ell_0, \ldots, \ell_{2n-1} = 0}^{\infty} \big(\ell_0 u_0 + \cdots \ell_{2n-1} u_{2n-1}\big)\}, \end{aligned}$$

where for $k = 0, 1, \ldots, 2n-1$ we have

$$\tau_k^{\ell_k} = \overbrace{\tau_k \circ \cdots \circ \tau_k}^{\ell_k} : \mathbb{C} \to \mathbb{C} : z \mapsto z + 2\ell_j u_k.$$

Proof. For each $k_0 = 0, 1, \ldots, 2n-1$ the center of the $2n$ regular stellated congruent n-gon $K_{k_0}^*$ formed by reflecting in an edge of K^* contained in the line ℓ_{k_0} is $\tau_{k_0}(0) = 2u_{k_0}$. Repeating the reflecting process in each edge of $K_{k_0}^*$ gives $2n$ congruent regular stellated n-gons $K_{k_0 k_1}^*$ with center at $\tau_{k_1}\big(\tau_{k_0}(0)\big) = 2(u_{k_1} + u_{k_0})$, where $k_1 = 0, 1, \ldots 2n-1$. Repeating this construction proves the lemma. □

The set $\mathbb{V}$ of vertices of the regular stellated n-gon K^* is

$$\{V_{2k} = C\mathrm{e}^{2k(\frac{1}{n}\pi i)},\ V_{2k+1} = D'\mathrm{e}^{(2k+1)(\frac{1}{n}\pi i)} \text{ for } 0 \leq k \leq n-1\},$$

see Figure 5. Clearly the set $\mathbb{V}$ is G invariant.

Corollary 11. *The set*

$$\begin{aligned} \mathbb{V}^+ = \{v_{\ell_0 \cdots \ell_{2n-1}} = \tau_0^{\ell_0} \circ \cdots \circ \tau_{2n-1}^{\ell_{2n-1}}(V) \big| \\ V \in \mathbb{V} \cup \{O\} \ \&\ (\ell_0, \ldots, \ell_{2n-1}) \in (\mathbb{Z}_{\geq 0})^{2n}\} \end{aligned} \tag{30}$$

is the collection of vertices and centers of the congruent regular stellated n-gons K^*, $K_{k_1}^*$, $K_{k_0 k_1}^*, \ldots$.

Proof. This follows immediately from Lemma 8. □

Corollary 12. *The union of $K^*, K^*_{k_0}, K^*_{k_0k_1}, \ldots K^*_{k_0k_1\cdots k_\ell}, \ldots$, where $\ell \geq 0$, $0 \leq j \leq \ell$, and $0 \leq k_j \leq 2n-1$, covers $\mathbb{C} \setminus \mathbb{V}^+$, that is,*

$$K^* \cup \bigcup_{\ell \geq 0} \bigcup_{0 \leq j \leq \ell} \bigcup_{0 \leq k_j \leq 2n-1} K^*_{k_0k_1\cdots k_\ell} = \mathbb{C} \setminus \mathbb{V}^+.$$

Proof. This follows immediately from $K^*_{k_0k_1\cdots k_\ell} = \tau_{k_\ell} \circ \cdots \circ \tau_{k_0}(K^*)$. $\square$

Let $\mathcal{T}$ be the abelian subgroup of the 2-dimensional Euclidean group $\mathrm{E}(2)$ generated by the translations τ_k (28) for $k = 0, 1, \ldots 2n-1$. It follows from Corollary 12 that the regular stellated n-gon K^* with its vertices and center removed is the fundamental domain for the action of the abelian group $\mathcal{T}$ on $\mathbb{C} \setminus \mathbb{V}^+$. The group $\mathcal{T}$ is isomorphic to the abelian subgroup $\mathfrak{T}$ of $(\mathbb{C}, +)$ generated by $\{2u_k\}_{k=0}^{2n-1}$.

Next we define the group $\mathfrak{G}$ and show that it acts freely, properly, and transitively on $\mathbb{C} \setminus \mathbb{V}^+$. Consider the group $\mathfrak{G} = G \ltimes \mathfrak{T} \subseteq G \times \mathfrak{T}$, which is the semidirect product of the dihedral group G, generated by the rotation R through $2\pi/n$ and the reflection U subject to the relations $R^n = e = U^2$ and $RU = UR^{-1}$, and the abelian group $\mathfrak{T}$. An element $(R^jU^\ell, 2u_k)$ of $\mathfrak{G}$ is the affine linear map

$$(R^jU^\ell, 2u_k) : \mathbb{C} \to \mathbb{C} : z \mapsto R^jU^\ell z + 2u_k.$$

Multiplication in $\mathfrak{G}$ is defined by

$$(R^jU^\ell, 2u_k) \cdot (R^{j'}U^{\ell'}, 2u_{k'}) = \big(R^{j+j'}U^{\ell+\ell'}, (R^jU^\ell)(2u_{k'}) + 2u_k\big), \tag{31}$$

which is the composition of the affine linear map $(R^{j'}U^{\ell'}, 2u_{k'})$ followed by $(R^jU^\ell, 2u_k)$. The mappings $G \to \mathfrak{G} : R^j \mapsto (R^jU^\ell, 0)$ and $\mathfrak{T} \to \mathfrak{G} : 2u_k \mapsto (e, 2u_k)$ are injective, which allows us to identify the groups G and $\mathfrak{T}$ with their image in $\mathfrak{G}$. Using (31) we may write an element $(R^jU^\ell, 2u_k)$ of $\mathfrak{G}$ as $(e, 2u_k) \cdot (R^jU^\ell, 0)$. So

$$(e, 2u_{(j+2k) \bmod 2n}) \cdot (R^kU^\ell, 0) = (R^kU^\ell, 2u_{(j+2k) \bmod 2n}),$$

For every $z \in \mathbb{C}$ we have

$$R^kU^\ell z + 2u_{(j+2k) \bmod 2n} = R^kU^\ell z + R^kU^\ell(2u_j), \quad \text{using (27)},$$

that is,

$$(R^kU^\ell, 2u_{(j+2k) \bmod 2n}) = (R^kU^\ell, R^kU^\ell(2u_j)) = (R^kU^\ell, 0) \cdot (e, 2u_j).$$

Hence

$$(e, 2u_{(j+2k) \bmod 2n}) \cdot (R^kU^\ell, 0) = (R^kU^\ell, 0) \cdot (e, 2u_j), \tag{32}$$

which is just Equation (29). The group $\mathfrak{G}$ acts on $\mathbb{C}$ as $\mathrm{E}(2)$ does, namely, by affine linear orthogonal mappings. Denote this action by

$$\psi : \mathfrak{G} \times \mathbb{C} \to \mathbb{C} : ((g, \tau), z) \mapsto \tau(g(z)).$$

Lemma 9. *The set $\mathbb{V}^+$ (30) is invariant under the $\mathfrak{G}$ action.*

Proof. Let $v \in \mathbb{V}^+$. Then for some $(\ell'_0, \ldots, \ell'_{2n-1}) \in \mathbb{Z}^{2n}_{\geq 0}$ and some $w \in \mathbb{V} \cup \{O\}$

$$v = \tau_0^{\ell'_0} \circ \cdots \circ \tau_{2n-1}^{\ell'_{2n-1}}(w) = \psi_{(e, 2u')}(w),$$

where $u' = \sum_{k=0}^{2n-1} \ell'_k u_k$. For $(R^jU^\ell, 2u) \in \mathfrak{G}$ with $j = 0, 1, \ldots, n-1$ and $\ell = 0, 1$ we have

$$\psi_{(R^jU^\ell, 2u)} v = \psi_{(R^jU^\ell, 2u)} \circ \psi_{(e, 2u')}(w) = \psi_{(R^jU^\ell, 2u) \cdot (e, 2u')}(w)$$

$$= \psi_{(R^jU^\ell, R^jU^\ell(2u')+2u)}(w) = \psi_{(e,2(R^jU^\ell u'+u))\cdot(R^jU^\ell,0)}(w)$$
$$= \psi_{(e,2(R^jU^\ell u'+u))}\big(\psi_{(R^jU^\ell,0)}(w)\big) = \psi_{(e,2(R^jU^\ell u'+u))}(w'), \tag{33}$$

where $w' = \psi_{(R^jU^\ell,0)}(w) = R^jU^\ell(w) \in \mathbb{V} \cup \{\mathrm{O}\}$. If $\ell = 0$, then

$$R^j u' = R^j(\sum_{k=0}^{2n-1} \ell'_k u_k) = \sum_{k=0}^{2n-1} \ell'_k R^j(u_k) = \sum_{k=0}^{2n-1} \ell'_k u_{(k+2j) \bmod 2n};$$

while if $\ell = 1$, then

$$R^j U(u') = \sum_{k=0}^{2n-1} \ell'_k R^j(U(u_k)) = \sum_{k=0}^{2n-1} \ell'_k R^j(u_{k'(k)}) = \sum_{k=0}^{2n-1} \ell'_k u_{(k'(k)+2j) \bmod 2n}.$$

Here $k'(k) = \begin{cases} 2n-k+1, & \text{if } k \text{ is even} \\ 2n-k-1, & \text{if } k \text{ is odd,} \end{cases}$ see Corollary 8. So $(e, 2(R^jU^\ell u' + u)) \in \mathfrak{T}$, which implies $\psi_{(e,2(R^jU^\ell u'+u))}(w') \in \mathbb{V}^+$, as desired. □

Lemma 10. *The action of $\mathfrak{G}$ on $\mathbb{C} \setminus \mathbb{V}^+$ is free.*

Proof. Suppose that for some $v \in \mathbb{C} \setminus \mathbb{V}^+$ and some $(R^jU^\ell, 2u) \in \mathfrak{G}$ we have $v = \psi_{(R^jU^\ell,2u)}(v)$. Then v lies in some $K^*_{k_0k_1\cdots k_\ell}$. So for some $v' \in K^*$ we have

$$v = \tau_0^{\ell'_0} \circ \cdots \tau_{2n-1}^{\ell'_{2n-1}}(v') = \psi_{(e,2u')}(v'),$$

where $u' = \sum_{j=0}^{2n-1} \ell'_j u_j$ for some $(\ell'_0, \ldots, \ell'_{2n-1}) \in (\mathbb{Z}_{\geq 0})^{2n}$. Thus,

$$\psi_{(e,2u')}(v') = \psi_{(R^jU^\ell,2u)\cdot(e,2u')}(v') = \psi_{(R^jU^\ell,2R^jU^\ell u'+2u)}(v').$$

This implies $R^jU^\ell = e$, that is, $j = \ell = 0$. So $2u' = 2R^jU^\ell u' + 2u = 2u' + 2u$, that is, $u = 0$. Hence $(R^jU^\ell, u) = (e, 0)$, which is the identity element of $\mathfrak{G}$. □

Lemma 11. *The action of $\mathcal{T}$ (and hence $\mathfrak{G}$) on $\mathbb{C} \setminus \mathbb{V}^+$ is transitive.*

Proof. Let $K^*_{k_0\cdots k_\ell}$ and $K^*_{k'_0\cdots k'_{\ell'}}$ lie in

$$\mathbb{C} \setminus \mathbb{V}^+ = K^* \cup \bigcup_{\ell \geq 0} \bigcup_{0 \leq j \leq \ell} \bigcup_{0 \leq k_j \leq 2n-1} K^*_{k_0k_1\cdots k_\ell}.$$

Since $K^*_{k_0\cdots k_\ell} = \tau_{k_\ell} \circ \cdots \circ \tau_{k_0}(K^*)$ and $K^*_{k'_0\cdots k'_{\ell'}} = \tau_{k'_{\ell'}} \circ \cdots \circ \tau_{k'_0}(K^*)$, it follows that $(\tau_{k'_{\ell'}} \circ \cdots \circ \tau_{k'_0}) \circ (\tau_{k_\ell} \circ \cdots \circ \tau_{k_0})^{-1}(K^*_{k_0\cdots k_\ell}) = K^*_{k'_0\cdots k'_{\ell'}}$. □

The action of $\mathfrak{G}$ on $\mathbb{C} \setminus \mathbb{V}^+$ is proper because $\mathfrak{G}$ is a discrete subgroup of E(2) with no accumulation points.

We now define an edge of $\mathbb{C} \setminus \mathbb{V}^+$ and what it means for an unordered pair of edges to be equivalent. We show that the group $\mathfrak{G}$ acts freely and properly on the identification space of equivalent edges.

Let E be an open edge of K^*. Since $E_{k_0\cdots k_\ell} = \tau_{k_0} \cdots \tau_{k_\ell}(E) \in K^*_{k_0\cdots k_\ell}$, it follows that $E_{k_0\cdots k_\ell}$ is an open edge of $K^*_{k_0\cdots k_\ell}$. Let

$$\mathfrak{E} = \{E_{k_0\cdots k_\ell} \big| \ell \geq 0,\ 0 \leq j \leq \ell \ \&\ 0 \leq k_j \leq 2n-1\}.$$

Then $\mathfrak{E}$ is the set of open edges of $\mathbb{C} \setminus \mathbb{V}^+$ by 12. Since $\tau_{k_\ell} \circ \cdots \circ \tau_{k_0}(0)$ is the center of $K^*_{k_0\cdots k_\ell}$, the element $(e, \tau_{k_\ell} \circ \cdots \circ \tau_{k_0}) \cdot (g, (\tau_{k_\ell} \circ \cdots \circ \tau_{k_0})^{-1})$ of $\mathfrak{G}$ is a rotation-reflection of

$K^*_{k_0\cdots k_\ell}$, which sends an edge of $K^*_{k_0\cdots k_\ell}$ to another edge of $g * K^*_{k_0\cdots k_\ell}$. Thus, $\mathfrak{G}$ sends $\mathfrak{E}$ into itself. For $j = 0, 1, \infty$ let $\mathfrak{E}^j_{k_0\cdots k_\ell}$ be the set of unordered pairs $[E_{k_0\cdots k_\ell}, E'_{k_0\cdots k_\ell}]$ of equivalent open edges of $K^*_{k_0\cdots k_\ell}$, that is, $E_{k_0\cdots k_\ell} \cap E'_{k_0\cdots k_\ell} = \varnothing$, so the open edges $E_{k_0\cdots k_\ell} = \tau_{k_0}\cdots\tau_{k_\ell}(E)$ and $E'_{k_0\cdots k_\ell} = \tau_{k_0}\cdots\tau_{k_\ell}(E')$ of $\mathrm{cl}(K^*_{k_0\cdots k_\ell})$ are not adjacent, which implies that the open edges E and E' of K^* are not adjacent, and for some generator $S^{(j)}_m$ of the group G^j of reflections with $j = 0, 1, \infty$ we have

$$E'_{k_0\cdots k_\ell} = (\tau_{k_0}\circ\cdots\circ\tau_{k_0})\big(S^{(j)}_m((\tau_{k_\ell}\circ\cdots\circ\tau_{k_0})^{-1}(E_{k_0\cdots k_\ell}))\big).$$

Let $\mathfrak{E}^j = \cup_{\ell\ge 0}\cup_{0\le j\le \ell}\cup_{0\le k_j\le 2n-1}\mathfrak{E}^j_{k_0\cdots k_\ell}$. So $\bigcup_{j=0,1,\infty}\mathfrak{E}^j$ is the set of unordered pairs of equivalent edges of $\mathbb{C}\setminus\mathbb{V}^+$. Define an action $*$ of $\mathfrak{G}$ on $\bigcup_{j=0,1,\infty}\mathcal{E}^j$ by

$$\begin{aligned}(g,\tau) * [E_{k_0\cdots k_\ell}, E'_{k_0\cdots k_\ell}] &= \big([(\tau'\circ\tau)(g(\tau')^{-1}(E_{k_0\cdots k_\ell})), (\tau'\circ\tau)(g((\tau')^{-1}(E'_{k_0\cdots k_\ell}))]\big)\\ &= [(g,\tau) * E_{k_0\cdots k_\ell}, (g,\tau) * E'_{k_0\cdots k_\ell}],\end{aligned}$$

where $\tau' = \tau_{k_\ell}\circ\cdots\tau_{k_0}$.

Define a relation $\sim$ on $\mathbb{C}\setminus\mathbb{V}^+$ as follows. We say that x and $y \in \mathbb{C}\setminus\mathbb{V}^+$ are related, $x \sim y$, if 1) $x \in F = \tau(E) \in \mathfrak{E}^j$ and $y \in F' = \tau(E') \in \mathfrak{E}^j$ such that $[F, F'] = [\tau(E), \tau(E')] \in \mathfrak{E}^0$, where $[E, E'] \in \mathcal{E}^j$ with $E' = S^{(j)}_m(E)$ for some $S^{(j)}_m \in G^j$ and $y = \tau\big(S^{(j)}_m(\tau^{-1}(x))\big)$ for some $j = 0, 1, \infty$, or 2) $x, y \in \big(\mathbb{C}\setminus\mathbb{V}^+\big)\setminus\mathfrak{E}$ and $x = y$. Then $\sim$ is an equivalence relation on $\mathbb{C}\setminus\mathbb{V}^+$. Let $(\mathbb{C}\setminus\mathbb{V}^+)^\sim$ be the set of equivalence classes and let Π be the map

$$\Pi : \mathbb{C}\setminus\mathbb{V}^+ \to (\mathbb{C}\setminus\mathbb{V}^+)^\sim : p \mapsto [p], \tag{34}$$

which assigns to every $p \in \mathbb{C}\setminus\mathbb{V}^+$ the equivalence class $[p]$ containing p.

Lemma 12. *$\Pi|_{K^*}$ is the map ρ (20).*

Proof. This follows immediately from the definition of the maps Π and ρ. □

Lemma 13. *The usual action of $\mathfrak{G}$ on $\mathbb{C}$, restricted to $\mathbb{C}\setminus\mathbb{V}^+$, is compatible with the equivalence relation $\sim$, that is, if $x, y \in \mathbb{C}\setminus\mathbb{V}$ and $x \sim y$, then $(g,\tau)(x) \sim (g,\tau)(y)$ for every $(g,\tau) \in \mathfrak{G}$.*

Proof. Suppose that $x \in F = \tau'(E)$, where $\tau' \in \mathcal{T}$. Then $y \in F' = \tau'(E')$, since $x \sim y$. So for some $S^{(j)}_m \in G^j$ with $j = 0, 1, \infty$, we have $(\tau')^{-1}(y) = S^{(j)}_m(\tau^{-1}(x))$. Let $(g,\tau) \in \mathfrak{G}$. Then

$$(g,\tau)\big((\tau')^{-1}(y)\big) = g((\tau')^{-1}(y)) + u_\tau = g\big(S^{(j)}_m(\tau^{-1}(x))\big) + u_\tau.$$

So $(g,\tau)(y) \in (g,\tau) * F'$. However, $(g,\tau)(x) \in (g,\tau) * F$ and $[(g,\tau) * F, (g,\tau) * F'] = (g,\tau) * [F, F']$. Hence $(g,\tau)(x) \sim (g,\tau)(y)$. □

Because of Lemma 13, the usual $\mathfrak{G}$-action on $\mathbb{C}\setminus\mathbb{V}^+$ induces an action of $\mathfrak{G}$ on $(\mathbb{C}\setminus\mathbb{V}^+)^\sim$.

Lemma 14. *The action of $\mathfrak{G}$ on $(\mathbb{C}\setminus\mathbb{V}^+)^\sim$ is free and proper.*

Proof. The following argument shows that it is free. Using Lemma A2 we see that an element of $\mathfrak{G}$, which lies in the isotropy group $\mathfrak{G}_{[F,F']}$ for $[F, F'] \in \mathfrak{E}^0$, interchanges the edge F with the equivalent edge F' and thus fixes the equivalence class $[p]$ for every $p \in F$. Hence the $\mathfrak{G}$ action on $(\mathbb{C}\setminus\mathbb{V}^+)^\sim$ is free. It is proper because $\mathfrak{G}$ is a discrete subgroup of the Euclidean group $\mathrm{E}(2)$ with no accumulation points. □

Theorem 4. *The $\mathfrak{G}$-orbit space $(\mathbb{C}\setminus\mathbb{V}^+)^\sim/\mathfrak{G}$ is holomorphically diffeomorphic to the G-orbit space $(K^*\setminus \mathrm{O})^\sim/G = \widetilde{\mathcal{S}}_{\mathrm{reg}}$.*

Proof. The claim follows because the fundamental domain of the $\mathfrak{G}$-action on $\mathbb{C} \setminus \mathbb{V}^+$ is $K^* \setminus \mathrm{O}$ is the fundamental domain of the G-action on $K^* \setminus \mathrm{O}$. Thus, $\Pi(\mathbb{C} \setminus \mathbb{V}^+)$ is a fundamental domain of the $\mathfrak{G}$-action on $(\mathbb{C} \setminus \mathbb{V}^+)^{\sim}$, which is equal to $\rho(K^* \setminus \mathrm{O}) = (K^* \setminus \mathrm{O})^{\sim}$ by Lemma 12. Hence the $\mathfrak{G}$-orbit space $(\mathbb{C} \setminus \mathbb{V}^+)^{\sim}/\mathfrak{G}$ is equal to the G-orbit space $\widetilde{\mathcal{S}}_{\mathrm{reg}}$. So the identity map from $\Pi(\mathbb{C} \setminus \mathbb{V}^+)$ to $(K^* \setminus \mathrm{O})^{\sim}$ induces a holomorphic diffeomorphism of orbit spaces. □

Because the group $\mathfrak{G}$ is a discrete subgroup of the 2-dimensional Euclidean group $\mathrm{E}(2)$, the Riemann surface $(\mathbb{C} \setminus \mathbb{V}^+)^{\sim}/\mathfrak{G}$ is an *affine* model of the affine Riemann surface $\mathcal{S}_{\mathrm{reg}}$.

6. The Developing Map and Geodesics

In this section, we show that the mapping

$$\delta : \mathcal{D} \subseteq \mathcal{S}_{\mathrm{reg}} \to Q \subseteq \mathbb{C} : (\xi, \eta) \to (F_Q \circ \widehat{\pi})(\xi, \eta)) \tag{35}$$

straightens the holomorphic vector field X (12) on the fundamental domain $\mathcal{D} \subseteq \mathcal{S}_{\mathrm{reg}}$, see [6] and Flaschka [7]. We also verify that X is the geodesic vector field for a flat Riemannian metric Γ on $\mathcal{D}$.

First we rewrite Equation (13) as

$$T_{(\xi,\eta)}\widehat{\pi}\big(X(\xi,\eta)\big) = \eta \frac{\partial}{\partial \xi}, \quad \text{for } (\xi,\eta) \in \mathcal{D}. \tag{36}$$

From the definition of the mapping F_Q (2) we get

$$\mathrm{d}z = \mathrm{d}F_Q = \frac{1}{\big(\xi^{n-n_0}(1-\xi)^{n-n_1}\big)^{1/n}}\,\mathrm{d}\xi = \frac{1}{\eta}\,\mathrm{d}\xi,$$

where we use the same complex nth root as in the definition of F_Q. This implies

$$\frac{\partial}{\partial z} = T_\xi F_Q\big(\eta \frac{\partial}{\partial \xi}\big), \quad \text{for } (\xi,\eta) \in \mathcal{D} \tag{37}$$

For each $(\xi,\eta) \in \mathcal{D}$ using (36) and (37) we get

$$T_{(\xi,\eta)}\delta\big(X(\xi,\eta)\big) = \big(T_\xi F_Q \circ T_{(\xi,\eta)}\widehat{\pi}\big)\big(X(\xi,\eta)\big) = T_\xi F_Q\big(\eta \frac{\partial}{\partial \xi}\big) = \frac{\partial}{\partial z}\Big|_{z=\delta(\xi,\eta)}.$$

So the holomorphic vector field X (12) on $\mathcal{D}$ and the holomorphic vector field $\frac{\partial}{\partial z}$ on Q are δ-related. Hence δ sends an integral curve of the vector field X starting at $(\xi,\eta) \in \mathcal{D}$ onto an integral curve of the vector field $\frac{\partial}{\partial z}$ starting at $z = \delta(\xi,\eta) \in Q$. Since an integral curve of $\frac{\partial}{\partial z}$ is a horizontal line segment in Q, we have proved

Theorem 5. *The holomorphic mapping δ (35) straightens the holomorphic vector field X (12) on the fundamental domain $\mathcal{D} \subseteq \mathcal{S}_{\mathrm{reg}}$.*

We can say more. Let $u = \mathrm{Re}\, z$ and $v = \mathrm{Im}\, z$. Then

$$\gamma = \mathrm{d}u \odot \mathrm{d}u + \mathrm{d}v \odot \mathrm{d}v = \mathrm{d}z \odot \overline{\mathrm{d}z} \tag{38}$$

is the flat Euclidean metric on $\mathbb{C}$. Its restriction $\gamma|_{\mathbb{C}\setminus\mathbb{V}^+}$ to $\mathbb{C} \setminus \mathbb{V}^+$ is invariant under the group $\mathfrak{G}$, which is a subgroup of the Euclidean group $\mathrm{E}(2)$.

Consider the flat Riemannian metric $\gamma|_Q$ on Q, where γ is the metric (38) on $\mathbb{C}$. Pulling back $\gamma|_Q$ by the mapping F_Q (2) gives a metric

$$\widetilde{\gamma} = F_Q^*(\gamma|_Q) = |\xi^{n-n_0}(1-\xi)^{n-n_1}|^{-2/n}\,\mathrm{d}\xi \odot \overline{\mathrm{d}\xi}$$

on $\mathbb{C} \setminus \{0,1\}$. Pulling the metric $\widetilde{\gamma}$ back by the projection mapping $\widetilde{\pi} : \mathbb{C}^2 \to \mathbb{C} : (\xi, \eta) \mapsto \xi$ gives

$$\widetilde{\Gamma} = \widetilde{\pi}^* \widetilde{\gamma} = |\xi^{n-n_0}(1-\xi)^{n-n_1}|^{-2/n}\, \mathrm{d}\xi \odot \overline{\mathrm{d}\xi}$$

on $\mathbb{C}^2$. Restricting $\widetilde{\Gamma}$ to the affine Riemann surface $\mathcal{S}_{\mathrm{reg}}$ gives $\Gamma = \frac{1}{\eta}\, \mathrm{d}\xi \odot \frac{1}{\overline{\eta}}\overline{\mathrm{d}\xi}$.

Lemma 15. Γ *is a flat Riemannian metric on* $\mathcal{S}_{\mathrm{reg}}$.

Proof. We compute. For every $(\xi, \eta) \in \mathcal{S}_{\mathrm{reg}}$ we have

$$\begin{aligned}
&\Gamma(\xi,\eta)\big(X(\xi,\eta), X(\xi,\eta)\big) = \\
&\quad = \tfrac{1}{\eta}\, \mathrm{d}\xi\big(\eta \tfrac{\partial}{\partial \xi} + \tfrac{n-n_0}{n} \tfrac{\xi(1-\xi)(1-\frac{2n-n_0-n_1}{n}\xi)}{\eta^{n-2}} \tfrac{\partial}{\partial \eta}\big) \cdot \tfrac{1}{\overline{\eta}}\overline{\mathrm{d}\xi}\big(\overline{\eta}\, \overline{\tfrac{\partial}{\partial \xi}} + \tfrac{n-n_0}{n} \overline{\tfrac{\xi(1-\xi)(1-\frac{2n-n_0-n_1}{n}\xi)}{\eta^{n-2}}}\, \overline{\tfrac{\partial}{\partial \eta}}\big) \\
&\quad = \tfrac{1}{\eta}\, \mathrm{d}\xi\big(\eta \tfrac{\partial}{\partial \xi}\big) \cdot \tfrac{1}{\overline{\eta}}\overline{\mathrm{d}\xi}\big(\overline{\eta}\, \overline{\tfrac{\partial}{\partial \xi}}\big) = 1.
\end{aligned}$$

Thus, Γ is a Riemannian metric on $\mathcal{S}_{\mathrm{reg}}$. It is flat by construction. □

Because $\mathcal{D}$ has nonempty interior and the map δ (35) is holomorphic, it can be analytically continued to the map

$$\delta_Q : \mathcal{S}_{\mathrm{reg}} \subseteq \mathbb{C}^2 \to Q \subseteq \mathbb{C} : (\xi, \eta) \mapsto F_Q\big(\widehat{\pi}(\xi, \eta)\big), \tag{39}$$

since $\delta = \delta_Q|_{\mathcal{D}}$. By construction $\delta_Q^*(\gamma|_Q) = \Gamma$. So the mapping δ_Q is an isometry of $(\mathcal{S}_{\mathrm{reg}}, \Gamma)$ onto $(Q, \gamma|_Q)$. In particular, the map δ is an isometry of $(\mathcal{D}, \Gamma|_{\mathcal{D}})$ onto $(Q, \gamma|_Q)$. Moreover, δ is a local holomorphic diffeomorphism, because for every $(\xi, \eta) \in \mathcal{D}$, the complex linear mapping $T_{(\xi,\eta)}\delta$ is an isomorphism, since it sends $X(\xi, \eta)$ to $\frac{\partial}{\partial z}\big|_{z=\delta(\xi,\eta)}$. Thus, δ is a developing map in the sense of differential geometry, see Spivak ([8], p. 97) note on §12 of Gauss [9]. The map δ is local because the integral curves of $\frac{\partial}{\partial z}$ on Q are only defined for a finite time, since they are horizontal line segments in Q. Thus, the integral curves of X (12) on $\mathcal{D}$ are defined for a finite time. Since the integral curves of $\frac{\partial}{\partial z}$ are geodesics on $(Q, \gamma|_Q)$, the image of a local integral curve of $\frac{\partial}{\partial z}$ under the local inverse of the mapping δ is a local integral curve of X. This latter local integral curve is a geodesic on $(\mathcal{D}, \Gamma|_{\mathcal{D}})$, since δ is an isometry. Thus, we have proved

Theorem 6. *The holomorphic vector field* X *(12) on the fundamental domain* $\mathcal{D}$ *is the geodesic vector field for the flat Riemannian metric* $\Gamma|_{\mathcal{D}}$ *on* $\mathcal{D}$.

Corollary 13. *The holomorphic vector field* X *on the affine Riemann surface* $\mathcal{S}_{\mathrm{reg}}$ *is the geodesic vector field for the flat Riemannian metric* Γ *on* $\mathcal{S}_{\mathrm{reg}}$.

Proof. The corollary follows by analytic continuation from the conclusion of Theorem 6, since $\operatorname{int} \mathcal{D}$ is a nonempty open subset of $\mathcal{S}_{\mathrm{reg}}$ and both the vector field X and the Riemannian metric Γ are holomorphic on $\mathcal{S}_{\mathrm{reg}}$. □

7. Discrete Symmetries and Billiard Motions

Let $\mathcal{G}$ be the group of homeomorphisms of the affine Riemann surface $\mathcal{S}$ (3) generated by the mappings

$$\mathcal{R} : \mathcal{S} \to \mathcal{S} : (\xi, \eta) \mapsto (\xi, \mathrm{e}^{2\pi i/n}\eta) \ \text{ and } \ \mathcal{U} : \mathcal{S} \to \mathcal{S} : (\xi, \eta) \mapsto (\overline{\xi}, \overline{\eta}).$$

Clearly, the relations $\mathcal{R}^n = \mathcal{U}^2 = e$ hold. For every $(\xi, \eta) \in \mathcal{S}$ we have

$$\mathcal{U}\mathcal{R}^{-1}(\xi, \eta) = \mathcal{U}(\xi, \mathrm{e}^{-2\pi i/n}\eta) = (\overline{\xi}, \mathrm{e}^{2\pi i/n}\overline{\eta}) = \mathcal{R}(\overline{\xi}, \overline{\eta}) = \mathcal{R}\mathcal{U}(\xi, \eta).$$

So the additional relation $\mathcal{U}\mathcal{R}^{-1} = \mathcal{R}\mathcal{U}$ holds. Thus, $\mathcal{G}$ is isomorphic to the dihedral group.

Lemma 16. *$\mathcal{G}$ is a group of isometries of $(\mathcal{S}_{\mathrm{reg}}, \Gamma)$.*

Proof. For every $(\xi, \eta) \in \mathcal{S}_{\mathrm{reg}}$ we get

$$\begin{aligned}
\mathcal{R}^*\Gamma(\xi,\eta)\big(X(\xi,\eta), X(\xi,\eta)\big) &= \Gamma\big(\mathcal{R}(\xi,\eta)\big)\big(T_{(\xi,\eta)}\mathcal{R}\big(X(\xi,\eta)\big), T_{(\xi,\eta)}\mathcal{R}\big(X(\xi,\eta)\big)\big) \\
&= \Gamma(\xi, \mathrm{e}^{2\pi i/n}\eta)\big(\mathrm{e}^{2\pi i/n}\eta\frac{\partial}{\partial\xi} + \tfrac{n-n_0}{n}\frac{\xi(1-\xi)(1-\frac{2n-n_0-n_1}{n-n_0}\xi)}{\eta^{n-2}}\mathrm{e}^{2\pi i/n}\frac{\partial}{\partial\eta}, \\
&\qquad \mathrm{e}^{2\pi i/n}\eta\frac{\partial}{\partial\xi} + \frac{n-n_0}{n}\frac{\xi(1-\xi)(1-\frac{2n-n_0-n_1}{n-n_0}\xi)}{\eta^{n-2}}\mathrm{e}^{2\pi i/n}\frac{\partial}{\partial\eta}\big) \\
&= \frac{1}{|\mathrm{e}^{2\pi i/n}\eta|^2}\,\mathrm{d}\xi\big(\mathrm{e}^{2\pi i/n}\eta\frac{\partial}{\partial\xi}\big)\cdot\overline{\mathrm{d}\xi}\big(\overline{\mathrm{e}^{2\pi i/n}\eta\frac{\partial}{\partial\xi}}\big) = 1 \\
&= \frac{1}{|\eta|^2}\,\mathrm{d}\xi\big(\eta\frac{\partial}{\partial\xi}\big)\cdot\overline{\mathrm{d}\xi}\big(\overline{\eta\frac{\partial}{\partial\xi}}\big) = \Gamma(\xi,\eta)\big(X(\xi,\eta), X(\xi,\eta)\big)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{U}^*\Gamma(\xi,\eta)\big(X(\xi,\eta), X(\xi,\eta)\big) &= \Gamma\big(\mathcal{U}(\xi,\eta)\big)\big(T_{(\xi,\eta)}\mathcal{U}\big(X(\xi,\eta)\big), T_{(\xi,\eta)}\mathcal{U}\big(X(\xi,\eta)\big)\big) \\
&= \frac{1}{|\eta|^2}\overline{\mathrm{d}\xi}\big(\overline{\eta\frac{\partial}{\partial\xi}}\big)\cdot\overline{\overline{\mathrm{d}\xi}}\big(\overline{\overline{\eta\frac{\partial}{\partial\xi}}}\big) = \Gamma(\xi,\eta)\big(X(\xi,\eta), X(\xi,\eta)\big).
\end{aligned}$$

□

Recall that the group G, generated by the linear mappings

$$R : \mathbb{C} \to \mathbb{C} : z \mapsto \mathrm{e}^{2\pi i/n}z \text{ and } U : \mathbb{C} \to \mathbb{C} : z \mapsto \overline{z},$$

is isomorphic to the dihedral group.

Lemma 17. *G is a group of isometries of $(\mathbb{C}, \gamma)$.*

Proof. This follows because R and U are Euclidean motions. □

We would like the developing map δ_Q (39) to intertwine the actions of $\mathcal{G}$ and G and the geodesic flows on $(\mathcal{S}_{\mathrm{reg}}, \Gamma)$ and $(Q, \gamma|_Q)$. There are several difficulties. The first is: the group G does *not* preserve the quadrilateral Q. To overcome this difficulty we extend the mapping δ_Q (39) to the mapping δ_{K^*} (17) of the affine Riemann surface $\mathcal{S}_{\mathrm{reg}}$ onto the regular stellated n-gon K^*.

Lemma 18. *The mapping δ_{K^*} (17) intertwines the action Φ (14) of $\mathcal{G}$ on $\mathcal{S}_{\mathrm{reg}}$ with the action*

$$\Psi : G \times K^* \to K^* : (g, z) \mapsto g(z) \tag{40}$$

of G on the regular stellated n-gon K^*.

Proof. From the definition of the mapping δ_{K^*} we see that for each $(\xi, \eta) \in \mathcal{D}$ we have $\delta_{K^*}\big(\mathcal{R}^j(\xi,\eta)\big) = R^j\delta_{K^*}(\xi,\eta)$ for every $j \in \mathbb{Z}$. By analytic continuation we see that the preceding equation holds for every $(\xi,\eta) \in \mathcal{S}_{\mathrm{reg}}$. Since $F_Q(\overline{\xi}) = \overline{F_Q(\xi)}$ by construction and $\widehat{\pi}(\overline{\xi}, \overline{\eta}) = \overline{\xi}$ (11), from the definition of the mapping δ (35) we get $\delta(\overline{\xi}, \overline{\eta}) = \overline{\delta(\xi,\eta)}$ for every $(\xi,\eta) \in \mathcal{D}$. In other words, $\delta_{K^*}\big(\mathcal{U}(\xi,\eta)\big) = U\delta_{K^*}(\xi,\eta)$ for every $(\xi,\eta) \in \mathcal{D}$. By analytic continuation we see that the preceding equation holds for all $(\xi,\eta) \in \mathcal{S}_{\mathrm{reg}}$. Hence on $\mathcal{S}_{\mathrm{reg}}$ we have

$$\delta_{K^*} \circ \Phi_g = \Psi_{\varphi(g)} \circ \delta_{K^*} \quad \text{for every } g \in \mathcal{G}. \tag{41}$$

The mapping $\varphi : \mathcal{G} \to G$ sends the generators $\mathcal{R}$ and $\mathcal{U}$ of the group $\mathcal{G}$ to the generators R and U of the group G, respectively. So it is an isomorphism. □

There is a second more serious difficulty: the integral curves of $\frac{\partial}{\partial z}$ run off the quadrilateral Q in finite time. We fix this by requiring that when an integral curve reaches a point P on the boundary ∂Q of Q, which is not a vertex, it undergoes a specular reflection at P. (If the integral curve reaches a vertex of Q in forward or backward time, then the motion ends). This motion can be continued as a straight line motion, which extends the motion on the original segment in Q.

To make this precise, we give Q the orientation induced from $\mathbb{C}$ and suppose that the incoming (and hence outgoing) straight line motion has the same orientation as ∂Q. If the incoming motion makes an angle α with respect to the inward pointing normal N to ∂Q at P, then the outgoing motion makes an angle α with the normal N, see Richens and Berry [2]. Specifically, if the incoming motion to P is an integral curve of $\frac{\partial}{\partial z}$, then the outgoing motion, after reflection at P, is an integral curve of $R^{-1}\frac{\partial}{\partial z} = \mathrm{e}^{-2\pi i/n}\frac{\partial}{\partial z}$. Thus, the outward motion makes a turn of $-2\pi/n$ at P towards the interior of Q, see Figure 10 (left). In Figure 10 (right) the incoming motion has the opposite orientation from ∂Q. This extended motion on Q is called a billiard motion. A billiard motion starting in the interior of $\mathrm{cl}(Q) \setminus (\mathrm{cl}(Q) \cap \mathbb{R})$ is defined for *all* time and remains in $\mathrm{cl}(Q)$ less its vertices, since each of the segments of the billiard motion is a straight line parallel to an edge of $\mathrm{cl}(Q)$ and does not hit a vertex of $\mathrm{cl}(Q)$, see Figure 11.

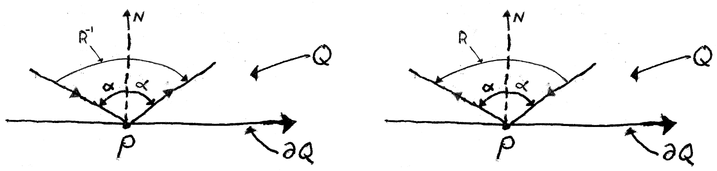

Figure 10. Reflection at a point P on ∂Q.

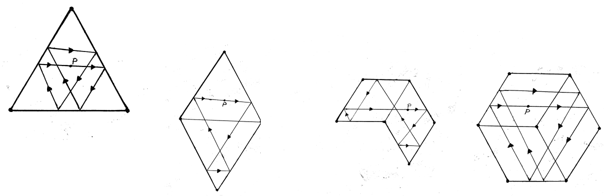

Figure 11. A periodic billiard motion in the equilateral triangle $T = T_{1,1,1}$ starting at P. First, extended by the reflection U to a periodic billiard motion in the quadrilateral $Q = T \cup U(T)$. Second, extended by the relection S to a periodic billiard motion in $Q \cup S(Q)$. Third, extended by the reflection SR to a periodic billiard motion in the stellated equilateral triangle $H = K^*_{1,1,1} = Q \cup S(Q)SR(S(Q))$.

We can do more. If we apply a reflection S in the edge of Q in its boundary ∂Q, which contains the reflection point P, to the initial reflected motion at P, see Figure 12.

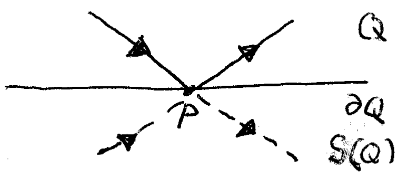

Figure 12. Continuation of a billiard motion in the quadrilateral Q to a billiard motion in the quarilateral $S(Q)$ obtained by the reflection S in an edge of Q.

The motion in $S(Q)$ when it reaches $\partial S(Q)$, et cetera, the extended motion becomes a billiard motion in the regular stellated n-gon $K^* = Q \cup \amalg_{0 \le k \le n-1} SR^k(Q)\big)$, see Figure 11. So we have verified

Theorem 7. *A billiard motion in the regular stellated n-gon K^*, which starts at a point in the interior of $K^* \setminus \mathrm{O}$ and does not hit a vertex of $\mathrm{cl}(K^*)$, is invariant under the action of the isometry subgroup $\widehat{G}$ of the isometry group G of $(K^*, \gamma|_{K^*})$ generated by the rotation R.*

Let $\widehat{\mathcal{G}}$ be the subgroup of $\mathcal{G}$ generated by the rotation $\mathcal{R}$. We now show

Lemma 19. *The holomorphic vector field X (12) on $\mathcal{S}_{\mathrm{reg}}$ is $\widehat{\mathcal{G}}$-invariant.*

Proof. We compute. For every $(\xi, \eta) \in \mathcal{S}_{\mathrm{reg}}$ and for $\mathcal{R} \in \widehat{\mathcal{G}}$ we have

$$\begin{aligned} T_{(\xi,\eta)}\Phi_{\mathcal{R}}\big(X(\xi,\eta)\big) &= \mathrm{e}^{2\pi i/n}\big[\eta\frac{\partial}{\partial\xi} + \tfrac{n-n_0}{n}\frac{\xi(1-\xi)(1-\frac{2n-n_0-n_1}{n}\xi)}{\eta^{n-2}}\frac{\partial}{\partial\eta}\big] \\ &= (\mathrm{e}^{2\pi i/n}\eta)\frac{\partial}{\partial\xi} + \tfrac{n-n_0}{n}\frac{\xi(1-\xi)(1-\frac{2n-n_0-n_1}{n}\xi)}{(\mathrm{e}^{2\pi i/n}\eta)^{n-2}}\frac{\partial}{\partial(\mathrm{e}^{2\pi i/n}\eta)} \\ &= X(\xi, \mathrm{e}^{2\pi i/n}\eta) = X\circ\Phi_{\mathcal{R}}(\xi,\eta). \end{aligned}$$

Hence for every $j \in \mathbb{Z}$ we get

$$T_{(\xi,\eta)}\Phi_{\mathcal{R}^j}\big(X(\xi,\eta)\big) = X\circ\Phi_{\mathcal{R}^j}(\xi,\eta) \tag{42}$$

for every $(\xi,\eta) \in \mathcal{S}_{\mathrm{reg}}$. In other words, the vector field X is invariant under the action of $\widehat{\mathcal{G}}$ on $\mathcal{S}_{\mathrm{reg}}$. □

Corollary 14. *For every $(\xi,\eta) \in \mathcal{D}$ we have*

$$X|_{\Phi_{\mathcal{R}^j}(\mathcal{D})} = T\Phi_{\mathcal{R}^j}\circ X|_{\mathcal{D}}. \tag{43}$$

Proof. Equation (43) is a rewrite of Equation (42). □

Corollary 15. *Every geodesic on $(\mathcal{S}_{\mathrm{reg}}, \Gamma)$ is $\widehat{\mathcal{G}}$-invariant.*

Proof. This follows immediately from the lemma. □

Lemma 20. *For every $(\xi,\eta) \in \mathcal{S}_{\mathrm{reg}}$ and every $j \in \mathbb{Z}$ we have*

$$T_{\Phi_{\mathcal{R}^j}(\xi,\eta)}\delta_{K^*}\big(X(\xi,\eta)\big) = \frac{\partial}{\partial z}\Big|_{\delta_{K^*}(\Phi_{\mathcal{R}^j}(\xi,\eta))=R^j z}. \tag{44}$$

Proof. From Equation (41) we get $\delta_{K^*}\circ\Phi_{\mathcal{R}} = \Psi_R\circ\delta_{K^*}$ on $\mathcal{S}_{\mathrm{reg}}$. Differentiating the preceding equation and then evaluating the result at $X(\xi,\eta) \in T_{(\xi,\eta)}\mathcal{S}_{\mathrm{reg}}$ gives

$$\big(T_{\Phi_{\mathcal{R}}(\xi,\eta)}\delta_{K^*}\circ T_{(\xi,\eta)}\Phi_{\mathcal{R}}\big)X(\xi,\eta) = \big(T_{\delta_{K^*}(\xi,\eta)}\Psi_R\circ T_{(\xi,\eta)}\delta_{K^*}\big)X(\xi,\eta)$$

for all $(\xi,\eta) \in \mathcal{S}_{\mathrm{reg}}$. When $(\xi,\eta) \in \mathcal{D}$, by definition $\delta_{K^*}(\xi,\eta) = \delta(\xi,\eta)$. So for every $(\xi,\eta) \in \mathcal{S}_{\mathrm{reg}}$

$$T_{(\xi,\eta)}\delta_{K^*}\big(X(\xi,\eta)\big) = T_{(\xi,\eta)}\delta\big(X(\xi,\eta)\big) = \frac{\partial}{\partial z}\Big|_{z=\delta(\xi,\eta)} = \frac{\partial}{\partial z}\Big|_{z=\delta_{K^*}(\xi,\eta)}.$$

Thus,

$$T_{\Phi_{\mathcal{R}}(\xi,\eta)}\delta_{K^*}\big(T_{(\xi,\eta)}\Phi_{\mathcal{R}}X(\xi,\eta)\big) = T_{\delta_{K^*}(\xi,\eta)}\Psi_R\big(\frac{\partial}{\partial z}\Big|_{z=\delta_{K^*}(\xi,\eta)}\big), \tag{45}$$

for every $(\xi,\eta) \in \mathcal{D}$. By analytic continuation (45) holds for every $(\xi,\eta) \in \mathcal{S}_{\mathrm{reg}}$. Now $T_{(\xi,\eta)}\Phi_{\mathcal{R}}$ sends $T_{(\xi,\eta)}\mathcal{S}_{\mathrm{reg}}$ to $T_{\Phi_{\mathcal{R}}(\xi,\eta)}\mathcal{S}_{\mathrm{reg}}$. Since $T_{(\xi,\eta)}\Phi_{\mathcal{R}}X(\xi,\eta) = \mathrm{e}^{2\pi i/n}X(\xi,\eta)$ for every

$(\xi,\eta) \in \mathcal{S}_{\text{reg}}$, it follows that $\mathrm{e}^{2\pi i/n} X(\xi,\eta)$ is in $T_{\Phi_{\mathcal{R}}(\xi,\eta)}\mathcal{S}_{\text{reg}}$. Furthermore, since $T_{\delta_{K^*}(\xi,\eta)}\Psi_R$ sends $T_{\delta_{K^*}(\xi,\eta)}K^*$ to $T_{\Psi_R(\delta_{K^*}(\xi,\eta))}K^*$, we get

$$T_{\delta_{K^*}(\xi,\eta)}\Psi_R\Big(\frac{\partial}{\partial z}\Big|_{z=\delta_{K^*}(\xi,\eta)}\Big) = R\frac{\partial}{\partial z}\Big|_{Rz=\Psi_R(\delta_{K^*}(\xi,\eta))}.$$

For every $(\xi,\eta) \in \mathcal{S}_{\text{reg}}$ we obtain

$$T_{\Phi_{\mathcal{R}}(\xi,\eta)}\delta_{K^*}\big(X(\xi,\eta)\big) = \frac{\partial}{\partial z}\Big|_{Rz=\Psi_{\mathcal{R}}(\delta_{K^*}(\xi,\eta))}, \tag{46}$$

that is, Equation (44) holds with $j = 0$. A similar calculation shows that Equation (46) holds with $\mathcal{R}$ replaces by $\mathcal{R}^j$. This verifies Equation (44). □

We now show

Theorem 8. *The image of a $\widehat{\mathcal{G}}$ invariant geodesic on $(\mathcal{S}_{\text{reg}},\Gamma)$ under the developing map δ_{K^*} (17) is a billiard motion in K^*, see Figure 13.*

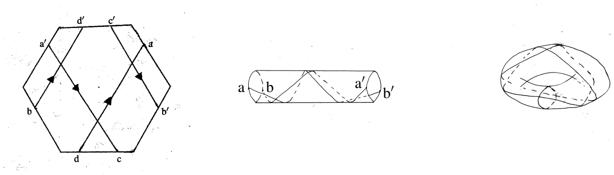

Figure 13. (**left**) A billiard motion in $K^* = K^*_{1,1,1}$. (**center**) The points c, c' and d, d' in K^* are identified, which results in motion on a cylinder. (**right**) After identifying the points a, a' and b, b' on the cylinder the motion becomes a periodic geodesic on $\widetilde{S}_{\text{reg}} = (K^* \setminus \{O\})^{\sim}/G$ on a smooth 2-torus less three points.

Proof. Because $\Phi_{\mathcal{R}^j}$ and Ψ_{R^j} are isometries of $(\mathcal{S}_{\text{reg}},\Gamma)$ and $(K^*,\gamma|_{K^*})$, respectively, it follows from equation (41) that the surjective map $\delta_{K^*} : (\mathcal{S}_{\text{reg}},\Gamma) \to (K^*,\gamma|_{K^*})$ (17) is an isometry. Hence δ_{K^*} is a local developing map. Using the local inverse of δ_{K^*} and Equation (44), it follows that a billiard motion in $\text{int}(K^* \setminus \text{O})$ is mapped onto a geodesic in $(\mathcal{S}_{\text{reg}},\Gamma)$, which is possibly broken at the points $(\xi_i,\eta_i) = \delta^{-1}_{K^*}(p_i)$. Here $p_i \in \partial K^*$ are the points where the billiard motion undergoes a reflection. However, the geodesic on $\mathcal{S}_{\text{reg}}$ is smooth at (ξ_i,η_i) since the geodesic vector field X is holomorphic on $\mathcal{S}_{\text{reg}}$. Thus, the image of the geodesic under the developing map δ_{K^*} is a billiard motion. □

Theorem 9. *Under the restriction of the mapping*

$$\nu = \sigma \circ \Pi : \mathbb{C} \setminus \mathbb{V}^+ \to (\mathbb{C} \setminus \mathbb{V}^+)^{\sim}/\mathfrak{G} = \widetilde{S}_{\text{reg}} \tag{47}$$

to $K^* \setminus \text{O}$ the image of a billiard motion λ_z is a smooth geodesic $\widehat{\lambda}_{\nu(z)}$ on $(\widetilde{S}_{\text{reg}},\widehat{\gamma})$, where $\nu^*(\widehat{\gamma}) = \gamma|_{\mathbb{C}\setminus\mathbb{V}^+}$.

Proof. Since the Riemannian metric γ on $\mathbb{C}$ is invariant under the group of Euclidean motions, the Riemannian metric $\gamma|_{K^*\setminus\text{O}}$ on $K^* \setminus \text{O}$ is $\widehat{G}$-invariant. Hence $\gamma_{K^*\setminus\text{O}}$ is invariant under the reflection S_m for $m \in \{0,1,\ldots,n-1\}$. So $\gamma|_{K^*\setminus\text{O}}$ pieces together to give a Riemannian metric $\gamma^{\sim}$ on the identification space $(K^* \setminus \text{O})^{\sim}$. In other words, the pull back of $\gamma^{\sim}$ under the map $\Pi|_{K^*\setminus\text{O}} : K^* \setminus \text{O} \to (K^* \setminus \text{O})^{\sim}$, which identifies equivalent edges of K^*, is the metric $\gamma|_{K^*\setminus\text{O}}$. Since $\Pi|_{K^*\setminus\text{O}}$ intertwines the G-action on $K^* \setminus \text{O}$ with the G-action on $(K^* \setminus \text{O})^{\sim}$, the metric $\gamma^{\sim}$ is $\widehat{G}$-invariant. It is flat because the metric γ is flat. So $\gamma^{\sim}$ induces a flat Riemannian metric $\widehat{\gamma}$ on the orbit space $(K^* \setminus \text{O})^{\sim}/G = \widetilde{S}_{\text{reg}}$. Since the billiard motion λ_z is a $\widehat{G}$-invariant broken geodesic on $(K^* \setminus \text{O}, \gamma_{K^*\setminus\text{O}})$, it gives rise to a *continuous* broken geodesic $\lambda^{\sim}_{\Pi(z)}$ on $((K^* \setminus \mathrm{O})^{\sim}, \gamma^{\sim})$, which is $\widehat{G}$-invariant. Thus, $\widehat{\lambda}_{\nu(z)} = \nu(\lambda_z)$ is a piecewise smooth geodesic on the smooth G-orbit space $((K^* \setminus \mathrm{O})^{\sim}/G = \widetilde{S}_{\mathrm{reg}}, \widehat{\gamma})$.

We need only show that $\widehat{\lambda}_{\nu(z)}$ is smooth. To see this we argue as follows. Let $s \subseteq K^*$ be a closed segment of a billiard motion γ_z, that does not meet a vertex of $\mathrm{cl}(K^*)$. Then s is a horizontal straight line motion in $\mathrm{cl}(K^*)$. Suppose that E_{k_0} is the edge of K^*, perpendicular to the direction u_{k_0}, which is first met by s and let P_{k_0} be the meeting point. Let S_{k_0} be the reflection in E_{k_0}. The continuation of the motion s at P_{k_0} is the horizontal line $RS_{k_0}(s)$ in $K^*_{k_0}$. Recall that $K^*_{k_0}$ is the translation of K^* by τ_{k_0}. Using a suitable sequence of reflections in the edges of a suitable $K^*_{k_0 \cdots k_\ell}$ each followed by a rotation R and then a translation in $\mathcal{T}$ corresponding to their origins, we extend s to a smooth straight line λ in $\mathbb{C} \setminus \mathbb{V}^+$, see Figure 14. The line λ is a geodesic in $(\mathbb{C} \setminus \mathbb{V}^+, \gamma|_{\mathbb{C} \setminus \mathbb{V}^+})$, which in K^* has image $\widehat{\lambda}_{\nu(z)}$ under the $\mathfrak{G}$-orbit map ν (47) that is a smooth geodesic on $(\widetilde{S}_{\mathrm{reg}}, \widehat{\gamma})$. The geodesic $\nu(\lambda)$ starts at $\nu(z)$. Thus, the smooth geodesic $\nu(\lambda)$ and the geodesic $\widehat{\lambda}_{\nu(z)}$ are equal. In other words, $\widehat{\lambda}_{\nu(z)}$ is a smooth geodesic. $\square$

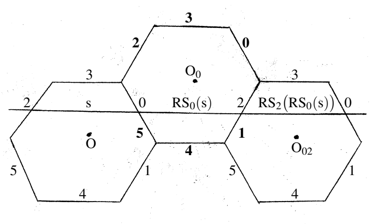

Figure 14. The billiard motion γ_z in the stellated regular 3-gon $K^*_{1,1,1}$ meets the edge 0, isreflected in this edge by S_0, and then is rotated by R. This gives an extended motion $RS_0\gamma_z$, which is a straight line that is the same as reflecting γ_z by U and then translating by τ_0.

Thus, the affine orbit space $\widetilde{S}_{\mathrm{reg}} = (\mathbb{C} \setminus \mathbb{V}^+)^{\sim}/\mathfrak{G}$ with flat Riemannian metric $\widehat{\gamma}$ is the *affine* analogue of the Poincaré model of the affine Riemann surface $\mathcal{S}_{\mathrm{reg}}$ as an orbit space of a discrete subgroup of $\mathrm{PGl}(2, \mathbb{C})$ acting on the unit disk in $\mathbb{C}$ with the Poincaré metric.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No data was used in this research.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A. Group Theoretic Properties

In this appendix we discuss some group theoretic properties of the set of equivalent edges of $\mathrm{cl}(K^*)$, which we use to determine the topology of $\widetilde{\mathcal{S}}_{\mathrm{reg}}$.

Let $\mathcal{E}$ be the set of unordered pairs $[E, E']$ of nonadjacent edges of $\mathrm{cl}(K^*)$. Define an action $\bullet$ of G on $\mathcal{E}$ by

$$g{\bullet}[E, E'] = [g(E), g(E')]$$

for every unordered pair $[E, E']$ of nonadjacent edges of $\mathrm{cl}(K^*)$. For every $g \in G$ the edges $g(E)$ and $g(E')$ are nonadjacent. This follows because the edges E and E' are nonadjacent and the elements of G are invertible mappings of $\mathbb{C}$ into itself. So $\varnothing = g(E \cap E') = g(E) \cap g(E')$. Thus, the mapping$\bullet$ is well defined. It is an action because for every g and $h \in G$ we have

$$\begin{aligned} g{\bullet}(h{\bullet}[E, E']) &= g{\bullet}[h(E), h(E')] = [g(h(E), g(h(E')] \\ &= [(gh)(E), (gh)(E')] = (gh){\bullet}[E, E']. \end{aligned}$$

Since $\mathcal{E} = \bigcup_{j=0,1,\infty} \mathcal{E}^j$, the action $\bullet$ of G on $\mathcal{E}$ induces an action $\cdot$ of the group G^j of reflections on the set $\mathcal{E}^j$ of equivalent edges of $\mathrm{cl}(K^*)$, which is defined by

$$g_j \cdot [E, S_k^{(j)}(E)] = [g_j(E), g_j(S_k^{(j)}(E))] = [g_j(E), (g_j S_k^{(j)} g_j^{-1})(g_j(E))],$$

for every $g_j \in G^j$, every edge E of $\mathrm{cl}(K^*)$, and every generator $S_k^{(j)}$ of G^j, where $k = 0, 1, \ldots, n-1$. Since $g_j S_k^{(j)} g_j^{-1} = S_r^{(j)}$ by Corollary 6, the mapping $\cdot$ is well defined.

Lemma A1. *The group G action $\bullet$ sends a G^j-orbit on $\mathcal{E}^j$ to another G^j-orbit on $\mathcal{E}^j$.*

Proof. Consider the G^j-orbit of $[E, S_m^{(j)}(E)] \in \mathcal{E}^j$. For every $g \in G$ we have

$$g \bullet (G^j \cdot [E, S_m^{(j)}(E)]) = (g G^j g^{-1}) \bullet (g \bullet [E, S_m^{(j)}(E)]) = G^j \cdot (g \bullet [E, S_m^{(j)}(E)]),$$

because G^j is a normal subgroup of G by Corollary 7. Since

$$g \bullet [E, S_m^{(j)}(E)] = [g(E), g(S_m^{(j)}(E))] = [g(E), g S_m^{(j)} g^{-1}(g(E))]$$

and $g S_m^{(j)} g^{-1} = S_r^{(j)}$ by Corollary 6, it follows that $g \bullet [E, S_m^{(j)}(E)] \in \mathcal{E}^j$. $\square$

Lemma A2. *For every $j = 0, 1, \infty$ and every $k = 0, 1, \ldots, n-1$ the isotropy group $G^j_{e^j_k}$ of the G^j action on $\mathcal{E}^j$ at $e^j_k = [E, S_k^{(j)}(E)]$ is $\langle S_k^{(j)} \,|\, (S_k^{(j)})^2 = e \rangle$.*

Proof. Every $g \in G^j_{e^j_k}$ satisfies

$$e^j_k = [E, S_k^{(j)}(E)] = g \cdot e^j_k = g \cdot [E, S_k^{(j)}(E)]$$

if and only if

$$[E, S_k^{(j)}(E)] = [g(E), g S_k^{(j)} g^{-1}(g(E))] = [g(E), S_r^{(j)}(g(E))]$$

if and only if one of the statements 1) $g(E) = E$ & $S_k^{(j)}(E) = S_r^{(j)}(g(E))$ or 2) $E = g(S_r^{(j)}(E))$ & $g(E) = S_k^{(j)}(E)$ holds. From $g(E) = E$ in 1) we get $g = e$ using Lemma 3. To see this we argue as follows. If $g \neq e$, then $g = R^p (S^{(j)})^\ell$ for some $\ell = 0, 1$ and some $p \in \{0, 1, \ldots, n-1\}$, see Equation (A1). Suppose that $g = R^p$ with $p \neq 0$. Then $g(E) \neq E$, which contradicts our hypothesis. Now suppose that $g = R^p S^{(j)}$. Then $E = g(E) = R^p S^{(j)}(E)$, which gives $R^{-p}(E) = S^{(j)}(E)$. Let A and B be end points of the edge E. Then the reflection $S^{(j)}$ sends A to B and B to A, while the rotation R^{-p} sends A to A and B to B. Thus, $R^{-p}(E) \neq S^{(j)}(E)$, which is a contradiction. Hence $g = e$. If $g(E) = S_k^{(j)}(E)$ in 2), then $(S_k^{(j)} g)(E) = E$. So $S_k^{(j)} g = e$ by Lemma 3, that is, $g = S_k^{(j)}$. $\square$

For every $j = 0, 1, \infty$ and every $m_j = 0, 1, \ldots, \frac{n}{d_j} - 1$ let $G^j_{e^j_{m_j d_j}} = \{ g_j \in G^j \,|\, g_j \cdot e^j_{m_j d_j} = e^j_{m_j d_j} \}$ be the isotropy group of the G^j action on $\mathcal{E}^j$ at $e^j_{m_j d_j} = [E, S_{m_j d_j}^{(j)}(E)]$. Since $G^j_{e^j_{m_j d_j}} = \langle S_{m_j d_j}^{(j)} \,|\, (S_{m_j d_j}^{(j)})^2 = e \rangle$ is an abelian subgroup of G^j, it is a normal subgroup. Thus, $H^j = G^j / G^j_{e^j_{m_j d_j}}$ is a subgroup of G^j of order $(2n/d_j)/2 = n/d_j$. This proves

Lemma A3. *For every $j = 0, 1, \infty$ and each $m_j = 0, 1, \ldots, \frac{n}{d_j} - 1$ the G^j-orbit of $e^j_{m_j d_j}$ in $\mathcal{E}^j$ is equal to the H^j-orbit of $e^j_{m_j d_j}$ in $\mathcal{E}^j$.*

Lemma A4. *For $j = 0, 1, \infty$ we have $H^j = \langle V = R^{d_j} \,|\, V^{n/d_j} = e \rangle$.*

Proof. Since

$$S^{(j)}_k = R^k S^{(j)} R^{-k} = R^k (R^{n_j} U) R^{-k} = R^{2k+n_j} U = R^{2k} S^{(j)}, \tag{A1}$$

we get $S^{(j)}_{m_j d_j} = R^{(2m_j + \frac{n_j}{d_j}) d_j} U = (R^{d_j})^{m_j} S^{(j)}$. Because the group G^j is generated by the reflections $S^{(j)}_k$ for $k = 0, 1, \ldots, n-1$, it follows that

$$G^j \subseteq \langle V = R^{d_j}, S^{(j)}_{m_j d_j} \,|\, V^{n/d_j} = e = (S^{(j)}_{m_j d_j})^2 \;\&\; V S^{(j)}_{m_j d_j} = S^{(j)}_{m_j d_j} V^{-1} \rangle = K_j.$$

K_j is a subgroup of G of order $2n/d_j$. Clearly the isotropy group $G^j_{e^j_{m_j d_j}} = \langle S^{(j)}_{m_j d_j} \,|\, (S^{(j)}_{m_j d_j})^2 = e \rangle$ is an abelian subgroup of K^j. Hence $H^j = G^j / G^j_{e^j_{m_j d_j}} \subseteq K^j / G^j_{e^j_{m_j d_j}} = L^j$, where L^j is a subgroup of K^j of order $(2n/d_j)/2 = n/d_j$. Thus, the group L^j has the same order as its subgroup H^j. So $H^j = L^j$. However, $L^j = \langle V = R^{d_j} \,|\, V^{n/d_j} = e \rangle$. □

Let $f^j_\ell = R^\ell \cdot e^j_0$. Then

$$\begin{aligned} f^j_\ell &= R^\ell \cdot e^j_0 = R^\ell \cdot [E, S^{(j)}(E)] \\ &= [R^\ell(E), R^\ell S^{(j)} R^{-\ell}(R^\ell(E))] = [R^\ell(E), S^{(j)}_\ell(R^\ell(E))]. \end{aligned}$$

So

$$\begin{aligned} V^m \cdot f^j_\ell &= V^m \cdot [R^\ell(E), R^\ell S^{(j)} R^{-\ell}(R^\ell(E))] \\ &= [V^m(R^\ell(E)), V^m S^{(j)}_\ell V^{-m}(V^m(R^\ell(E))] \\ &= [R^{m d_j + \ell}(E), S^{(j)}_{m d_j + \ell}(E)] = e^j_{m d_j + \ell}. \end{aligned}$$

This proves

$$\bigcup_{\ell_j=0}^{d_j-1} H^j \cdot f^j_{\ell_j} = \bigcup_{\ell_j=0}^{d_j-1} \bigcup_{m_j=0}^{\frac{n}{d_j}-1} V^{m_j} \cdot f^j_{\ell_j} = \bigcup_{k=0}^{n-1} e^j_k, \tag{A2}$$

since every $k \in \{0, 1, \ldots, n-1\}$ may be written uniquely as $m_j d_j + \ell_j$ for some $m_j \in \{0, 1, \ldots, \frac{n}{d_j} - 1\}$ and some $\ell_j \in \{0, 1, \ldots, d_j - 1\}$.

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