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Mathematical Properties on the Hyperbolicity of Interval Graphs

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Abstract: Gromov hyperbolicity is an interesting geometric property, and so it is natural to study it in the context of geometric graphs. In particular, we are interested in interval and indifference graphs, which are important classes of intersection and Euclidean graphs, respectively. Interval graphs (with a very weak hypothesis) and indifference graphs are hyperbolic. In this paper, we give a sharp bound for their hyperbolicity constants. The main result in this paper is the study of the hyperbolicity constant of every interval graph with edges of length 1. Moreover, we obtain sharp estimates for the hyperbolicity constant of the complement of any interval graph with edges of length 1.

Keywords: interval graphs; indifference graphs; euclidean graphs; geometric graphs; Gromov hyperbolicity; geodesics

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1. Introduction

The focus of the first works on Gromov hyperbolic spaces were finitely generated groups [1]. Initially, the main application of hyperbolic spaces were the automatic groups (see, e.g., [2]). This concept appears also in some algorithmic problems (see [3] and the references therein). Besides, they are useful in the study of secure transmission of information on the internet [4].

In [5], the equivalence of the hyperbolicity of graphs and negatively curved surfaces was proved. The study of hyperbolic graphs is a topic of increasing interest (see, e.g., [4–28] and the references therein).

If $\gamma : [a,b] \to X$ is a continuous curve in the metric space (X,d), γ is a *geodesic* if $L_X(\gamma|_{[t,s]}) = d(\gamma(t),\gamma(s)) = |t-s|$ for every $t,s \in [a,b]$. We say that X is a *geodesic metric space* if, for every $x,y \in X$, there exists a geodesic in X joining them. Let us denote by $[xy]$ any geodesic joining x and y (this notation is very convenient although it is ambiguous, recall that we do not assume uniqueness of geodesics). Consequently, any geodesic metric space is connected.

$G = (V(G),E(G))$ will denote a non-trivial ($V(E) \neq \emptyset$) simple graph such that we have defined a length function, denoted by L_G or L, on the edges $L_G : E(G) \to \mathbb{R}_+$; the length of a path $\eta = \{e_1,e_2,\ldots,e_k\}$ is defined as $L_G(\eta) = \sum_{i=1}^{k} L_G(e_i)$. We assume that $\ell(G) := \sup\{L_G(e) \mid e \in E(G)\} < \infty$. In order to consider a graph G as a geodesic metric space, identify (by an isometry $\mathcal{I}$) any edge $uv \in E(G)$ with the interval $[0,L_G(uv)]$ in the real line; then, the real interval $[0,L_G(uv)]$ is isometric to the edge uv (considered as a graph with a single edge). If $x,y \in uv$ and η_{xy} denotes the segment contained in uv joining x and y, we define the length of η_{xy} as $L_G(\eta_{xy}) = |\mathcal{I}(x) - \mathcal{I}(y)|$. Thus, the points in G are the vertices $u \in V(G)$ and, in addition, the points in the interior of any edge $uv \in E(G)$.

We denote by d_G or d the natural distance of the graph G. If x, y belong to different connected components of G, then let us define $d_G(x, y) = \infty$. In Section 3, we just consider graphs with every edge of length 1. Otherwise, if a graph G has edges with different lengths, then we also assume that it is locally finite. These properties guarantee that any connected component of G is a geodesic metric space.

If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, the union of three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ is a *geodesic triangle* that will be denoted by $T = \{x_1, x_2, x_3\}$ and we will say that x_1, x_2 and x_3 are the vertices of T; we can also write $T = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$. The triangle T is *δ-thin* if any side of T is contained in the δ-neighborhood of the union of the two other sides. Let us denote by $\delta(T)$ the sharp thin constant of the geodesic triangle T, i.e., $\delta(T) := \inf\{\delta \geq 0 \mid T \text{ is } \delta\text{-thin}\}$. We say that the space X is *δ-hyperbolic* if every geodesic triangle in X is δ-thin. Let us define:

$$\delta(X) := \sup\{\delta(T) \mid T \text{ is a geodesic triangle in } X\}.$$

The geodesic metric space X is *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$; then, X is hyperbolic if and only if $\delta(X) < \infty$. If Y is the union of geodesic metric spaces $\{Y_i\}_{i \in I}$, we define its hyperbolicity constant by $\delta(Y) := \sup_{i \in I} \delta(Y_i)$, and we say that Y is hyperbolic if $\delta(Y) < \infty$.

To relate hyperbolicity with other properties of graphs is an interesting problem. The papers [6,9,28] prove, respectively, that chordal, k-chordal and edge-chordal graphs are hyperbolic; these results are improved in [23]. In addition, several authors have proved results on hyperbolicity for some particular classes of graphs (see, e.g., [21,29–31]).

A geometric graph is a graph in which the vertices or edges are associated with geometric objects. Two of the main classes of geometric graphs are Euclidean graphs and intersection graphs. A graph is *Euclidean* if the vertices are points in $\mathbb{R}^n$ and the length of each edge connecting two vertices is the Euclidean distance between them (this makes a lot of sense with the cities and roads analogy commonly used to describe graphs). An *intersection graph* is a graph in which each vertex corresponds with a set, and two vertices are connected by an edge if and only if their corresponding sets have non-empty intersection. In this paper, we work with interval graphs (a class of intersection graphs) and indifference graphs (a class of Euclidean graphs).

We say that G is an *interval graph* if it is the intersection graph of a family of intervals in $\mathbb{R}$: there is a vertex for each interval in the family, and an edge joins two vertices if and only if the their corresponding intervals intersect. Usually, we consider that every edge of an interval graph has length 1, but we also consider interval graphs whose edges have different lengths. It is well-known that interval graphs are always chordal graphs [32,33]. The complements of interval graphs also have interesting properties: they are comparability graphs [34], and the comparability relations are the interval orders [32]. The theory of interval graphs was developed focused on its applications by researchers at the RAND Corporation's mathematics department (pp. ix–10, [35]).

An *indifference graph* is an interval graph whose vertices correspond to a set of intervals with length 1, and the length of the corresponding edge to two unit intervals that intersect is the distance between their midpoints. In addition, we can see an indifference graph as an Euclidean graph in $\mathbb{R}$ constructed by taking the vertex set as a subset of $\mathbb{R}$ and two vertices are connected by an edge if and only if they are within one unit from each other. Since it is a Euclidean graph, the length of each edge connecting two vertices is the Euclidean distance between them. Indifference graphs possess several interesting properties: connected indifference graphs have Hamiltonian paths [36]; an indifference graph has a Hamiltonian cycle if and only if it is biconnected [37]. In the same direction, we consider indifference graphs since for these graphs we can remove one of the hypothesis of a main theorem on interval graphs (compare Theorem 8 and Corollary 6).

We would like to mention that Ref. [38] collects very rich results, especially those concerning path properties, about interval graphs and unit interval graphs. It is well-known that interval graphs (with a very weak hypothesis) and indifference graphs are hyperbolic. One of the main results in this paper is Theorem 8, which provides a sharp upper bound of the hyperbolicity constant of interval graphs verifying a very weak hypothesis. This result allows for obtaining bounds for the hyperbolicity constant of every indifference graph (Corollary 6) and the hyperbolicity constant of every interval graph with edges of length 1 (Corollary 7). Moreover, Theorem 10 provides sharp bounds for the hyperbolicity constant of the complement of any interval graph with edges of length 1. Note that it is not usual to obtain such precise bounds for large classes of graphs. The main result in this paper is Theorem 9, which allows for computing the hyperbolicity constant of every interval graph with edges of length 1, by using geometric criteria.

2. Previous Results

We collect some previous results that will be useful along the paper.

A *cycle* is a path with different vertices, unless the last vertex, which is equal to the first one.

Lemma 1. *([39] Lemma 2.1) Let us consider a geodesic metric space X. If every geodesic triangle in X that is a cycle is δ-thin, then X is δ-hyperbolic.*

Corollary 1. *In any geodesic metric space X,*

$$\delta(X) = \sup\{\delta(T) \mid T \text{ is a geodesic triangle that is a cycle}\}.$$

Recall that a *chordal graph* is one in which all cycles of four or more vertices have a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle.

If C is a cycle in G and $v \in V(G)$, we denote by $\deg_C(v)$ the degree of the vertex v in the subgraph Γ induced by $V(C)$ (note that Γ could contain edges that are not contained in C, and thus it is possible to have $\deg_C(v) > 2$).

Lemma 2. *([9] Lemma 2.2) Consider a chordal graph G and a cycle C in G with $a, v, b \in C \cap V(G)$ and $av, vb \in E(G)$. If $ab \notin E(G)$, then $\deg_C(v) \geq 3$.*

Corollary 2. *Consider a cycle C in a chordal graph G and v_1, v_2, v_3 consecutive vertices in C. If $\deg_C(v_2) = 2$, then $v_1v_3 \in E(G)$. Consequently, if C has at least four vertices, then $\deg_C(v_1) \geq 3$ and $\deg_C(v_3) \geq 3$.*

Let $J(G)$ be the set of vertices and midpoints of edges in G. Consider the set $\mathbb{T}_1$ of geodesic triangles T in G that are cycles and such that the three vertices of the triangle T belong to $J(G)$, and denote by $\delta_1(G)$ the infimum of the constants λ such that every triangle in $\mathbb{T}_1$ is λ-thin.

Theorem 1. *([40] Theorem 2.5) For every graph G with edges of length 1, we have $\delta_1(G) = \delta(G)$.*

The next result will narrow the possible values for the hyperbolicity constant.

Theorem 2. *([40] Theorem 2.6) If G is a hyperbolic graph with edges of length 1, then $\delta(G)$ is a multiple of $1/4$.*

Theorem 3. *([40] Theorem 2.7) If G is a hyperbolic graph with edges of length 1, then there exists a geodesic triangle $T \in \mathbb{T}_1$ such that $\delta(T) = \delta(G)$.*

In the following theorems, we study the graphs G with $\delta(G) < 1$.

Theorem 4. *([41] Theorem 11) If G is a graph with edges of length 1 with $\delta(G) < 1$, then we have either $\delta(G) = 0$ or $\delta(G) = 3/4$. Furthermore,*

- *$\delta(G) = 0$ if and only if G is a tree.*
- *$\delta(G) = 3/4$ if and only if G is not a tree and every cycle in G has length 3.*

Corollary 3. *A graph G with edges of length 1 satisfies $\delta(G) \geq 1$ if and only if there exists a cycle in G with length at least 4.*

In order to characterize from a geometric viewpoint the interval graphs with hyperbolicity constant 1, we need the following result, which is a direct consequence of Theorems 2 and 4, and ([7] Theorem 4.14).

Theorem 5. *Let G be any graph with edges of length 1. We have $\delta(G) = 1$ if and only if $\delta(G) \notin \{0, 3/4\}$ and, for every cycle C in G and every $x, y \in C \cap J(G)$, we have $d(x, y) \leq 2$.*

Theorems 4 and 5 have the following consequences.

Corollary 4. *Let G be any graph with edges of length 1. We have $\delta(G) \leq 1$ if and only if, for every cycle C in G and every $x, y \in C \cap J(G)$, we have $d(x, y) \leq 2$.*

By Theorems 2 and 4, and ([7] Theorems 4.14 and 4.21), we have the following result.

Theorem 6. *Let G be any graph with edges of length 1. If there exists a cycle in G with $p, q \in V(G)$ and $d(p, q) \geq 3$, then $\delta(G) \geq 3/2$.*

We will also need this last result.

Theorem 7. *([41] Theorem 30) If G is any graph with edges of length 1 and n vertices, then $\delta(G) \leq n/4$.*

3. Interval Graphs and Hyperbolicity

Given a cycle C in an interval graph G, let $\{v_1, \ldots, v_k\}$ be the vertices in G with

$$C = v_1 v_2 \cup \cdots \cup v_{k-1} v_k \cup v_k v_1.$$

Denote by $\{I_1, \ldots, I_k\}$ the corresponding intervals to $\{v_1, \ldots, v_k\}$. If $I_j = [a_j, b_j]$, then let us define the *minimal interval of* C as the interval $I_{j_1} = [a_{j_1}, b_{j_1}]$ with $a_{j_1} \leq a_j$ for every $1 \leq j \leq k$ and $b_{j_1} > b_j$ if $a_j = a_{j_1}$ with $1 \leq j \leq k$ and $j \neq j_1$, and the *maximal interval of* C as the interval $I_{j_2} = [a_{j_2}, b_{j_2}]$ with $b_{j_2} \geq b_j$ for every $1 \leq j \leq k$ and $a_{j_2} < a_j$ if $b_j = b_{j_2}$ with $1 \leq j \leq k$ and $j \neq j_2$. If $i \in \mathbb{Z} \setminus \{1, 2, \ldots, k\}$, $1 \leq j \leq k$ and $i = j \,(\mathrm{mod}\, k)$, then we define $v_i := v_j$ and $I_i := I_j$.

If H is a subgraph of G and $w \in V(H)$, we denote by $\deg_H(w)$ the degree of the vertex w in the subgraph induced by $V(H)$.

For any graph G,

$$\operatorname{diam} V(G) := \sup \{ d_G(v, w) \mid v, w \in V(G) \},$$
$$\operatorname{diam} G := \sup \{ d_G(x, y) \mid x, y \in G \},$$

i.e., $\operatorname{diam} V(G)$ is the diameter of the set of vertices of G, and $\operatorname{diam} G$ is the diameter of the whole graph G (recall that in order to have a geodesic metric space, G must contain both the vertices and the points in the interior of any edge of G).

The following result is well-known.

Lemma 3. *For any geodesic triangle T in a graph G, we have $\delta(T) \leq (\operatorname{diam} T)/2 \leq L(T)/4$.*

Corollary 5. *The inequalities*

$$\delta(G) \leq \frac{1}{2} \operatorname{diam} G \leq \frac{1}{2} \left(\operatorname{diam} V(G) + \ell(G) \right)$$

hold for every graph G.

A graph G is *length-proper* if every edge is a geodesic. A large class of length-proper graphs are the graphs with edges of length 1. Another important class of length-proper graphs are the following geometric graphs: consider a discrete set V in an Euclidean space (or in a metric space) where we consider two points connected by an edge if some criterium is satisfied. If we define the length of an edge as the distance between its vertices, then we obtain a length-proper graph.

It is well-known that every interval graph is chordal. Hence, every length-proper interval graph is hyperbolic. The following result is one of the main theorems in this paper, since it provides a sharp inequality for the hyperbolicity constant of any length-proper interval graph. Recall that $\ell(G) := \sup\{L_G(e) \mid e \in E(G)\}$.

Theorem 8. *Every length-proper interval graph G satisfies the sharp inequality*

$$\delta(G) \leq \frac{3}{2}\,\ell(G).$$

Proof. Consider a geodesic triangle $T = \{x, y, z\}$ that is a cycle in G and $p \in [xy]$. Assume first that T satisfies the following property:

$$\text{if } a, b \in V(G) \cap [xy] \text{ and } ab \in E(G), \text{ then } ab \subseteq [xy]. \tag{1}$$

Consider the consecutive vertices $\{v_1, \dots, v_k\}$ in the cycle T, and their corresponding intervals $\{I_1, \dots, I_k\}$. As before, we denote by I_{j_1} and I_{j_2} the minimal and maximal intervals, respectively.

If $k < 4$, then $L(T) \leq 3\ell(G)$ and Lemma 3 gives:

$$d(p, [xz] \cup [zy]) \leq \frac{1}{4}\,L(T) \leq \frac{3}{4}\,\ell(G). \tag{2}$$

Assume now that $k \geq 4$.

Case (A). Assume that $p \in V(G)$. Let $a, b \in V(G)$ with $ap, bp \in E(G)$ and $ap \cup bp \subset T$.

Case $(A.1)$. If $ab \notin E(G)$, then Lemma 2 gives $\deg_T(p) \geq 3$, and there exists $q \in V(G) \cap T$ with $pq \in E(G)$ such that pq is not contained in T. By (1), $q \in [xz] \cup [zy]$ and so:

$$d(p, [xz] \cup [zy]) \leq d(p, q) = L(pq) \leq \ell(G). \tag{3}$$

Case $(A.2)$. If $ab \in E(G)$, then ab is not contained in T, since T is a cycle and $k \geq 4$. By (1), $\{a, b\}$ is not contained in $[xy]$, and:

$$d(p, [xz] \cup [zy]) \leq \max\{d(p, a),\, d(p, b)\} = \max\{L(pa),\, L(pb)\} \leq \ell(G). \tag{4}$$

Case (B). Assume that $p \notin V(G)$. Let $a, b \in V(G)$ with $p \in ab \subset T$ and $d(p, a) \leq L(ab)/2 \leq \ell(G)/2$. Corollary 2 gives that we have $\deg_T(a) \geq 3$ or $\deg_T(b) \geq 3$.

Case $(B.1)$. Assume that $\deg_T(a) \geq 3$.

Case $(B.1.1)$. If $a \notin [xy]$, then:

$$d(p, [xz] \cup [zy]) \leq d(p, a) \leq \frac{1}{2}\,\ell(G). \tag{5}$$

Case $(B.1.2)$. Assume that $a \in [xy]$. Since $\deg_T(a) \geq 3$, there exists $q \in V(G) \cap T$ with $aq \in E(G)$ such that aq is not contained in T. By (1), $q \in [xz] \cup [zy]$ and so:

$$\begin{aligned} d(p, [xz] \cup [zy]) &\leq d(p, a) + d(a, [xz] \cup [zy]) \leq d(p, a) + d(a, q) \\ &= d(p, a) + L(aq) \leq \frac{1}{2}\,\ell(G) + \ell(G) = \frac{3}{2}\,\ell(G). \end{aligned} \tag{6}$$

Case $(B.2)$. Assume that $\deg_T(a) = 2$ and $\deg_T(b) \geq 3$. Let $\alpha \neq b$ with $\alpha \in V(G)$, $\alpha a \in E(G)$ and $\alpha a \subset T$. Corollary 2 gives that we have $\alpha b \in E(G)$. By (1), we have that $\{\alpha, b\}$ is not contained in $[xy]$, and:

$$\begin{aligned} d(p,[xz]\cup[zy]) &\leq \max\{d(p,\alpha),\, d(p,b)\} \leq \max\{d(p,a)+d(a,\alpha),\, d(p,b)\} \\ &\leq \max\left\{\frac{1}{2}\,\ell(G)+\ell(G),\, \ell(G)\right\} = \frac{3}{2}\,\ell(G). \end{aligned} \tag{7}$$

Inequalities (2)–(7) give in every case $d(p,[xz]\cup[zy]) \leq 3\ell(G)/2$.

Consider now a geodesic triangle $T = \{x,y,z\} = \{[xy],[xz],[yz]\}$ that does not satisfy property (1). We are going to obtain a new geodesic γ joining x and y such that the geodesic triangle $T' = \{\gamma,[xz],[yz]\}$ satisfies (1).

Let us define inductively a finite sequence of geodesics $\{g_0, g_1, g_2, \ldots, g_r\}$ joining x and y in the following way:

If $j = 0$, then $g_0 := [xy]$.

Assume that $j \geq 1$. If the geodesic triangle $\{g_{j-1},[xz],[yz]\}$ satisfies (1), then $r = j-1$ and the sequence stops. If $\{g_{j-1},[xz],[yz]\}$ does not satisfy (1), then there exists $a, b \in V(G)\cap[xy]$ such that $ab \in E(G)$ and ab is not contained in $[xy]$. Denote by $[ab]$ the geodesic joining a and b contained in g_{j-1}. Let us define $g_j := (g_{j-1}\setminus[ab])\cup ab$. Note that $g_j\cap V(G) \subset g_{j-1}\cap V(G)$ and $|g_j\cap V(G)| < |g_{j-1}\cap V(G)|$.

Since $|g_j\cap V(G)| < |g_{j-1}\cap V(G)|$ for any $j \geq 1$, this sequence must finish with some geodesic g_r such that the geodesic triangle $T' := \{g_r,[xz],[yz]\}$ satisfies (1). Thus, define $\gamma := g_r$. Hence,

$$g_r\cap V(G) \subset g_{r-1}\cap V(G) \subset \cdots \subset g_1\cap V(G) \subset g_0\cap V(G),$$

and so $\gamma\cap V(G) \subset [xy]\cap V(G)$.

Let us consider $p \in [xy] \subset T$.

If $p \in \gamma \subset T'$, then, by applying the previous argument to the geodesic triangle T', we obtain $d(p,[xz]\cup[zy]) \leq 3\ell(G)/2$. Assume that $p \notin \gamma$.

Since $\gamma\cap V(G) \subset [xy]\cap V(G)$, there exist $v, w \in \gamma\cap V(G)$ with $vw \in E(G)$ such that, if $[vw]$ denotes the geodesic joining v and w contained in $[xy]$, then:

$$p \in [vw], \qquad [vw]\cap vw = \{v,w\}.$$

Since vw and $[vw]$ are geodesics, we have $L(vw) = L([vw])$. Thus, we can define $p' \in \gamma$ as the point in vw with $d(p',v) = d(p,v)$ and $d(p',w) = d(p,w)$. By applying the previous argument to p' and T', we obtain $d(p',[xz]\cup[zy]) \leq 3\ell(G)/2$. Since p' belongs to the edge vw, we have $d(p',[xz]\cup[zy]) = d(p',v)+d(v,[xz]\cup[zy])$ or $d(p',[xz]\cup[zy]) = d(p',w)+d(w,[xz]\cup[zy])$. By symmetry, we can assume that $d(p',[xz]\cup[zy]) = d(p',v)+d(v,[xz]\cup[zy])$. Since $d(p',v) = d(p,v)$, we have:

$$d(p,[xz]\cup[zy]) \leq d(p,v)+d(v,[xz]\cup[zy]) = d(p',v)+d(v,[xz]\cup[zy]) = d(p',[xz]\cup[zy]) \leq \frac{3}{2}\,\ell(G).$$

Finally, Corollary 1 gives $\delta(G) \leq 3\ell(G)/2$.

Proposition 1 below shows that the inequality is sharp. □

Note that, if we remove the hypothesis $\ell(G) < \infty$, then there are non-hyperbolic length-proper interval graphs: if Γ is any graph such that every cycle in Γ has exactly three vertices and $\sup\{L(C) \mid C \text{ is a cycle in } \Gamma\} = \infty$, then Γ is a non-hyperbolic chordal graph. Some of these graphs Γ are length-proper interval graphs.

Recall that every indifference graph is an Euclidean graph. Hence, every indifference graph G is a length-proper graph and $\ell(G) \leq 1$.

Theorem 8 has the following direct consequence.

Corollary 6. *Every indifference graph G satisfies the inequality:*

$$\delta(G) \le \frac{3}{2}\,\ell(G) \le \frac{3}{2}\,.$$

4. Interval Graphs with Edges of Length 1

Along this section, we just consider graphs with edges of length 1. This is a very usual class of graphs. Note that every graph G with edges of length 1 is a length-proper graph with $\ell(G) = 1$.

The goal of this section is to compute the precise value of the hyperbolicity constant of every interval graph with edges of length 1 (see Theorem 9). We wish to emphasize that it is unusual to be able to compute the hyperbolicity constant of every graph in a large class of graphs. Let us start with a direct consequence of Theorem 8.

Corollary 7. *Every interval graph G with edges of length* 1 *satisfies the inequality:*

$$\delta(G) \le \frac{3}{2}\,.$$

First of all, we characterize the interval graphs with edges of length 1 and $\delta(G) = 3/2$ in Proposition 1 below. Furthermore, Proposition 1 shows that the inequality in Theorem 8 is sharp.

Let G be an interval graph. We say that G has the $(3/2)$-*intersection property* if there exists two disjoint intervals I' and I'' corresponding to vertices in a cycle C in G such that there is no corresponding interval I to a vertex in G with $I \cap I' \neq \emptyset$ and $I \cap I'' \neq \emptyset$.

Proposition 1. *An interval graph G with edges of length* 1 *satisfies* $\delta(G) = 3/2$ *if and only if G has the* $(3/2)$-*intersection property.*

Proof. Assume that G has the $(3/2)$-intersection property. Thus, there exist two disjoint corresponding intervals I' and I'' to vertices in a cycle C in G such that there is no corresponding interval I to a vertex in G with $I \cap I' \neq \emptyset$ and $I \cap I'' \neq \emptyset$. If v' and v'' are the corresponding vertices to I' and I'', respectively, then $v', v'' \in C$ and $d(v', v'') \ge 3$. Thus, Theorem 6 gives $\delta(G) \ge 3/2$ and, since $\delta(G) \le 3/2$ by Corollary 7, we conclude $\delta(G) = 3/2$.

Assume now that G does not have the $(3/2)$-intersection property. Seeking for a contradiction, assume that $\delta(G) = 3/2$. By Theorem 3, there exist a geodesic triangle $T = \{x, y, z\}$ that is a cycle in G and $p \in [xy]$ such that $d(p, [xz] \cup [zy]) = \delta(T) = \delta(G) = 3/2$ and $x, y, z \in J(G)$. Since $d(p, \{x, y\}) \ge d(p, [xz] \cup [zy]) = 3/2$, we have $d(x, y) \ge 3$. Since G does not have the $(3/2)$-intersection property, for each two disjoint corresponding intervals I' and I'' to vertices in the cycle T, there exists a corresponding interval I to a vertex in G with $I \cap I' \neq \emptyset$ and $I \cap I'' \neq \emptyset$. If v' and v'' are the corresponding vertices to I' and I'', respectively, then $v', v'' \in T$ and $d(v', v'') = 2$. We conclude that $\operatorname{diam}(T \cap V(G)) \le 2$ and $\operatorname{diam} T \le 3$. Since $d(x, y) \ge 3$ with $x, y \in J(G)$, we have $\operatorname{diam}(T \cap V(G)) = 2$, $\operatorname{diam} T = 3$, $d(x, y) = 3$, $L([xy])/2 = d(p, x) = d(p, y) = d(p, [xz] \cup [zy]) = \delta(T) = \delta(G) = 3/2$ and p is the midpoint of $[xy]$. Thus $x, y \in J(G) \setminus V(G)$ and $p \in V(G)$. If $x \in x_1x_2 \in E(G)$ and $y \in y_1y_2 \in E(G)$, then $d(\{x_1, x_2\}, \{y_1, y_2\}) = 2$. Let $I_{x_1}, I_{x_2}, I_{y_1}, I_{y_2}, I_p$ be the corresponding intervals to the vertices x_1, x_2, y_1, y_2, p, respectively. We can assume that $x_1, y_1 \in [xy]$ and thus $I_{x_1} \cap I_p \neq \emptyset$ and $I_{y_1} \cap I_p \neq \emptyset$ since $d(x_1, y_1) = 2$, $I_{x_1} \cap I_{y_1} = \emptyset$. Thus, there exists $\zeta \in I_p \setminus (I_{x_1} \cup I_{y_1})$. Since $[xy] \cap V(G) = \{x_1, p, y_1\}$ and T is a cycle containing x_1, p, y_1, by continuity, there exists a corresponding interval J to a vertex $v \in ([xz] \cup [zy]) \cap V(G)$ with $\zeta \in J$. Thus, $pv \in E(G)$ and $3/2 = d(p, [xz] \cup [zy]) \le d(p, v) = 1$, which is a contradiction. Hence, $\delta(G) \neq 3/2$. □

Corollary 7 and Theorems 2 and 4 give that $\delta(G) \in \{0, 3/4, 1, 5/4, 3/2\}$ for every interval graph G with edges of length 1. Proposition 1 characterizes the interval graphs with edges of length 1 and

$\delta(G) = 3/2$. In order to characterize the interval graphs with the other values of the hyperbolicity constant, we need some definitions.

Let G be an interval graph.

We say that G has the 0-*intersection property* if, for every three corresponding intervals I', I'' and I''' to vertices in G, we have $I' \cap I'' \cap I''' = \emptyset$.

G has the $(3/4)$-*intersection property* if it does not have the 0-intersection property and for every four corresponding intervals I', I'', I''' and I'''' to vertices in G we have $I' \cap I'' \cap I''' = \emptyset$ or $I' \cap I'' \cap I'''' = \emptyset$.

By a *couple* of intervals in a cycle C of G, we mean the union of two non-disjoint intervals whose corresponding vertices belong to C. We say that G has the 1-*intersection property* if it does not have the 0 and $(3/4)$-*intersection properties* and, for every cycle C in G, each interval and a couple of corresponding intervals to vertices in C are not disjoint.

One can check that every chordal graph that has a cycle with length of at least four has a cycle with length four and, since this cycle has a chord, it also has a cycle with length three.

Next, we provide a characterization of the interval graphs with hyperbolicity constant 0. It is well-known that these are the caterpillar trees, see [42], but we prefer to characterize them by the 0-intersection property in Proposition 2 below, since it looks similar to the other intersection properties.

Proposition 2. *An interval graph G with edges of length 1 satisfies $\delta(G) = 0$ if and only if G has the 0-intersection property.*

Proof. By Theorem 4, $\delta(G) = 0$ if and only if G is a tree. Since every interval graph is chordal, G is not a tree if and only if it contains a cycle with length 3, and this last condition holds if and only if there exist three corresponding intervals I', I'' and I''' to vertices in G with $I' \cap I'' \cap I''' \neq \emptyset$. Hence, G has a cycle if and only if it does not have the 0-intersection property. □

Proposition 3. *An interval graph G with edges of length 1 satisfies $\delta(G) = 3/4$ if and only if G has the $(3/4)$-intersection property.*

Proof. By Theorem 4, $\delta(G) = 3/4$ if and only if G is not a tree and every cycle in G has length 3. Proposition 2 gives that G is not a tree if and only if G does not have the 0-intersection property. Therefore, it suffices to show that every cycle in G has length 3, if and only if for every four corresponding intervals I', I'', I''' and I'''' to vertices in G, we have $I' \cap I'' \cap I''' = \emptyset$ or $I' \cap I'' \cap I'''' = \emptyset$.

Since every interval graph is chordal, G has a cycle with length at least 4 if and only if it has a cycle C with length 4 and this cycle has at least a chord.

Assume first that there exists such a cycle C. If I', I'', I''', I'''' are the corresponding intervals to the vertices in C and I', I'' corresponds to vertices with a chord, and then $I' \cap I'' \cap I''' \neq \emptyset$ and $I' \cap I'' \cap I'''' \neq \emptyset$.

Assume now that there are corresponding intervals I', I'', I''', I'''' to the vertices v', v'', v''', v'''' in G with $I' \cap I'' \cap I''' \neq \emptyset$ and $I' \cap I'' \cap I'''' \neq \emptyset$. Thus, $v'v''', v''v''' \in E(G)$ and $v'v'''', v''v'''' \in E(G)$, and so $v'v''' \cup v'''v'' \cup v''v'''' \cup v''''v'$ is a cycle in G with length 4. □

Proposition 4. *An interval graph G with edges of length 1 satisfies $\delta(G) = 1$ if and only if G has the 1-intersection property.*

Proof. By Theorem 5, $\delta(G) = 1$ if and only if $\delta(G) \notin \{0, 3/4\}$, and, for every cycle C in G and every $x, y \in C \cap J(G)$, we have $d(x,y) \leq 2$. Propositions 2 and 3 give that $\delta(G) \notin \{0, 3/4\}$ if and only if G does not have the 0 and $(3/4)$-intersection properties. Therefore, it suffices to show that for every cycle C in G, we have $d(x,y) \leq 2$ for every $x, y \in C \cap J(G)$ if and only if each interval and couple of corresponding intervals to vertices in C are not disjoint.

Fix a cycle C in G. Each interval and couple of corresponding intervals to vertices in C are not disjoint if and only if $d(x,y) \leq 3/2$ for every $x \in C \cap V(G)$ and $y \in C \cap (J(G) \setminus V(G))$. Since every point in $C \cap (J(G) \setminus V(G))$ has a point in $C \cap V(G)$ at distance $1/2$, this last condition is equivalent to $d(x,y) \leq 2$ for every $x, y \in C \cap J(G)$. □

Finally, we collect the previous geometric characterizations in the following theorem. Note that the characterization of $\delta(G) = 5/4$ in Theorem 9 is much simpler than the one in [7]. Recall that to characterize the graphs with hyperbolicity $3/2$ is a very difficult task, as it was shown in ([7] Remark 4.19).

Theorem 9. *Every interval graph G with edges of length 1 is hyperbolic and $\delta(G) \in \{0, 3/4, 1, 5/4, 3/2\}$. Furthermore,*

- *$\delta(G) = 0$ if and only if G has the 0-intersection property.*
- *$\delta(G) = 3/4$ if and only if G has the $(3/4)$-intersection property.*
- *$\delta(G) = 1$ if and only if G has the 1-intersection property.*
- *$\delta(G) = 5/4$ if and only if G does not have the 0, $3/4$, 1 and $(3/2)$-intersection properties.*
- *$\delta(G) = 3/2$ if and only if G has the $(3/2)$-intersection property.*

Complement of Interval Graphs

The *complement* $\overline{G}$ of the graph G is defined as the graph with $V(\overline{G}) = V(G)$ and such that $e \in E(\overline{G})$ if and only if $e \notin E(G)$. Recall that, for every disconnected graph G, we define $\delta(G)$ as the supremum of $\delta(G_i)$, where G_i varies in the set of connected components of G.

We consider that the length of the edges of every complement graph is 1.

If Γ is a subgraph of G, we consider in Γ the *inner metric* obtained by the restriction of the metric in G, that is:

$$d_\Gamma(v,w) := \inf\{L(\gamma) \mid \gamma \subset \Gamma \text{ is a continuous curve joining } v \text{ and } w\} \geq d_G(v,w)\,.$$

Note that the inner metric d_Γ is the usual metric if we consider the subgraph Γ as a graph.

Since the complements of interval graphs belong to the class of comparability graphs [34], it is natural to also study the hyperbolicity constant of complements of interval graphs. In order to do it, we need some preliminary results and the following technical lemma.

Lemma 4. *Let G be an interval graph with edges of length 1, $V(G) = \{v_1, \ldots, v_r\}$ and corresponding intervals $\{I_1, \ldots, I_r\}$. We have $\operatorname{diam} V(G) = 2$ if and only if there exists an interval I_i with $I_j \cap I_i \neq \emptyset$ for every $1 \leq j \leq r$ and $\operatorname{diam} V(G') \geq 2$, where G' is the corresponding interval graph to $\{I_1, \ldots, I_r\} \setminus I_i$. Furthermore, if this is the case, then $\delta(\overline{G}) = \delta(\overline{G'})$.*

Proof. Assume that $\operatorname{diam} V(G) = 2$. Let $[a_j, b_j] = I_j$ for $1 \leq j \leq r$. Consider integers $1 \leq i_1, i_2 \leq r$ satisfying:

$$b_{i_1} \leq b_j, \qquad a_j \leq a_{i_2}, \qquad \text{for every } 1 \leq j \leq r. \tag{8}$$

Since $\operatorname{diam} V(G) = 2$, we have $b_{i_1} < a_{i_2}$. Thus, $d_G(v_{i_1}, v_{i_2}) = 2$ and there exists i with $v_i v_{i_1}, v_i v_{i_2} \in E(G)$. Hence, $I_{i_1} \cap I_i \neq \emptyset$ and $I_{i_2} \cap I_i \neq \emptyset$. Thus, (8) gives $I_j \cap I_i \neq \emptyset$ for every $1 \leq j \leq r$, and we deduce $d_G(v_j, v_i) \leq 1$ for every $1 \leq j \leq r$.

Seeking for a contradiction assume that $\operatorname{diam} V(G') \leq 1$. Thus, $d_G(v_j, v_{j'}) \leq d_{G'}(v_j, v_{j'}) \leq 1$ for every $1 \leq j, j' \leq r$ with $j, j' \neq i$. Furthermore, we have proved $d_G(v_j, v_i) \leq 1$ for every $1 \leq j \leq r$. Therefore, $d_G(v_j, v_{j'}) \leq 1$ for every $1 \leq j, j' \leq r$ and we conclude $\operatorname{diam} V(G) \leq 1$, which is a contradiction. Hence, $\operatorname{diam} V(G') \geq 2$.

The converse implication is well-known.

Finally, since $v_j v_i \in E(G)$ for every $1 \leq j \leq r$ with $j \neq i$, we have $\overline{G} = \{v_i\} \cup \overline{G'}$ and:

$$\delta(\overline{G}) = \max\{\delta(\{v_i\}), \delta(\overline{G'})\} = \max\{0, \delta(\overline{G'})\} = \delta(\overline{G'}).$$

□

Note that it is not usual to obtain such close lower and upper bounds for a large class of graphs. Some inequalities are not difficult to prove; the most difficult cases are the upper bound when $\operatorname{diam} V(G) = 2$ (recall that this is the more difficult case in the study of the complement of a graph), and the lower bound when $\operatorname{diam} V(G) \geq 4$.

Theorem 10. *Let G be any interval graph.*

- *If* $\operatorname{diam} V(G) = 1$, *then* $\delta(\overline{G}) = 0$.
- *If* $2 \leq \operatorname{diam} V(G) \leq 3$, *then* $0 \leq \delta(\overline{G}) \leq 2$.
- *If* $\operatorname{diam} V(G) \geq 4$, *then* $5/4 \leq \delta(\overline{G}) \leq 3/2$.

Furthermore, the lower bounds on $\delta(\overline{G})$ *are sharp.*

Proof. If $\operatorname{diam} V(G) = 1$, then G is a complete graph. Thus, $\overline{G}$ is a union of isolated vertices and $\delta(\overline{G}) = 0$.

Let us prove now the upper bounds.

It is well-known that if $\operatorname{diam} V(G) \geq 3$, then $\overline{G}$ is connected and $\operatorname{diam} V(\overline{G}) \leq 3$. Therefore, Corollary 5 gives $\delta(\overline{G}) \leq 2$.

If $\operatorname{diam} V(G) \geq 4$, then ([43] Theorem 2.14) gives $\delta(\overline{G}) \leq 3/2$.

Assume now that $\operatorname{diam} V(G) = 2$. By Lemma 4, there exists an interval graph G' with $|V(G')| = |V(G)| - 1$, $\operatorname{diam} V(G') \geq 2$ and $\delta(\overline{G}) = \delta(\overline{G'})$. Let us define inductively a finite sequence of interval graphs $\{G^{(0)}, G^{(1)}, G^{(2)}, \dots, G^{(k)}\}$ with:

$$\delta(\overline{G^{(0)}}) = \delta(\overline{G^{(1)}}) = \delta(\overline{G^{(2)}}) = \cdots = \delta(\overline{G^{(k)}}),$$

$$|V(G^{(j)})| = |V(G^{(j-1)})| - 1, \qquad \text{for } 0 < j \leq k,$$

$$\operatorname{diam} V(G^{(j)}) \geq 2, \qquad \text{for } 0 \leq j \leq k,$$

in the following way:

If $j = 0$, then $G^{(0)} := G$.

If $j = 1$, then $G^{(1)} := G'$.

Assume that $j > 1$. If $\operatorname{diam} V(G^{(j-1)}) \geq 3$, then $k = j - 1$ and the sequence stops. If $\operatorname{diam} V(G^{(j-1)}) = 2$, then Lemma 4 provides an interval graph $(G^{(j-1)})'$ with:

$$|V((G^{(j-1)})')| = |V(G^{(j-1)})| - 1, \quad \operatorname{diam} V((G^{(j-1)})') \geq 2, \quad \delta(\overline{G^{(j-1)}}) = \delta(\overline{(G^{(j-1)})'}),$$

and we define $G^{(j)} := (G^{(j-1)})'$.

Since $|V(G^{(j)})| = |V(G^{(j-1)})| - 1$ for $0 < j \leq k$ and the diameter of a graph with just a vertex is 0, this sequence must finish with some graph $G^{(k)}$ satisfying $\operatorname{diam} V(G^{(k)}) \geq 3$. Thus,

$$\delta(\overline{G}) = \delta(\overline{G^{(0)}}) = \delta(\overline{G^{(1)}}) = \cdots = \delta(\overline{G^{(k)}}) \leq 2.$$

We prove now that $\delta(\overline{G}) \geq 5/4$ if $\operatorname{diam} V(G) \geq 4$. Let us fix any graph G with $\operatorname{diam} V(G) \geq 4$. Thus, there exists a geodesic $[v_0 v_4] = v_0v_1 \cup v_1v_2 \cup v_2v_3 \cup v_3v_4$ in G. If Γ is the subgraph of $\overline{G}$ induced by $\{v_0, v_1, v_2, v_3, v_3, v_4\}$, then $E(\Gamma) = \{v_0v_2, v_0v_3, v_0v_4, v_1v_3, v_1v_4, v_2v_4\}$. Consider the cycle $C := v_0v_2 \cup v_2v_4 \cup v_4v_1 \cup v_1v_3 \cup v_3v_0$ in Γ. If p is the midpoint of v_0v_2, then $d_\Gamma(v_1, p) = 5/2$ and so Corollary 4 gives $\delta(\Gamma) > 1$. Therefore, Theorem 2 gives $\delta(\Gamma) \geq 5/4$. Since Γ is an induced subgraph of $\overline{G}$, if g is a path in $\overline{G}$ joining v_i and v_j $(0 \leq i, j \leq 4)$ and g is not contained in Γ, then $L_{\overline{G}}(g) \geq 2$. Since $\operatorname{diam}_{\overline{G}} V(\Gamma) = 2$, we have $d_\Gamma(v_i, v_j) = d_{\overline{G}}(v_i, v_j)$ for every $0 \leq i, j \leq 4$; consequently, $d_\Gamma(x, y) = d_{\overline{G}}(x, y)$ for every $x, y \in \Gamma$, i.e., Γ is an isometric subgraph of $\overline{G}$. Hence, the geodesic triangles in Γ are also geodesic triangles in $\overline{G}$, and we have $\delta(\overline{G}) \geq \delta(\Gamma) \geq 5/4$.

Let us show now that the lower bounds on $\delta(\overline{G})$ are sharp. Recall that the path graph with n vertices P_n is a graph with $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $E(P_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}$.

Consider the path graph with four vertices $G = P_4$. Since $\overline{G} = P_4$, we have $\operatorname{diam} V(G) = 3$ and $\delta(\overline{G}) = 0$.

Consider the path graph with five vertices $G = P_5$. Since $\operatorname{diam} V(G) = 4$, we have $\delta(\overline{G}) \geq 5/4$. Note that $\overline{G}$ has five vertices and thus Theorem 7 gives $\delta(\overline{G}) \leq 5/4$. Hence, we conclude $\delta(\overline{G}) = 5/4$. $\square$

Corollary 8. *If G is any interval graph with edges of length* 1*, then*

$$\delta(G)\,\delta(\overline{G}) \leq \begin{cases} 0, & \text{if } \operatorname{diam} V(G) = 1, \\ 3, & \text{if } 2 \leq \operatorname{diam} V(G) \leq 3, \\ 9/4, & \text{if } \operatorname{diam} V(G) \geq 4. \end{cases}$$

Note that we can not improve the trivial lower bound $\delta(G)\delta(\overline{G}) \geq 0$, since it is attained if G is any tree.

Corollary 9. *If G is any interval graph with edges of length* 1*, then*

$$\delta(G) + \delta(\overline{G}) \leq \begin{cases} 3/2, & \text{if } \operatorname{diam} V(G) = 1, \\ 7/2, & \text{if } 2 \leq \operatorname{diam} V(G) \leq 3, \\ 3, & \text{if } \operatorname{diam} V(G) \geq 4. \end{cases}$$

In addition, $\delta(G) + \delta(\overline{G}) \geq 5/4$ *for every graph G with* $\operatorname{diam} V(G) \geq 4$.

5. Conclusions

Gromov hyperbolicity is an interesting geometric property, and so it is natural to study it in the context of geometric graphs. In this work we deal with interval and indifference graphs, which are important classes of intersection and Euclidean graphs, respectively. It is well-known that interval graphs (with a very weak hypothesis) and indifference graphs are hyperbolic. One of our main results is Theorem 8, which provides a sharp upper bound of the hyperbolicity constant of interval graphs verifying a very weak hypothesis. This result allows for obtaining bounds for the hyperbolicity constant of every indifference graph (Corollary 6) and the hyperbolicity constant of every interval graph with edges of length 1 (Corollary 7). Moreover, Theorem 10 provides sharp bounds for the hyperbolicity constant of the complement of any interval graph with edges of length 1. Note that it is not usual to obtain such precise bounds for large classes of graphs. Our main result is Theorem 9, which provides the hyperbolicity constant of every interval graph with edges of length 1, by using geometric criteria.

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