symmetry

MDPI

Article

Solving Multi-Objective Matrix Games with Fuzzy Payoffs through the Lower Limit of the Possibility Degree

Dong Qiu *, Yumei Xing and Shuqiao Chen

College of Science, Chongqing University of Post and Telecommunication, Chongqing 400065, China; x_ym2015@163.com (Y.X.); cicanary@163.com (S.C.)

* Correspondence: dongqiumath@163.com; Tel.: +86-15123126186; Fax: +86-28-83206036

Received: 20 June 2017; Accepted: 18 July 2017; Published: 25 July 2017

Abstract: In this article, we put forward the multi-objective matrix game model based on fuzzy payoffs. In order to solve the game model, we first discuss the relationship of two fuzzy numbers via the lower limit$-\frac{1}{2}$ of the possibility degree. Then, utilizing this relationship, we conclude that the equilibrium solution of this game model and the optimal solution of multicriteria linear optimization problems are of equal value. Finally, to illustrate the effectiveness and correctness of the obtained model, an example is provided.

Keywords: multi-objective game; possibility degree; equilibrium solution; multicriteria linear optimization

1. Introduction

The multicriteria zero sum game is a generalization of the standard zero sum game model. The multicriteria zero sum game is also known as the multi-objective matrix game as it can be represented by multiple payoffs. Along with the collision of distinct decision makers in the social and corporate circumstance, much research in recent years has focused on multiple objective matrix game problems.

The notions of maxmin and minmax values were first used to discuss the multi-objective game model in [1]. Zeleny [2] studied the matrix game based on multiple payoff through notions of compromise solutions and a decomposition of parametric spaces. Ghose et al. [3] proposed the concepts of Pareto-optimal, Pareto saddle points and security levels of the multicriteria matrix game and analyzed the existence of Pareto saddle points of this game problem. Afterwards, the same game model was discussed by Fernandez et al. [4] and they proved that efficient solutions of multicriteria linear optimization problems and Pareto-optimal security strategies (POSS) for each Player are of equal value. Meanwhile, they obtained the set of all POSS through alternative ways.

The fuzzy set theory was initially introduced in 1965 by Zadeh [5]. The fuzziness occurring in the game problems is categorized as the fuzzy game problems. Single objective fuzzy game problems and related problems have attracted a wide range of research [6–11]. Therefore, fuzzy games theory has been extensively studied in some fields, such as economics, engineering and management science [12–19]. In order to deal with the fuzzy matrix games problem, a method of robust ranking is formulated by Bhaumik [20]. In terms of fuzzy games problems, Tan et al. [21] presented a concept of the potential function. Furthermore, they also reached a conclusion that the solution of fuzzy games and the marginal value of potential functions are equivalent. In [19], in order to solve the game problem quickly, the gradient iterative algorithm was proposed. Cevikle et al. [22] utilized the fuzzy relation method to find the solution of matrix games in terms of fuzzy goals and fuzzy payoffs. Chakeri et al. [23] used fuzzy logic to determine the priority of the payoff based on the linguistic preference relation and proposed the notion of linguistic Nash equilibrium. Fuzzy preference relation has been widely used in fuzzy game theory [24–27]. At the same time, they [24] utilize the same method [23] to determine the priority of the payoff based on fuzzy preference relation. In order to deal with this game model, a new approach is put forward. Moreover, Sharifian et al. [28] also applied fuzzy linguistic preference relation to fuzzy game theory.

Although the research on single objective fuzzy matrix games has become increasingly widespread, there are still few conclusions in the multicriteria case. The major contributions in this aspect have been studied in [18,29–32]. Sakawa et al. [32] discussed the fuzzy multicriteria games model with fuzzy goals according to the theory of maxmin value. In order to solve multiple decision-making problems, a model of fuzzy multiple matrix games is presented by Peldschus et al. [18]. Subsequently, Chen [30] found that the equilibrium solution of multiple matrix games based on fuzzy payoffs is equivalent to the solution of the fuzzy multi-objective attribute decision-making problem. Inspired by [3,4], Aggarwal et al. [29] applied the notions of POSS and security levels of apiece players to research the multicriteria matrix game in terms of fuzzy goals and demonstrated that this game problem and fuzzy multiple objective linear optimization problems are of equal value. Taking elicitation from [29,33,34], we can take inspiration and put forward a new model of the multiple objective matrix game based on fuzzy payoffs according to the lower limit$-\frac{1}{2}$ of the possibility degree.

The outline of this article is as follows: The background of this paper is introduced in Section 1. Section 2 introduces some basic definitions and recalls some results concerning crisp multi-objective matrix games and the fuzzy numbers. Furthermore, we discuss the relationship of two fuzzy numbers via the lower limit$-\frac{1}{2}$ of the possibility degree. In Section 3, The multiple objective matrix game model based on fuzzy payoffs is considered. We conclude that the equilibrium solution of this game model and the optimal solution of multi-objective linear optimization problems are of equal value. In Section 4, a small numerical example is given.

2. Preliminaries

In this section, we begin to depict a crisp multiple objective matrix game in [29]. For this, we recall some definitions.

Definition 1. [3] *The set of mixed strategies for Player* I *is denoted by*

$$S^m = \{x = (x_1, x_2, \cdots, x_m)^T \in \mathbb{R}^m | \sum_{i=1}^{m} x_i = 1, x_i \geq 0, i = 1, 2, \cdots, m.\} \tag{1}$$

Similarly, The set of mixed strategies for Player II *is denoted by*

$$S^n = \{y = (y_1, y_2, \cdots, y_n)^T \in \mathbb{R}^n | \sum_{j=1}^{n} y_j = 1, y_j \geq 0, j = 1, 2, \cdots, n.\} \tag{2}$$

where x^T is the transposition of x, $\mathbb{R}^m$ and $\mathbb{R}^n$ are m- and n-dimensional Euclidean spaces.

Multiple payoff matrixes of multicriteria matrix games are taken as follows [3]:

$$A^1 = \begin{pmatrix} a_{11}^1 & \cdots & a_{1n}^1 \\ \vdots & \ddots & \vdots \\ a_{m1}^1 & \cdots & a_{mn}^1 \end{pmatrix}, \cdots, A^r = \begin{pmatrix} a_{11}^r & \cdots & a_{1n}^r \\ \vdots & \ddots & \vdots \\ a_{m1}^r & \cdots & a_{mn}^r \end{pmatrix}. \tag{3}$$

In order not to lose generality, we suppose that Player I and Player II are maximized players.

A multiple objective matrix game (MOG) [29] is defined by

$$MOG = (S^m, S^n, A^k(1,2,\cdots,r)).$$

Definition 2. [3] *When Player* I *chooses a mixed strategy* $x \in S^m$ *and Player* II *chooses a mixed strategy* $y \in S^n$*, a vector*

$$\begin{array}{l} E\left(x,y,A\right) = x^T A y = \left[E_1\left(x,y\right), E_2\left(x,y\right), \cdots, E_r\left(x,y\right)\right] \\ = \left[x^T A^1 y, x^T A^2 y, \cdots, x^T A^r y\right] \end{array} \tag{4}$$

is called an expected payoff of Player I*. As the multi-objective game* (MOG) *is zero-sum, the payoff for Player* II *is* $-x^T Ay$*.*

Definition 3. $(x^*, y^*) \in S^m \times S^n$ *is called a solution of the* (MOG) *model if*

$$x^* A^k y \geq (V^k)^*, \ \forall y \in S^n,$$

$$x A^k y^* \leq (V^k)^*, \ \forall x \in S^m.$$

Here, x^* *and* y^* *are called the equilibrium solution for Player I and Player II, respectively. Furthermore,* $(V^k)^*(k = 1, 2, \cdots, r)$ *are called the values of* (MOG)*.*

Given a multi-objective game (MOG), its solution can be obtained by solving the following pair of primal-dual multiple objective linear optimization problems $(MOGLP)$ and $(MOGLD)$.

$$\begin{array}{lll} (MOGLP) & max \ (V^1, V^2, \cdots, V^r) & \\ & such \ that \ \sum\limits_{i=1}^{m} (a_{ij})^k x_i \geq V^k, (k = 1, 2, \cdots, r, j = 1, 2, \cdots, n), \\ & \qquad \forall x \in S^m, \forall y \in S^n, \\ (MOGLD) & min \ (W^1, W^2, \cdots, W^r) & \\ & such \ that \ \sum\limits_{j=1}^{n} (a_{ij})^k y_j \leq W^k, (k = 1, 2, \cdots, r, i = 1, 2, \cdots, m), \\ & \qquad \forall x \in S^m, \forall y \in S^n. \end{array}$$

The following notations, definitions and results will be needed in the sequel.

We denote $\mathcal{K}_C$ as the family of all bounded closed intervals in $\mathbb{R}$ [35], that is,

$$\mathcal{K}_C = \{[a_L, a_R] | a_L, a_R \in \mathbb{R} \ and \ a_L \leq a_R\}.$$

A fuzzy set $\widetilde{x}$ of $\mathbb{R}$ is characterized by a membership function $\mu_{\widetilde{x}} : \mathbb{R} \to [0,1]$ [5]. For each such fuzzy set $\widetilde{x}$, we denote by $[\widetilde{x}]^\alpha = \{x \in \mathbb{R} : \mu_{\widetilde{x}}(x) \geq \alpha\}$ for any $\alpha \in (0,1]$, its α-level set. We define the set $[\widetilde{x}]^0$ by $[\widetilde{x}]^0 = \overline{\bigcup_{\alpha \in (0,1]} [\widetilde{x}]^\alpha}$, where $\overline{A}$ denotes the closure of a crisp set A. A fuzzy number $\widetilde{x}$ is a fuzzy set with non-empty bounded closed level sets $[\widetilde{x}]^\alpha = [\widetilde{x}_L(\alpha), \widetilde{x}_R(\alpha)]$ for all $\alpha \in [0,1]$, where $[\widetilde{x}_L(\alpha), \widetilde{x}_R(\alpha)]$ denotes a closed interval with the left end point $\widetilde{x}_L(\alpha)$ and the right end point $\widetilde{x}_R(\alpha)$ [36]. We denote the class of fuzzy numbers by $\mathscr{F}$.

Definition 4. [37] *Let* $\widetilde{x}$ *and* $\widetilde{y}$ *be fuzzy numbers. It is said that* $\widetilde{x}$ *precedes* $\widetilde{y}$ $(\widetilde{x} \preceq \widetilde{y})$*, if* $\widetilde{x}_R(\alpha) \leq \widetilde{x}_R(\alpha)$ *and* $\widetilde{y}_L(\alpha) \leq \widetilde{y}_L(\alpha)$*,* $\alpha \in [0,1]$*.*

Definition 5. [5] *Let* $\widetilde{x}$ *be fuzzy numbers, If the membership function* $u_{\widetilde{x}}(x)$ *of the fuzzy number* $\widetilde{x}$ *is denoted by*

$$u_{\widetilde{x}}(x) = \begin{cases} 0, & x < \acute{a}, x > \grave{a}, \\ \frac{x - \acute{a}}{a - \acute{a}}, & \acute{a} \leq x \leq a, \\ \frac{\grave{a} - x}{\grave{a} - a}, & a < x \leq \grave{a}. \end{cases}$$

Then, $\widetilde{x}$ is called a triangular fuzzy number. Furthermore, the triangular fuzzy number $\widetilde{x}$ is presented by $\widetilde{x} = (\acute{a}, a, \grave{a})$.

Furthermore, the α-level set of the triangular fuzzy number $\widetilde{x} = (\acute{a}, a, \grave{a})$ is the closed interval [5]

$$[\widetilde{x}]^{\alpha} = [\widetilde{x}_L(\alpha), \widetilde{x}_R(\alpha)] = [(a - \acute{a})\alpha + \acute{a}, -(\grave{a} - a)\alpha + \grave{a}], \ \ \alpha \in (0, 1]. \tag{5}$$

Definition 6. [38] *Let $\widetilde{x}_i$ be fuzzy numbers and $b_i \geq 0$ $(i = 1, 2, \cdots, n)$ be real numbers. Then, $\sum_{i=1}^{n} \widetilde{x}_i b_i$ is a fuzzy number.*

We define the new relationship of two fuzzy numbers.

Definition 7. *Let $\widetilde{x}$ and $\widetilde{y}$ be fuzzy numbers. The width of $[\widetilde{x}]^{\alpha} = [\widetilde{x}_L(\alpha), \widetilde{x}_R(\alpha)]$ and $[\widetilde{y}]^{\alpha} = [\widetilde{y}_L(\alpha), \widetilde{y}_R(\alpha)]$ respectively are given by*

$$w(\widetilde{x}) = \widetilde{x}_R(\alpha) - \widetilde{x}_L(\alpha), \ \ w(\widetilde{y}) = \widetilde{y}_R(\alpha) - \widetilde{y}_L(\alpha), \ \alpha \in [0, 1].$$

We say that $p(\widetilde{x} \preceq \widetilde{y})$ is possibility a degree of $\widetilde{x} \preceq \widetilde{y}$, where

$$p(\widetilde{x} \preceq \widetilde{y}) = 1 - max\{min\{1, \frac{\widetilde{x}_R(\alpha) - \widetilde{y}_L(\alpha)}{w(\widetilde{x}) + w(\widetilde{y})}\}, 0\}.$$

Definition 8. *Let $\widetilde{x}$ and $\widetilde{y}$ be fuzzy numbers. If $p(\widetilde{x} \preceq \widetilde{y}) \geq \frac{1}{2}$, we say that $\widetilde{x}$ precedes $\widetilde{y}$ $(\widetilde{x} \preceq \widetilde{y})$. Furthermore, $\frac{1}{2}$ is the lower limit of the possibility degree of $\widetilde{x} \preceq \widetilde{y}$. That is, $\widetilde{x}$ precedes $\widetilde{y}$ with the possibility degree not less than $\frac{1}{2}$.*

Theorem 1. *Let $\widetilde{x}$ and $\widetilde{y}$ be fuzzy numbers. Then $\widetilde{x} \preceq \widetilde{y}$ if and only if $p(\widetilde{x} \preceq \widetilde{y}) \geq \frac{1}{2}$.*

Proof. Sufficiency: Since $p(\widetilde{x} \preceq \widetilde{y}) \geq \frac{1}{2}$. According to the above Definition 8, we get $\widetilde{x} \preceq \widetilde{y}$. □

Necessity: Since $\widetilde{x} \preceq \widetilde{y}$. Then, By Definition 4, we have

$$\widetilde{x}_R(\alpha) \leq \widetilde{y}_R(\alpha) \quad and \quad \widetilde{x}_L(\alpha) \leq \widetilde{y}_L(\alpha) \ \ \alpha \in [0, 1].$$

Hence,

$$\frac{\widetilde{x}_R(\alpha) + \widetilde{x}_L(\alpha)}{2} \leq \frac{\widetilde{y}_R(\alpha) + \widetilde{y}_L(\alpha)}{2} \quad \alpha \in [0, 1].$$

$$\frac{\widetilde{x}_R(\alpha) - \widetilde{y}_L(\alpha)}{2} \leq \frac{\widetilde{y}_R(\alpha) - \widetilde{x}_L(\alpha)}{2} \quad \alpha \in [0, 1].$$

$$\widetilde{x}_R(\alpha) - \widetilde{y}_L(\alpha) \leq \frac{\widetilde{y}_R(\alpha) - \widetilde{y}_L(\alpha) + \widetilde{x}_R(\alpha) - \widetilde{x}_L(\alpha)}{2} \quad \alpha \in [0, 1].$$

$$\frac{\widetilde{x}_R(\alpha) - \widetilde{y}_L(\alpha)}{\widetilde{y}_R(\alpha) - \widetilde{y}_L(\alpha) + \widetilde{x}_R(\alpha) - \widetilde{x}_L(\alpha)} \leq \frac{1}{2} \quad \alpha \in [0, 1].$$

Thus,

$$\min\{1, \frac{\widetilde{x}_R(\alpha) - \widetilde{y}_L(\alpha)}{\widetilde{y}_R(\alpha) - \widetilde{y}_L(\alpha) + \widetilde{x}_R(\alpha) - \widetilde{x}_L(\alpha)}\} \leq \frac{1}{2} \quad \alpha \in [0, 1].$$

$$\max\{\min\{1, \frac{\widetilde{x}_R(\alpha) - \widetilde{y}_L(\alpha)}{\widetilde{y}_R(\alpha) - \widetilde{y}_L(\alpha) + \widetilde{x}_R(\alpha) - \widetilde{x}_L(\alpha)}\}, 0\} \leq \frac{1}{2} \quad \alpha \in [0, 1].$$

$$1 - \max\{\min\{1, \frac{\widetilde{x}_R(\alpha) - \widetilde{y}_L(\alpha)}{\widetilde{y}_R(\alpha) - \widetilde{y}_L(\alpha) + \widetilde{x}_R(\alpha) - \widetilde{x}_L(\alpha)}\}, 0\} \geq \frac{1}{2} \quad \alpha \in [0, 1].$$

Therefore, by Definition 7, we have that

$$1 - \max\{\min\{1, \frac{\widetilde{x}_R(\alpha) - \widetilde{y}_L(\alpha)}{w(\widetilde{y}) + w(\widetilde{x})}\}, 0\} \geq \frac{1}{2} \quad \alpha \in [0,1].$$

That is to say,

$$p(\widetilde{x} \preceq \widetilde{y}) \geq \frac{1}{2}.$$

3. A Generalized Model for a Multi-Objective Fuzzy Matrix Game

S^m and S^n are given in Section 2. Suppose that the elements of $\widetilde{A}^k$ $(k = 1, 2, \cdots, r)$ are fuzzy numbers. Let $\widetilde{V}_0^k$ and $\widetilde{W}_0^k$ be the aspiration levels as fuzzy numbers of Player I and Player II, respectively. Therefore, the multiple objective matrix game based on fuzzy payoffs, denoted by $MOFP$, can be presented as

$$MOFP = (S^m, S^n, \widetilde{A}^k, \widetilde{V}_0^k, \widetilde{W}_0^k, (k = 1, 2, \cdots, r)).$$

Now, we have the following definition to define the solution of $MOFP$.

Definition 9. *Let $(x, y) \in S^m \times S^n$. If $x \in S^m$ and $y \in S^n$ satisfy the following the conditions:*

$$(\widetilde{V}_0^k)^* \preceq {x^*}^T \widetilde{A}^k y, \ \forall y \in S^n,$$

$$x \widetilde{A}^k y^* \preceq (\widetilde{W}_0^k)^*, \ \forall x \in S^m.$$

Then, $(x^*, y^*) \in S^m \times S^n$ is called the equilibrium solution of $(MOFP)$.

In order to obtain the equilibrium solution of $(MOFP)$, we conclude the following the theorem.

Theorem 2. *Let $\alpha \in (0,1]$ be fixed. $(x, ((\widetilde{V}_0^1)_L(\alpha) + (\widetilde{V}_0^1)_R(\alpha), \cdots, (\widetilde{V}_0^r)_L(\alpha) + (\widetilde{V}_0^r)_R(\alpha)))$ and $(y, ((\widetilde{W}_0^1)_L(\alpha) + (\widetilde{W}_0^1)_R(\alpha), \cdots, (\widetilde{W}_0^r)_L(\alpha) + (\widetilde{W}_0^r)_R(\alpha)))$ are optimal solutions of multi-objective linear optimization problems $(MOCLP1)$ and $(MOCLD1)$ if and only if (x, y) is the equilibrium solution of $(MOFP)$.*

$$\begin{array}{lll}
(MOCLP1) & max & [(\widetilde{V}_0^1)_L(\alpha) + (\widetilde{V}_0^1)_R(\alpha), \cdots, (\widetilde{V}_0^r)_L(\alpha)) + (\widetilde{V}_0^r)_R(\alpha)] \\
 & such\ \ that & (\widetilde{V}_0^k)_L(\alpha) + (\widetilde{V}_0^k)_R(\alpha) \leq \sum\limits_{i=1}^{m} [(\widetilde{a}_{ij}^k)_L(\alpha) + (\widetilde{a}_{ij}^k)_R(\alpha)] x_i, \\
 & & 0 \leq \alpha \leq 1, \\
 & & \forall x \in S^m, \forall, \ (k = 1, 2, \cdots, r; j = 1, 2, \cdots, n). \\
(MOCLD1) & min & [(\widetilde{W}_0^1)_L(\alpha) + (\widetilde{W}_0^1)_R(\alpha), \cdots, (\widetilde{W}_0^r)_L(\alpha) + (\widetilde{W}_0^r)_R(\alpha)] \\
 & such\ \ that & \sum\limits_{j=1}^{n} [(\widetilde{a}_{ij}^k)_L(\alpha) + (\widetilde{a}_{ij}^k)_R(\alpha)] y_j \leq (\widetilde{W}_0^k)_L(\alpha) + (\widetilde{W}_0^k)_R(\alpha), \\
 & & 0 \leq \alpha \leq 1, \\
 & & \forall y \in S^n, \ (k = 1, 2, \cdots, r; i = 1, 2, \cdots, m).
\end{array}$$

Proof. By utilizing Definition 9, we obtain that (x, y) is the equilibrium solution of $(MOFP)$ if and only if (x, y) is the optimal solution of multiple objective fuzzy optimization problems $(MOFP1)$ and $(MOFD1)$. □

$$\begin{array}{ll}
(MOFP1) & Find\ \ x \in S^m\ \ such\ \ that \\
 & \quad \widetilde{V}_0^k \preceq (x^T \widetilde{A}^k y), \quad \forall y \in S^n, \\
(MOFD1) & Find\ \ y \in S^n\ \ such\ \ that \\
 & \quad (x^T \widetilde{A}^k y) \preceq \widetilde{W}_0^k, \quad \forall x \in S^m, (k = 1, 2, \cdots, r).
\end{array}$$

Furthermore, by Theorem 1, the problems $(MOFP1)$ and $(MOFD1)$ respectively are equivalent to

$$\begin{array}{ll}
(MOP1) & Find\ \ x \in S^m\ \ such\ \ that \\
 & \quad p(\widetilde{V}_0^k \ \preceq\ (x^T \widetilde{A}^k y)) \geq \frac{1}{2}, \quad \forall y \in S^n, \\
(MOD1) & Find\ \ y \in S^n\ \ such\ \ that \\
 & \quad p((x^T \widetilde{A}^k y)\ \preceq\ \widetilde{W}_0^k) \geq \frac{1}{2}, \quad \forall x \in S^m, (k = 1, 2, \cdots, r).
\end{array}$$

By Definition 7, the problems $(MOP1)$ and $(MOD1)$ can be rewritten as $(MOP2)$ and $(MOD2)$, respectively.

$$\begin{array}{lll}
(MOP2) & max & [((\widetilde{V}_0^1)_L(\alpha), \cdots, (\widetilde{V}_0^r)_L(\alpha)), ((\widetilde{V}_0^1)_R(\alpha), \cdots, (\widetilde{V}_0^r)_R(\alpha))] \\
 & such\ \ that & 1 - \max\{\min\{1, \frac{(\widetilde{V}_0^k)_R(\alpha) - (x^T \widetilde{A}_L^k(\alpha) y)}{(\widetilde{V}_0^k)_R(\alpha) - (\widetilde{V}_0^k)_L(\alpha) + x^T(\widetilde{A}_R^k(\alpha) - \widetilde{A}_L^k(\alpha)) y}\}, 0\} \geq \frac{1}{2}, \\
 & & 0 \leq \alpha \leq 1, \\
 & & \forall x \in S^m, \forall y \in S^n,\ (k = 1, 2, \cdots, r). \\
(MOD2) & min & [((\widetilde{W}_0^1)_L(\alpha), \cdots, (\widetilde{W}_0^r)_L(\alpha)), ((\widetilde{W}_0^1)_R(\alpha), \cdots, (\widetilde{W}_0^r)_R(\alpha))] \\
 & such\ \ that & 1 - \max\{\min\{1, \frac{(x^T \widetilde{A}_R^k(\alpha) y) - (\widetilde{W}_0^k)_L(\alpha)}{x^T(\widetilde{A}_R^k(\alpha) - \widetilde{A}_L^k(\alpha)) y + (\widetilde{W}_0^k)_R(\alpha) - (\widetilde{W}_0^k)_L(\alpha)}\}, 0\} \geq \frac{1}{2}, \\
 & & 0 \leq \alpha \leq 1, \\
 & & \forall x \in S^m, \forall y \in S^n,\ (k = 1, 2, \cdots, r).
\end{array}$$

That is equivalent to

$$\begin{array}{lll}
(MOP3) & max & [((\widetilde{V}_0^1)_L(\alpha), \cdots, (\widetilde{V}_0^r)_L(\alpha)), ((\widetilde{V}_0^1)_R(\alpha), \cdots, (\widetilde{V}_0^r)_R(\alpha))] \\
 & such\ \ that & \max\{\min\{1, \frac{(\widetilde{V}_0^k)_R(\alpha) - (x^T \widetilde{A}_L^k(\alpha) y)}{(\widetilde{V}_0^k)_R(\alpha) - (\widetilde{V}_0^k)_L(\alpha) + x^T(\widetilde{A}_R^k(\alpha) - \widetilde{A}_L^k(\alpha)) y}\}, 0\} \leq \frac{1}{2}, \\
 & & 0 \leq \alpha \leq 1, \\
 & & \forall x \in S^m, \forall y \in S^n,\ (k = 1, 2, \cdots, r). \\
(MOD3) & min & [((\widetilde{W}_0^1)_L(\alpha), \cdots, (\widetilde{W}_0^r)_L(\alpha)), ((\widetilde{W}_0^1)_R(\alpha), \cdots, (\widetilde{W}_0^r)_R(\alpha))] \\
 & such\ \ that & \max\{\min\{1, \frac{(x^T \widetilde{A}_R^k(\alpha) y) - (\widetilde{W}_0^k)_L(\alpha)}{x^T(\widetilde{A}_R^k(\alpha) - \widetilde{A}_L^k(\alpha)) y + (\widetilde{W}_0^k)_R(\alpha) - (\widetilde{W}_0^k)_L(\alpha)}\}, 0\} \leq \frac{1}{2}, \\
 & & 0 \leq \alpha \leq 1, \\
 & & \forall x \in S^m, \forall y \in S^n,\ (k = 1, 2, \cdots, r).
\end{array}$$

That is to say,

$$\begin{array}{lll}
(MOP4) & max & [((\widetilde{V}_0^1)_L(\alpha), \cdots, (\widetilde{V}_0^r)_L(\alpha)), ((\widetilde{V}_0^1)_R(\alpha), \cdots, (\widetilde{V}_0^r)_R(\alpha))] \\
 & such\ \ that & \frac{(\widetilde{V}_0^k)_R(\alpha) - (x^T \widetilde{A}_L^k(\alpha) y)}{(\widetilde{V}_0^k)_R(\alpha) - (\widetilde{V}_0^k)_L(\alpha) + (x^T \widetilde{A}_R^k(\alpha) y) - (x^T \widetilde{A}_L^k(\alpha) y)} \leq \frac{1}{2}, \\
 & & 0 \leq \alpha \leq 1, \\
 & & \forall x \in S^m, \forall y \in S^n,\ (k = 1, 2, \cdots, r). \\
(MOD4) & min & [((\widetilde{W}_0^1)_L(\alpha), \cdots, (\widetilde{W}_0^r)_L(\alpha)), ((\widetilde{W}_0^1)_R(\alpha), \cdots, (\widetilde{W}_0^r)_R(\alpha))] \\
 & such\ \ that & \frac{(x^T \widetilde{A}_R^k(\alpha) y) - (\widetilde{W}_0^k)_L(\alpha)}{x^T(\widetilde{A}_R^k(\alpha) - \widetilde{A}_L^k(\alpha)) y + (\widetilde{W}_0^k)_R(\alpha) - (\widetilde{W}_0^k)_L(\alpha)} \leq \frac{1}{2}, \\
 & & 0 \leq \alpha \leq 1, \\
 & & \forall x \in S^m, \forall y \in S^n,\ (k = 1, 2, \cdots, r).
\end{array}$$

By arranging the models $(MOP4)$ and $(MOD4)$, we have

$$\begin{array}{lll}(MOP5) & max & [((\widetilde{V}_0^1)_L(\alpha),\cdots,(\widetilde{V}_0^r)_L(\alpha)),((\widetilde{V}_0^1)_R(\alpha),\cdots,(\widetilde{V}_0^r)_R(\alpha))] \\ & such\ that & (\widetilde{V}_0^k)_L(\alpha)+(\widetilde{V}_0^k)_R(\alpha)\leq x^T(\widetilde{A}_R^k(\alpha)+\widetilde{A}_L^k(\alpha))y, \\ & & 0\leq\alpha\leq 1, \\ & & \forall x\in S^m,\forall y\in S^n,\ (k=1,2,\cdots,r). \\ (MOD5) & min & [((\widetilde{W}_0^1)_L(\alpha),\cdots,(\widetilde{W}_0^r)_L(\alpha)),((\widetilde{W}_0^1)_R(\alpha),\cdots,(\widetilde{W}_0^r)_R(\alpha))] \\ & such\ that & x^T(\widetilde{A}_R^k(\alpha)+\widetilde{A}_L^k(\alpha))y\leq(\widetilde{W}_0^k)_L(\alpha)+(\widetilde{W}_0^k)_R(\alpha), \\ & & 0\leq\alpha\leq 1, \\ & & \forall x\in S^m,\forall y\in S^n,\ (k=1,2,\cdots,r).\end{array}$$

Since S^m and S^n are convex polytopes. Furthermore, the problems $(MOP5)$ and $(MOD5)$ are crisp multiple objective linear optimization problems; it is sufficient to consider only the extreme points of S^m and S^n. Thus, the problems $(MOP5)$ and $(MOD5)$ can be converted into

$$\begin{array}{lll}(MOCLP) & max & [((\widetilde{V}_0^1)_L(\alpha),\cdots,(\widetilde{V}_0^r)_L(\alpha)),((\widetilde{V}_0^1)_R(\alpha),\cdots,(\widetilde{V}_0^r)_R(\alpha))] \\ & such\ that & (\widetilde{V}_0^k)_L(\alpha)+(\widetilde{V}_0^k)_R(\alpha)\leq\sum_{i=1}^m[(\widetilde{a}_{ij}^k)_L(\alpha)+(\widetilde{a}_{ij}^k)_R(\alpha)]x_i, \\ & & 0\leq\alpha\leq 1, \\ & & \forall x\in S^m,\ (k=1,2,\cdots,r;j=1,2,\cdots,n). \\ (MOCLD) & min & [((\widetilde{W}_0^1)_L(\alpha),\cdots,(\widetilde{W}_0^r)_L(\alpha)),((\widetilde{W}_0^1)_R(\alpha),\cdots,(\widetilde{W}_0^r)_R(\alpha))] \\ & such\ that & \sum_{j=1}^n[(\widetilde{a}_{ij}^k)_L(\alpha)+(\widetilde{a}_{ij}^k)_R(\alpha)]y_j\leq(\widetilde{W}_0^k)_L(\alpha)+(\widetilde{W}_0^k)_R(\alpha), \\ & & 0\leq\alpha\leq 1, \\ & & \forall y\in S^n,\ (k=1,2,\cdots,r;i=1,2,\cdots,m).\end{array}$$

That is equal to

$$\begin{array}{lll}(MOCLP1) & max & [(\widetilde{V}_0^1)_L(\alpha)+(\widetilde{V}_0^1)_R(\alpha),\cdots,(\widetilde{V}_0^r)_L(\alpha))+(\widetilde{V}_0^r)_R(\alpha)] \\ & such\ that & (\widetilde{V}_0^k)_L(\alpha)+(\widetilde{V}_0^k)_R(\alpha)\leq\sum_{i=1}^m[(\widetilde{a}_{ij}^k)_L(\alpha)+(\widetilde{a}_{ij}^k)_R(\alpha)]x_i, \\ & & 0\leq\alpha\leq 1, \\ & & \forall x\in S^m,\ (k=1,2,\cdots,r;j=1,2,\cdots,n). \\ (MOCLD1) & min & [(\widetilde{W}_0^1)_L(\alpha)+(\widetilde{W}_0^1)_R(\alpha),\cdots,(\widetilde{W}_0^r)_L(\alpha)+(\widetilde{W}_0^r)_R(\alpha)] \\ & such\ that & \sum_{j=1}^n[(\widetilde{a}_{ij}^k)_L(\alpha)+(\widetilde{a}_{ij}^k)_R(\alpha)]y_j\leq(\widetilde{W}_0^k)_L(\alpha)+(\widetilde{W}_0^k)_R(\alpha), \\ & & 0\leq\alpha\leq 1, \\ & & \forall y\in S^n,\ (k=1,2,\cdots,r;i=1,2,\cdots,m).\end{array}$$

Remark 1. *When the elements of $\widetilde{A}^k$, $\widetilde{V}_0^k$ and $\widetilde{W}_0^k (k=1,2,\cdots,r)$ are crisp numbers, the MOFP model reduces the MOG model, and the optimization problems $MOCLP1$ and $MOCLD1$ become the optimization problems $MOGLP$ and $MOGLD$.*

Theorem 3. *Let $\alpha\in(0,1]$ be fixed. If $(x,((\widetilde{V}_0^1)_L(\alpha)+(\widetilde{V}_0^1)_R(\alpha),\cdots,(\widetilde{V}_0^r)_L(\alpha)+(\widetilde{V}_0^r)_R(\alpha)))$ and $(y,((\widetilde{W}_0^1)_L(\alpha)+(\widetilde{W}_0^1)_R(\alpha),\cdots,(\widetilde{W}_0^r)_L(\alpha)+(\widetilde{W}_0^r)_R(\alpha)))$ are the optimal solutions of $(MOCLP1)$ and $(MOCLD1)$, then,*

$$\sum_{k=1}^r[(\widetilde{V}_0^k)_L(\alpha)+(\widetilde{V}_0^k)_R(\alpha)]\leq\sum_{k=1}^r[(\widetilde{W}_0^k)_L(\alpha)+(\widetilde{W}_0^k)_R(\alpha)].$$

Proof. By Theorem 2, we obtain

$$(\widetilde{V}_0^k)_L(\alpha)+(\widetilde{V}_0^k)_R(\alpha)\leq\sum_{i=1}^m[(\widetilde{a}_{ij}^k)_L(\alpha)+(\widetilde{a}_{ij}^k)_R(\alpha)]x_i \tag{6}$$

and
$$\sum_{j=1}^{n}[(\widetilde{a}_{ij}^{k})_L(\alpha)+(\widetilde{a}_{ij}^{k})_R(\alpha)]y_j \le (\widetilde{W}_0^k)_L(\alpha)+(\widetilde{W}_0^k)_R(\alpha) \tag{7}$$

Since $\forall x \in S^m, \forall y \in S^n$, we have

$$\begin{aligned}(\widetilde{V}_0^k)_L(\alpha)+(\widetilde{V}_0^k)_R(\alpha) &= \sum_{j=1}^{n}[(\widetilde{V}_0^k)_L(\alpha)+(\widetilde{V}_0^k)_R(\alpha)]y_j \\ &\le \sum_{j=1}^{n}\sum_{i=1}^{m}[(\widetilde{a}_{ij}^{k})_L(\alpha)+(\widetilde{a}_{ij}^{k})_R(\alpha)]x_i y_j\end{aligned} \tag{8}$$

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{n}[(\widetilde{a}_{ij}^{k})_L(\alpha)+(\widetilde{a}_{ij}^{k})_R(\alpha)]x_i y_j &\le \sum_{i=1}^{m}[(\widetilde{W}_0^k)_L(\alpha)+(\widetilde{W}_0^k)_R(\alpha)]x_i \\ &= (\widetilde{W}_0^k)_L(\alpha)+(\widetilde{W}_0^k)_R(\alpha)\end{aligned} \tag{9}$$

Therefore, observe that

$$(\widetilde{V}_0^k)_L(\alpha)+(\widetilde{V}_0^k)_R(\alpha) \le (\widetilde{W}_0^k)_L(\alpha)+(\widetilde{W}_0^k)_R(\alpha)$$

Then,
$$\sum_{k=1}^{r}[(\widetilde{V}_0^k)_L(\alpha)+(\widetilde{V}_0^k)_R(\alpha)] \le \sum_{k=1}^{r}[(\widetilde{W}_0^k)_L(\alpha)+(\widetilde{W}_0^k)_R(\alpha)].$$

□

Theorem 4. *Let $\alpha \in (0,1]$ be fixed. If the elements of $\widetilde{A}^k$ $(k=1,2,\cdots,r)$ are triangular fuzzy numbers, then $(x,((\widetilde{V}_0^1)_L(\alpha)+(\widetilde{V}_0^1)_R(\alpha),\cdots,(\widetilde{V}_0^r)_L(\alpha)+(\widetilde{V}_0^r)_R(\alpha)))$ and $(y,((\widetilde{W}_0^1)_L(\alpha)+(\widetilde{W}_0^1)_R(\alpha),\cdots,(\widetilde{W}_0^r)_L(\alpha)+(\widetilde{W}_0^r)_R(\alpha)))$ are the optimal solutions of $(MOCLP2)$ and $(MOCLD2)$ if and only if (x,y) is the equilibrium solution of $(MOFP)$.*

$$\begin{array}{lll}(MOCLP2) & max & [(\widetilde{V}_0^1)_L(\alpha)+(\widetilde{V}_0^1)_R(\alpha),\cdots,(\widetilde{V}_0^r)_L(\alpha))+(\widetilde{V}_0^r)_R(\alpha)] \\ & such\ \ that & (\widetilde{V}_0^k)_L(\alpha)+(\widetilde{V}_0^k)_R(\alpha) \le \sum_{i=1}^{m}[(\acute{a}_{ij}^k+\grave{a}_{ij}^k)(1-\alpha)+2\alpha a_{ij}^k]x_i \\ & & 0 \le \alpha \le 1, \\ & & \forall x \in S^m,\ (k=1,2,\cdots,r; j=1,2,\cdots,n). \\ (MOCLD2) & min & [(\widetilde{W}_0^1)_L(\alpha)+(\widetilde{W}_0^1)_R(\alpha),\cdots,(\widetilde{W}_0^r)_L(\alpha)+(\widetilde{W}_0^r)_R(\alpha)] \\ & such\ \ that & \sum_{j=1}^{n}[(\acute{a}_{ij}^k+\grave{a}_{ij}^k)(1-\alpha)+2\alpha a_{ij}^k]y_j \le (\widetilde{W}_0^k)_L(\alpha)+(\widetilde{W}_0^k)_R(\alpha), \\ & & 0 \le \alpha \le 1, \\ & & \forall y \in S^n,\ (k=1,2,\cdots,r; i=1,2,\cdots,m).\end{array}$$

Proof. Since the elements of $\widetilde{A}^k$ $(k=1,2,\cdots,r)$ are triangular fuzzy numbers, using (5), we have

$$[\widetilde{a}_{ij}^k]^\alpha = [(\widetilde{a}_{ij}^k)_L(\alpha),(\widetilde{a}_{ij}^k)_R(\alpha)] = [(a_{ij}^k-\acute{a}_{ij}^k)\alpha+\acute{a}_{ij}^k, -(\grave{a}_{ij}^k-a_{ij}^k)\alpha+\grave{a}_{ij}^k],\ \ \alpha \in (0,1].$$

Hence,
$$(\widetilde{a}_{ij}^k)_L(\alpha)+(\widetilde{a}_{ij}^k)_R(\alpha) = (\acute{a}_{ij}^k+\grave{a}_{ij}^k)(1-\alpha)+2\alpha a_{ij}^k$$

By utilizing Theorem 2, we have that $(x, ((\widetilde{V}_0^1)_L(\alpha) + (\widetilde{V}_0^1)_R(\alpha), \cdots, (\widetilde{V}_0^r)_L(\alpha) + (\widetilde{V}_0^r)_R(\alpha)))$ and $(y, ((\widetilde{W}_0^1)_L(\alpha) + (\widetilde{W}_0^1)_R(\alpha), \cdots, (\widetilde{W}_0^r)_L(\alpha) + (\widetilde{W}_0^r)_R(\alpha)))$ are optimal solutions of $(MOCLP2)$ and $(MOCLD2)$ if and only if (x, y) is the equilibrium solution of $(MOFP)$.

$$
\begin{array}{lll}
(MOCLP2) & max & [(\widetilde{V}_0^1)_L(\alpha) + (\widetilde{V}_0^1)_R(\alpha), \cdots, (\widetilde{V}_0^r)_L(\alpha)) + (\widetilde{V}_0^r)_R(\alpha)] \\
& such\ that & (\widetilde{V}_0^k)_L(\alpha) + (\widetilde{V}_0^k)_R(\alpha) \leq \sum_{i=1}^{m} [(\acute{a}_{ij}^k + \grave{a}_{ij}^k)(1-\alpha) + 2\alpha a_{ij}^k] x_i \\
& & 0 \leq \alpha \leq 1, \\
& & \forall x \in S^m,\ (k = 1, 2, \cdots, r; j = 1, 2 \cdots, n). \\
(MOCLD2) & min & [(\widetilde{W}_0^1)_L(\alpha) + (\widetilde{W}_0^1)_R(\alpha), \cdots, (\widetilde{W}_0^r)_L(\alpha) + (\widetilde{W}_0^r)_R(\alpha)] \\
& such\ that & \sum_{j=1}^{n} [(\acute{a}_{ij}^k + \grave{a}_{ij}^k)(1-\alpha) + 2\alpha a_{ij}^k] y_j \leq (\widetilde{W}_0^k)_L(\alpha) + (\widetilde{W}_0^k)_R(\alpha), \\
& & 0 \leq \alpha \leq 1, \\
& & \forall y \in S^n,\ (k = 1, 2, \cdots, r; i = 1, 2 \cdots, m).
\end{array}
$$

□

4. Example

In order to illustrate the effectiveness and correctness of the obtained model, we consider multiple payoffs of the multiple objective matrix game based on fuzzy payoffs $(MOFP)$ that are taken as

$$
\tilde{A}^1 = \begin{bmatrix} (2.90, 3.00, 3.10) & (7.75, 8.00, 8.20) \\ (3.50, 4.00, 4.20) & (0.90, 1.00, 1.20) \end{bmatrix}
$$

and

$$
\tilde{A}^2 = \begin{bmatrix} (4.80, 5.00, 5.50) & (1.80, 2.00, 2.75) \\ (5.75, 6.00, 6.40) & (0.75, 1.00, 1.20) \end{bmatrix}.
$$

In order to solve the game for a given α, by Theorem 4, we have to solve the following problems $(MOGCLP3)$ and $(MOGCLD3)$.

$$
\begin{array}{lll}
(MOCLP3) & max & [(\widetilde{V}_0^1)_L(\alpha) + (\widetilde{V}_0^1)_R(\alpha), (\widetilde{V}_0^2)_L(\alpha)) + (\widetilde{V}_0^2)_R(\alpha)] \\
& such\ that & (\widetilde{V}_0^1)_L(\alpha) + (\widetilde{V}_0^1)_R(\alpha) \leq 6x_1 + (7.7 + 0.3\alpha)x_2 \\
& & (\widetilde{V}_0^1)_L(\alpha) + (\widetilde{V}_0^1)_R(\alpha) \leq (15.95 + 0.05\alpha)x_1 + (2.1 - 0.1\alpha)x_2 \\
& & (\widetilde{V}_0^2)_L(\alpha) + (\widetilde{V}_0^2)_R(\alpha) \leq (10.3 - 0.3\alpha)x_1 + (12.15 - 0.15\alpha)x_2 \\
& & (\widetilde{V}_0^2)_L(\alpha) + (\widetilde{V}_0^2)_R(\alpha) \leq (4.55 - 0.55\alpha)x_1 + (1.95 + 0.05\alpha)x_2 \\
& & 0 \leq \alpha \leq 1, \\
& & x_1 + x_2 = 1,\ x_1 \geq 0,\ x_2 \geq 0,\ (k = 1, 2, \cdots, r). \\
(MOCLD3) & min & [(\widetilde{W}_0^1)_L(\alpha) + (\widetilde{W}_0^1)_R(\alpha), (\widetilde{W}_0^2)_L(\alpha) + (\widetilde{W}_0^2)_R(\alpha)] \\
& such\ that & 6y_1 + (15.95 + 0.05\alpha)y_2 \leq (\widetilde{W}_0^1)_L(\alpha) + (\widetilde{W}_0^1)_R(\alpha), \\
& & (7.7 + 0.3\alpha)y_1 + (2.1 - 0.1\alpha)y_2 \leq (\widetilde{W}_0^1)_L(\alpha) + (\widetilde{W}_0^1)_R(\alpha), \\
& & (10.3 - 0.3\alpha)y_1 + (4.55 - 0.55\alpha)y_2 \leq (\widetilde{W}_0^2)_L(\alpha) + (\widetilde{W}_0^2)_R(\alpha), \\
& & (12.15 - 0.15\alpha)y_1 + (1.95 + 0.05\alpha)y_2 \leq (\widetilde{W}_0^2)_L(\alpha) + (\widetilde{W}_0^2)_R(\alpha), \\
& & 0 \leq \alpha \leq 1, \\
& & y_1 + y_2 = 1,\ y_1 \geq 0,\ y_2 \geq 0,\ (k = 1, 2, \cdots, r).
\end{array}
$$

By solving the above problems $(MOGCLP3)$ and $(MOGCLD3)$, particularly, let $\alpha^* = 1$, then we can obtain that $((x_1^* = 0.20, x_2^* = 0.80), ((\widetilde{V}_0^1)_L(\alpha^*) + (\widetilde{V}_0^1)_R(\alpha^*) = 4.80, (\widetilde{V}_0^2)_L(\alpha^*) + (\widetilde{V}_0^2)_R(\alpha^*) = 2.40))$ is the optimal solution of $(MOGCLP3)$ and $((y_1^* = 0.80, y_2^* = 0.20), ((\widetilde{W}_0^1)_L(\alpha^*) + (\widetilde{W}_0^1)_R(\alpha^*) = 8.00, (\widetilde{W}_0^2)_L(\alpha^*) + (\widetilde{W}_0^2)_R(\alpha^*) = 10.00))$ is optimal solution of $(MOGCLD3)$. By Theorem 4, we have that $((x_1^* = 0.20, x_2^* = 0.80), (y_1^* = 0.80, y_2^* = 0.20))$ is the equilibrium solution of $(MOFP)$.

5. Conclusions

In this paper, we proposed the multicriteria matrix game model based on fuzzy payoffs. In order to solve the game model, we first discussed the relationship of two fuzzy numbers via the lower limit$-\frac{1}{2}$ of the possibility degree. Then, utilizing this relationship, we conclude that the equilibrium solution of this game model and optimal solutions of a pair of multiple objective linear optimization problems are of equal value. We will use other more effective methods to study the matrix game in the future.

Acknowledgments: This work was supported by The National Natural Science Foundations of China (Grant Nos. 11671001 and 61472056).

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: The authors declare that they have no competing interests.

References

1. Blackwell, D. An analog of the minimax theorem for vector payoffs. *Pac. J. Math.* **1956**, *6*, 1–8.
2. Zeleny, M. Games with multiple payoffs. *Int. J. Game Theory* **1975**, *4*, 179–191.
3. Ghose, D.; Prasad, U.R. Solution concepts in two-person multicriteria games. *J. Optim. Theory Appl.* **1989**, *63*, 167–189.
4. Fernández, F.R.; Puerto, J. Vector linear programming in zero-sum multicriteria matrix games. *J. Optim. Theory Appl.* **1996**, *89*, 115–127.
5. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353.
6. Bector, C.R.; Chandra, S.; Vidyottama, V. Matrix games with fuzzy goals and fuzzy linear programming duality. *Fuzzy Optim. Decis. Mak.* **2004**, *3*, 255–269.
7. Bector, C.R.; Chandra, S. *Fuzzy Mathematical Programming and Fuzzy Matrix Games*; Springer: Berlin, Germany, 2005.
8. Chen, B.S.; Tseng, C.S.; Uang, H.J. Fuzzy differential games for nonlinear stochastic systems: Suboptimal approach. *IEEE Trans. Fuzzy Syst.* **2002**, *10*, 222–233.
9. Garagic, D.; Cruz, J.B., Jr. An Approach to Fuzzy Noncooperative Nash Games. *J. Optim. Theory Appl.* **2003**, *118*, 475–491.
10. Vijay, V.; Chandra, S.; Bector, C.R. Matrix games with fuzzy goals and fuzzy payoffs. *Omega* **2005**, *33*, 425–429.
11. Žilinskas, J.; Bector, C.R.; Chandra, S. Fuzzy Mathematical Programming and Fuzzy Matrix Games. *Interfaces* **2007**, *37*, 388–389.
12. Baumgarten, C. Old Game, New Rules: Rethinking the Form of Physics. *Symmetry* **2016**, *8*, 30.
13. Byun, S.S.; Gil, J.M. Fair Dynamic Spectrum Allocation Using Modified Game Theory for Resource-Constrained Cognitive Wireless Sensor Networks. *Symmetry* **2017**, *9*, 73.
14. Deng, X.; Jiang, W.; Zhang, J. Zero-sum matrix game with payoffs of Dempster-Shafer belief structures and its applications on sensors. *Sensors* **2017**, *17*, 922.
15. Khanzadi, M.; Turskis, Z.; Ghodrati, A.G.; Chalekaee, A. A model of discrete zero-sum two-person matrix games with grey numbers to solve dispute resolution problems in construction. *J. Civ. Eng. Manag.* **2017**, *23*, 824–835.
16. Medineckiene, M.; Zavadskas, E.K.; Turskis, Z. Dwelling selection by applying fuzzy game theory. *Arch. Civ. Mech. Eng.* **2011**, *11*, 681–697.
17. Park, Y.; Hyun, S. Characterizations of Network Structures Using Eigenmode Analysis. *Symmetry* **2015**, *7*, 962–975.
18. Peldschus, F.; Zavadskas, E.K. Fuzzy matrix games multi-criteria model for decision-making in engineering. *Informatica* **2005**, *16*, 107–120.
19. Ye, F.; Dai, J.; Li, Y. Game Algorithm for Resource Allocation Based on Intelligent Gradient in HetNet. *Symmetry* **2017**, *9*, 34.
20. Bhaumik, A.; Roy, S.K.; Li, D.F. Analysis of triangular intuitionistic fuzzy matrix games using robust ranking. *J. Intell. Fuzzy Syst.* **2017**, *33*, 327–336.
21. Tan, C.; Jiang, Z.Z.; Chen, X.; Ip, W.H. A Banzhaf function for a fuzzy game. *IEEE Trans. Fuzzy Syst.* **2014**, *22*, 1489–1502.

22. Cevikel, A.C.; Ahlatcioğlu, M. Solutions for fuzzy matrix games. *Comput. Math. Appl.* **2010**, *60*, 399–410.
23. Chakeri, A.; Habibi, J.; Heshmat, Y. Fuzzy type-2 Nash equilibrium. In Proceedings of the 2008 International Conference on Computational Intelligence for Modelling Control and Automation, Vienna, Austria, 10–12 December 2008; pp. 398–402.
24. Chakeri, A.; Dariani, A.N.; Lucas, C. How can fuzzy logic determine game equilibriums better? In Proceedings of the 4th International IEEE Conference on Intelligent Systems (IS'08), Varna, Bulgaria, 6–8 September 2008; pp. 251–256.
25. Chakeri, A.; Sadati, N.; Sharifian, S. Fuzzy Nash equilibrium in fuzzy games using ranking fuzzy numbers. In Proceedings of the 2010 IEEE International Conference on Fuzzy Systems (FUZZ), Barcelona, Spain, 18–23 July 2010; pp. 1–5.
26. Chakeri, A.; Sheikholeslam, F. Fuzzy Nash equilibriums in crisp and fuzzy games. *IEEE Trans. Fuzzy Syst.* **2013**, *21*, 171–176.
27. Chakeri, A.; Sadati, N.; Dumont, G.A. Nash Equilibrium Strategies in Fuzzy Games. In *Game Theory Relaunched*; InTech: Rijeka, Croatia, 2013; doi:10.5772/54677.
28. Sharifian, S.; Chakeri, A.; Sheikholeslam, F. Linguisitc representation of Nash equilibriums in fuzzy games. In Proceedings of the IEEE 2010 Annual Meeting of the North American Fuzzy Information Processing Society (NAFIPS), Toronto, ON, Canada, 12–14 July 2010; pp. 1–6.
29. Aggarwal, A.; Khan, I. Solving multi-objective fuzzy matrix games via multi-objective linear programming approach. *Kybernetika* **2016**, *52*, 153–168.
30. Chen, Y.W.; Larbani, M. Two-person zero-sum game approach for fuzzy multiple attribute decision making problems. *Fuzzy Sets Syst.* **2006**, *157*, 34–51.
31. Li, D.; Cheng, C. Fuzzy multiobjective programming methods for fuzzy constrained matrix games with fuzzy numbers. *Int. J. Uncertain. Fuzziness Knowl.-Based Syst.* **2002**, *10*, 385–400.
32. Sakawa, M.; Nishizaki, I. Max-min solutions for fuzzy multiobjective matrix games. *Fuzzy Sets Syst.* **1994**, *67*, 53–69.
33. Qiu, D.; Dong, R.; Chen, S.; Li, A. On an Optimization Method Based on Z-Numbers and the Multi-Objective Evolutionary Algorithm. *Intell. Autom. Soft Comput.* **2017**, 1–4, doi:10.1080/10798587.2017.1327153.
34. Sun, J.; Gong, D.; Sun, X. Solving interval multi-objective optimization problems using evolutionary algorithms with lower limit of possibility degree. *Chin. J. Electron.* **2013**, *22*, 269–272.
35. Diamond, P.; Kloeden, P. Metric spaces of fuzzy sets. *Fuzzy Sets Syst.* **1990**, *35*, 241–249.
36. Dubois, D.J. *Fuzzy Sets and Systems: Theory and Applications*; Academic Press: Pittsburgh, PA, USA, 1980.
37. Osuna-Gómez, R.; Chalco-Cano, Y.; Rufián-Lizana, A.; Hernández-Jiménez, B. Necessary and sufficient conditions for fuzzy optimality problems. *Fuzzy Sets Syst.* **2016**, *296*, 112–123.
38. Li, L.; Liu, S.; Zhang, J. On fuzzy generalized convex mappings and optimality conditions for fuzzy weakly univex mappings. *Fuzzy Sets Syst.* **2015**, *280*, 107–132.