

Article

On Classification of Symmetry Reductions for the Eikonal Equation

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Abstract: We study the relationship between the classification of three-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$ and the types of symmetry reduction of the eikonal equation to ordinary differential equations (ODEs).

Keywords: symmetry reduction; invariant solutions; eikonal equation; classification of Lie algebras; nonconjugate subalgebras; Poincaré group $P(1,4)$

1. Introduction

The eikonal equations in spaces of different dimensions and different types have many applications in the geometric optics, acoustics of inhomogeneous media, theoretical physics, *etc.* The details on this theme can be found in [1–7] (see also the references therein).

Sheng, Guha and Gonzalez (see [8,9] and the references therein) proposed new effective methods for solving many problems of computational optics.

The complete group classification of the eikonal equations for a two- and three-dimensional nonhomogeneous medium is carried out in [1–4]. In those papers the exact solutions for equations under investigation are also obtained.

In [5], a group classification of generalized eikonal equations was performed. The paper contains a list of all non-equivalent (with respect to equivalence group) equations with symmetry extensions of the kernel. New nonlinear equations of first-order with wide symmetry groups are found.

In [6], the method of propagating waves in multidimensional media is studied in order to find cases of integrability in an explicit form of the eikonal equation.

Many important results concerning symmetry, reduction, and exact solutions of the eikonal equations can be found in the monograph of Fushchych, Shtelen, and Serov [10].

In this paper, we consider the eikonal equation of the form as follows:

$$\left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\partial u}{\partial x_2}\right)^2 - \left(\frac{\partial u}{\partial x_3}\right)^2 = 1 \quad (1)$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3) \in M(1,3)$.

In 1982, Fushchych and Shtelen [11] proved that the maximally extensive local (in sense of Lie) invariance group of Equation (1) was the conformal group $C(1,4)$ of the $(4+1)$ -dimensional Poincaré-Minkowski space with the metric

$$s^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2, \quad x_4 = u \quad (2)$$

Some exact multiparametrical solutions of Equation (1) are obtained. A procedure of generating new exact solutions from the known ones is also presented.

In [12], using subalgebras of rank 3 of the Lie algebra of the group $C(1,4)$, the ansatzes, which reduce the eikonal equation to ODEs, were constructed. Taking into account the solutions of the reduced equations, wide classes of exact solutions of the eikonal equation were found.

The Lie algebra of the group $C(1,4)$ contains, as subalgebras, the Lie algebras of the following groups: Poincaré group $P(1,4)$, extended Poincaré group $\tilde{P}(1,4)$, and the optical group $Opt(3)$ [13,14].

In [15], the subalgebras of rank 3 of the Lie algebra of the group $Opt(3)$, which are nonequivalent to subalgebras of the Lie algebra of the group $\tilde{P}(1,4)$, were used for symmetry reduction of the Equation (1) to ODEs.

In [13], the symmetry reduction was made and some exact solutions for the eikonal equation were constructed using subalgebras of the Lie algebra of the group $\tilde{P}(1,4)$.

In the papers [16–22], we studied the eikonal equation using the subgroup structure of the proper orthochronous group $P(1,4)$. By using nonconjugate subalgebras of ranks 1, 2, and 3, we performed the symmetry reduction of the Equation (1) to differential equations with less a number of independent variables. Taking into account the solutions of the reduced equations, some invariant solutions of the eikonal equation were constructed.

However, it turned out that the reduced equations, obtained with the help of nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ of the given rank, were of different types. Let us present some examples.

It is known that if we reduce nonlinear partial differential equations (PDEs) we obtain, as a rule, nonlinear reduced equations. But, in the cases of the eikonal equation, Euler-Lagrange-Born-Infeld equation, and Monge-Ampere equation in the space $M(1,3) \times R(u)$ we obtained, in some cases, linear ODEs instead of the nonlinear ODEs [16,21,22]. Here, and in what follows, $R(u)$ is a real number axis of the dependent variable u .

It is also known (see, for example [23,24]) that if we reduce PDEs using the nonconjugate subalgebras with a given rank of the Lie algebras of their symmetry groups, we obtain, as a rule, reduced equations with the same number of independent variables. However, in the case of symmetry reduction of the nonlinear wave equation in the space $M(1,4) \times R(u)$ we obtained, in some cases, functional equations, ODEs, two- and three-dimensional PDEs instead of ODEs, two-, three- and four-dimensional reduced PDEs, correspondingly [16,25].

Grundland, Harnad, and Winternitz [26] were the first to point out and investigate the similar phenomenon.

Recently, Nikitin and Kuriksha [27,28] used the three-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,3)$ for symmetry reductions of equations of axion electrodynamics. They obtained four types of reductions. The results obtained cannot be explained in the usual approach (using only the ranks or dimensions of the considered subalgebras).

It means that using only the rank of nonconjugate subalgebras of the Lie algebras of the symmetry groups of the above mentioned equations under investigation, we cannot explain differences in the properties of their reduced equations.

It is known that the nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ of the same rank may have different structural properties. Therefore, in order to try to explain some of the differences in the properties of the above mentioned reduced equations, we suggest to try to investigate the relationship between structural properties of nonconjugate subalgebras of the same rank of the Lie algebra of the group $P(1,4)$ and the properties of the reduced equations corresponding to them.

In the present paper, we plan to present some of the results concerning the relationship between the structural properties of three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ and the types of symmetry reduction of the eikonal equation to ODEs.

2. The Lie Algebra of the Group $P(1,4)$ and Its Nonconjugate Subalgebras

The group $P(1,4)$ is a group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. This group has many applications in the theoretical and mathematical physics (see, for example, [29,30]).

The Lie algebra of the group $P(1,4)$ is given by the 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ ($\mu, \nu = 0, 1, 2, 3, 4$) and P_μ ($\mu = 0, 1, 2, 3, 4$), satisfying the commutation relations

$$[P_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, P_\sigma] = g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu \quad (3)$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho} \quad (4)$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$.

In the following, we will use new basis elements

$$G = M_{04}, \quad L_1 = M_{23}, \quad L_2 = -M_{13}, \quad L_3 = M_{12} \quad (5)$$

$$P_a = M_{a4} - M_{0a}, \quad C_a = M_{a4} + M_{0a}, \quad (a = 1, 2, 3) \quad (6)$$

$$X_0 = \frac{1}{2}(P_0 - P_4), \quad X_k = P_k \quad (k = 1, 2, 3), \quad X_4 = \frac{1}{2}(P_0 + P_4) \quad (7)$$

Let us consider the following representation for the Lie algebra of the group $P(1,4)$:

$$P_0 = \frac{\partial}{\partial x_0}, \quad P_1 = -\frac{\partial}{\partial x_1}, \quad P_2 = -\frac{\partial}{\partial x_2}, \quad P_3 = -\frac{\partial}{\partial x_3} \quad (8)$$

$$P_4 = -\frac{\partial}{\partial u}, \quad M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \quad x_4 \equiv u \quad (9)$$

It means that the group $P(1,4)$ acts on the space $M(1,3) \times R(u)$. More details about this type of representations can be found in [10,31,32].

In order to describe nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$, we used a method proposed by Patera, Winternitz, and Zassenhaus in [33].

The subgroup structure of the group $P(1,4)$ was studied in [34–38]. One of the nontrivial consequences of the description of the non-conjugate subalgebras of the Lie algebra of the group $P(1,4)$ is that the Lie algebra of the group $P(1,4)$ contains, as subalgebras, the Lie algebra of the Poincaré group $P(1,3)$ and the Lie algebra of the extended Galilei group $\tilde{G}(1,3)$ [39], i.e., it naturally unites the Lie algebras of the symmetry groups of relativistic and non-relativistic physics.

In the present work, we use the complete list of nonconjugate (up to $P(1,4)$ -conjugation) subalgebras of the Lie algebra of the group $P(1,4)$ as given by Fushchich, Barannik, and Barannik in [40].

Taking into account the complete classification of real structures of Lie algebras of a dimension less or equal four obtained by Mubarakzhanov in [41], we classify all the three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ into classes of isomorphic subalgebras (see, [42]).

3. Results

In this section, we present the results obtained while studying the relationship between the structural properties of the three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ and those obtained through the symmetry reduction of the eikonal equation to ODEs.

In the paper, the symbol $A_{r,j}^a$ denotes the j th Lie algebra of dimension r and a is a continuous parameter for the algebra. It should be indicated that the notation $A_{r,j}^a$ corresponds to those used in the paper by Patera *et al.* [43]. In what follows, for the given specific Lie algebra, we write only nonzero commutation relations [41,43].

3.1. Lie Algebras of the Type $3A_1$

By $3A_1$ we denote the real Lie algebra of dimension three [44]. The Lie algebras of the type $3A_1$ are Abelian.

The Lie algebra of the group $P(1, 4)$ contains 31 nonconjugate subalgebras of the type $3A_1$. Below, we present the results obtained for those subalgebras.

3.1.1. Reductions to Algebraic Equations

The invariants of five subalgebras allow us to construct the ansatzes, which reduce the eikonal equation to algebraic equations.

$$(1) \quad \langle P_1 - \gamma X_3, \gamma > 0 \rangle \oplus \langle P_2 - X_2 - \delta X_3, \delta \neq 0 \rangle \oplus \langle X_4 \rangle.$$

Ansatz:

$$x_3(x_0 + u)^2 - (\gamma x_1 + x_2\delta - x_3)(x_0 + u) - \gamma x_1 = \varphi(\omega), \quad \omega = x_0 + u \quad (10)$$

Reduced equation:

$$\omega^4 + 2\omega^3 + (\gamma^2 + \delta^2 + 1)\omega^2 + 2\gamma^2\omega + \gamma^2 = 0 \quad (11)$$

Solution of the eikonal equation:

$$(x_0 + u)^4 + 2(x_0 + u)^3 + (\gamma^2 + \delta^2 + 1)(x_0 + u)^2 + 2\gamma^2(x_0 + u) + \gamma^2 = 0 \quad (12)$$

$$(2) \quad \langle P_1 - \gamma X_3, \gamma > 0 \rangle \oplus \langle P_2 - X_2 \rangle \oplus \langle X_4 \rangle.$$

Ansatz:

$$x_3(x_0 + u)^2 - (\gamma x_1 - x_3)(x_0 + u) - \gamma x_1 = \varphi(\omega), \quad \omega = x_0 + u \quad (13)$$

Reduced equation:

$$(\omega + 1)^2(\omega^2 + \gamma^2) = 0 \quad (14)$$

Solutions of the eikonal equation:

$$u = -1 - x_0, \quad (x_0 + u)^2 + \gamma^2 = 0 \quad (15)$$

$$(3) \quad \langle P_1 \rangle \oplus \langle P_2 - X_2 - \delta X_3, \delta > 0 \rangle \oplus \langle X_4 \rangle.$$

Ansatz:

$$x_3(x_0 + u) - x_2\delta + x_3 = \varphi(\omega), \quad \omega = x_0 + u \quad (16)$$

Reduced equation:

$$\omega^2 + 2\omega + \delta^2 - 1 = 0 \quad (17)$$

Solution of the eikonal equation:

$$(x_0 + u)^2 + 2(x_0 + u) + \delta^2 + 1 = 0 \quad (18)$$

As we see, the left hand sides of the Ansatzes (1)–(3) are polinomials in invariant $\omega = x_0 + u$. The reduced equations are also polinomials in ω , but with the constant coefficients. The solutions of the eikonal equation are also polinomials in variable $x_0 + u$ with the constant coefficients.

$$(4) \quad \langle P_1 - X_3 \rangle \oplus \langle P_2 \rangle \oplus \langle X_4 \rangle.$$

Ansatz:

$$x_3 - \frac{x_1}{x_0 + u} = \varphi(\omega), \quad \omega = x_0 + u \quad (19)$$

Reduced equation:

$$(1 + \omega^2)\omega^2 = 0 \quad (20)$$

Solutions of the reduced equation:

$$1 + \omega^2 = 0, \quad \omega = 0 \quad (21)$$

Solutions of the eikonal equation:

$$1 + (x_0 + u)^2 = 0, \quad u = -x_0 \quad (22)$$

$$(5) \quad \langle P_3 - X_2 \rangle \oplus \langle X_1 \rangle \oplus \langle X_4 \rangle.$$

Ansatz:

$$x_2 - \frac{x_3}{x_0 + u} = \varphi(\omega), \quad \omega = x_0 + u \quad (23)$$

Reduced equation:

$$(\omega^2 + 1)\omega^2 = 0 \quad (24)$$

Solutions of the reduced equation:

$$\omega^2 + 1 = 0, \quad \omega = 0 \quad (25)$$

Solutions of the eikonal equation:

$$(x_0 + u)^2 + 1 = 0, \quad u = -x_0 \quad (26)$$

3.1.2. Reductions to Linear ODEs

The invariants of six subalgebras allow us to construct the ansatzes, which reduce the eikonal equation to linear ODEs.

$$(1) \quad \langle P_1 \rangle \oplus \langle P_2 \rangle \oplus \langle X_3 \rangle.$$

Ansatz:

$$x_0^2 - x_1^2 - x_2^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u \quad (27)$$

Reduced equation:

$$\omega \varphi' - \varphi = 0 \quad (28)$$

Solution of the reduced equation:

$$\varphi(\omega) = c_1 \omega \quad (29)$$

Solution of the eikonal equation:

$$x_0^2 - x_1^2 - x_2^2 - u^2 = c_1(x_0 + u) \quad (30)$$

$$(2) \quad \langle P_3 \rangle \oplus \langle X_1 \rangle \oplus \langle X_2 \rangle.$$

Ansatz:

$$x_0^2 - x_3^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u \quad (31)$$

Reduced equation:

$$\omega \varphi' - \varphi = 0 \quad (32)$$

Solution of the reduced equation:

$$\varphi(\omega) = c_1 \omega \quad (33)$$

Solution of the eikonal equation:

$$x_0^2 - x_3^2 - u^2 = c_1(x_0 + u) \quad (34)$$

$$(3) \quad \langle P_1 \rangle \oplus \langle P_2 \rangle \oplus \langle P_3 \rangle.$$

Ansatz:

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u \quad (35)$$

Reduced equation:

$$\omega \varphi' - \varphi = 0 \quad (36)$$

Solution of the reduced equation:

$$\varphi(\omega) = c_1 \omega \quad (37)$$

Solution of the eikonal equation:

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 = c_1(x_0 + u) \quad (38)$$

Let us note, that in the cases (1)–(3) we obtained the same reduced equations.

$$(4) \quad \langle P_1 \rangle \oplus \langle P_2 - X_2 \rangle \oplus \langle X_3 \rangle.$$

Ansatz:

$$\frac{x_0^2 - x_1^2 - u^2}{x_0 + u} - \frac{x_2^2}{x_0 + u + 1} = \varphi(\omega), \quad \omega = x_0 + u \quad (39)$$

Reduced equation:

$$\varphi'(\omega + 1)^4 \omega^4 = 0 \quad (40)$$

Solutions of the reduced equation:

$$\varphi(\omega) = c_1, \quad \omega + 1 = 0, \quad \omega = 0 \quad (41)$$

Solutions of the eikonal equation:

$$\frac{x_0^2 - x_1^2 - u^2}{x_0 + u} - \frac{x_2^2}{x_0 + u + 1} = c_1, \quad u = -1 - x_0, \quad u = -x_0 \quad (42)$$

$$(5) \quad \langle P_1 \rangle \oplus \langle P_2 - \alpha X_2, \alpha > 0 \rangle \oplus \langle P_3 - \gamma X_3, \gamma \neq 0 \rangle.$$

Ansatz:

$$2u + \frac{x_1^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + \alpha} + \frac{x_3^2}{x_0 + u + \gamma} = \varphi(\omega), \quad \omega = x_0 + u \quad (43)$$

The reduced equation:

$$\omega^4 (\omega + \gamma)^4 (\omega + \alpha)^4 (\varphi' - 1) = 0 \quad (44)$$

Solutions of the reduced equation:

$$\omega = 0, \quad \omega + \gamma = 0, \quad \omega + \alpha = 0, \quad \varphi(\omega) = \omega + c \quad (45)$$

Solutions on the eikonal equation:

$$u = -x_0, \quad u = -x_0 - \gamma, \quad u = -x_0 - \alpha \quad (46)$$

$$2u + \frac{x_1^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + \alpha} + \frac{x_3^2}{x_0 + u + \gamma} = x_0 + u + c \quad (47)$$

$$(6) \quad \langle P_1 \rangle \oplus \langle P_2 - \alpha X_2, \alpha > 0 \rangle \oplus \langle P_3 \rangle.$$

Ansatz:

$$2u + \frac{x_1^2 + x_3^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + \alpha} = \varphi(\omega), \quad \omega = x_0 + u \quad (48)$$

The reduced equation:

$$\omega^4(\omega + \alpha)^4(\varphi' - 1) = 0 \quad (49)$$

Solutions of the reduced equation:

$$\omega = 0, \quad \omega + \alpha = 0, \quad \varphi(\omega) = \omega + c \quad (50)$$

Solutions on the eikonal equation:

$$u = -x_0, \quad u = -x_0 - \alpha, \quad 2u + \frac{x_1^2 + x_3^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + \alpha} = x_0 + u + c \quad (51)$$

3.1.3. Reductions to Equations, Which Can Be Split on Two Linear ODEs

Taking into account the invariants of nine nonconjugate subalgebras, we constructed the ansatzes, which reduced the eikonal equation to those, which could be split on two linear ODEs.

$$(1) \quad \langle X_1 \rangle \oplus \langle X_2 \rangle \oplus \langle X_4 \rangle.$$

Ansatz:

$$x_0 + u = \varphi(\omega), \quad \omega = x_3 \quad (52)$$

Reduced equation:

$$(\varphi')^2 = 0 \quad (53)$$

Solution of the reduced equation:

$$\varphi(\omega) = c_1 \quad (54)$$

Solution of the eikonal equation:

$$u = c_1 - x_0 \quad (55)$$

$$(2) \quad \langle P_1 \rangle \oplus \langle P_2 - X_2 \rangle \oplus \langle X_4 \rangle.$$

Ansatz:

$$x_0 + u = \varphi(\omega), \quad \omega = x_3 \quad (56)$$

Reduced equation:

$$(\varphi')^2 = 0 \quad (57)$$

Solution of the reduced equation:

$$\varphi(\omega) = c_1 \quad (58)$$

Solution of the eikonal equation:

$$u = c_1 - x_0 \quad (59)$$

$$(3) \quad \langle L_3 \rangle \oplus \langle X_3 \rangle \oplus \langle X_4 \rangle.$$

Ansatz:

$$x_0 + u = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2} \quad (60)$$

Reduced equation:

$$(\varphi')^2 = 0 \quad (61)$$

Solution of the reduced equation:

$$\varphi(\omega) = c_1 \quad (62)$$

Solution of the eikonal equation:

$$u = c_1 - x_0 \quad (63)$$

$$(4) \quad \langle P_3 \rangle \oplus \langle X_1 \rangle \oplus \langle X_4 \rangle.$$

Ansatz:

$$x_0 + u = \varphi(\omega), \quad \omega = x_2 \quad (64)$$

Reduced equation:

$$(\varphi')^2 = 0 \quad (65)$$

Solution of the reduced equation:

$$\varphi(\omega) = c_1 \quad (66)$$

Solution of the eikonal equation:

$$u = c_1 - x_0 \quad (67)$$

$$(5) \quad \langle L_3 \rangle \oplus \langle P_3 \rangle \oplus \langle X_4 \rangle.$$

Ansatz:

$$x_0 + u = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2} \quad (68)$$

Reduced equation:

$$(\varphi')^2 = 0 \quad (69)$$

Solution of the reduced equation:

$$\varphi(\omega) = c_1 \quad (70)$$

Solution of the eikonal equation:

$$u = c_1 - x_0 \quad (71)$$

$$(6) \quad \langle P_1 \rangle \oplus \langle P_2 \rangle \oplus \langle X_4 \rangle.$$

Ansatz:

$$x_0 + u = \varphi(\omega), \quad \omega = x_3 \quad (72)$$

Reduced equation:

$$(\varphi')^2 = 0 \quad (73)$$

Solution of the reduced equation:

$$\varphi(\omega) = c_1 \quad (74)$$

Solution of the eikonal equation:

$$u = c_1 - x_0 \quad (75)$$

Let us note that, in the cases (1)–(6), we obtained the same reduced equations. The solutions of the eikonal equation are also the same.

$$(7) \quad \langle G \rangle \oplus \langle X_2 \rangle \oplus \langle X_1 \rangle.$$

Ansatz:

$$(x_0^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = x_3 \quad (76)$$

Reduced equation:

$$(\varphi' - 1)(\varphi' + 1) = 0 \quad (77)$$

Solutions of the reduced equation:

$$\varphi(\omega) = \varepsilon\omega + c_1, \varepsilon = \pm 1 \quad (78)$$

Solutions of the eikonal equation:

$$(x_0^2 - u^2)^{1/2} = \varepsilon x_3 + c_1, \varepsilon = \pm 1 \quad (79)$$

$$(8) \quad \langle G \rangle \oplus \langle L_3 \rangle \oplus \langle X_3 \rangle.$$

Ansatz:

$$(x_0^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2} \quad (80)$$

Reduced equation:

$$(\varphi' - 1)(\varphi' + 1) = 0 \quad (81)$$

Solutions of the reduced equation:

$$\varphi(\omega) = \varepsilon\omega + c_1, \varepsilon = \pm 1 \quad (82)$$

Solutions of the eikonal equation:

$$(x_0^2 - u^2)^{1/2} = \varepsilon(x_1^2 + x_2^2)^{1/2} + c_1, \varepsilon = \pm 1 \quad (83)$$

$$(9) \quad \langle P_3 - 2X_0 \rangle \oplus \langle X_1 \rangle \oplus \langle X_2 \rangle.$$

Ansatz:

$$\frac{1}{6}(x_0 + u)^3 + x_3(x_0 + u) + x_0 - u = \varphi(\omega), \quad \omega = (x_0 + u)^2 + 4x_3 \quad (84)$$

Reduced equation:

$$16(\varphi')^2 - \omega = 0 \quad (85)$$

Solutions of the reduced equation:

$$\varphi(\omega) = \frac{\varepsilon}{6}\omega^{3/2} + c_1, \varepsilon = \pm 1 \quad (86)$$

Solutions of the eikonal equation:

$$\frac{1}{6}(x_0 + u)^3 + x_3(x_0 + u) + x_0 - u = \frac{\varepsilon}{6} \left((x_0 + u)^2 + 4x_3 \right)^{3/2} + c_1, \varepsilon = \pm 1 \quad (87)$$

3.1.4. Reductions to Nonlinear ODEs

From the invariants of five nonconjugate subalgebras we constructed the ansatzes, which reduced the eikonal equation to nonlinear ODEs.

$$(1) \quad \langle G + \alpha X_3, \alpha > 0 \rangle \oplus \langle X_1 \rangle \oplus \langle X_2 \rangle.$$

Ansatz:

$$x_3 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = x_0^2 - u^2 \quad (88)$$

Reduced equation:

$$4\omega(\varphi')^2 + 4\alpha\varphi' - 1 = 0 \quad (89)$$

Solutions of the reduced equation:

$$\varphi(\omega) = \varepsilon(\alpha^2 + \omega)^{1/2} - \varepsilon\alpha \operatorname{artanh} \frac{(\alpha^2 + \omega)^{1/2}}{\alpha} - \frac{\alpha}{2} \ln(\omega) + c_1, \quad \varepsilon = \pm 1 \quad (90)$$

Solutions of the eikonal equation:

$$x_3 - \alpha \ln(x_0 + u) = \varepsilon(\alpha^2 + x_0^2 - u^2)^{1/2} - \varepsilon\alpha \operatorname{artanh} \frac{(\alpha^2 + x_0^2 - u^2)^{1/2}}{\alpha} - \frac{\alpha}{2} \ln(x_0^2 - u^2) + c_1, \quad \varepsilon = \pm 1 \quad (91)$$

$$(2) \quad \langle L_3 \rangle \oplus \langle P_3 + C_3 \rangle \oplus \langle X_0 + X_4 \rangle.$$

Ansatz:

$$(x_3^2 + u^2)^{1/2} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2} \quad (92)$$

Reduced equation:

$$(1 + (\varphi')^2)\varphi^2 = 0 \quad (93)$$

Solutions of the reduced equation:

$$\varphi(\omega) = i\varepsilon\omega + c_1, \quad \varepsilon = \pm 1; \quad \varphi = 0 \quad (94)$$

Solutions of the eikonal equation:

$$(x_3^2 + u^2)^{1/2} = i\varepsilon(x_1^2 + x_2^2)^{1/2} + c_1, \quad \varepsilon = \pm 1; \quad x_3^2 + u^2 = 0 \quad (95)$$

$$(3) \quad \langle L_3 + \alpha(X_0 + X_4), \alpha > 0 \rangle \oplus \langle X_3 \rangle \oplus \langle X_4 \rangle.$$

Ansatz:

$$x_0 + u + \alpha \arctan \frac{x_2}{x_1} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2} \quad (96)$$

Reduced equation:

$$\omega^2(\varphi')^2 + \alpha^2 = 0 \quad (97)$$

Solutions of the reduced equation:

$$\varphi(\omega) = i\varepsilon\alpha \ln(\omega) + c_1, \quad \varepsilon = \pm 1 \quad (98)$$

Solutions of the eikonal equation:

$$x_0 + u + \alpha \arctan \frac{x_2}{x_1} = i\frac{\varepsilon\alpha}{2} \ln(x_1^2 + x_2^2) + c_1, \quad \varepsilon = \pm 1 \quad (99)$$

$$(4) \quad \langle P_3 - 2X_0 \rangle \oplus \langle X_1 \rangle \oplus \langle X_4 \rangle.$$

Ansatz:

$$(x_0 + u)^2 + 4x_3 = \varphi(\omega), \quad \omega = x_2 \quad (100)$$

Reduced equation:

$$(\varphi')^2 + 16 = 0 \quad (101)$$

Solutions of the reduced equation:

$$\varphi(\omega) = 4i\varepsilon\omega + c_1, \quad \varepsilon = \pm 1 \quad (102)$$

Solutions of the eikonal equation:

$$(x_0 + u)^2 + 4x_3 = 4i\varepsilon x_2 + c_1, \quad \varepsilon = \pm 1 \quad (103)$$

$$(5) \quad \langle L_3 \rangle \oplus \langle -P_3 + 2X_0 \rangle \oplus \langle 2X_4 \rangle.$$

Ansatz:

$$(x_0 + u)^2 + 4x_3 = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2} \quad (104)$$

Reduced equation:

$$(\varphi')^2 + 16 = 0 \quad (105)$$

Solutions of the reduced equation:

$$\varphi(\omega) = 4i\varepsilon\omega + c_1, \varepsilon = \pm 1 \quad (106)$$

Solutions of the eikonal equation:

$$(x_0 + u)^2 + 4x_3 = 4i\varepsilon(x_1^2 + x_2^2)^{1/2} + c_1, \varepsilon = \pm 1 \quad (107)$$

3.1.5. There Are No Reductions

From the invariants of remaining six nonconjugate subalgebras, it is impossible to construct the ansatzes, which reduce the eikonal equation.

Below, we present those subalgebras as well as their invariants.

- (1) $\langle L_3 \rangle \oplus \langle X_4 - X_0 \rangle \oplus \langle X_3 \rangle : x_0, (x_1^2 + x_2^2)^{1/2};$
- (2) $\langle L_3 \rangle \oplus \langle X_0 + X_4 \rangle \oplus \langle X_4 - X_0 \rangle : x_3, (x_1^2 + x_2^2)^{1/2};$
- (3) $\langle X_1 \rangle \oplus \langle X_2 \rangle \oplus \langle X_4 - X_0 \rangle : x_0, x_3;$
- (4) $\langle X_0 \rangle \oplus \langle X_1 \rangle \oplus \langle X_4 \rangle : x_2, x_3;$
- (5) $\langle L_3 + \alpha(X_0 + X_4), \alpha > 0 \rangle \oplus \langle X_3 \rangle \oplus \langle X_4 - X_0 \rangle : (x_1^2 + x_2^2)^{1/2}, x_0 + \alpha \arctan \frac{x_2}{x_1};$
- (6) $\langle L_3 + \alpha X_3, \alpha > 0 \rangle \oplus \langle X_0 + X_4 \rangle \oplus \langle X_4 - X_0 \rangle : (x_1^2 + x_2^2)^{1/2}, x_3 + \alpha \arctan \frac{x_1}{x_2}.$

3.2. Lie Algebras of the Type $A_2 \oplus A_1$

$$[e_1, e_2] = e_2 \quad (108)$$

By A_1 we denote the real Lie algebra of dimension one [44]. It is known that there are only two different types of the real two-dimensional Lie algebras: decomposable $A_1 \oplus A_1 \equiv 2A_1$ and indecomposable A_2 [41]. The Lie algebras of the type $2A_1$ are Abelian. Bases elements (e_1 and e_2) of the Lie algebras of type A_2 satisfy the commutation relations: $[e_1, e_2] = e_2$ [43]. The Lie algebras of the type A_2 are solvable ([41,43]).

The Lie algebra of the group $P(1, 4)$ contains 10 nonconjugate subalgebras of the Type $A_2 \oplus A_1$. Below, we present the results obtained for those subalgebras.

3.2.1. Reductions to Equations, Which Can Be Split on Two Linear ODEs

Taking into account the invariants of two nonconjugate subalgebras, we constructed the ansatzes, which reduced the eikonal equation to those, which could be split on two linear ODEs.

$$(1) \quad \langle -G, P_3 \rangle \oplus \langle X_1 \rangle.$$

Ansatz:

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = x_2 \quad (109)$$

Reduced equation:

$$(\varphi' - 1)(\varphi' + 1)\varphi^2 = 0 \quad (110)$$

Solutions of the reduced equation:

$$\varphi(\omega) = \varepsilon\omega + c_1, \quad \varphi = 0, \quad \varepsilon = \pm 1 \quad (111)$$

Solutions of the eikonal equation:

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varepsilon x_2 + c_1, \quad x_0^2 - x_3^2 - u^2 = 0, \quad \varepsilon = \pm 1 \quad (112)$$

$$(2) \quad \langle -G, P_3 \rangle \oplus \langle L_3 \rangle.$$

Ansatz:

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2} \quad (113)$$

Reduced equation:

$$(\varphi' - 1)(\varphi' + 1)\varphi^2 = 0 \quad (114)$$

Solutions of the reduced equation:

$$\varphi(\omega) = \varepsilon\omega + c_1, \quad \varphi = 0, \quad \varepsilon = \pm 1 \quad (115)$$

Solutions of the eikonal equation:

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varepsilon(x_1^2 + x_2^2)^{1/2} + c_1, \quad x_0^2 - x_3^2 - u^2 = 0, \quad \varepsilon = \pm 1 \quad (116)$$

3.2.2. Reductions to Nonlinear ODEs

From the invariants of five nonconjugate subalgebras we constructed the ansatzes, which reduced the eikonal equation to non-linear ODEs.

$$(1) \quad \langle -(G + \alpha X_2), P_3, \alpha > 0 \rangle \oplus \langle X_1 \rangle.$$

Ansatz:

$$x_2 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = (x_0^2 - x_3^2 - u^2)^{1/2} \quad (117)$$

Reduced equation:

$$\omega \left(\omega(\varphi')^2 + 2\alpha\varphi' - \omega \right) = 0 \quad (118)$$

Solutions of the reduced equation:

$$\omega = 0, \quad \varphi(\omega) = \varepsilon(\alpha^2 + \omega^2)^{1/2} - \varepsilon\alpha \operatorname{artanh} \frac{\alpha}{(\alpha^2 + \omega^2)^{1/2}} - \alpha \ln(\omega) + c_1, \quad \varepsilon = \pm 1 \quad (119)$$

Solutions of the eikonal equation:

$$x_0^2 - x_3^2 - u^2 = 0 \quad (120)$$

$$\begin{aligned} x_2 - \alpha \ln(x_0 + u) &= \varepsilon(x_0^2 - x_3^2 - u^2 + \alpha^2)^{1/2} \\ &\quad - \varepsilon\alpha \operatorname{artanh} \frac{\alpha}{(x_0^2 - x_3^2 - u^2 + \alpha^2)^{1/2}} \\ &\quad - \frac{\alpha}{2} \ln(x_0^2 - x_3^2 - u^2) + c_1 \\ \varepsilon &= \pm 1 \end{aligned} \quad (121)$$

$$(2) \quad \langle -\frac{1}{\lambda}L_3 - G, 2X_4, \lambda > 0 \rangle \oplus \langle X_3 \rangle.$$

Ansatz:

$$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2} \quad (122)$$

Reduced equation:

$$\omega^2(\varphi')^2 + \lambda^2 = 0 \quad (123)$$

Solutions of the reduced equation:

$$\varphi(\omega) = i\varepsilon\lambda \ln(\omega) + c_1, \quad \varepsilon = \pm 1 \quad (124)$$

Solutions of the eikonal equation:

$$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = i\frac{\varepsilon\lambda}{2} \ln(x_1^2 + x_2^2) + c_1, \quad \varepsilon = \pm 1 \quad (125)$$

$$(3) \quad \langle -(G + \alpha X_2), X_4, \alpha > 0 \rangle \oplus \langle X_1 \rangle.$$

Ansatz:

$$x_2 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = x_3 \quad (126)$$

Reduced equation:

$$(\varphi')^2 + 1 = 0 \quad (127)$$

Solutions of the reduced equation:

$$\varphi(\omega) = i\varepsilon\omega + c_1, \quad \varepsilon = \pm 1 \quad (128)$$

Solutions of the eikonal equation:

$$x_2 - \alpha \ln(x_0 + u) = i\varepsilon x_3 + c_1, \quad \varepsilon = \pm 1 \quad (129)$$

$$(4) \quad \langle -(G + \alpha X_3), X_4, \alpha > 0 \rangle \oplus \langle L_3 + \beta X_3, \beta > 0 \rangle.$$

Ansatz:

$$x_3 - \alpha \ln(x_0 + u) + \beta \arctan \frac{x_1}{x_2} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2} \quad (130)$$

Reduced equation:

$$\omega^2(\varphi')^2 + \omega^2 + \beta^2 = 0 \quad (131)$$

Solutions of the reduced equation:

$$\varphi(\omega) = i\varepsilon(\omega^2 + \beta^2)^{1/2} - i\varepsilon\beta \operatorname{artanh} \frac{\beta}{(\omega^2 + \beta^2)^{1/2}} + c_1, \quad \varepsilon = \pm 1 \quad (132)$$

Solutions of the eikonal equation:

$$x_3 - \alpha \ln(x_0 + u) + \beta \arctan \frac{x_1}{x_2} = -i\varepsilon\beta \operatorname{artanh} \frac{\beta}{(x_1^2 + x_2^2 + \beta^2)^{1/2}} + i\varepsilon(x_1^2 + x_2^2 + \beta^2)^{1/2} + c_1 \quad (133)$$

$$\varepsilon = \pm 1 \quad (134)$$

$$(5) \quad \langle -(G + \alpha X_3), X_4, \alpha > 0 \rangle \oplus \langle L_3 \rangle.$$

Ansatz:

$$x_3 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2} \quad (135)$$

Reduced equation:

$$(\varphi')^2 + 1 = 0 \quad (136)$$

Solutions of the reduced equation:

$$\varphi(\omega) = i\varepsilon\omega + c_1, \quad \varepsilon = \pm 1 \quad (137)$$

Solutions of the eikonal equation:

$$x_3 - \alpha \ln(x_0 + u) = i\varepsilon(x_1^2 + x_2^2)^{1/2} + c_1, \quad \varepsilon = \pm 1 \quad (138)$$

3.2.3. There Are No Reductions

From the invariants of remaining three nonconjugate subalgebras, it is impossible to construct the ansatzes, which reduce the eikonal equation.

Below, we present those subalgebras as well as their invariants.

- (1) $\langle -G, X_4 \rangle \oplus \langle X_1 \rangle : x_2, x_3;$
- (2) $\langle -G, X_4 \rangle \oplus \langle L_3 \rangle : x_3, (x_1^2 + x_2^2)^{1/2};$
- (3) $\langle -G, X_4 \rangle \oplus \langle L_3 + \alpha X_3, \alpha > 0 \rangle : (x_1^2 + x_2^2)^{1/2}, x_3 + \alpha \arctan \frac{x_1}{x_2}.$

3.3. Lie Algebras of the Type $A_{3,1}$

$$[e_2, e_3] = e_1 \quad (139)$$

Lie algebras of the type $A_{3,1}$ are nilpotent [43].

The Lie algebra of the group $P(1,4)$ contains 17 nonconjugate subalgebras of the type $A_{3,1}$. Below, we present the results obtained for those subalgebras.

3.3.1. Reductions to Algebraic Equations

Taking into account the invariants of seven nonconjugate subalgebras, we constructed the ansatzes, which reduced the eikonal equation to algebraic equations.

- (1) $\langle 4X_4, P_1 - X_2 - \gamma X_3, P_2 + X_1 - \mu X_2 - \delta X_3, \gamma > 0, \delta \neq 0, \mu > 0 \rangle.$

Ansatz:

$$x_3(x_0 + u)^2 - (\gamma x_1 + x_2\delta - \mu x_3)(x_0 + u) + (\delta - \gamma\mu)x_1 - x_2\gamma + x_3 = \varphi(\omega), \quad \omega = x_0 + u \quad (140)$$

Reduced equation:

$$\omega^4 + 2\mu\omega^3 + (\gamma^2 + \mu^2 + \delta^2 + 2)\omega^2 + 2\mu(\gamma^2 + 1)\omega + (\gamma\mu - \delta)^2 + \gamma^2 + 1 = 0 \quad (141)$$

Solution of the eikonal equation:

$$(x_0 + u)^4 + 2\mu(x_0 + u)^3 + (\gamma^2 + \mu^2 + \delta^2 + 2)(x_0 + u)^2 + 2\mu(\gamma^2 + 1)(x_0 + u) + (\gamma\mu - \delta)^2 + \gamma^2 + 1 = 0 \quad (142)$$

- (2) $\langle 4X_4, P_1 - X_2 - \gamma X_3, P_2 + X_1 - \mu X_2, \gamma > 0, \mu > 0 \rangle.$

Ansatz:

$$x_3(x_0 + u)^2 - (\gamma x_1 - \mu x_3)(x_0 + u) - \gamma\mu x_1 - x_2\gamma + x_3 = \varphi(\omega), \quad \omega = x_0 + u \quad (143)$$

Reduced equation:

$$\omega^4 + 2\mu\omega^3 + (\gamma^2 + \mu^2 + 2)\omega^2 + 2\mu(\gamma^2 + 1)\omega + \gamma^2(\mu^2 + 1) + 1 = 0 \quad (144)$$

Solution of the eikonal equation:

$$(x_0 + u)^4 + 2\mu(x_0 + u)^3 + (\gamma^2 + \mu^2 + 2)(x_0 + u)^2 + 2\mu(\gamma^2 + 1)(x_0 + u) + \gamma^2(\mu^2 + 1) + 1 = 0 \quad (145)$$

$$(3) \quad \langle 4X_4, P_1 - X_2, P_2 + X_1 - \mu X_2 - \delta X_3, \delta > 0, \mu \neq 0 \rangle.$$

Ansatz:

$$x_3(x_0 + u)^2 - (x_2\delta - \mu x_3)(x_0 + u) + \delta x_1 + x_3 = \varphi(\omega), \quad \omega = x_0 + u \quad (146)$$

Reduced equation:

$$\omega^4 + 2\mu\omega^3 + (\delta^2 + \mu^2 + 2)\omega^2 + 2\mu\omega + \delta^2 + 1 = 0 \quad (147)$$

Solution of the eikonal equation:

$$(x_0 + u)^4 + 2\mu(x_0 + u)^3 + (\delta^2 + \mu^2 + 2)(x_0 + u)^2 + 2\mu(x_0 + u) + \delta^2 + 1 = 0 \quad (148)$$

$$(4) \quad \langle 4X_4, P_1 - X_2, P_2 + X_1 - \delta X_3, \delta > 0 \rangle.$$

Ansatz:

$$x_3(x_0 + u)^2 - x_2\delta(x_0 + u) + \delta x_1 + x_3 = \varphi(\omega), \quad \omega = x_0 + u \quad (149)$$

Reduced equation:

$$(\omega^2 + 1)(\omega^2 + \delta^2 + 1) = 0 \quad (150)$$

Solutions of the eikonal equation:

$$(x_0 + u)^2 + 1 = 0, (x_0 + u)^2 + \delta^2 + 1 = 0 \quad (151)$$

$$(5) \quad \langle 4X_4, P_1 - X_2 - \beta X_3, P_2 + X_1, \beta > 0 \rangle.$$

Ansatz:

$$x_3(x_0 + u)^2 - \beta x_1(x_0 + u) - \beta x_2 + x_3 = \varphi(\omega), \quad \omega = x_0 + u \quad (152)$$

Reduced equation:

$$(\omega^2 + 1)(\omega^2 + \beta^2 + 1) = 0 \quad (153)$$

Solutions of the eikonal equation:

$$(x_0 + u)^2 + 1 = 0, (x_0 + u)^2 + \beta^2 + 1 = 0 \quad (154)$$

As we see, the left hand sides of the Ansatzes (1)–(5) are polinomials in invariant $\omega = x_0 + u$. The reduced equations are also polinomials in variable ω , but with the constant coefficients. The solutions of the eikonal equation are also polinomials in variable $x_0 + u$ with the constant coefficients.

$$(6) \quad \langle 4X_4, P_1 - X_2, P_2 + X_1 - \mu X_2, \mu \neq 0 \rangle.$$

Ansatz:

$$x_3(x_0 + u)^2 + \mu x_3(x_0 + u) + x_3 = \varphi(\omega), \quad \omega = x_0 + u \quad (155)$$

Reduced equation:

$$(\omega^2 + \mu\omega + 1)^2 = 0 \quad (156)$$

Solution of the eikonal equation:

$$u = -\frac{1}{2} \left(\mu + (\mu^2 - 4)^{1/2} \right) - x_0 \quad (157)$$

$$(7) \quad \langle 2\mu X_4, P_3 - X_2, X_1 + \mu X_3, \mu > 0 \rangle.$$

Ansatz:

$$x_2 - \frac{x_3 - \mu x_1}{x_0 + u} = \varphi(\omega), \quad \omega = x_0 + u \quad (158)$$

Reduced equation:

$$\omega^2(\omega^2 + \mu^2 + 1) = 0 \quad (159)$$

Solutions of the reduced equation:

$$\omega = 0, \quad \omega^2 + \mu^2 + 1 = 0 \quad (160)$$

Solutions of the eikonal equation:

$$u = -x_0, \quad (x_0 + u)^2 + \mu^2 + 1 = 0 \quad (161)$$

3.3.2. Reductions to Equations, Which Can Be Split on Two Linear ODEs

Taking into account the invariants of five nonconjugate subalgebras, we constructed the ansatzes, which reduced the eikonal equation to those, which could be split on two linear ODEs.

$$(1) \quad \langle 2\mu X_4, P_3, X_1 + \mu X_3, \mu > 0 \rangle.$$

Ansatz:

$$x_0 + u = \varphi(\omega), \quad \omega = x_2 \quad (162)$$

Reduced equation:

$$(\varphi')^2 = 0 \quad (163)$$

Solution of the reduced equation:

$$\varphi(\omega) = c_1 \quad (164)$$

Solution of the eikonal equation:

$$u = c_1 - x_0 \quad (165)$$

$$(2) \quad \langle 2X_4, P_3 - L_3, X_3 \rangle.$$

Ansatz:

$$x_0 + u = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2} \quad (166)$$

Reduced equation:

$$(\varphi')^2 = 0 \quad (167)$$

Solution of the reduced equation:

$$\varphi(\omega) = c_1 \quad (168)$$

Solution of the eikonal equation:

$$u = c_1 - x_0 \quad (169)$$

$$(3) \quad \langle 2X_4, P_3 - X_1, X_3 \rangle.$$

Ansatz:

$$x_0 + u = \varphi(\omega), \quad \omega = x_2 \quad (170)$$

Reduced equation:

$$(\varphi')^2 = 0 \quad (171)$$

Solution of the reduced equation:

$$\varphi(\omega) = c_1 \quad (172)$$

Solution of the eikonal equation:

$$u = c_1 - x_0 \quad (173)$$

(4) $\langle -2\alpha X_4, L_3 + \alpha X_3, P_3, \alpha > 0 \rangle$.

Ansatz:

$$x_0 + u = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2} \quad (174)$$

Reduced equation:

$$(\varphi')^2 = 0 \quad (175)$$

Solution of the reduced equation:

$$\varphi(\omega) = c_1 \quad (176)$$

Solution of the eikonal equation:

$$u = c_1 - x_0 \quad (177)$$

(5) $\langle 4X_4, P_1 - X_2, P_2 + X_1 \rangle$.

Ansatz:

$$x_0 + u = \varphi(\omega), \quad \omega = x_3 \quad (178)$$

Reduced equation:

$$(\varphi')^2 = 0 \quad (179)$$

Solution of the reduced equation:

$$\varphi(\omega) = c_1 \quad (180)$$

Solution of the eikonal equation:

$$u = c_1 - x_0 \quad (181)$$

Let us note that, in the cases (1)–(5), we obtained the same reduced equation. The solutions of the eikonal equation are also the same.

3.3.3. Reductions to Nonlinear ODEs

From the invariants of four nonconjugate subalgebras, we constructed the ansatzes, which reduced the eikonal equation to nonlinear ODEs.

(1) $\langle 2\mu X_4, P_3 - 2X_0, X_1 + \mu X_3, \mu > 0 \rangle$.

Ansatz:

$$(x_0 + u)^2 + 4x_3 - 4\mu x_1 = \varphi(\omega), \quad \omega = x_2 \quad (182)$$

Reduced equation:

$$(\varphi')^2 + 16(\mu^2 + 1) = 0 \quad (183)$$

Solutions of the reduced equation:

$$\varphi(\omega) = 4i\varepsilon(\mu^2 + 1)^{1/2}\omega + c_1, \quad \varepsilon = \pm 1 \quad (184)$$

Solutions of the eikonal equation:

$$u = 2 \left(i\epsilon x_2 \sqrt{\mu^2 + 1} + \mu x_1 - x_3 + c_1 \right)^{1/2} - x_0, \quad \epsilon = \pm 1 \quad (185)$$

(2) $\langle 2X_4, P_3 - L_3 - 2\alpha X_0, X_3, \alpha > 0 \rangle$.

Ansatz:

$$2\alpha \arctan \frac{x_1}{x_2} - x_0 - u = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2} \quad (186)$$

Reduced equation:

$$\omega^2(\varphi')^2 + 4\alpha^2 = 0 \quad (187)$$

Solutions of the reduced equation:

$$\varphi(\omega) = 2i\epsilon\alpha \ln(\omega) + c_1, \quad \epsilon = \pm 1 \quad (188)$$

Solutions of the eikonal equation:

$$u = 2\alpha \arctan \frac{x_1}{x_2} + i\epsilon\alpha \ln(x_1^2 + x_2^2) - x_0 + c_1, \quad \epsilon = \pm 1 \quad (189)$$

(3) $\langle -2\beta X_4, L_3 + \beta X_3, P_3 - 2X_0, \beta > 0 \rangle$.

Ansatz:

$$\beta \arctan \frac{x_1}{x_2} + \frac{1}{4}(x_0 + u)^2 + x_3 = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2} \quad (190)$$

Reduced equation:

$$\omega^2(\varphi')^2 + \omega^2 + \beta^2 = 0 \quad (191)$$

Solutions of the reduced equation:

$$\varphi(\omega) = i\epsilon \sqrt{\omega^2 + \beta^2} - i\epsilon\beta \operatorname{artanh} \frac{\beta}{\sqrt{\omega^2 + \beta^2}} + c_1, \quad \epsilon = \pm 1 \quad (192)$$

Solutions of the eikonal equation:

$$\beta \arctan \frac{x_1}{x_2} + \frac{1}{4}(x_0 + u)^2 = i\epsilon \sqrt{x_1^2 + x_2^2 + \beta^2} - i\epsilon\beta \operatorname{artanh} \frac{\beta}{\sqrt{x_1^2 + x_2^2 + \beta^2}} - x_3 + c_1, \quad \epsilon = \pm 1 \quad (193)$$

(4) $\langle 2X_4, P_3, X_3 \rangle$.

Ansatz:

$$x_2 = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_0 + u, \quad \omega_2 = x_1. \quad (194)$$

Reduced equation:

$$\varphi_2^2 + 1 = 0 \quad (195)$$

Solution of the reduced equation:

$$\varphi(\omega_1, \omega_2) = i\omega_2 + f(\omega_1) \quad (196)$$

where f is an arbitrary function.

Solution of the eikonal equation:

$$x_2 = ix_1 + f(x_0 + u) \quad (197)$$

where f is an arbitrary function.

3.3.4. There Are No Reductions

From the invariants of remaining one nonconjugate subalgebra, it is impossible to construct the ansatz, which reduces the eikonal equation.

Below, we present the subalgebra as well as its invariants.

$$\langle 2X_4, P_3 - 2X_0, X_3 \rangle : x_1, x_2 \quad (198)$$

3.4. Lie Algebras of the Type $A_{3,2}$

$$[e_1, e_3] = e_1, [e_2, e_3] = e_1 + e_2 \quad (199)$$

The Lie algebras of the type $A_{3,2}$ are solvable [41,43].

The Lie algebra of the group $P(1,4)$ contains three nonconjugate subalgebras of the type $A_{3,2}$. Below, we present the results obtained for those subalgebras.

3.4.1. Reductions to Nonlinear ODEs

From the invariants of two nonconjugate subalgebras, we constructed the ansatzes, which reduced the eikonal equation to nonlinear ODEs.

$$(1) \quad \langle 2\beta X_4, P_3, G + \alpha X_1 + \beta X_3, \alpha > 0, \beta > 0 \rangle.$$

Ansatz:

$$x_1 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = x_2 \quad (200)$$

Reduced equation:

$$(\varphi')^2 + 1 = 0 \quad (201)$$

Solutions of the reduced equation:

$$\varphi(\omega) = i\varepsilon\omega + c_1, \quad \varepsilon = \pm 1 \quad (202)$$

Solutions of the eikonal equation:

$$x_1 - \alpha \ln(x_0 + u) = i\varepsilon x_2 + c_1, \quad \varepsilon = \pm 1 \quad (203)$$

$$(2) \quad \langle 2\alpha X_4, \lambda P_3, \frac{1}{\lambda} L_3 + G + \frac{\alpha}{\lambda} X_3, \alpha > 0, \lambda > 0 \rangle.$$

Ansatz:

$$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2} \quad (204)$$

Reduced equation:

$$\omega^2 (\varphi')^2 + \lambda^2 = 0 \quad (205)$$

Solutions of the reduced equation:

$$\varphi(\omega) = i\varepsilon\lambda \ln(\omega) + c_1, \quad \varepsilon = \pm 1 \quad (206)$$

Solutions of the eikonal equation:

$$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = i\varepsilon \frac{\lambda}{2} \ln(x_1^2 + x_2^2) + c_1, \quad \varepsilon = \pm 1 \quad (207)$$

3.4.2. There Are No Reductions

From the invariants of remaining one nonconjugate subalgebra, it is impossible to construct the ansatz, which reduces the eikonal equation.

Below, we present the subalgebra as well as its invariants.

$$\langle 2\alpha X_4, P_3, G + \alpha X_3, \alpha > 0 \rangle : x_1, x_2 \quad (208)$$

3.5. Lie Algebras of the Type $A_{3,3}$

$$[e_1, e_3] = e_1, [e_2, e_3] = e_2 \quad (209)$$

The Lie algebras of the type $A_{3,3}$ are solvable [41,43].

The Lie algebra of the group $P(1,4)$ contains five nonconjugate subalgebras of the type $A_{3,3}$. Below, we present the results obtained for those subalgebras.

3.5.1. Reductions to Equations, Which Can Be Split on Two Linear ODEs

Taking into account the invariants of one nonconjugate subalgebra, we constructed the ansatz, which reduced the eikonal equation to those, which could be split on two linear ODEs.

$$\langle P_1, P_2, G \rangle \quad (210)$$

Ansatz:

$$(x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = x_3 \quad (211)$$

Reduced equation:

$$(\varphi' - 1)(\varphi' + 1)\varphi^2 = 0 \quad (212)$$

Solutions of the reduced equation:

$$\varphi(\omega) = \varepsilon\omega + c, \quad \varepsilon = \pm 1, \quad \varphi = 0 \quad (213)$$

Solutions of the eikonal equation:

$$(x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2} = \varepsilon x_3 + c_1, \quad \varepsilon = \pm 1, \quad x_0^2 - x_1^2 - x_2^2 - u^2 = 0 \quad (214)$$

3.5.2. Reductions to Nonlinear ODEs

From the invariants of three subalgebras, we constructed the ansatzes, which reduced the eikonal equation to nonlinear ODEs.

$$(1) \quad \langle P_1, P_2, G + \alpha X_3, \alpha > 0 \rangle.$$

Ansatz:

$$x_3 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = x_0^2 - x_1^2 - x_2^2 - u^2 \quad (215)$$

The reduced equation:

$$4\omega (\varphi')^2 + 4\alpha\varphi' - 1 = 0 \quad (216)$$

Solutions of the reduced equation:

$$\varphi(\omega) = \varepsilon(\alpha^2 + \omega)^{1/2} - i\varepsilon\alpha \arctan \frac{(\alpha^2 + \omega)^{1/2}}{i\alpha} - \frac{\alpha}{2} \ln \omega + c, \quad \varepsilon = \pm 1 \quad (217)$$

Solutions of the eikonal equation:

$$x_3 - \alpha \ln(x_0 + u) = \varepsilon (x_0^2 - x_1^2 - x_2^2 - u^2 + \alpha^2)^{1/2} - i\varepsilon \alpha \arctan \frac{(x_0^2 - x_1^2 - x_2^2 - u^2 + \alpha^2)^{1/2}}{i\alpha} - \frac{\alpha}{2} \ln(x_0^2 - x_1^2 - x_2^2 - u^2) + c, \quad \varepsilon = \pm 1$$

$$(2) \quad \langle P_3, X_4, \frac{1}{\lambda} L_3 + G, \lambda > 0 \rangle.$$

Ansatz:

$$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2} \quad (218)$$

Reduced equation:

$$\omega^2 (\varphi')^2 + \lambda^2 = 0 \quad (219)$$

Solutions of the reduced equation:

$$\varphi(\omega) = i\varepsilon \lambda \ln(\omega) + c, \quad \varepsilon = \pm 1 \quad (220)$$

Solutions of the eikonal equation:

$$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = i\varepsilon \frac{\lambda}{2} \ln(x_1^2 + x_2^2) + c, \quad \varepsilon = \pm 1 \quad (221)$$

$$(3) \quad \langle P_3, X_4, G + \alpha X_1, \alpha > 0 \rangle.$$

Ansatz:

$$x_1 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = x_2 \quad (222)$$

Reduced equation:

$$(\varphi')^2 + 1 = 0 \quad (223)$$

Solutions of the reduced equation:

$$\varphi(\omega) = i\varepsilon \omega + c, \quad \varepsilon = \pm 1 \quad (224)$$

Solutions of the eikonal equation:

$$x_1 - \alpha \ln(x_0 + u) = i\varepsilon x_2 + c_1, \quad \varepsilon = \pm 1 \quad (225)$$

3.5.3. There Are No Reductions

From the invariants of remaining one nonconjugate subalgebra, it is impossible to construct the ansatz, which reduces the eikonal equation.

Below, we present the subalgebra as well as its invariants.

$$\langle P_3, X_4, G \rangle : x_1, x_2 \quad (226)$$

3.6. Lie Algebras of the Type $A_{3,4}$

$$[e_1, e_3] = e_1, [e_2, e_3] = -e_2 \quad (227)$$

The Lie algebras of the type $A_{3,4}$ are solvable [41,43].

The Lie algebra of the group $P(1,4)$ contains four nonconjugate subalgebras of the type $A_{3,4}$.

3.6.1. There Are No Reductions

From the invariants of all four nonconjugate subalgebras, it is impossible to construct the ansatzes, which reduce the eikonal equation.

Below, we present those subalgebras as well as their invariants.

- (1) $\langle X_4, X_0, G \rangle: x_1, x_2, x_3;$
- (2) $\langle X_0, -X_4, -\frac{1}{\lambda}L_3 - G, \lambda > 0 \rangle: x_3, (x_1^2 + x_2^2)^{1/2};$
- (3) $\langle X_0, X_4, -(G + \alpha X_1), \alpha > 0 \rangle: x_2, x_3;$
- (4) $\langle X_0, X_4, -\frac{L_3}{\lambda} - G - \frac{\alpha}{\lambda}X_3, \alpha > 0, \lambda > 0 \rangle: (x_1^2 + x_2^2)^{1/2}, x_3 + \alpha \arctan \frac{x_1}{x_2}.$

3.7. Lie Algebras of the Type $A_{3,5}^a$

$$[e_1, e_3] = e_1, [e_2, e_3] = ae_2, (0 < |a| < 1) \quad (228)$$

The Lie algebra of the group $P(1, 4)$ contains no nonconjugate subalgebras of the type $A_{3,5}^a$.

3.8. Lie Algebras of the Type $A_{3,6}$

$$[e_1, e_3] = -e_2, [e_2, e_3] = e_1 \quad (229)$$

The Lie algebras of the type $A_{3,6}$ are solvable [41,43].

The Lie algebra of the group $P(1, 4)$ contains 18 nonconjugate subalgebras of the type $A_{3,6}$.

Below, we present the results obtained for those subalgebras.

3.8.1. Reductions to Linear ODEs

The invariants of four subalgebras allow us to construct the ansatzes, which reduce the eikonal equation to linear ODEs.

- (1) $\langle P_1 - X_1, P_2 - X_2, -P_3 + L_3 \rangle.$

Ansatz:

$$\frac{x_1^2 + x_2^2}{x_0 + u + 1} + \frac{x_3^2}{x_0 + u} + 2u = \varphi(\omega), \quad \omega = x_0 + u \quad (230)$$

Reduced equation:

$$\omega^4(\omega + 1)^4(\varphi' - 1) = 0 \quad (231)$$

Solutions of the reduced equation:

$$\omega + 1 = 0, \quad \omega = 0, \quad \varphi(\omega) = \omega + c_1 \quad (232)$$

Solutions of the eikonal equation:

$$u = -1 - x_0, \quad u = -x_0, \quad \frac{x_1^2 + x_2^2}{x_0 + u + 1} + \frac{x_3^2}{x_0 + u} + 2u = x_0 + u + c_1 \quad (233)$$

- (2) $\langle P_1, -P_2, -(L_3 + \alpha X_3), \alpha > 0 \rangle.$

Ansatz:

$$x_0^2 - x_1^2 - x_2^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u \quad (234)$$

Reduced equation:

$$\omega\varphi' - \varphi = 0 \quad (235)$$

Solution of the reduced equation:

$$\varphi(\omega) = c\omega \quad (236)$$

Solution of the eikonal equation:

$$x_0^2 - x_1^2 - x_2^2 - u^2 = c(x_0 + u) \quad (237)$$

(3) $\langle X_1, -X_2, P_3 - L_3 \rangle$.

Ansatz:

$$x_0^2 - x_3^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u \quad (238)$$

Reduced equation:

$$\omega\varphi' - \varphi = 0 \quad (239)$$

Solution of the reduced equation:

$$\varphi(\omega) = c_1\omega \quad (240)$$

Solution of the eikonal equation:

$$x_0^2 - x_3^2 - u^2 = c_1(x_0 + u) \quad (241)$$

(4) $\langle P_1, P_2, -P_3 + L_3 \rangle$.

Ansatz:

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u \quad (242)$$

Reduced equation:

$$\omega\varphi' - \varphi = 0 \quad (243)$$

Solution of the reduced equation:

$$\varphi(\omega) = c_1\omega \quad (244)$$

Solution of the eikonal equation:

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 = c_1(x_0 + u) \quad (245)$$

3.8.2. Reductions to Equations, Which Can Be Split on Two Linear ODEs

Taking into account the invariants of seven nonconjugate subalgebras, we constructed the ansatzes, which reduced the eikonal equation to those, which could be split on two linear ODEs.

(1) $\langle X_1, -X_2, -(L_3 + 2X_4) \rangle$.

Ansatz:

$$x_0 + u = \varphi(\omega), \quad \omega = x_3 \quad (246)$$

Reduced equation:

$$(\varphi')^2 = 0 \quad (247)$$

Solution of the reduced equation:

$$\varphi(\omega) = c_1 \quad (248)$$

Solution of the eikonal equation:

$$u = c_1 - x_0 \quad (249)$$

$$(2) \langle P_1, P_2, L_3 + 2X_4 \rangle.$$

Ansatz:

$$x_0 + u = \varphi(\omega), \quad \omega = x_3 \quad (250)$$

Reduced equation:

$$(\varphi')^2 = 0 \quad (251)$$

Solution of the reduced equation:

$$\varphi(\omega) = c_1 \quad (252)$$

Solution of the eikonal equation:

$$u = c_1 - x_0 \quad (253)$$

$$(3) \langle X_1, X_2, L_3 + \frac{1}{2}(P_3 + C_3) \rangle.$$

Ansatz:

$$(x_3^2 + u^2)^{1/2} = \varphi(\omega), \quad \omega = x_0 \quad (254)$$

Reduced equation:

$$(\varphi' - 1)(\varphi' + 1)\varphi^2 = 0 \quad (255)$$

Solutions of the reduced equation:

$$\varphi(\omega) = \varepsilon\omega + c_1, \quad \varepsilon = \pm 1, \quad \varphi = 0 \quad (256)$$

Solutions of the eikonal equation:

$$(x_3^2 + u^2)^{1/2} = \varepsilon x_0 + c, \quad \varepsilon = \pm 1, \quad x_3^2 + u^2 = 0 \quad (257)$$

$$(4) \langle -X_1, X_2, -L_3 - \frac{\lambda}{2}(P_3 + C_3), 0 < \lambda < 1 \rangle.$$

Ansatz:

$$(x_3^2 + u^2)^{1/2} = \varphi(\omega), \quad \omega = x_0 \quad (258)$$

Reduced equation:

$$(\varphi' - 1)(\varphi' + 1)\varphi^2 = 0 \quad (259)$$

Solutions of the reduced equation:

$$\varphi(\omega) = \varepsilon\omega + c_1, \quad \varepsilon = \pm 1, \quad \varphi = 0 \quad (260)$$

Solutions of the eikonal equation:

$$(x_3^2 + u^2)^{1/2} = \varepsilon x_0 + c_1, \quad \varepsilon = \pm 1, \quad x_3^2 + u^2 = 0 \quad (261)$$

$$(5) \langle -X_1, X_2, -(L_3 + \lambda G), \lambda > 0 \rangle.$$

Ansatz:

$$(x_0^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = x_3 \quad (262)$$

Reduced equation:

$$(\varphi' - 1)(\varphi' + 1) = 0 \quad (263)$$

Solutions of the reduced equation:

$$\varphi(\omega) = \varepsilon\omega + c_1, \quad \varepsilon = \pm 1 \quad (264)$$

Solutions of the eikonal equation:

$$(x_0^2 - u^2)^{1/2} = \varepsilon x_3 + c_1, \varepsilon = \pm 1 \quad (265)$$

$$(6) \quad \langle X_1, -X_2, -(L_3 + \alpha X_3), \alpha > 0 \rangle.$$

Ansatz:

$$u = \varphi(\omega), \quad \omega = x_0 \quad (266)$$

Reduced equation:

$$(\varphi' - 1)(\varphi' + 1) = 0 \quad (267)$$

Solutions of the reduced equation:

$$\varphi(\omega) = \varepsilon \omega + c_1, \varepsilon = \pm 1 \quad (268)$$

Solutions of the eikonal equation:

$$u = \varepsilon x_0 + c_1, \varepsilon = \pm 1 \quad (269)$$

$$(7) \quad \langle X_1, -X_2, P_3 - L_3 - 2\alpha X_0, \alpha > 0 \rangle.$$

Ansatz:

$$(x_0 + u)^3 + 6\alpha x_3(x_0 + u) + 6\alpha^2(x_0 - u) = \varphi(\omega), \quad \omega = (x_0 + u)^2 + 4x_3\alpha \quad (270)$$

Reduced equation:

$$4(\varphi')^2 - 9\omega = 0 \quad (271)$$

Solutions of the reduced equation:

$$\varphi(\omega) = \varepsilon \omega^{3/2} + c, \varepsilon = \pm 1 \quad (272)$$

Solutions of the eikonal equation:

$$(x_0 + u)^3 + 6\alpha x_3(x_0 + u) + 6\alpha^2(x_0 - u) = \varepsilon \left((x_0 + u)^2 + 4\alpha x_3 \right)^{3/2} + c_1, \varepsilon = \pm 1 \quad (273)$$

3.8.3. Reductions to Nonlinear ODEs

From the invariants of three nonconjugate subalgebras, we constructed the ansatzes, which reduced the eikonal equation to nonlinear ODEs.

$$(1) \quad \langle X_1, -X_2, -L_3 - \frac{1}{2}(P_3 + C_3) - \alpha(X_0 + X_4), \alpha > 0 \rangle.$$

Ansatz:

$$\alpha \arctan \frac{x_3}{u} - x_0 = \varphi(\omega), \quad \omega = (x_3^2 + u^2)^{1/2} \quad (274)$$

Reduced equation:

$$\omega^4 \left(\omega^2 (\varphi')^2 - \omega^2 + \alpha^2 \right) = 0 \quad (275)$$

Solutions of the reduced equation:

$$\omega = 0, \quad \varphi(\omega) = \varepsilon \sqrt{\omega^2 - \alpha^2} - i\varepsilon \alpha \operatorname{arctanh} \frac{i\alpha}{\sqrt{\omega^2 - \alpha^2}} + c, \varepsilon = \pm 1 \quad (276)$$

Solutions of the eikonal equation:

$$x_3^2 + u^2 = 0, \alpha \arctan \frac{x_3}{u} - x_0 = \varepsilon \sqrt{x_3^2 + u^2 - \alpha^2} - i\varepsilon \alpha \operatorname{artanh} \frac{i\alpha}{\sqrt{x_3^2 + u^2 - \alpha^2}} + c, \varepsilon = \pm 1 \quad (277)$$

$$(2) \langle X_1, X_2, L_3 + \frac{\lambda}{2}(P_3 + C_3) + \alpha(X_0 + X_4), \alpha > 0, 0 < \lambda < 1 \rangle.$$

Ansatz:

$$\alpha \arctan \frac{x_3}{u} - \lambda x_0 = \varphi(\omega), \quad \omega = (x_3^2 + u^2)^{1/2} \quad (278)$$

Reduced equation:

$$\omega^4 \left(\omega^2 (\varphi')^2 - \lambda^2 \omega^2 + \alpha^2 \right) = 0 \quad (279)$$

Solutions of the reduced equation:

$$\omega = 0, \quad \varphi(\omega) = \varepsilon \sqrt{\lambda^2 \omega^2 - \alpha^2} - i\varepsilon \alpha \operatorname{artanh} \frac{i\alpha}{\sqrt{\lambda^2 \omega^2 - \alpha^2}} + c, \varepsilon = \pm 1 \quad (280)$$

Solutions of the eikonal equation:

$$\alpha \arctan \frac{x_3}{u} - \lambda x_0 = \varepsilon \sqrt{\lambda^2 (x_3^2 + u^2) - \alpha^2} - i\varepsilon \alpha \operatorname{artanh} \frac{i\alpha}{\sqrt{\lambda^2 (x_3^2 + u^2) - \alpha^2}} + c$$

$$\varepsilon = \pm 1, \quad x_3^2 + u^2 = 0$$

$$(3) \langle X_1, X_2, L_3 + \lambda G + \alpha X_3, \alpha > 0, \lambda > 0 \rangle.$$

Ansatz:

$$\lambda x_3 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = (x_0^2 - u^2)^{1/2} \quad (281)$$

Reduced equation:

$$\omega(\omega(\varphi')^2 + 2\alpha\varphi' - \lambda^2\omega) = 0 \quad (282)$$

Solutions of the reduced equation:

$$\omega = 0, \quad \varphi(\omega) = \varepsilon \sqrt{\lambda^2 \omega^2 + \alpha^2} - \varepsilon \alpha \operatorname{artanh} \frac{\alpha}{\sqrt{\lambda^2 \omega^2 + \alpha^2}} - \alpha \ln(\omega) + c_1, \varepsilon = \pm 1. \quad (283)$$

Solutions of the eikonal equation:

$$x_0^2 - u^2 = 0 \quad (284)$$

$$\lambda x_3 - \alpha \ln(x_0 + u) = \varepsilon \sqrt{\lambda^2 (x_0^2 - u^2) + \alpha^2} - \varepsilon \alpha \operatorname{artanh} \frac{\alpha}{\sqrt{\lambda^2 (x_0^2 - u^2) + \alpha^2}} - \frac{\alpha}{2} \ln(x_0^2 - u^2) + c_1 \quad (285)$$

$$\varepsilon = \pm 1 \quad (286)$$

3.8.4. Reductions to PDEs

From the invariants of two nonconjugate subalgebras, we constructed the ansatzes, which reduced the eikonal equation to PDEs.

$$(1) \langle X_1, X_2, L_3 \rangle.$$

Ansatz:

$$u = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_0, \quad \omega_2 = x_3 \quad (287)$$

Reduced equation:

$$\varphi_1^2 - \varphi_2^2 - 1 = 0 \quad (288)$$

Solutions of the reduced equation:

$$\varphi(\omega_1, \omega_2) = \varepsilon \sqrt{c_2^2 + 1} \omega_1 + c_2 \omega_2 + c_1 + c_2, \varepsilon = \pm 1 \quad (289)$$

Solutions of the eikonal equation:

$$u = \varepsilon \sqrt{c_2^2 + 1} x_0 + c_2 x_3 + c_1 + c_2, \varepsilon = \pm 1 \quad (290)$$

(2) $\langle P_1, P_2, L_3 \rangle$.

Ansatz:

$$x_3 = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_0 + u, \quad \omega_2 = x_0^2 - x_1^2 - x_2^2 - u^2 \quad (291)$$

The reduced equation:

$$4\omega_2 \varphi_2^2 + 4\omega_1 \varphi_1 \varphi_2 - 1 = 0 \quad (292)$$

Solutions of the reduced equation:

$$\varphi(\omega_1, \omega_2) = c_1 \ln \omega_1 - \varepsilon (\omega_2 + c_1^2)^{1/2} + i \varepsilon c_1 \arctan \frac{\sqrt{\omega_2 + c_1^2}}{i c_1} - \frac{c_1}{2} \ln \omega_2 + c_2, \varepsilon = \pm 1 \quad (293)$$

Solutions on the eikonal equation:

$$\begin{aligned} x_3 = & c_1 \ln(x_0 + u) - \varepsilon (x_0^2 - x_1^2 - x_2^2 - u^2 + c_1^2)^{1/2} \\ & + i \varepsilon c_1 \arctan \frac{\sqrt{x_0^2 - x_1^2 - x_2^2 - u^2 + c_1^2}}{i c_1} \\ & - \frac{c_1}{2} \ln(x_0^2 - x_1^2 - x_2^2 - u^2) + c_2 \\ & \varepsilon = \pm 1 \end{aligned} \quad (294)$$

As we see, in the above two cases, the reduced equations are PDEs. The reason is that the subalgebras corresponding to them have the rank two. Therefore, they have three invariants. As a rule, the ansatzes, which can be constructed with the help of those invariants, reduce the eikonal equation to PDEs.

3.8.5. There Are No Reductions

From the invariants of remaining two nonconjugate subalgebras, it is impossible to construct the ansatzes, which reduce the eikonal equation.

Below, we present those subalgebras as well as their invariants.

- (1) $\langle -X_3, X_4 - X_0, -\frac{L_3}{\lambda} - \frac{1}{2}(P_3 + C_3), 0 < \lambda < 1 \rangle : x_0, (x_1^2 + x_2^2)^{1/2};$
- (2) $\langle X_3, X_4 - X_0, \frac{L_3}{\lambda} + \frac{1}{2}(P_3 + C_3) + \frac{\alpha}{\lambda}(X_0 + X_4), \alpha > 0, 0 < \lambda < 1 \rangle : (x_1^2 + x_2^2)^{1/2},$
 $\alpha \arctan \frac{x_1}{x_2} - x_0.$

3.9. Lie Algebras of the Type $A_{3,7}^a$

$$[e_1, e_3] = a e_1 - e_2, [e_2, e_3] = e_1 + a e_2, (a > 0). \quad a = c : \quad (295)$$

The Lie algebras of the type $A_{3,7}^a$ are solvable [41,43].

The Lie algebra of the group $P(1,4)$ contains two nonconjugate subalgebras of the type $A_{3,7}^a$. Below, we present the results obtained for those subalgebras.

3.9.1. Reductions to Equations, Which Can Be Split on Two Linear ODEs

Taking into account the invariants of one nonconjugate subalgebra, we constructed the ansatz, which reduced the eikonal equation to those, which could be split on two linear ODEs.

$$\langle P_1, P_2, L_3 + \lambda G, \lambda > 0 \rangle \quad (296)$$

Ansatz:

$$(x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = x_3 \quad (297)$$

Reduced equation:

$$(\varphi' - 1)(\varphi' + 1)\varphi^2 = 0 \quad (298)$$

Solutions of the reduced equation:

$$\varphi(\omega) = \varepsilon\omega + c_1, \quad \varepsilon = \pm 1, \quad \varphi = 0 \quad (299)$$

Solutions of the eikonal equation:

$$(x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2} = \varepsilon x_3 + c_1, \quad \varepsilon = \pm 1, \quad x_0^2 - x_1^2 - x_2^2 - u^2 = 0 \quad (300)$$

3.9.2. Reductions to Nonlinear ODEs

From the invariants of one nonconjugate subalgebra, we constructed the ansatz, which reduced the eikonal equation to nonlinear ODEs.

$$\langle P_1, P_2, L_3 + \lambda G + \alpha X_3, \alpha > 0, \lambda > 0 \rangle \quad (301)$$

Ansatz:

$$\lambda x_3 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = x_0^2 - x_1^2 - x_2^2 - u^2 \quad (302)$$

The reduced equation:

$$4\omega (\varphi')^2 + 4\alpha\varphi' - \lambda^2 = 0 \quad (303)$$

Solutions of the reduced equation:

$$\varphi(\omega) = \varepsilon(\lambda^2\omega + \alpha^2)^{1/2} - i\varepsilon\alpha \arctan \frac{\sqrt{\lambda^2\omega + \alpha^2}}{i\alpha} - \frac{\alpha}{2} \ln \omega + c, \quad \varepsilon = \pm 1 \quad (304)$$

Solutions of the eikonal equation:

$$\begin{aligned} \lambda x_3 - \alpha \ln(x_0 + u) &= \varepsilon(\lambda^2(x_0^2 - x_1^2 - x_2^2 - u^2) + \alpha^2)^{1/2} \\ &\quad - i\varepsilon\alpha \arctan \frac{\sqrt{\lambda^2(x_0^2 - x_1^2 - x_2^2 - u^2) + \alpha^2}}{i\alpha} \\ &\quad - \frac{\alpha}{2} \ln(x_0^2 - x_1^2 - x_2^2 - u^2) + c \\ \varepsilon &= \pm 1 \end{aligned}$$

3.10. Lie Algebras of the Type $A_{3,8}$

$$[e_1, e_3] = -2e_2, [e_1, e_2] = e_1, [e_2, e_3] = e_3 \quad (305)$$

The Lie algebras of the type $A_{3,8}$ are semisimple [43].

The Lie algebra of the group $P(1,4)$ contains one nonconjugate subalgebra of the type $A_{3,8}$. Below, we present the results obtained for this subalgebra.

3.10.1. Reductions to PDEs

From the invariants of one nonconjugate subalgebra, we constructed the ansatz, which reduced the eikonal equation to PDEs.

$$\langle P_3, G, -C_3 \rangle \quad (306)$$

Ansatz:

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_1, \quad \omega_2 = x_2 \quad (307)$$

Reduced equation:

$$\varphi^2(\varphi_1^2 + \varphi_2^2 - 1) = 0 \quad (308)$$

Solutions of the reduced equation:

$$\varphi(\omega_1, \omega_2) = 0, \quad \varphi(\omega_1, \omega_2) = \varepsilon \sqrt{1 - c_2^2} \omega_1 + c_2 \omega_2 + c_1 + c_2, \quad \varepsilon = \pm 1 \quad (309)$$

Solutions of the eikonal equation:

$$x_0^2 - x_3^2 - u^2 = 0, \quad (x_0^2 - x_3^2 - u^2)^{1/2} = \varepsilon(1 - c_2^2)^{1/2} x_1 + c_2 x_2 + c_1 + c_2, \quad \varepsilon = \pm 1 \quad (310)$$

As we see, the reduced equation is PDEs. As above, the reason is that the corresponding subalgebra has rank two.

3.11. Lie Algebras of the Type $A_{3,9}$

$$[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2 \quad (311)$$

The Lie algebras of the type $A_{3,9}$ are semisimple [43].

The Lie algebra of the group $P(1,4)$ contains two nonconjugate subalgebras of the type $A_{3,9}$. Below, we present the results obtained for those subalgebras.

3.11.1. Reductions to Equations, Which Can Be Split on Two Linear ODEs

Taking into account the invariants of one nonconjugate subalgebra, we constructed the ansatz, which reduced the eikonal equation to those, which could be split on two linear ODEs.

$$\left\langle -\frac{1}{2} \left(L_3 + \frac{1}{2} (P_3 + C_3) \right), \frac{1}{2} \left(L_2 + \frac{1}{2} (P_2 + C_2) \right), \frac{1}{2} \left(L_1 + \frac{1}{2} (P_1 + C_1) \right) \right\rangle \quad (312)$$

Ansatz:

$$(x_1^2 + x_2^2 + x_3^2 + u^2)^{1/2} = \varphi(\omega), \quad \omega = x_0 \quad (313)$$

Reduced equation:

$$(\varphi' - 1)(\varphi' + 1)\varphi^2 = 0 \quad (314)$$

Solutions of the reduced equation:

$$\varphi(\omega) = \varepsilon \omega + c_1, \quad \varepsilon = \pm 1, \quad \varphi = 0 \quad (315)$$

Solutions of the eikonal equation:

$$(x_1^2 + x_2^2 + x_3^2 + u^2)^{1/2} = \varepsilon x_0 + c_1, \quad \varepsilon = \pm 1, \quad x_1^2 + x_2^2 + x_3^2 + u^2 = 0 \quad (316)$$

3.11.2. Reductions to PDEs

From the invariants of one nonconjugate subalgebra, we constructed the ansatz, which reduced the eikonal equation to PDEs.

$$\langle -L_3, -L_2, -L_1 \rangle \quad (317)$$

Ansatz:

$$u = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_0, \quad \omega_2 = (x_1^2 + x_2^2 + x_3^2)^{1/2} \quad (318)$$

Reduced equation:

$$\varphi_1^2 - \varphi_2^2 - 1 = 0 \quad (319)$$

Solutions of the reduced equation:

$$\varphi(\omega_1, \omega_2) = \varepsilon \sqrt{c_2^2 + 1} \omega_1 + c_2 \omega_2 + c_1 + c_2, \quad \varepsilon = \pm 1 \quad (320)$$

Solutions of the eikonal equation:

$$u = \varepsilon (c_2^2 + 1)^{1/2} x_0 + c_2 (x_1^2 + x_2^2 + x_3^2)^{1/2} + c_1 + c_2, \quad \varepsilon = \pm 1 \quad (321)$$

As we see, the reduced equation is PDEs. As above, the reason is that the corresponding subalgebra has rank two.

4. Conclusions

The relationship is studied between the structural properties of three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ and the types of the symmetry reductions of the eikonal equation to ODEs.

Below, we present some consequences resulting from that relationship.

- Reductions to algebraic equations are induced by some subalgebras of the types $3A_1, A_{3,1}$;
- Reductions to linear ODEs are induced by some subalgebras of the types $3A_1, A_{3,6}$;
- Reductions to equations, which can be split on two linear ODEs, are induced by some subalgebras of the types $3A_1, A_2 \oplus A_1, A_{3,1}, A_{3,3}, A_{3,6}, A_{3,7}^a, A_{3,9}$;
- Reductions to nonlinear ODEs are induced by subalgebras of the type $A_{3,2}$ as well as by some subalgebras of the types $3A_1, A_2 \oplus A_1, A_{3,1}, A_{3,3}, A_{3,6}, A_{3,7}^a$;
- Reductions to PDEs are induced by subalgebras of the type $A_{3,8}$ as well as by some subalgebras of the types $A_{3,6}, A_{3,9}$;
- From invariants of subalgebras of the type $A_{3,4}$ as well as of some subalgebras of the types $3A_1, A_2 \oplus A_1, A_{3,1}, A_{3,2}, A_{3,3}, A_{3,6}$, we cannot construct ansatzes reducing the eikonal equation;
- There are no nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ of the type $A_{3,5}$.

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