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Relaxation Limit of the Aggregation Equation with Pointy Potential

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Abstract: This work was devoted to the study of a relaxation limit of the so-called aggregation equation with a pointy potential in one-dimensional space. The aggregation equation is today widely used to model the dynamics of a density of individuals attracting each other through a potential. When this potential is pointy, solutions are known to blow up in final time. For this reason, measure-valued solutions have been defined. In this paper, we investigated an approximation of such measure-valued solutions thanks to a relaxation limit in the spirit of Jin and Xin. We study the convergence of this approximation and give a rigorous estimate of the speed of convergence in one dimension with the Newtonian potential. We also investigated the numerical discretization of this relaxation limit by uniformly accurate schemes.

Keywords: aggregation equation; relaxation limit; scalar conservation law; finite volume scheme

MSC: 35L65; 65M12; 35D30

Citation: Fabrèges, B.; Lagoutière, F.; Tran Tien, S.; Vauchelet, N Relaxation Limit of the Aggregation Equation with Pointy Potential. *Axioms* **2021**, *10*, 108. https://doi.org/10.3390/axioms10020108

Academic Editor: Giampiero Palatucci

Received: 9 April 2021
Accepted: 26 May 2021
Published: 28 May 2021

1. Introduction

The so-called aggregation equation has been widely used to model the dynamics of a population of individuals in interaction. Let $W : \mathbb{R} \to \mathbb{R}$, sufficiently smooth, be the interaction potential governing the population. Then, in one dimension in space, the dynamics of the density of individuals, denoted by ρ, is governed by the following equation, for $t > 0$ and $x \in \mathbb{R}$:

$$\partial_t \rho + \partial_x(a[\rho]\rho) = 0, \qquad \text{with} \quad a[\rho] = -W' * \rho. \tag{1}$$

Such equations appear in many applications in population dynamics: for instance, to describe the collective migration of cells by swarming, the motion of bacteria by chemotaxis, the crowd motion, the flocking of birds, or fishes school, see, e.g., [1–7]. From a mathematical point of view, these equations have been widely studied. When the potential W is not smooth enough, it is known that weak solutions may blow up in finite time [8,9]. Thus, the existence of weak (measure) solutions has been investigated in, e.g., [10,11].

In this paper, we consider a relaxation limit in the spirit of Jin–Xin [12] of the aggregation equation in one space dimension on $\mathbb{R}$. It is now well-established that such modifications allow regularizing the solutions. For a given $c > \|a\|_\infty$, we introduce the system:

$$\partial_t \rho + \partial_x \sigma = 0, \tag{2a}$$
$$\partial_t \sigma + c^2 \partial_x \rho = \frac{1}{\varepsilon}(a[\rho]\rho - \sigma) \tag{2b}$$
$$a[\rho] = -W' * \rho \tag{2c}$$

This system is complemented by initial data ρ_0 and $\sigma_0 := a[\rho_0]\rho_0$. It is clear, at least formally, that when $\varepsilon \to 0$, the solution ρ of system (2) converges to the one of the aggregation equations (1) (and it is actually only true if $c > \|a\|_\infty$). We mention that the aggregation equation may also be derived thanks to a hydrodynamical limit of kinetic equations [6,7,13].

The aim of this work was to study the convergence as $\varepsilon \to 0$ of the relaxation system (2) towards the aggregation equation. More precisely, we establish a precise estimate of the speed of convergence, and we also illustrate with some numerical simulations. These estimates are obtained only in the case of the Newtonian potential in one dimension $W(x) = \frac{1}{2}|x|$. Indeed, in this particular case, we may link the aggregation equation to a scalar conservation law [14,15]. The same link holds for the relaxation system (2)—denoting:

$$u(t,x) = \frac{1}{2} - \int_{-\infty}^{x} \rho(t,dy), \qquad v(t,x) = \frac{1}{2} - \int_{-\infty}^{x} \sigma(t,dy),$$

where the notation $\int \rho(t,dy)$ stands for the integral with respect to the probability measure $\rho(t)$, then we verify easily that:

$$u = -W' * \rho, \qquad \rho = -\partial_x u,$$

so that $a[\rho] = u$. Then, integrating (2), we deduce that (u,v) is a solution to:

$$\partial_t u + \partial_x vs. = 0 \tag{3a}$$
$$\partial_t vs. + c^2 \partial_x u = \frac{1}{\varepsilon}\Big(\frac{1}{2}u^2 - v\Big), \tag{3b}$$

which is complemented with the initial data $u_0 = \frac{1}{2} - \int_{-\infty}^{x} \rho_0(dy)$, and $v_0 = \frac{1}{2} - \int_{-\infty}^{x} \sigma_0(dy)$. Clearly, as $\varepsilon \to 0$, we expect that the solution of the above system converges to the solution of the following Burgers equation:

$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0.$$

Introducing the quantities $a = v - cu$ and $b = v + cu$, (3) is equivalent to the diagonalized system:

$$\partial_t a - c\partial_x a = \frac{1}{\varepsilon}\Big(\frac{1}{2}\Big(\frac{b-a}{2c}\Big)^2 - \frac{a+b}{2}\Big) \tag{4a}$$
$$\partial_t b + c\partial_x b = \frac{1}{\varepsilon}\Big(\frac{1}{2}\Big(\frac{b-a}{2c}\Big)^2 - \frac{a+b}{2}\Big). \tag{4b}$$

We will adapt the techniques developed in [16] to obtain convergence estimates for our system.

In order to illustrate this convergence result, numerical discretizations of the relaxation system (2) are investigated. The schemes we propose are such that they are uniform with respect to ε, that is they satisfy the so-called asymptotic preserving (AP) property [17]. Therefore, such schemes in the limit $\varepsilon \to 0$ must be consistent with the aggregation equation. The numerical simulations of solutions of the aggregation equation for pointy potentials have been studied by several authors, see, e.g., [11,13,18–22]. In particular, some authors pay attention to recover the correct behavior of the numerical solutions after the blow-up time. To do so, particular attention must be paid to the definition of the product $a[\rho]\rho$ when ρ is a measure.

In this article, we propose two discretizations of the relaxation system which satisfy the AP property. In a first approach, we propose a simple splitting algorithm where we split the transport part and the right hand side in system (2). It results in a numerical scheme which is very simple to implement and for which we easily verify the AP property. The second approach relies on a well-balanced discretization in the spirit of [20,23]. This scheme is more expensive to implement than the first scheme, but its numerical solution has less diffusion, as it is illustrated by our numerical results.

The outline of the paper is the following. In Section 2, after recalling some useful notations, we prove our main result: an estimation of the speed of convergence in the Wasserstein W_1 distance with respect to ε of the solutions of the relaxation system (2) towards the solution of the aggregation Equation (1) in the case $W(x) = \frac{1}{2}|x|$. The numerical discretization is investigated in Section 3. Two numerical schemes verifying the AP property are proposed. The first scheme is based on a splitting algorithm, whereas the second scheme relies on a well-balanced discretization. Numerical results and comparisons are provided in Section 4.

2. Convergence Result

2.1. Notations

Before stating and proving our main results, we first recall some useful notations and results. Since we are dealing with conservation laws (in which the total mass is conserved), we will work in some space of probability measures, namely the Wasserstein space of order $p \geq 1$, which is the space of probability measures with a finite order p moment:

$$\mathcal{P}_p(\mathbb{R}^N) = \left\{ \mu \text{ nonnegative Borel measure}, \mu(\mathbb{R}^N) = 1, \int |x|^p \mu(dx) < \infty \right\}.$$

This space is endowed with the Wasserstein distance defined by (see, e.g., [24,25])

$$W_p(\mu,\nu) = \inf_{\gamma \in \Gamma(\mu,\nu)} \left\{ \int |y-x|^p \, \gamma(dx,dy) \right\}^{1/p}, \tag{5}$$

where $\Gamma(\mu,\nu)$ is the set of measures on $\mathbb{R}^N \times \mathbb{R}^N$ with marginals μ and ν, meaning that:

$$\Gamma(\mu,\nu) = \left\{ \gamma \in \mathcal{P}_p(\mathbb{R}^N \times \mathbb{R}^N);\ \forall\, \xi \in C_0(\mathbb{R}^N), \int_{\mathbb{R}^{2N}} \xi(y_0)\gamma(dy_0,dy_1) = \int_{\mathbb{R}^N} \xi(y_0)\mu(dy_0), \right.$$
$$\left. \int_{\mathbb{R}^{2N}} \xi(y_1)\gamma(dy_0,dy_1) = \int_{\mathbb{R}^N} \xi(y_1)\nu(dy_1) \right\},$$

with $C_0(\mathbb{R}^N)$, the set of continuous functions on $\mathbb{R}^N$ that vanish at infinity. From a simple minimization argument, we know that in the definition of W_p, the infimum is actually a minimum. A map that realizes the minimum in the definition (5) of W_p is called an optimal transport plan, the set of which is denoted by $\Gamma_0(\mu,\nu)$.

In the one-dimensional framework, we may simplify these definitions. Indeed, any probability measure μ on the real line $\mathbb{R}$ can be described in terms of its cumulative distribution function $F_\mu(x) = \mu((-\infty,x))$, which is a right-continuous and non-decreasing function with $F_\mu(-\infty) = 0$ and $F_\mu(+\infty) = 1$. Then, we can define the generalized inverse F_μ^{-1} of F_μ (or monotone rearrangement of μ) by $F_\mu^{-1}(z) := \inf\{x \in \mathbb{R}/F_\mu(x) > z\}$, it is a right-continuous and non-decreasing function as well, defined on $[0,1]$. We have for every non-negative Borel map ξ:

$$\int_{\mathbb{R}} \xi(x)\mu(dx) = \int_0^1 \xi(F_\mu^{-1}(z))\, dz.$$

In particular, $\mu \in \mathcal{P}_p(\mathbb{R})$ if and only if $F_\mu^{-1} \in L^p(0,1)$. Moreover, in the one-dimensional setting, there exists a unique optimal transport plan realizing the minimum in (5). More precisely, if μ and ν belong to $\mathcal{P}_p(\mathbb{R})$, with monotone rearrangements F_μ^{-1} and F_ν^{-1}, then $\Gamma_0(\mu,\nu) = \{(F_\mu^{-1}, F_\nu^{-1})_\# \mathbb{L}_{(0,1)}\}$ where $\mathbb{L}_{(0,1)}$ is the restriction of the Lebesgue measure on $(0,1)$. Thus, we have the explicit expression of the Wasserstein distance (see [24,26,27]):

$$W_p(\mu,\nu) = \left(\int_0^1 |F_\mu^{-1}(z) - F_\nu^{-1}(z)|^p \, dz \right)^{1/p}, \tag{6}$$

and the map $\mu \mapsto F_\mu^{-1}$ is an isometry between $\mathcal{P}_p(\mathbb{R})$ and the convex subset of (essentially) non-decreasing functions of $L^p(0,1)$.

2.2. Convergence Estimates

Let us first consider the limit $\varepsilon \to 0$ for the system (3). Compactness methods were used in [28] to get L^1_{loc} convergence in space. However, in order to pass to the aggregation equation, one may want global L^1 convergence, which we prove in the following theorem, along the lines of Katsoulakis and Tzavaras [16].

Theorem 1. *Let $u_0 \in L^\infty \cap BV(\mathbb{R})$, $c > \|u_0\|_{L^\infty}$ and set $v_0 = \frac{u_0^2}{2}$. There exists a constant $C > 0$ such that, for any $\varepsilon > 0$, denoting by $(u^\varepsilon, v^\varepsilon)$ the solution to (3) with initial data (u_0, v_0), the following estimate holds:*

$$\forall T > 0, \qquad \|u(T) - u^\varepsilon(T)\|_{L^1} \le C\,TV(u_0)(\sqrt{\varepsilon T} + \varepsilon),$$

where u is the entropy solution to the Burgers equation with initial datum u_0.

Proof. Denote $(a^\varepsilon, b^\varepsilon)$ the solution to (4), and $G(a,b) = \frac{1}{2}\left(\frac{b-a}{2c}\right)^2 - \frac{a+b}{2}$.

So as to obtain entropy inequalities on $(a^\varepsilon, b^\varepsilon)$, we need monotonicity properties on G. One can check that $G(a^\varepsilon, b^\varepsilon)$ is decreasing with respect to a^ε and b^ε if the so-called subcharacteristic condition $|u^\varepsilon| < c$ holds. Up to a slight modification of the nonlinear term $f(u^\varepsilon) = \frac{(u^\varepsilon)^2}{2}$ in (3), which does not affect the value of $(a^\varepsilon, b^\varepsilon)$:

$$f(u) := \begin{cases} -\|u_0\|u - \dfrac{\|u_0\|^2}{2}, & \text{if } u \le -\|u_0\|, \\ \dfrac{u^2}{2}, & \text{if } -\|u_0\| \le u \le \|u_0\|, \\ \|u_0\|u - \dfrac{\|u_0\|^2}{2}, & \text{if } \|u_0\| \le u, \end{cases}$$

the choice $c > \|u_0\|_{L^\infty}$ ensures that the subcharacteristic condition and the bound $\|u^\varepsilon(t)\|_{L^\infty} \le \|u_0\|_{L^\infty}$ holds for all time.

Now, obtaining entropy inequalities on $(a^\varepsilon, b^\varepsilon)$ consists of making a comparison with constant state solutions to (4). Namely, letting $m = \|u_0\|_{L^\infty}\left(\frac{\|u_0\|_{L^\infty}}{2} - c\right)$, $M = \|u_0\|_{L^\infty}\left(\frac{\|u_0\|_{L^\infty}}{2} + c\right)$ and $h(a) = a + 2c^2 - 2c\sqrt{c^2 + 2a}$, we have $G(k, h(k)) = 0$ for all $k \in [m, M]$, and therefore $(k, h(k))$ is a solution to (4). Thus, the following system holds:

$$\partial_t(a^\varepsilon - k) - c\partial_x(a^\varepsilon - k) = \frac{1}{\varepsilon}\Big(G(a^\varepsilon, b^\varepsilon) - G(k, h(k))\Big), \tag{7a}$$

$$\partial_t(b^\varepsilon - h(k)) + c\partial_x(b^\varepsilon - h(k)) = \frac{1}{\varepsilon}\Big(G(a^\varepsilon, b^\varepsilon) - G(k, h(k))\Big). \tag{7b}$$

Multiplying (7a) by $\operatorname{sgn}(a^\varepsilon - k)$, (7b) by $\operatorname{sgn}(b^\varepsilon - h(k))$ and summing yields:

$$\begin{aligned}
&\partial_t\Big(|a^\varepsilon - k| + |b^\varepsilon - h(k)|\Big) - c\partial_x\Big(|a^\varepsilon - k| - |b^\varepsilon - h(k)|\Big)\\
&= \frac{1}{\varepsilon}\Big(\operatorname{sgn}(a^\varepsilon - k) + \operatorname{sgn}(b^\varepsilon - h(k))\Big)\Big(G(a^\varepsilon, b^\varepsilon) - G(k, h(k))\Big).
\end{aligned}$$

Hence, using the monotonicity of G, we obtain the following entropy inequalities on $(a^\varepsilon, b^\varepsilon)$:

$$\partial_t\Big(|a^\varepsilon - k| + |b^\varepsilon - h(k)|\Big) - c\partial_x\Big(|a^\varepsilon - k| - |b^\varepsilon - h(k)|\Big) \le 0. \tag{8}$$

We now turn to proving the entropy inequalities on u^ε. Straightforward computations yield the existence of a constant $C > 0$ such that, for all $a, b \in [m, M]$, one has $|h(a) - b| \le C|G(a, b)|$. We therefore work on the variable $w^\varepsilon := \frac{h(a^\varepsilon) - a^\varepsilon}{2c}$ in the first place. Let $\kappa \in \big[-\|u_0\|_{L^\infty}, \|u_0\|_{L^\infty}\big]$, and $k \in [m, M]$ such that $\kappa = \frac{h(k) - k}{2c}$. We have:

$$|w^\varepsilon - \kappa| = \frac{1}{2c}\Big(|h(a^\varepsilon) - h(k)| + |a^\varepsilon - k|\Big) = \frac{1}{2c}\Big(|a^\varepsilon - k| + |b^\varepsilon - h(k)| + r_1^\varepsilon\Big), \tag{9}$$

where $r_1^\varepsilon = |h(a^\varepsilon) - h(k)| - |b^\varepsilon - h(k)|$ verifies $|r_1^\varepsilon| \le |h(a^\varepsilon) - b^\varepsilon| \le C|G(a^\varepsilon, b^\varepsilon)|$. Thus, we are left to control $|G(a^\varepsilon, b^\varepsilon)|$. To do so, we formally differentiate this quantity and use (4):

$$\begin{aligned}
\partial_t|G(a^\varepsilon, b^\varepsilon)| &= \Big(\partial_t a^\varepsilon \partial_a G(a^\varepsilon, b^\varepsilon) + \partial_t b^\varepsilon \partial_b G(a^\varepsilon, b^\varepsilon)\Big)\operatorname{sgn}(G(a^\varepsilon, b^\varepsilon)),\\
&= \frac{1}{\varepsilon}\Big(\partial_a G(a^\varepsilon, b^\varepsilon) + \partial_b G(a^\varepsilon, b^\varepsilon)\Big)|G(a^\varepsilon, b^\varepsilon)|\\
&\quad - c\operatorname{sgn}(G(a^\varepsilon, b^\varepsilon))\Big(\partial_x a^\varepsilon \partial_a G(a^\varepsilon, b^\varepsilon) + \partial_x b^\varepsilon \partial_b G(a^\varepsilon, b^\varepsilon)\Big),\\
&\le \frac{1}{\varepsilon}\sup_{[m,M]^2}\Big(\partial_a G + \partial_b G\Big)|G(a^\varepsilon, b^\varepsilon)| + c\sup_{[m,M]^2}\Big(|\partial_a G| + |\partial_b G|\Big)\Big(|\partial_x a^\varepsilon| + |\partial_x b^\varepsilon|\Big).
\end{aligned}$$

Integrating in space gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|G(a^\varepsilon, b^\varepsilon)\|_{L^1} \le -\frac{A}{\varepsilon}\|G(a^\varepsilon, b^\varepsilon)\|_{L^1} + B\Big(TV(a_0) + TV(b_0)\Big),$$

where $A = -\sup_{[m,M]^2}(\partial_a G + \partial_b G)$ and $B = c\sup_{[m,M]^2}(|\partial_a G| + |\partial_b G|)$ are positive constants which do not depend on ε nor on time. A Gronwall lemma then gives:

$$\|G(a^\varepsilon(t), b^\varepsilon(t))\|_{L^1} \le C\Big(TV(a_0) + TV(b_0)\Big)\varepsilon, \tag{10}$$

where we still denote $C = B/A$ as a constant independent of time and of ε.

In addition, since, $G(a, h(a)) = 0$, one has $\frac{1}{2}\left(\frac{h(a) - a}{2c}\right)^2 = \frac{1}{2}(h(a) + a)$ and therefore:

$$\begin{aligned}
\operatorname{sgn}(w^\varepsilon - \kappa)\left(\frac{(w^\varepsilon)^2}{2} - \frac{\kappa^2}{2}\right) &= \frac{1}{2}\operatorname{sgn}\Big(h(a^\varepsilon) - h(k) - (a^\varepsilon - k)\Big)\Big(h(a^\varepsilon) + a^\varepsilon - (h(k) + k)\Big),\\
&= \frac{1}{2}\Big(|h(a^\varepsilon) - h(k)| - |a^\varepsilon - k|\Big),\\
&= \frac{1}{2}\Big(|b - h(k)| - |a^\varepsilon - k| + r_2^\varepsilon\Big),
\end{aligned} \tag{11}$$

with $|r_2^\varepsilon| \le C|G(a^\varepsilon, b^\varepsilon)|$. Differentiating (9) in time and (11) in space, and using (8) thus yields:

$$\partial_t|w^\varepsilon - \kappa| + \partial_x\operatorname{sgn}(w^\varepsilon - \kappa)\left(\frac{(w^\varepsilon)^2}{2} - \frac{\kappa^2}{2}\right) \le \frac{1}{2c}\Big(\partial_t r_1^\varepsilon + c\partial_x r_2^\varepsilon\Big). \tag{12}$$

Then, we estimate $\|u(t) - w^\varepsilon(t)\|_{L^1}$ using Kuznetsov's doubling of variables technique (see, e.g., [29] for scalar conservation laws with viscosity and [30] for a more general formalism) in order to combine (12) with Kruzkov inequalities on the entropy solution u, that read:

$$\partial_t |u - \kappa| + \partial_x \operatorname{sgn}(u - \kappa)(f(u) - f(\kappa)) \leq 0. \tag{13}$$

Writing, respectively, (13) at point (s, x) for $\kappa = w^\varepsilon(t, y)$ and (12) at point (t, y) for $\kappa = u(s, x)$, we obtain:

$$\partial_s |u(s,x) - w^\varepsilon(t,y)| + \partial_x \operatorname{sgn}(u(s,x) - w^\varepsilon(t,y))\left(\frac{u(s,x)^2}{2} - \frac{(w^\varepsilon(t,y))^2}{2}\right) \leq 0, \tag{14a}$$

$$\begin{aligned} &\partial_t |w^\varepsilon(t,y) - u(s,x)| + \partial_y \operatorname{sgn}(w^\varepsilon(t,y) - u(s,x))\left(\frac{(w^\varepsilon(t,y))^2}{2} - \frac{u(s,x)^2}{2}\right) \\ &\qquad \leq \frac{1}{2c}\Big(\partial_t r_1^\varepsilon(t,y) + c\partial_y r_2^\varepsilon(t,y)\Big). \end{aligned} \tag{14b}$$

Now, let $\omega_\alpha(t) = \frac{1}{\alpha}\omega\left(\frac{t}{\alpha}\right)$ and $\Omega_\beta(x) = \frac{1}{\beta}\Omega\left(\frac{x}{\beta}\right)$ be two mollyfing kernels. Setting $g(s,t,x,y) = \omega_\alpha(s-t)\Omega_\beta(x-y)$ and testing (14a) and (14b) against $g(\cdot, t, \cdot, y)\mathbb{1}_{[0,T]}$ and $g(s, \cdot, x, \cdot)\mathbb{1}_{[0,T]}$, respectively, and integrating over $[0, T] \times \mathbb{R}$, we obtain on the one hand:

$$\begin{aligned} &\iiiint \partial_s g(s,t,x,y)|u(s,x) - w^\varepsilon(t,y)| \,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\ &+ \iiiint \partial_x g(s,t,x,y) \operatorname{sgn}(u(s,x) - w^\varepsilon(t,y))\left(\frac{u(s,x)^2}{2} - \frac{(w^\varepsilon(t,y))^2}{2}\right) \mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\ &- \iiint g(T,t,x,y)|u(T,x) - w^\varepsilon(t,y)| \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\ &+ \iiint g(0,t,x,y)|u(0,x) - w^\varepsilon(t,y)| \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \geq 0, \end{aligned} \tag{15}$$

and on the other hand:

$$\begin{aligned} &\iiiint \partial_t g(s,t,x,y)|w^\varepsilon(t,y) - u(s,x)| \,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\ &+ \iiiint \partial_y g(s,t,x,y) \operatorname{sgn}(w^\varepsilon(t,y) - u(s,x))\left(\frac{(w^\varepsilon(t,y))^2}{2} - \frac{u(s,x)^2}{2}\right) \mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\ &- \iiint g(s,T,x,y)|w^\varepsilon(T,y) - u(s,x)| \,\mathrm{d}s\mathrm{d}x\mathrm{d}y + \iiint g(s,0,x,y)|w^\varepsilon(0,y) - u(s,x)| \,\mathrm{d}s\mathrm{d}x\mathrm{d}y \\ &\geq \frac{1}{2c}\Bigg(\iiiint \partial_t g(s,t,x,y) r_1^\varepsilon(t,y) \,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y + c\iiiint \partial_y g(s,t,x,y) r_2^\varepsilon(t,y) \,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\ &- \iiint g(s,T,x,y) r_1^\varepsilon(T,y) \,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y + \iiint g(s,0,x,y) r_1^\varepsilon(0,y) \,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y\Bigg) =: \mathrm{RHS}. \end{aligned} \tag{16}$$

Now, since $|\cdot|$ is even, and $\partial_s g = -\partial_t g$ and $\partial_x g = -\partial_y g$, we deduce by adding (15) and (16):

$$\begin{aligned} &- \iiint g(T,t,x,y)|u(T,x) - w^\varepsilon(t,y)| \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\ &+ \iiint g(0,t,x,y)|u(0,x) - w^\varepsilon(t,y)| \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\ &- \iiint g(s,T,x,y)|u(s,x) - w^\varepsilon(T,y)| \,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y \\ &+ \iiint g(s,0,x,y)|u(s,x) - w^\varepsilon(0,y)| \,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y \geq \mathrm{RHS}. \end{aligned} \tag{17}$$

Then, we write:

$$\begin{aligned}\|u(T)-w^{\varepsilon}(T)\|_{L^1} &= \iiint \omega_\alpha(T-t)\Omega_\beta(x-y)|u(T,y)-w^{\varepsilon}(T,y)|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\ &+ \iiint \omega_\alpha(s-T)\Omega_\beta(x-y)|u(T,y)-w^{\varepsilon}(T,y)|\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y, \\ &=: I_1+I_2.\end{aligned} \tag{18}$$

A triangle inequality gives for I_1:

$$\begin{aligned}I_1 &\le \iiint \omega_\alpha(T-t)\Omega_\beta(x-y)|u(T,y)-u(T,x)|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\ &+ \iiint \omega_\alpha(T-t)\Omega_\beta(x-y)|u(T,x)-w^{\varepsilon}(t,y)|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\ &+ \iiint \omega_\alpha(T-t)\Omega_\beta(x-y)|w^{\varepsilon}(t,y)-w^{\varepsilon}(T,y)|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\ &=: T_1+T_2+T_3.\end{aligned}$$

with $T_1 \le C\beta\cdot TV(u_0)$, the second term T_2 appearing in (17) and for the last one we write:

$$T_3 \le \int_{\mathbb{R}} \Omega_\beta(x-y)\int_0^T \omega_\alpha(T-t)\int_{\mathbb{R}} |w^{\varepsilon}(t,y)-w^{\varepsilon}(T,y)|\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}x,$$

and then we use the fact that w^{ε} is uniformely Lipschitz in $L^1(\mathbb{R})$ with respect to ε. Indeed, one has $\partial_t w^{\varepsilon} = \frac{\partial_t a^{\varepsilon}(h'(a^{\varepsilon})-1)}{2c}$ with $h'(a^{\varepsilon})-1$ being uniformely bounded with respect to ε as a^{ε} stays in the compact set $[m,M]$ for all time. In addition, estimating $\|\partial_t a^{\varepsilon}(t)\|_{L^1}$ can be done reusing (4) and (10):

$$\|\partial_t a^{\varepsilon}(t)\|_{L^1} \le c\|\partial_x a^{\varepsilon}(t)\|_{L^1} + \frac{1}{\varepsilon}\|G(a^{\varepsilon}(t),b^{\varepsilon}(t))\|_{L^1} \le C(TV(a_0)+TV(b_0)).$$

with $C>0$ still independent of time and of ε. Hence, $\|\partial_t w^{\varepsilon}(t)\|_{L^1} \le C(TV(a_0)+TV(b_0))$ and $T_3 \le \alpha C(TV(a_0)+TV(b_0))$. All in all, we get for I_1:

$$\begin{aligned}I_1 &\le \iiint \omega_\alpha(T-t)\Omega_\beta(x-y)|u(T,x)-w^{\varepsilon}(t,y)|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y + C\beta\cdot TV(u_0) \\ &+ \alpha C(TV(a_0)+TV(b_0)).\end{aligned}$$

Moreover, similarly, for I_2:

$$I_2 \le \iiint \omega_\alpha(s-T)\Omega_\beta(x-y)|u(s,x)-w^{\varepsilon}(T,y)|\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y + C(\alpha+\beta)TV(u_0).$$

Returning to (18), we obtain:

$$\|u(T)-w^{\varepsilon}(T)\|_{L^1} \le \iiint \omega_\alpha(t)\Omega_\beta(x-y)|u(0,x)-w^{\varepsilon}(t,y)|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \tag{19}$$

$$+ \iiint \omega_\alpha(s)\Omega_\beta(x-y)|u(s,x)-w^{\varepsilon}(0,y)|\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y - RHS \tag{20}$$

$$+ \alpha C(TV(a_0)+TV(b_0)) + C(\alpha+\beta)TV(u_0). \tag{21}$$

However, using a triangle inequality, one can show that:

$$\iiint \omega_\alpha(t)\Omega_\beta(x-y)|u_0(x)-w^{\varepsilon}(t,y)|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \le C\beta\cdot TV(u_0) + \alpha C(TV(a_0)+TV(b_0)),$$

and similarly:

$$\iiint \omega_\alpha(s)\Omega_\beta(x-y)|u(s,x)-w^{\varepsilon}(0,y)|\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y \le C(\alpha+\beta)TV(u_0).$$

We then bound from above the term RHS using inequality $\|r_i^\varepsilon(t)\|_{L^1} \leq C(TV(a_0)+TV(b_0))\varepsilon$ for $i=1,2$:

$$\begin{aligned}\Big|\mathrm{RHS}\Big| = \frac{1}{2c}\Big|\frac{1}{\alpha}\iiiint \omega'\left(\frac{s-t}{\alpha}\right)\Omega_\beta(x-y)r_1^\varepsilon(t,y)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\ + \frac{c}{\beta}\iiiint \omega_\alpha(s-t)\Omega'(x-y)r_2^\varepsilon(t,y)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\ - \iiint \omega_\alpha(s-T)\Omega_\beta(x-y)r_1^\varepsilon(T,y)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y \\ + \iiint \omega_\alpha(s)\Omega_\beta(x-y)r_1^\varepsilon(0,y)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y\Big|, \\ \leq C\left(\frac{T}{\alpha}+\frac{T}{\beta}+1\right)\cdot(TV(a_0)+TV(b_0))\varepsilon.\end{aligned}$$

Finally, we obtain:

$$\begin{aligned}\|u(T)-w^\varepsilon(T)\|_{L^1} \leq & C\left(\frac{T}{\alpha}+\frac{T}{\beta}+1\right)(TV(a_0)+TV(b_0))\varepsilon \\ & + C(\alpha+\beta)TV(u_0)+\alpha C(TV(a_0)+TV(b_0)),\end{aligned}$$

which, after optimizing the values of α and β and noticing that $TV(a_0), TV(b_0) \leq C\cdot TV(u_0)$, gives:

$$\|u(T)-w^\varepsilon(T)\|_{L^1} \leq CTV(u_0)(\sqrt{\varepsilon T}+\varepsilon),$$

and this inequality, along with $|h(a)-b| \leq C|G(a,b)|$ and (10) gives in turn the result. □

Denoting $\rho = -\partial_x u$, the convergence of $u^\varepsilon(t)$ towards $u(t)$ in $L^1(\mathbb{R})$ ensures that $\rho(t)$ is a probability measure. Indeed, since for all $\varepsilon > 0$, $\rho^\varepsilon = -\partial_x u^\varepsilon$ is a non-negative distribution, so is ρ. The Riesz–Markov theorem then ensures that ρ can be represented by a non-negative Borel measure. In addition, almost everywhere, for $t \geq 0$, $u^\varepsilon(t)$ is a non-increasing function taking values in $[0,1]$ and hence converges to a certain limit when x goes to $+\infty$. The same holds true for the limit function $u(t)$. However, since $u^\varepsilon(t)-u(t) \in L^1(\mathbb{R})$, then $u^\varepsilon(t,x)-u(t,x)$ must vanish as x goes to $+\infty$. Therefore, the total mass of $\rho(t)$ is 1.

Then, passing to the relaxation system (2) for the aggregation Equation (1) can be done by using (6) with $p=1$. As a consequence, Theorem 1 translates as follows for the aggregation.

Theorem 2. *Let $\rho_0 \in \mathcal{P}_2(\mathbb{R})$, $c > 1/2$ and set $\sigma_0 = a[\rho_0]\rho_0$. There exists a constant $C > 0$ such that, for any $\varepsilon > 0$, denoting $(\rho^\varepsilon, \sigma^\varepsilon)$ the solution to (2) with initial data (ρ_0, σ_0), one has:*

$$\forall T > 0, \qquad W_1(\rho(T), \rho^\varepsilon(T)) \leq C(\sqrt{\varepsilon T}+\varepsilon),$$

where $\rho \in C([0,+\infty), \mathcal{P}_2(\mathbb{R}))$ is the unique solution (1) with initial datum ρ_0.

3. Numerical Discretization

Hereafter, we denote Δt the time step and we introduce a Cartesian mesh of size Δx. We denote $t^n = n\Delta t$ for $n \in \mathbb{N}$ and $x_j = j\Delta x$ for $j \in \mathbb{Z}$. In this section, we extend our framework and consider the aggregation Equation (1) with arbitrary pointy potentials W, which satisfy the following conditions:

(i) W is even and $W(0)=0$;
(ii) $W \in \mathcal{C}^1(\mathbb{R}\setminus\{0\})$;
(iii) W is λ-convex, i.e., there exists $\lambda \in \mathbb{R}$ such that $W(x) - \lambda\frac{|x|^2}{2}$ is convex;
(iv) W is a_∞-lipschitz continuous for some $a_\infty \geq 0$.

In this framework, the convergence of ρ^ε towards ρ for a slightly different problem has also been studied in [7]. Adapting the argument, the convergence still holds provided the sub-characteristic condition $c > a_\infty$ is verified. However, for such general potentials, the authors were not able to obtain the estimates of the speed of convergence as stated in Theorem 2.

In this section, we propose some numerical schemes able to capture the limit $\varepsilon \to 0$, thus satisfying the so-called asymptotic preserving (AP) property. We consider two approaches, the first one based on a splitting algorithm, and the second one based on a well-balanced discretization.

3.1. A Splitting Algorithm

A first simple approach to discretize the system (2) is to use a splitting method. Such a method is known to be convergent and easy to implement but introduces numerical diffusion.

Notice that the system (2) rewrites, with $\mu = \sigma - c\rho$, $\nu = \sigma + c\rho$, as

$$\partial_t \mu - c\partial_x \mu = \frac{1}{\varepsilon}\Big(a\Big[\frac{\nu-\mu}{2c}\Big]\Big(\frac{\nu-\mu}{2c}\Big) - \frac{\mu+\nu}{2}\Big) \tag{22a}$$

$$\partial_t \nu + c\partial_x \nu = \frac{1}{\varepsilon}\Big(a\Big[\frac{\nu-\mu}{2c}\Big]\Big(\frac{\nu-\mu}{2c}\Big) - \frac{\mu+\nu}{2}\Big). \tag{22b}$$

The idea of the method is to solve in a first step on $(t^n, t^n + \Delta t)$ the system:

$$\partial_t \mu = \frac{1}{\varepsilon}\Big(a\Big[\frac{\nu-\mu}{2c}\Big]\Big(\frac{\nu-\mu}{2c}\Big) - \frac{\mu+\nu}{2}\Big)$$
$$\partial_t \nu = \frac{1}{\varepsilon}\Big(a\Big[\frac{\nu-\mu}{2c}\Big]\Big(\frac{\nu-\mu}{2c}\Big) - \frac{\mu+\nu}{2}\Big),$$

with initial data $(\mu(t^n), \nu(t^n)) = (\mu^n, \nu^n)$. We obtain $\mu_j^{n+\frac12} = \mu(t^n + \Delta t, x_j)$ and $\nu_j^{n+\frac12} = \nu(t^n + \Delta t, x_j)$. Notice that this system may be solved explicitly. Indeed, by adding and subtracting the two equations, we deduce after an integration:

$$\nu_j^{n+\frac12} - \mu_j^{n+\frac12} = \nu_j^n - \mu_j^n \tag{23a}$$

$$\mu_j^{n+\frac12} + \nu_j^{n+\frac12} = (\mu_j^n + \nu_j^n)e^{-\Delta t/\varepsilon} + a\Big[\frac{\nu^n-\mu^n}{2c}\Big]\Big(\frac{\nu^n-\mu^n}{2c}\Big)(1 - e^{-\Delta t/\varepsilon}). \tag{23b}$$

Then, in a second step, we discretize by a classical finite volume upwind scheme the system:

$$\partial_t \mu - c\partial_x \mu = 0, \qquad \partial_t \nu + c\partial_x \nu = 0.$$

That is:

$$\mu_j^{n+1} = \mu_j^{n+\frac12} + c\frac{\Delta t}{\Delta x}(\mu_{j+1}^{n+\frac12} - \mu_j^{n+\frac12}), \tag{24a}$$

$$\nu_j^{n+1} = \nu_j^{n+\frac12} - c\frac{\Delta t}{\Delta x}(\nu_j^{n+\frac12} - \nu_{j-1}^{n+\frac12}). \tag{24b}$$

Coming back to the variables ρ and σ, we obtain:

$$\nu_j^{n+\frac12} = c\rho_j^n + \sigma_j^n e^{-\Delta x/\varepsilon} + a_j^n \rho_j^n (1 - e^{-\Delta t/\varepsilon}),$$
$$\mu_j^{n+\frac12} = -c\rho_j^n + \sigma_j^n e^{-\Delta x/\varepsilon} + a_j^n \rho_j^n (1 - e^{-\Delta t/\varepsilon}),$$

with $a_j^n = -\sum_{k\neq j} W'(x_j - x_k)\rho_k^n$. Then, the splitting algorithm reads:

$$\begin{aligned}
\rho_j^{n+1} &= \rho_j^n - \frac{1}{2}\frac{\Delta t}{\Delta x}(\mu_{j+1}^{n+\frac{1}{2}} + \nu_j^{n+\frac{1}{2}} - \mu_j^{n+\frac{1}{2}} - \nu_{j-1}^{n+\frac{1}{2}}) \\
&= \rho_j^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\Big((\sigma_{j+1}^n - \sigma_{j-1}^n)e^{-\Delta t/\varepsilon} \\
&\quad + (1 - e^{-\Delta t/\varepsilon})(a_{j+1}^n\rho_{j+1}^n - a_{j-1}^n\rho_{j-1}^n) - c(\rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n)\Big),
\end{aligned} \tag{25}$$

and:

$$\begin{aligned}
\sigma_j^{n+1} &= \sigma_j^{n+\frac{1}{2}} + \frac{c}{2}\frac{\Delta t}{\Delta x}(\sigma_{j+1}^n - 2\sigma_j^n + \sigma_{j-1}^n)e^{-\Delta t/\varepsilon} \\
&\quad + \frac{c}{2}\frac{\Delta t}{\Delta x}\Big((a_{j+1}^n\rho_{j+1}^n - 2a_j^n\rho_j^n + a_{j-1}^n\rho_{j-1}^n)(1 - e^{-\Delta t/\varepsilon}) - c(\rho_{j+1}^n - \rho_{j-1}^n)\Big) \\
&= \sigma_j^n e^{-\Delta t/\varepsilon} + a_j^n\rho_j^n(1 - e^{-\Delta t/\varepsilon}) + \frac{c}{2}\frac{\Delta t}{\Delta x}(\sigma_{j+1}^n - 2\sigma_j^n + \sigma_{j-1}^n)e^{-\Delta t/\varepsilon} \\
&\quad + \frac{c}{2}\frac{\Delta t}{\Delta x}\Big((a_{j+1}^n\rho_{j+1}^n - 2a_j^n\rho_j^n + a_{j-1}^n\rho_{j-1}^n)(1 - e^{-\Delta t/\varepsilon}) - c(\rho_{j+1}^n - \rho_{j-1}^n)\Big).
\end{aligned} \tag{26}$$

Lemma 1. *For any $\varepsilon > 0$, if both the CFL condition $\frac{c\Delta t}{\Delta x} \leq 1$ and the subcharacteristic condition $c \geq a_\infty$ hold, then the splitting scheme (23) and (24) is L^1-stable:*

$$\forall n \in \mathbb{N}, \qquad \sum_{j\in\mathbb{Z}}\Big(|\mu_j^{n+1}| + |\nu_j^{n+1}|\Big) \leq \sum_{j\in\mathbb{Z}}\Big(|\mu_j^n| + |\nu_j^n|\Big).$$

Proof. We have:

$$\begin{aligned}
\mu_j^{n+\frac{1}{2}} &= \frac{1}{2}\left(e^{-\Delta t/\varepsilon}\Big(1 + \frac{a_j^n}{c}\Big) + 1 - \frac{a_j^n}{c}\right)\mu_j^n - \frac{1 - e^{-\Delta t/\varepsilon}}{2}\left(1 - \frac{a_j^n}{c}\right)\nu_j^n, \\
\nu_j^{n+\frac{1}{2}} &= -\frac{1 - e^{-\Delta t/\varepsilon}}{2}\left(1 + \frac{a_j^n}{c}\right)\mu_j^n + \frac{1}{2}\left(e^{-\Delta t/\varepsilon}\Big(1 - \frac{a_j^n}{c}\Big) + 1 + \frac{a_j^n}{c}\right)\nu_j^n.
\end{aligned}$$

Under the condition $c \geq a_\infty$, in the expression of $\mu_j^{n+\frac{1}{2}}$, the coefficient in front of μ_j^n is non-negative and the one in front of ν_j^n is non-positive. Similarly, in $\nu_j^{n+\frac{1}{2}}$, the coefficient of μ_j^n is non-positive and the one in front of ν_j^n is non-negative. Taking the absolute value and adding up therefore yields:

$$\left|\mu_j^{n+\frac{1}{2}}\right| + \left|\nu_j^{n+\frac{1}{2}}\right| \leq \left|\mu_j^n\right| + \left|\nu_j^n\right|.$$

It remains to remark that, provided the CFL condition $\frac{c\Delta t}{\Delta x} \leq 1$ is verified, (24) gives:

$$\begin{aligned}
\sum_{j\in\mathbb{Z}}\Big(|\mu_j^{n+1}| + |\nu_j^{n+1}|\Big) &\leq \left(1 - \frac{c\Delta t}{\Delta x}\right)\sum_{j\in\mathbb{Z}}\left(\left|\mu_j^{n+\frac{1}{2}}\right| + \left|\nu_j^{n+\frac{1}{2}}\right|\right) \\
&\quad + \frac{c\Delta t}{\Delta x}\sum_{j\in\mathbb{Z}}\left|\mu_{j+1}^{n+\frac{1}{2}}\right| + \frac{c\Delta t}{\Delta x}\sum_{j\in\mathbb{Z}}\left|\nu_{j-1}^{n+\frac{1}{2}}\right|, \\
&\leq \left(1 - \frac{c\Delta t}{\Delta x}\right)\sum_{j\in\mathbb{Z}}\Big(|\mu_j^n| + |\nu_j^n|\Big) + \frac{c\Delta t}{\Delta x}\sum_{j\in\mathbb{Z}}\left|\mu_j^{n+\frac{1}{2}}\right| + \frac{c\Delta t}{\Delta x}\sum_{j\in\mathbb{Z}}\left|\nu_j^{n+\frac{1}{2}}\right|, \\
&\leq \sum_{j\in\mathbb{Z}}\Big(|\mu_j^n| + |\nu_j^n|\Big).
\end{aligned}$$

□

Note that similar schemes have also been studied in [31] and proved convergent at a rate of $\sqrt{\Delta x}$.

Let us now verify the AP property. When $\varepsilon \to 0$, we verify that the equation on ρ (25) converges to the following Rusanov discretization of (1) (see [21] for numerical simulations using the Rusanov scheme):

$$\rho_j^{n+1} = \rho_j^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(a_{j+1}^n \rho_{j+1}^n - a_{j-1}^n \rho_{j-1}^n\right) + \frac{c\Delta t}{2\Delta x}(\rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n), \tag{27a}$$

$$a_j^n = -\sum_{k \neq j} W'(x_j - x_k)\rho_k^n. \tag{27b}$$

This limiting scheme provides a consistent discretization of (1). Indeed, a similar scheme has been extensively studied in [11] using compactness arguments and the following convergence result was proven:

Lemma 2. *Assume $\rho_0 \in \mathcal{P}_2(\mathbb{R})$ and that the stability conditions $c\frac{\Delta t}{\Delta x} \leq 1$ and $c \geq a_\infty$ are satisfied. Let $T > 0$ and suppose we initialize the scheme (27) with $\rho_j^0 = \dfrac{1}{\Delta x}\rho_0(C_j)$ where $C_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$. Then, denoting $\rho_{\Delta x}$ the reconstruction given by the scheme (27), that is:*

$$\rho_{\Delta x}(t) = \sum_{n \in \mathbb{N}} \sum_{j \in \mathbb{Z}} \rho_j^n \mathbb{1}_{[t^n, t^{n+1})}(t)\delta_{x_j},$$

then $\rho_{\Delta x}$ converges weakly in the sense of measures on $[0, T] \times \mathbb{R}$ towards the solution ρ of Equation (1), as Δx goes to 0.

It has been also proven in [32] that the scheme (27) converges at a rate of $\sqrt{\Delta x}$.

3.2. Well-Balanced Discretization

Although the splitting method provides a simple way to obtain a discretization which is uniform with respect to the parameter ε, the resulting scheme has strong numerical diffusion and may not have good large time behavior. Then, well-balanced schemes have been introduced. A scheme is said to be well-balanced when it conserves equilibria. The method proposed in this section comes from [20].

Let us assume that, for some $n \in \mathbb{N}$, the approximation $(\mu_j^n, \nu_j^n)_{j\in\mathbb{Z}}$ of $(\mu(t^n, x_j), \nu(t^n, x_j))_{j\in\mathbb{Z}}$ solution of (22) is known. We construct an approximation at time t^{n+1} using a finite volume upwind discretization of (22), with the discretization of the source terms $H_{\mu,j}^n, H_{\nu,j}^n$ to be prescribed right afterwards:

$$\mu_j^{n+1} = \mu_j^n + c\frac{\Delta t}{\Delta x}(\mu_{j+1}^n - \mu_j^n) + \frac{\Delta t}{\varepsilon}H_{\mu,j}^n \tag{28a}$$

$$\nu_j^{n+1} = \nu_j^n - c\frac{\Delta t}{\Delta x}(\nu_j^n - \nu_{j-1}^n) + \frac{\Delta t}{\varepsilon}H_{\nu,j}^n. \tag{28b}$$

In order to preserve equilibria, we set :

$$H_{\mu,j}^n = \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} H(\overline{\mu}, \overline{\nu})\,\mathrm{d}x, \qquad H(\mu,\nu) = a\left[\frac{\nu-\mu}{2c}\right]\left(\frac{\nu-\mu}{2c}\right) - \frac{\mu+\nu}{2}, \tag{29}$$

where $(\overline{\mu}, \overline{\nu})$ solve the stationary system with incoming boundary conditions, on (x_{j-1}, x_j):

$$-c\partial_x\overline{\mu} = \frac{1}{\varepsilon}H(\overline{\mu},\overline{\nu}) \tag{30a}$$
$$c\partial_x\overline{\nu} = \frac{1}{\varepsilon}H(\overline{\mu},\overline{\nu}) \tag{30b}$$
$$\overline{\mu}(x_j) = \mu_j^n, \qquad \overline{\nu}(x_{j-1}) = \nu_{j-1}^n. \tag{30c}$$

In addition, in the same fashion, $H_{\nu,j}^n = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} H(\tilde{\mu},\tilde{\nu})\,\mathrm{d}x$, where $(\tilde{\mu},\tilde{\nu})$ is the solution of the stationary system on (x_j, x_{j+1}):

$$-c\partial_x\tilde{\mu} = \frac{1}{\varepsilon}H(\tilde{\mu},\tilde{\nu}) \tag{31a}$$
$$c\partial_x\tilde{\mu} = \frac{1}{\varepsilon}H(\tilde{\mu},\tilde{\nu}) \tag{31b}$$
$$\tilde{\mu}(x_{j+1}) = \mu_{j+1}^n, \qquad \tilde{\nu}(x_j) = \nu_j^n, \tag{31c}$$

Reporting Equations (30b) and (31a) into the discretization of the source term, we obtain $H_{\nu,j}^n = \frac{c\varepsilon}{\Delta x}(\overline{\nu}(x_j) - \nu_{j-1})$ and $H_{\mu,j}^n = -\frac{c\varepsilon}{\Delta x}(\mu_j^n - \tilde{\mu}(x_j))$. Hence, one may rewrite the scheme (28) as

$$\mu_j^{n+1} = \mu_j^n + c\frac{\Delta t}{\Delta x}(\tilde{\mu}(x_j) - \mu_j^n) \tag{32a}$$
$$\nu_j^{n+1} = \nu_j^n - c\frac{\Delta t}{\Delta x}(\nu_j^n - \overline{\nu}(x_j)). \tag{32b}$$

Remark that the stationary system:

$$-c\partial_x\mu = \frac{1}{\varepsilon}H(\mu,\nu), \qquad c\partial_x\nu = \frac{1}{\varepsilon}H(\mu,\nu), \tag{33}$$

is equivalent to:

$$\partial_x\sigma = 0, \qquad c^2\partial_x\rho = \frac{1}{\varepsilon}(a[\rho]\rho - \sigma). \tag{34}$$

Therefore, denoting $\sigma_{j+\frac{1}{2}} = \dfrac{\tilde{\mu}+\tilde{\nu}}{2}$ and $\sigma_{j-\frac{1}{2}} = \dfrac{\overline{\mu}+\overline{\nu}}{2}$, which are constant, respectively, on (x_j, x_{j+1}) and (x_{j-1}, x_j), one has:

$$\tilde{\mu}(x_j) = 2\sigma_{j+\frac{1}{2}} - \nu_j^n, \qquad \overline{\nu}(x_j) = 2\sigma_{j-\frac{1}{2}} - \mu_j^n. \tag{35}$$

Thus, it turns out that the scheme can be rewritten only in terms of the discretized unknowns and of $\sigma_{j\pm\frac{1}{2}}$:

$$\mu_j^{n+1} = \mu_j^n - c\frac{\Delta t}{\Delta x}(\mu_j^n + \nu_j^n) + \frac{2c\Delta t}{\Delta x}\sigma_{j+\frac{1}{2}}, \tag{36a}$$
$$\nu_j^{n+1} = \nu_j^n - c\frac{\Delta t}{\Delta x}(\mu_j^n + \nu_j^n) + \frac{2c\Delta t}{\Delta x}\sigma_{j-\frac{1}{2}}. \tag{36b}$$

Or equivalently:

$$\rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{\Delta x}(\sigma_{j+\frac{1}{2}} - \sigma_{j-\frac{1}{2}}), \tag{37a}$$
$$\sigma_j^{n+1} = \sigma_j^n - c\frac{\Delta t}{\Delta x}(2\sigma_j^n - \sigma_{j+\frac{1}{2}} - \sigma_{j-\frac{1}{2}}). \tag{37b}$$

However, solving the stationary systems (30) and (31) involves the resolution of a nonlinear and nonlocal ODE. Instead, we propose an approximation in the spirit of [20].

We replace the nonlinear term in (30a)–(30b) by $a^n_{j-\frac{1}{2}} \cdot \frac{\bar{\nu}-\bar{\mu}}{2c}$, where $a^n_{j-\frac{1}{2}}$ stands for a fixed and consistent discretization of $a\left[\frac{\bar{\nu}-\bar{\mu}}{2c}\right]$ on the interval (x_{j-1}, x_j), to be specified afterwards. Similarly, we will replace the nonlinear term in (31a)–(31b) by $a^n_{j+\frac{1}{2}} \cdot \frac{\tilde{\nu}-\tilde{\mu}}{2c}$ with $a^n_{j+\frac{1}{2}}$ defined accordingly. In the following, we detail the construction for the problem (30a)–(30b) on (x_{j-1}, x_j).

Obviously, the definition of $a^n_{j-\frac{1}{2}}$ should be taken with care [11,20]. In [32], the authors showed that, when discretizing the product $a[\rho]\rho$, if $a[\rho]$ and ρ were not evaluated at the same point, then the resulting scheme produces the wrong dynamics. To take this into account, we will split ρ into one contribution coming from the left and one contribution coming from the right, i.e., we set $\bar{\rho} = \rho_L + \rho_R$ and $\bar{\sigma} = \sigma_L + \sigma_R$ where $\rho_L(\Delta x) = 0$ and $\rho_R(0) = 0$. This implies that $\bar{\rho}(\Delta x) = \rho_R(\Delta x)$ and $\bar{\rho}(0) = \rho_L(0)$.

More precisely, we solve the two following boundary value problem, on $(0, \Delta x)$:

$$\varepsilon c^2 \frac{d}{dx}\rho_L = a^n_{j-\frac{1}{2},L}\rho_L - \sigma_L, \qquad \rho_L(\Delta x) = 0, \tag{38a}$$

$$\varepsilon c^2 \frac{d}{dx}\rho_R = a^n_{j-\frac{1}{2},R}\rho_R - \sigma_R, \qquad \rho_R(0) = 0, \tag{38b}$$

We may explicitly solve these linear systems, and since $\rho_L(0) = \bar{\rho}(0)$ and $\rho_R(\Delta x) = \bar{\rho}(\Delta x)$, we obtain the relations:

$$\sigma_L = \bar{\rho}(0)\kappa^n_{j-\frac{1}{2},L}, \qquad \sigma_R = \bar{\rho}(\Delta x)\kappa^n_{j-\frac{1}{2},R}. \tag{39}$$

with:

$$\kappa^n_{j-\frac{1}{2},L} = \frac{a^n_{j-\frac{1}{2},L}}{1-\exp(-a^n_{j-\frac{1}{2},L}\Delta x/(\varepsilon c^2))}, \qquad \kappa^n_{j-\frac{1}{2},R} = \frac{a^n_{j-\frac{1}{2},R}}{1-\exp(a^n_{j-\frac{1}{2},R}\Delta x/(\varepsilon c^2))}. \tag{40}$$

Notice that we have:

$$\kappa^n_{j-\frac{1}{2},L} \to (a^n_{j-\frac{1}{2},L})_+, \qquad \kappa^n_{j-\frac{1}{2},R} \to -(a^n_{j-\frac{1}{2},R})_-, \qquad \text{when } \varepsilon \to 0, \tag{41}$$

where we denote $a_+ = \max(0, a) \geq 0$ and $a_- = \max(0, -a) \geq 0$—the positive and negative negative part of a. Using the boundary conditions in (30), we have:

$$\bar{\rho}(0) = \frac{\nu^n_{j-1} - \bar{\mu}(0)}{2c}, \qquad \bar{\rho}(\Delta x) = \frac{\bar{\nu}(\Delta x) - \mu^n_j}{2c}. \tag{42}$$

with (39) and the fact that $\bar{\sigma} = \sigma_L + \sigma_R$ is constant on $[0, \Delta x]$, we obtain the following 2×2 system on the unknowns $\bar{\mu}(0), \bar{\nu}(\Delta x)$:

$$\mu^n_j + \bar{\nu}(\Delta x) = \bar{\mu}(0) + \nu^n_{j-1}, \tag{43a}$$

$$\mu^n_j + \bar{\nu}(\Delta x) = \frac{\nu^n_{j-1} - \bar{\mu}(0)}{2c}\kappa^n_{j-\frac{1}{2},L} + \frac{\bar{\nu}(\Delta x) - \mu^n_j}{2c}\kappa^n_{j-\frac{1}{2},R} \tag{43b}$$

Solving this system yields:

$$\bar{\mu}(0) = -\nu^n_{j-1}\frac{c - \kappa^n_{j-\frac{1}{2},R} - \kappa^n_{j-\frac{1}{2},L}}{c - \kappa^n_{j-\frac{1}{2},R} + \kappa^n_{j-\frac{1}{2},L}} - \mu^n_j \frac{\kappa^n_{j-\frac{1}{2},R}}{c - \kappa^n_{j-\frac{1}{2},R} + \kappa^n_{j-\frac{1}{2},L}}, \tag{44a}$$

$$\bar{\nu}(\Delta x) = \nu^n_{j-1}\frac{\kappa^n_{j-\frac{1}{2},L}}{c - \kappa^n_{j-\frac{1}{2},R} + \kappa^n_{j-\frac{1}{2},L}} - \mu^n_j \frac{c + \kappa^n_{j-\frac{1}{2},R} + \kappa^n_{j-\frac{1}{2},L}}{c - \kappa^n_{j-\frac{1}{2},R} + \kappa^n_{j-\frac{1}{2},L}}. \tag{44b}$$

From which we deduce with (42):

$$\rho^n_{j-\frac{1}{2},L} := \overline{\rho}(0) = \frac{1}{c}\left(\frac{(c-\kappa^n_{j-\frac{1}{2},R})\nu^n_{j-1} + \kappa^n_{j-\frac{1}{2},R}\mu^n_j}{c+\kappa^n_{j-\frac{1}{2},L} - \kappa_{j-\frac{1}{2},R}}\right) \tag{45a}$$

$$\rho^n_{j-\frac{1}{2},R} := \overline{\rho}(\Delta x) = \frac{1}{c}\left(\frac{\kappa^n_{j-\frac{1}{2},L}\nu^n_{j-1} - (c+\kappa^n_{j-\frac{1}{2},L})\mu^n_j}{c+\kappa^n_{j-\frac{1}{2},L} - \kappa_{j-\frac{1}{2},R}}\right) \tag{45b}$$

and with (39):

$$\overline{\sigma}_{j-\frac{1}{2}} := \sigma_L + \sigma_R = \rho^n_{j-\frac{1}{2},L}\kappa^n_{j-\frac{1}{2},L} + \rho^n_{j-\frac{1}{2},R}\kappa^n_{j-\frac{1}{2},R} = \frac{\nu^n_{j-1}\kappa^n_{j-\frac{1}{2},L} - \mu^n_j\kappa^n_{j-\frac{1}{2},R}}{c-\kappa^n_{j-\frac{1}{2},R} + \kappa^n_{j-\frac{1}{2},L}}, \tag{46}$$

(the above quantities are well-defined since $\kappa^n_{j-\frac{1}{2},L} \geq 0$ and $\kappa^n_{j-\frac{1}{2},R} \leq 0$). Injecting into (37), it gives the following scheme:

$$\mu^{n+1}_j = \left(1 - \frac{c\Delta t}{\Delta x}\right)\mu^n_j - \frac{c\Delta t}{\Delta x}\frac{c-\kappa^n_{j+\frac{1}{2},R} - \kappa^n_{j+\frac{1}{2},L}}{c-\kappa^n_{j+\frac{1}{2},R} + \kappa^n_{j+\frac{1}{2},L}}\nu^n_j - \frac{2c\Delta t}{\Delta x}\frac{\kappa^n_{j+\frac{1}{2},R}}{c-\kappa^n_{j+\frac{1}{2},R} + \kappa^n_{j+\frac{1}{2},L}}\mu^n_{j+1}, \tag{47a}$$

$$\nu^{n+1}_j = \left(1 - \frac{c\Delta t}{\Delta x}\right)\nu^n_j - \frac{c\Delta t}{\Delta x}\frac{c+\kappa^n_{j-\frac{1}{2},R} + \kappa^n_{j-\frac{1}{2},L}}{c-\kappa^n_{j-\frac{1}{2},R} + \kappa^n_{j-\frac{1}{2},L}}\mu^n_j + \frac{2c\Delta t}{\Delta x}\frac{\kappa^n_{j-\frac{1}{2},L}}{c-\kappa^n_{j-\frac{1}{2},R} + \kappa^n_{j-\frac{1}{2},L}}\nu^n_{j-1}, \tag{47b}$$

where the coefficients $\kappa^n_{j-\frac{1}{2},L/R}$ are defined in (40). Equivalently, for the variable (ρ,σ), the scheme reads:

$$\rho^{n+1}_j = \rho^n_j - \frac{\Delta t}{\Delta x}\left(\frac{\nu^n_j\kappa^n_{j+\frac{1}{2},L} - \mu^n_{j+1}\kappa^n_{j+\frac{1}{2},R}}{c-\kappa^n_{j+\frac{1}{2},R} + \kappa^n_{j+\frac{1}{2},L}} - \frac{\nu^n_{j-1}\kappa^n_{j-\frac{1}{2},L} - \mu^n_j\kappa^n_{j-\frac{1}{2},R}}{c-\kappa^n_{j-\frac{1}{2},R} + \kappa^n_{j-\frac{1}{2},L}}\right) \tag{48a}$$

$$\sigma^{n+1}_j = \sigma^n_j - c\frac{\Delta t}{\Delta x}\left(2\sigma^n_j - \frac{\nu^n_j\kappa^n_{j+\frac{1}{2},L} - \mu^n_{j+1}\kappa^n_{j+\frac{1}{2},R}}{c-\kappa^n_{j+\frac{1}{2},R} + \kappa^n_{j+\frac{1}{2},L}} - \frac{\nu^n_{j-1}\kappa^n_{j-\frac{1}{2},L} - \mu^n_j\kappa^n_{j-\frac{1}{2},R}}{c-\kappa^n_{j-\frac{1}{2},R} + \kappa^n_{j-\frac{1}{2},L}}\right), \tag{48b}$$

where we recall that $\mu^n_j = \sigma^n_j - c\rho^n_j$ and $\nu^n_j = \sigma^n_j + c\rho^n_j$.

It remains to define the velocities $a^n_{j-\frac{1}{2},L/R}$ used in (38) and in (40). We take:

$$a^n_{j-\frac{1}{2},L/R} = -\sum_{k\neq j} W'(x_j - x_k)\rho^n_{k-\frac{1}{2},L/R}.$$

However, this discretization implies the resolution of a nonlinear problem, since the quantities $\rho^n_{k-\frac{1}{2},L/R}$ depends nonlinearly on $a^n_{j-\frac{1}{2},L/R}$.

Then, we implement a fixed point method initialized with $a^{n,(0)}_{j-\frac{1}{2},L} := a^n_{j-1}$ and $a^{n,(0)}_{j-\frac{1}{2},R} := a^n_j$. Solving, on each cell (x_{j-1}, x_j), the system of ODEs (38) with these values for the velocities gives two sequences, $(\rho^{(1)}_{j-\frac{1}{2},L})_{j\in\mathbb{Z}}$ and $(\rho^{(1)}_{j-\frac{1}{2},R})_{j\in\mathbb{Z}}$. Then, we assign the next value of the velocity to $a^{n,(1)}_{j-\frac{1}{2},L/R} := -\sum_{k\neq j} W'(x_j - x_k)\rho^{(1)}_{k-\frac{1}{2},L/R}$, which allows us to compute new values for the left and right densities, $(\rho^{(2)}_{j-\frac{1}{2},L})_{j\in\mathbb{Z}}$ and $(\rho^{(2)}_{j-\frac{1}{2},R})_{j\in\mathbb{Z}}$, through (38). We iterate until $W_2(\rho^{(i)}_L, \rho^{(i+1)}_L)$ and $W_2(\rho^{(i)}_R, \rho^{(i+1)}_R)$ pass below a certain threshold. Notice that the velocities $a^{n,(i)}_{j-\frac{1}{2},L/R}$ always remain bounded by a_∞. In practice, only a few iterations are needed.

The resulting scheme is consistent for any $\varepsilon > 0$ and stable under standard stability conditions, as shown by the following lemmas.

Lemma 3 (L^1 stability)**.** *Under the CFL condition $\frac{c\Delta t}{\Delta x} \leq 1$ and the subcharacteristic condition $c \geq a_\infty$, there holds that the sequence $(\mu_j^n, \nu_j^n)_{j,n}$ defined by the scheme (47), verifies the following L^1 stability property:*

$$\forall n \in \mathbb{N}, \qquad \sum_{j\in\mathbb{Z}} \Big(|\mu_j^{n+1}| + |\nu_j^{n+1}|\Big) \leq \sum_{j\in\mathbb{Z}} \Big(|\mu_j^n| + |\nu_j^n|\Big).$$

Proof. In each combination of (47), the first coefficient is non-negative under the CFL condition $\frac{c\Delta t}{\Delta x} \leq 1$, and so is the last one since $\kappa^n_{j\pm\frac12,L} \geq 0$ and $\kappa^n_{j\pm\frac12,R} \leq 0$. Moreover, under the subcharacteristic condition $c \geq a_\infty$, it holds that $-c \leq \kappa_{j\pm\frac12,R} + \kappa_{j\pm\frac12,R} \leq c$ so the remaining coefficient is non-positive. Thus, applying the triangle inequality and re-indexing the sums appropriately:

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \Big(|\mu_j^{n+1}| + |\nu_j^{n+1}|\Big) &\leq \sum_{j\in\mathbb{Z}} \Big(1 - \frac{c\Delta t}{\Delta x}\Big)|\mu_j^n| + \sum_{j\in\mathbb{Z}} \frac{c\Delta t}{\Delta x} \frac{c - \kappa^n_{j+\frac12,R} - \kappa^n_{j+\frac12,L}}{c - \kappa^n_{j+\frac12,R} + \kappa^n_{j+\frac12,L}} |\nu_j^n| \\
&\quad - \sum_{j\in\mathbb{Z}} \frac{2c\Delta t}{\Delta x} \frac{\kappa^n_{j+\frac12,R}}{c - \kappa^n_{j+\frac12,R} + \kappa^n_{j+\frac12,L}} |\mu^n_{j+1}| + \sum_{j\in\mathbb{Z}} \Big(1 - \frac{c\Delta t}{\Delta x}\Big)|\nu_j^n| \\
&\quad + \sum_{j\in\mathbb{Z}} \frac{c\Delta t}{\Delta x} \frac{c + \kappa^n_{j+\frac12,R} + \kappa^n_{j+\frac12,L}}{c - \kappa^n_{j+\frac12,R} + \kappa^n_{j+\frac12,L}} |\mu^n_{j+1}| + \frac{2c\Delta t}{\Delta x} \frac{\kappa^n_{j+\frac12,L}}{c - \kappa^n_{j+\frac12,R} + \kappa^n_{j+\frac12,L}} |\nu_j^n|, \\
&\leq \Big(1 - \frac{c\Delta t}{\Delta x}\Big) \sum_{j\in\mathbb{Z}} \Big(|\mu_j^n| + |\nu_j^n|\Big) + \frac{c\Delta t}{\Delta x} \sum_{j\in\mathbb{Z}} |\mu^n_{j+1}| + \frac{c\Delta t}{\Delta x} \sum_{j\in\mathbb{Z}} |\nu_j^n|, \\
&\leq \sum_{j\in\mathbb{Z}} \Big(|\mu_j^n| + |\nu_j^n|\Big).
\end{aligned}$$

This concludes the proof. □

Lemma 4 (Consistency for smooth solutions)**.** *Assume that, for all $j \in \mathbb{Z}$, we have $a^n_{j-\frac12,L/R} = -\sum_{k\neq j} W'(x_j - x_k)\rho_{k-\frac12,L/R}$. Then, for any $\varepsilon > 0$, the scheme (37) is consistent with (2) provided that the solutions are smooth enough.*

Proof. For $j \in \mathbb{Z}$, one has, using the Taylor expansions as $\Delta x \to 0$:

$$\begin{aligned}
\frac{\kappa^n_{j-\frac12,L}}{c - \kappa^n_{j-\frac12,R} + \kappa^n_{j-\frac12,L}} &= \frac12 - \frac{1}{4\varepsilon c^2}\left(c - \frac{a^n_{j-\frac12,L} + a^n_{j-\frac12,R}}{2}\right)\Delta x + O(\Delta x^2), \\
\frac{\kappa^n_{j-\frac12,R}}{c - \kappa^n_{j-\frac12,R} + \kappa^n_{j-\frac12,L}} &= -\frac12 + \frac{1}{4\varepsilon c^2}\left(c + \frac{a^n_{j-\frac12,L} + a^n_{j-\frac12,R}}{2}\right)\Delta x + O(\Delta x^2).
\end{aligned}$$

Thus:

$$\begin{aligned}
\sigma_{j-\frac12} = \frac{\sigma^n_{j-1} + \sigma^n_j}{2} + c\frac{\rho^n_{j-1} - \rho^n_j}{2} - \frac{1}{4\varepsilon c^2}\Bigg(&\left(c - \frac{a^n_{j-\frac12,L} + a^n_{j-\frac12,R}}{2}\right)(\sigma^n_{j-1} + c\rho^n_{j-1}) \\
&+ \left(c + \frac{a^n_{j-\frac12,L} + a^n_{j-\frac12,R}}{2}\right)(\sigma^n_j - c\rho^n_j)\Bigg)\Delta x + O(\Delta x^2).
\end{aligned}$$

In particular, $\sigma_{j-\frac12}$ is clearly consistent with $\sigma(t^n, x_{j-\frac12})$ as long as the solution (ρ, σ) is smooth enough to perform standard consistency analysis for finite differences. This shows that (37a) is consistent with $\partial_t\rho + \partial_x\sigma = 0$. As for the consistency of (37b) with $\partial_t\sigma + c^2\partial_x\rho = \frac{1}{\varepsilon}(a[\rho]\rho - \sigma)$, we write:

$$\begin{aligned}\sigma_{j+\frac12} + \sigma_{j-\frac12} - 2\sigma_j^n &= \frac{\sigma_{j+1}^n - 2\sigma_j^n + \sigma_{j-1}^n}{2} + c\frac{\rho_{j-1}^n - \rho_{j+1}^n}{2} - \frac{\Delta x}{4\varepsilon c^2}\Bigg[c(\sigma_{j-1}^n + 2\sigma_j^n + \sigma_{j+1}^n) \\ &+ \frac{a_{j-\frac12,L}^n + a_{j-\frac12,R}^n}{2}(\sigma_j^n - \sigma_{j-1}^n) + \frac{a_{j+\frac12,L}^n + a_{j+\frac12,R}^n}{2}(\sigma_{j+1}^n - \sigma_j^n) + c^2(\rho_{j-1}^n - \rho_{j+1}^n) \\ &- c\left(\frac{a_{j-\frac12,L}^n + a_{j-\frac12,R}^n}{2}\rho_{j-1}^n + \frac{a_{j-\frac12,L}^n + a_{j-\frac12,R}^n + a_{j+\frac12,L}^n + a_{j+\frac12,R}^n}{2}\rho_j^n + \frac{a_{j+\frac12,L}^n + a_{j+\frac12,R}^n}{2}\rho_{j+1}^n\right)\Bigg] \\ &+ O(\Delta x^2).\end{aligned}$$

Using Taylor expansions, we have, for smooth solutions $\sigma(t^n, x_{j+1}) - 2\sigma(t^n, x_j) + \sigma(t^n, x_{j-1}) = O(\Delta x^2)$, $\rho(t^n, x_{j-1}) - \rho(t^n, x_{j+1}) = O(\Delta x)$, $\sigma(t^n, x_j) - \sigma(t^n, x_{j-1}) = O(\Delta x)$ and $\sigma(t^n, x_{j+1}) - \sigma(t^n, x_j) = O(\Delta x)$. Along with the bound $|a_{j\pm\frac12,L/R}^n| \le a_\infty$, this implies:

$$\begin{aligned}\sigma_{j+\frac12} + \sigma_{j-\frac12} - 2\sigma_j^n = c\frac{\rho_{j-1}^n - \rho_{j+1}^n}{2} - \frac{1}{4\varepsilon c^2}\Bigg[&c(\sigma_{j-1}^n + 2\sigma_j^n + \sigma_{j+1}^n) \\ &- c\left(\frac{a_{j-\frac12,L}^n + a_{j-\frac12,R}^n}{2}\rho_{j-1}^n + \frac{a_{j-\frac12,L}^n + a_{j-\frac12,R}^n + a_{j+\frac12,L}^n + a_{j+\frac12,R}^n}{2}\rho_j^n\right. \\ &\left. + \frac{a_{j+\frac12,L}^n + a_{j+\frac12,R}^n}{2}\rho_{j+1}^n\right)\Bigg]\Delta x + O(\Delta x^2).\end{aligned}$$

Clearly, $c\frac{\rho_{j-1}^n - \rho_{j+1}^n}{2}$ and $c(\sigma_{j-1}^n + 2\sigma_j^n + \sigma_{j+1}^n)$ are consistent with an accuracy of $O(\Delta x^2)$ and $O(\Delta x)$, respectively, with $-c\partial_x\rho(t^n, x_j)$ and $4c\sigma(t^n, x_j)$. For the remaining terms, let us recall that, with the notations of (42):

$$\rho_{j-\frac12,L} = \frac{\nu_{j-1}^n - \overline{\mu}(0)}{2c} = \frac{\nu_{j-1}^n - \sigma_{j-\frac12}}{c}, \qquad \rho_{j-\frac12,R} = \frac{\overline{\nu}(\Delta x) - \mu_j^n}{2c} = \frac{\sigma_{j-\frac12} - \mu_j}{c}.$$

Hence, $\rho_{j-\frac12,L} + \rho_{j-\frac12,R} = \dfrac{\nu_{j-1}^n - \mu_j^n}{c} = \dfrac{\sigma_{j-1}^n - \sigma_j^n}{c} + \rho_{j-1}^n + \rho_j^n$. Since $\sigma(t^n, x_{j-1}) - \sigma(t^n, x_j) = O(\Delta x)$, and assuming that:

$$a_{j-\frac12,L/R}^n = -\sum_{k\neq j} W'(x_j - x_k)\rho_{k-\frac12,L/R}$$

we deduce that $a_{j-\frac12,L}^n + a_{j-\frac12,R}^n$ is consistent with $a[\rho(t^n)](x_{j-1}) + a[\rho(t^n)](x_j)$ with accuracy $O(\Delta x)$. It follows that $\sigma_{j+\frac12} + \sigma_{j-\frac12} - 2\sigma_j^n$ is consistent with $-\partial_x\rho(t^n, x_j) - \frac{1}{\varepsilon}\Big(\sigma(t^n, x_j) - a[\rho(t^n)](x_j)\rho(t^n, x_j)\Big)$, again with accuracy $O(\Delta x)$, and this concludes the proof. □

The stability conditions in Lemma 3 are independent on ε, we recover in the limit $\varepsilon \to 0$, using (41), the scheme of [20]:

$$\rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{\Delta x}\left(\frac{\nu_j^n(a_{j+\frac12,L}^n)_+ + \mu_{j+1}^n(a_{j+\frac12,R}^n)_-}{c + (a_{j+\frac12,R}^n)_- + (a_{j+\frac12,L}^n)_+} - \frac{\nu_{j-1}^n(a_{j-\frac12,L}^n)_+ + \mu_j^n(a_{j-\frac12,R}^n)_-}{c + (a_{j-\frac12,R}^n)_- + (a_{j-\frac12,L}^n)_+}\right) \tag{49a}$$

$$\begin{aligned}\sigma_j^{n+1} = \sigma_j^n - c\frac{\Delta t}{\Delta x}\Bigg(2\sigma_j^n &- \frac{\nu_j^n(a_{j+\frac12,L}^n)_+ + \mu_{j+1}^n(a_{j+\frac12,R}^n)_-}{c + (a_{j+\frac12,R}^n)_- + (a_{j+\frac12,L}^n)_+} \\ &- \frac{\nu_{j-1}^n(a_{j-\frac12,L}^n)_+ + \mu_j^n(a_{j-\frac12,R}^n)_-}{c + (a_{j-\frac12,R}^n)_- + (a_{j-\frac12,L}^n)_+}\Bigg),\end{aligned} \tag{49b}$$

which is stable under the conditions $\frac{c\Delta t}{\Delta x} \leq 1$ and $c \geq a_\infty$. Notice that with the notation in (46), Equation (49a) may be rewritten as

$$\rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{\Delta x}\Big(\rho^n_{j+\frac{1}{2},L}(a^n_{j+\frac{1}{2},L})_+ - \rho^n_{j+\frac{1}{2},R}(a^n_{j+\frac{1}{2},R})_- \\ - \rho^n_{j-\frac{1}{2},L}(a^n_{j-\frac{1}{2},L})^+ + \rho^n_{j-\frac{1}{2},R}(a^n_{j-\frac{1}{2},R})_-\Big).$$

4. Numerical Experiments

We present some numerical illustrations for the two schemes described in the previous section. In addition to the potential $W(x) = \frac{|x|}{2}$, we also consider the smooth potential $W(x) = \frac{x^2}{2}$.

Numerical tests are conducted on the domain $[-1,1]$ with the inital data $\rho_0 = \frac{1}{2}\delta_{-0.5} + \frac{1}{2}\delta_{0.5}$, $\sigma_0 = a[\rho_0]\rho_0$ and both schemes are initialized with:

$$\rho_j^0 = \frac{1}{\Delta x}\rho_0(C_j), \qquad \sigma_j^0 = \frac{1}{\Delta x}\sigma_0(C_j).$$

Figure 1 shows that both schemes recover the correct dynamics in the limit $\varepsilon \to 0$: for the potential $W(x) = \frac{|x|}{2}$, one can compute the exact velocity of both Dirac masses for the aggregation Equation (1) and see that they should be located, respectively, in $x = -0.2$ and $x = 0.2$ in final time $T = 1.2$.

This test is set up with $\varepsilon = 10^{-7}$, on a Cartesian mesh of $[-1,1]$ with 1500 cells, $c = 1$ and the CFL $c\dfrac{\Delta t}{\Delta x} = 0.9$. Both schemes (27) and (49) display the correct velocity for the Dirac masses, but one can notice that the Rusanov scheme (27) shows more numerical diffusion. Note that both schemes are written in conservation form, they preserve the total mass of ρ, which is also verified numerically.

We then investigated the order of convergence when Δx goes to 0 with ε fixed, in Wasserstein distance W_1 (the numerical results are the same for W_2).

After performing tests for several values of ε, it appears that the convergence rate does not depend on the size of ε. Therefore, as an example, we propose simulations in final time $T = 0.5$, with the same intial data and stability parameters as above, and with $\varepsilon = 2 \times 10^{-6}$ for Figure 2 and with $\varepsilon = 10^{-2}$ for Figure 3:

For a fixed value of ε, both schemes seem to converge with order $1/2$ with respect to Δx for the smooth potential $W(x) = \frac{x^2}{2}$ (see Figure 2) whereas they seem to be of order 1 for the potential $W(x) = \frac{|x|}{2}$ (see Figure 3). This can be explained as both schemes possess some numerical diffusion which is somehow counterbalanced by the aggregation phenomenon in the case of a pointy potential, as already observed in [21]. Due to the link with the Burgers equation, this superconvergence phenomenon is directly linked to the results of Després [33], which should be rigorously extended to our case (the mere extension to the upwind scheme of [11] for the aggregation is not straightforward).

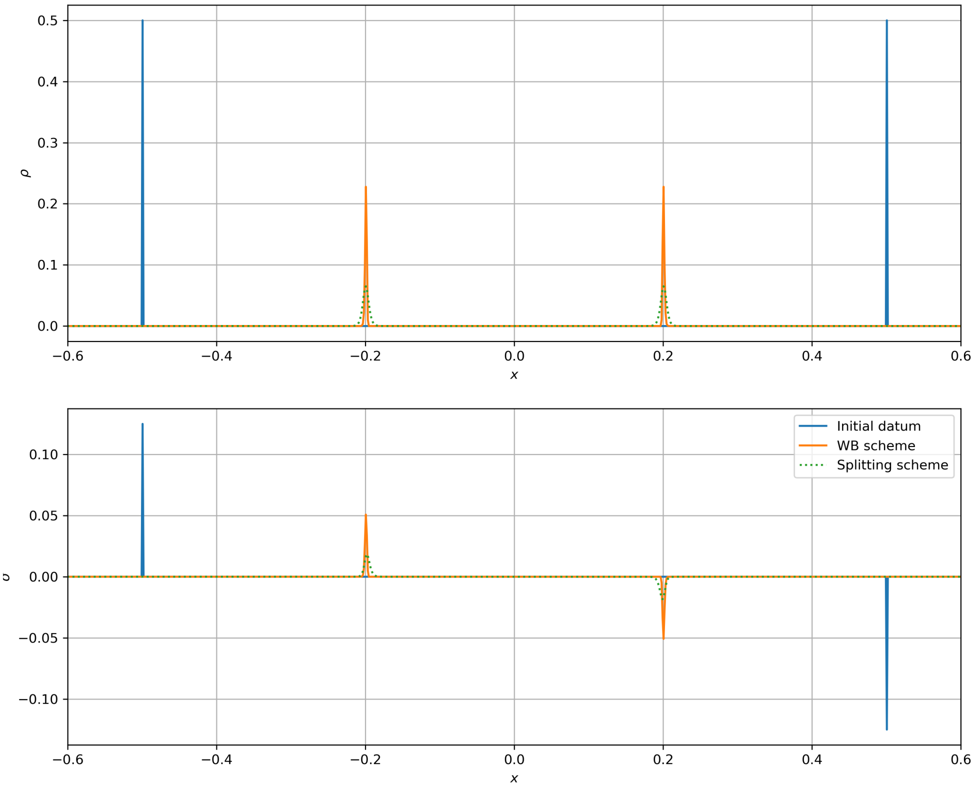

Figure 1. Dynamics of two Dirac masses for the potential $W(x) = \frac{|x|}{2}$ in time $T = 1.2$.

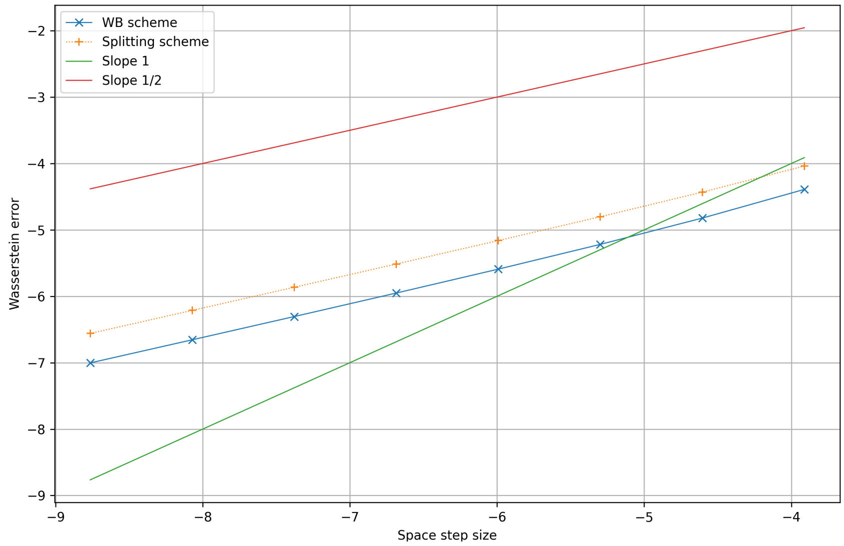

Figure 2. Order of convergence of the splitting scheme and the well-balanced scheme for the smooth potential $W(x) = \frac{x^2}{2}$.

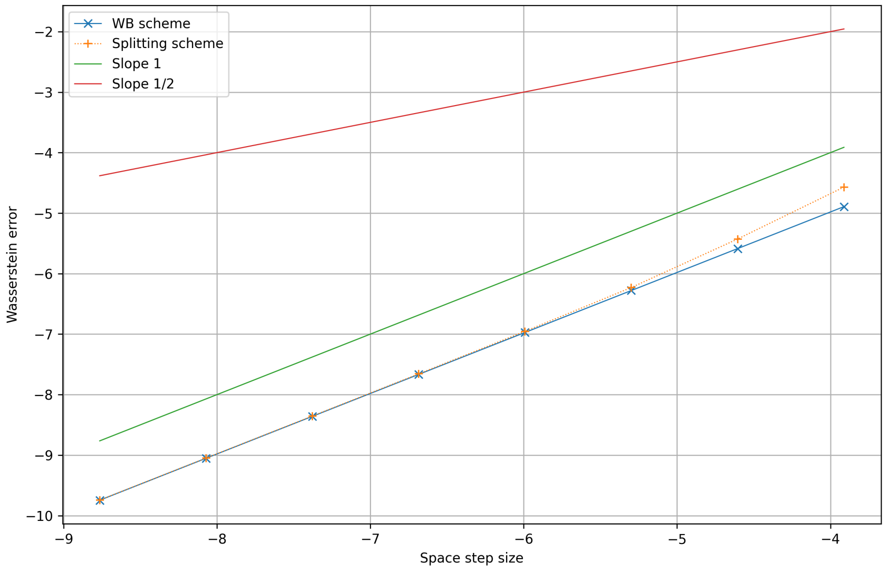

Figure 3. Order of convergence of the splitting scheme and the well-balanced scheme for the pointy potential $W(x) = \frac{|x|}{2}$.

Finally, we also verified the well-balanced property of the scheme (48) by computing the W_1 distance between the approximated solution at time $T = 0.5$ and the stationary solution of (2) given by

$$\rho(t,x) = \rho_0(x) := \frac{1}{8\varepsilon c^2}\left(1 - \tanh^2\left(\frac{x}{4\varepsilon c^2}\right)\right).$$

The test is conducted with $\varepsilon = 2 \times 10^{-4}$, with the exact boundary conditions given by the above formula, and for several values of Δx. As we show in Figure 4, the scheme (48) preserves well the above equilibrium for any Δx (although we replaced the resolution of the systems (30) and (31) with linear systems, see (38)), while for the splitting scheme, we recover the linear convergence towards ρ_0 which is, in this case, the exact solution.

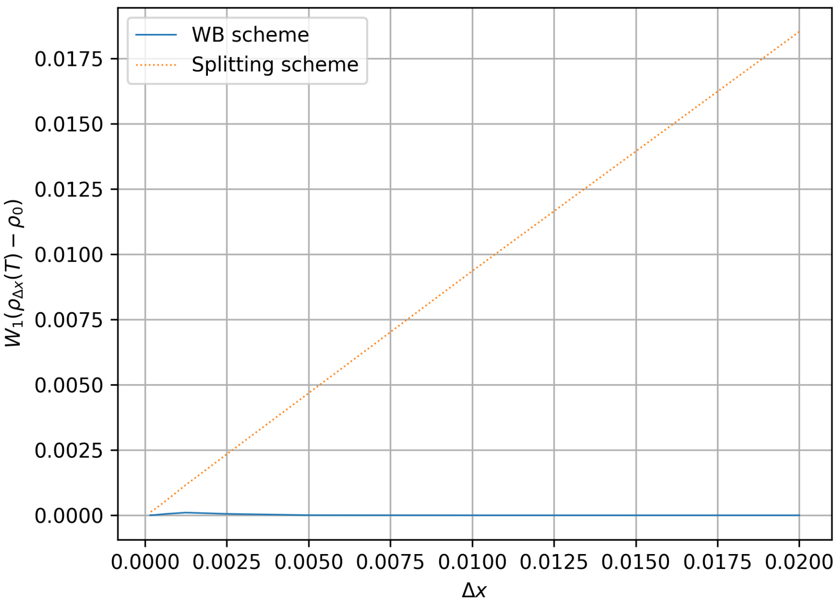

Figure 4. Distance to the equilibrium for the splitting scheme and the well-balanced scheme and for the pointy potential $W(x) = \frac{|x|}{2}$.

Author Contributions: B.F., F.L., S.T.T. and N.V. contributed equally in writing this article. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of Open Access Journals
AS	asymptotic preserving

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