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## Estrada Index of Random Bipartite Graphs

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**Abstract:** The Estrada index of a graph  $G$  of  $n$  vertices is defined by  $EE(G) = \sum_{i=1}^n e^{\lambda_i}$ , where  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the eigenvalues of  $G$ . In this paper, we give upper and lower bounds of  $EE(G)$  for almost all bipartite graphs by investigating the upper and lower bounds of the spectrum of random matrices. We also formulate an exact estimate of  $EE(G)$  for almost all balanced bipartite graphs.

**Keywords:** Estrada index; random graph; eigenvalues; limit spectral distribution

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### 1. Introduction

The topological structures of many social, biological, and technological systems can be characterized by the connectivity properties of the interaction pathways (edges) between system components (vertices) [1]. Starting with the Königsberg seven-bridge problem in 1736, graphs with bidirectional or symmetric edges have ideally epitomized structures of various complex systems, and have developed into one of the mainstays of the modern discrete mathematics and network theory. Formally, a simple graph  $G$  consists of a vertex set  $V = \{1, 2, \dots, n\}$  and an edge set  $E \subseteq V \times V$ . The adjacency matrix of  $G$  is a symmetric  $(0, 1)$ -matrix  $A(G) = (a_{ij}) \in \mathbb{R}^{n \times n}$ , where  $a_{ij} = a_{ji} = 1$  if vertices  $i$  and  $j$  are adjacent, and  $a_{ij} = a_{ji} = 0$  otherwise. It is well-known in algebraic graph theory that  $A(G)$  has exactly  $n$  real eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  due to its symmetry. They are usually called the spectrum (eigenvalues) of  $G$  itself [2].

A spectral graph invariant, the Estrada index  $EE(G)$  of  $G$ , is defined as

$$EE(G) = \sum_{i=1}^n e^{\lambda_i}.$$

This quantity was introduced by Estrada [3] in 2000. It has noteworthy chemical applications, such as quantifying the degree of folding of long-chain molecules and the Shannon entropy [3–7]. The Estrada

index provides a remarkable measure of subgraph centrality as well as fault tolerance in the study of complex networks [1,8–10]. Building upon varied symmetric features in graphs, mathematical properties of this invariant can be found in, e.g., [11–18].

The Estrada index can be readily calculated once the eigenvalues are known. However, it is notoriously difficult to compute the eigenvalues of a large matrix even for  $(0, 1)$ -matrix  $A(G)$ . In the past few years, researchers managed to establish a number of lower and upper bounds to estimate this invariant (see [13] for an updated survey). A common drawback is that only few classes of graphs attain the equalities of those bounds. Therefore, one may naturally wonder the typical behavior of the invariant  $EE(G)$  for most graphs with respect to other graph parameters such as the number of vertices  $n$ .

The classical Erdős–Rényi random graph model  $\mathcal{G}_n(p)$  includes the edges between all pairs of vertices independently at random with probability  $p$  [19]. It has symmetric, bell-shaped degree distribution, which is shared by many other random graph models. Regarding the Estrada index, Chen *et al.* [20] showed the following result: Let  $G_n(p) \in \mathcal{G}_n(p)$  be a random graph with a constant  $p \in (0, 1)$ , then

$$EE(G_n(p)) = e^{np}(e^{O(\sqrt{n})} + o(1)) \quad a.s. \quad (1)$$

Here, we say that a certain property  $\mathcal{P}$  holds in  $\mathcal{G}_n(p)$  almost surely (a.s.) if the probability that a random graph  $G_n(p)$  has the property  $\mathcal{P}$  converges to 1 as  $n$  tends to infinity. Therefore, the result (1) presents an analytical estimate of the Estrada index for almost all graphs.

Our motivation in this paper is to investigate the Estrada index of random bipartite graphs, which is a natural bipartite version of Erdős–Rényi random graphs. Bipartite graphs appear in a range of applications in timetabling, communication networks and computer science, where components of the systems are endowed with two different attributes and symmetric relations are only established between these two parts [21–23]. Formally, a bipartite graph is a graph whose vertices can be divided into two disjoint sets  $V_1$  and  $V_2$  such that every edge connects a vertex in  $V_1$  to a vertex in  $V_2$ . A bipartite graph is a graph that does not contain any odd-length cycles; (chemical) trees are bipartite graphs. The random bipartite graph model is denoted by  $\mathcal{G}_{n_1, n_2}(p)$ , where  $n_i = |V_i|$  for  $i = 1, 2$ , satisfying  $n_1 + n_2 = n$ .

The authors in [20] posed the following conjecture pertaining to bipartite graphs.

**Conjecture 1.** Let  $G_{n_1, n_2}(p) \in \mathcal{G}_{n_1, n_2}(p)$  be a random bipartite graph with a constant  $p \in (0, 1)$ . Then

$$EE(G_{n_1, n_2}(p)) = e^{\frac{n}{2}p}(e^{O(\sqrt{n})} + o(1)) \quad a.s. \quad (2)$$

if and only if  $\lim_{n \rightarrow \infty} n_2/n_1 = 1$ .

In this paper, by means of the symmetry in  $G_{n_1, n_2}(p)$  and the spectral distribution of random matrix, we obtain lower and upper bounds for  $EE(G_{n_1, n_2}(p))$ . For ease of analysis, we assume that  $\lim_{n \rightarrow \infty} n_2/n_1 := y \in (0, 1]$ . We establish the estimate

**Theorem 1.**

$$e^{n_2 p}(e^{O(\sqrt{n})} + o(1)) \leq EE(G_{n_1, n_2}(p)) \leq e^{n_1 p}(e^{O(\sqrt{n})} + O(1)) \quad a.s. \quad (3)$$

Thus a weak version of Conjecture 1 follows readily:

$$EE(G_{n_1, n_2}(p)) = e^{\frac{n}{2}p}(e^{O(\sqrt{n})} + O(1)) \quad a.s.$$

holds, provided  $\lim_{n \rightarrow \infty} n_2/n_1 = 1$  (i.e.,  $y = 1$ ).

## 2. Examples

In this section, we give some analytical and numerical examples to demonstrate that our estimate (3) is valid, and is better than some existing bounds obtained by algebraic approaches.

**Example 1.** Let  $G$  be a bipartite graph with vertex set  $V = V_1 \cup V_2$ ,  $|V| = |V_1| + |V_2| = n_1 + n_2 = n$  and  $m$  edges. In [11] and [18], it was proved that

$$\sqrt{n^2 + 4m} \leq EE(G) \leq n - 2 + 2 \cosh(\sqrt{m}), \quad (4)$$

where the equality on the left-hand side of (4) holds if and only if  $G$  is an empty graph, while the equality on the right-hand side of (4) holds if and only if  $G$  is a union of a complete bipartite graph and some isolated vertices. If  $G \in \mathcal{G}_{n_1, n_2}(p)$ , then the number of edges is  $m = n_1 n_2 p$  a.s. [19]. Hence, the lower bounds in (4) becomes

$$\sqrt{n^2 + 4m} = \sqrt{n^2 + 4n_1 n_2 p} \ll e^{n_2 p} (e^{O(\sqrt{n})} + o(1)) \quad a.s.$$

The upper bound in (4) becomes

$$\begin{aligned} n - 2 + 2 \cosh(\sqrt{m}) &= n - 2 + e^{\sqrt{n_1 n_2 p}} + e^{-\sqrt{n_1 n_2 p}} = e^{n_1 p} (e^{n_1 (\sqrt{y p} - p)} + o(1)) \\ &= e^{n_1 p} \cdot \left( e^{n \left( \frac{\sqrt{y p} - p}{1+y} \right)} + o(1) \right) \\ &\gg e^{n_1 p} (e^{O(\sqrt{n})} + O(1)), \quad a.s. \end{aligned}$$

if  $n_2/n_1 \rightarrow y > p$ .

**Example 2.** In [15], the authors bounded the Estrada index of a graph  $G$  of order  $n$  by its graph energy  $E(G)$ :

$$EE(G) \leq n - 1 + e^{\frac{E(G)}{2}},$$

with equality if and only if  $E(G) = 0$ . If  $G \in \mathcal{G}_{n_1, n_2}(p)$ , it can be shown that [24]  $E(G) = (2n_2 \sqrt{n_1} C(p) n^{-3/2} + o(1)) n^{3/2}$  a.s., where  $C(p) > 0$  is a constant depending on  $p$ . Hence, we have

$$EE(G) \leq n - 1 + e^{\left( 2n_2 \sqrt{n_1} C(p) n^{-\frac{3}{2}} + o(1) \right) n^{\frac{3}{2}}}, \quad a.s. \quad (5)$$

We calculate the upper bound in (5) as

$$\begin{aligned} &n - 1 + e^{\left( 2n_2 \sqrt{n_1} C(p) n^{-\frac{3}{2}} + o(1) \right) n^{\frac{3}{2}}} \\ &= e^{n_1 p} \cdot \left( e^{\left( 2y C(p) n_1^{\frac{3}{2}} n^{-\frac{3}{2}} + o(1) \right) n^{\frac{3}{2}} - n_1 p} + o(1) \right) \\ &= e^{n_1 p} \cdot \left( e^{\left( 2y(1+y)^{-\frac{3}{2}} C(p) + o(1) \right) n^{\frac{3}{2}} - np(1+y)^{-1}} + o(1) \right) \\ &\gg e^{n_1 p} (e^{O(\sqrt{n})} + O(1)). \quad a.s. \end{aligned}$$

**Example 3.** Let  $G$  be a bipartite graph with  $n$  vertices and  $m \geq 1$  edges. Denote by  $n_0$  the number of zero eigenvalues in the spectrum of  $G$ . It was proved in [25] and [12] that

$$EE(G) \geq n_0 + (n - n_0) \cosh \left( \sqrt{\frac{2m}{n - n_0}} \right), \tag{6}$$

where the equality holds if and only if  $n - n_0$  is even, and  $G$  consists of copies of complete bipartite graphs  $K_{a_i, b_i}$ ,  $i = 1, \dots, (n - n_0)/2$ , such that all products  $a_i b_i$  are mutually equal. If  $G \in \mathcal{G}_{n_1, n_2}(p)$ , then the number of edges is  $m = n_1 n_2 p$  a.s. Hence, the lower bound in (6) becomes

$$\begin{aligned} n_0 + (n - n_0) \cosh \left( \sqrt{\frac{2m}{n - n_0}} \right) &= n_0 + \frac{1}{2}(n - n_0) \left( e^{\sqrt{\frac{2n_1 n_2 p}{n - n_0}}} + e^{-\sqrt{\frac{2n_1 n_2 p}{n - n_0}}} \right) \\ &\leq e^{n_2 p} \left( \frac{n}{2} e^{\sqrt{\frac{2n_1 n_2 p}{n}} - n_2 p} + o(1) \right) \\ &= e^{n_2 p} \left( \frac{n}{2} e^{\frac{\sqrt{2y n p} - y n p}{1+y}} + o(1) \right) \\ &\ll e^{n_2 p} (e^{O(\sqrt{n})} + o(1)). \quad a.s. \end{aligned}$$

**Example 4.** An upper bound for  $EE(G_{n_1, n_2}(p))$  was derived in [20] as

$$EE(G_{n_1, n_2}(p)) \leq e^{np} (e^{O(\sqrt{n})} + o(1)) \quad a.s. \tag{7}$$

Since

$$\frac{e^{n_1 p} (e^{O(\sqrt{n})} + O(1))}{e^{np} (e^{O(\sqrt{n})} + o(1))} = (1 + o(1)) e^{-\frac{y n p}{1+y}} = o(1),$$

our upper bound in Theorem 1 is better than that in (7).

In Table 1, we compare the estimates for  $EE(G_{n_1, n_2}(1/2))$  in Theorem 1 with numerical value obtained by Matlab software. The results are in good agreement with the theory.

**Table 1.** Estrada index of  $G_{n_1, n_2}(p)$  with  $n_1 = 3000$  and  $p = 1/2$ . Numerical results are based on average over 10 independent runs.

$n_2$	$y$	Theoretical lower bound	Numerical value	Theoretical upper bound
3000	1	$e^{1500} (e^{O(77.46)} + o(1))$	$e^{1543.72}$	$e^{1500} (e^{O(77.46)} + O(1))$
2700	0.9	$e^{1350} (e^{O(75.50)} + o(1))$	$e^{1528.29}$	$e^{1500} (e^{O(75.50)} + O(1))$
2400	0.8	$e^{1200} (e^{O(73.48)} + o(1))$	$e^{1497.03}$	$e^{1500} (e^{O(73.48)} + O(1))$
2100	0.7	$e^{1050} (e^{O(71.41)} + o(1))$	$e^{1456.85}$	$e^{1500} (e^{O(71.41)} + O(1))$

### 3. Proof of Theorem 1

This section is devoted to the proof of Theorem 1, which heavily relies on the symmetry in  $G_{n_1, n_2}(p)$ .

Throughout the paper, we shall understand  $p \in (0, 1)$  as a constant. Let  $M = (m_{ij}) \in \mathbb{R}^{n \times n}$  be a random matrix, where the entries  $m_{ij}$  ( $i \leq j$ ) are independent and identically distributed with

$m_{ij} = m_{ji}$ . We denote by the eigenvalues of  $M$  by  $\lambda_1(M), \lambda_2(M), \dots, \lambda_n(M)$  and its empirical spectral distribution by

$$\begin{aligned}\Phi_M(x) &= \frac{1}{n} \cdot \#\{\lambda_i(M) \mid \lambda_i(M) \leq x, i = 1, 2, \dots, n\} \\ &= \frac{1}{n} \sum_{i=1}^n 1_{\{\lambda_i(M) \leq x\}}.\end{aligned}$$

**Lemma 1.** (Marčenko–Pastur Law [26]) *Let  $X = (x_{ij}) \in \mathbb{R}^{n_2 \times n_1}$  be a random matrix, where the entries  $x_{ij}$  are independent and identically distributed with mean zero and variance  $p(1-p)$ . Suppose that  $n_1, n_2$  are functions of  $n$ , and  $\lim_{n \rightarrow \infty} n_2/n_1 = y \in (0, \infty)$ . Then, with probability 1, the empirical spectral distribution  $\Phi_{\frac{1}{n_1}XX^T}(x)$  converges weakly to the Marčenko–Pastur Law  $F_y$  as  $n \rightarrow \infty$ , where  $F_y$  has the density*

$$f_y(x) = \frac{1}{2\pi p(1-p)xy} \sqrt{(b-x)(x-a)} 1_{\{a \leq x \leq b\}}$$

and has a point mass  $1 - 1/y$  at the origin if  $y > 1$ , where  $a = p(1-p)(1 - \sqrt{y})^2$  and  $b = p(1-p)(1 + \sqrt{y})^2$ .

The above result formulates the limit spectral distribution of  $\frac{1}{n_1}XX^T$ , which will be a key ingredient of our later derivation for  $EE(G_{n_1, n_2}(p))$ . The main approach employed to prove the assertion is called moment approach. It can be shown that for each  $k \in \mathbb{N}$ ,

$$\lim_{n \rightarrow \infty} \int_0^\infty x^k d\Phi_{\frac{1}{n_1}XX^T}(x) = \int_0^\infty x^k dF_y(x) \quad a.s. \quad (8)$$

We refer the reader to the seminal survey by Bai [26] for further details on the moment approach and the Marčenko–Pastur Law-like results.

The following two lemmas will be needed.

**Lemma 2.** (In Page 219 [27]), *Let  $\mu$  be a measure. Suppose that functions  $g_n, h_n, f_n$  converge almost everywhere to functions  $g, h, f$ , respectively, and that  $g_n \leq f_n \leq h_n$  almost everywhere. If  $\int g_n d\mu \rightarrow \int g d\mu$  and  $\int h_n d\mu \rightarrow \int h d\mu$ , then  $\int f_n d\mu \rightarrow \int f d\mu$ .*

**Lemma 3.** (Weyl's inequality [28]) *Let  $X \in \mathbb{R}^{n \times n}, Y \in \mathbb{R}^{n \times n}$  and  $Z \in \mathbb{R}^{n \times n}$  be symmetric matrices such that  $X = Y + Z$ . Suppose their eigenvalues are ordered as  $\lambda_1(X) \geq \lambda_2(X) \geq \dots \geq \lambda_n(X)$ ,  $\lambda_1(Y) \geq \lambda_2(Y) \geq \dots \geq \lambda_n(Y)$ , and  $\lambda_1(Z) \geq \lambda_2(Z) \geq \dots \geq \lambda_n(Z)$ , respectively. Then*

$$\lambda_i(Y) + \lambda_n(Z) \leq \lambda_i(X) \leq \lambda_i(Y) + \lambda_1(Z),$$

for any  $1 \leq i \leq n$ .

Recall that the random bipartite graph  $\mathcal{G}_{n_1, n_2}(p)$  consists of all bipartite graphs with vertex set  $V = V_1 \cup V_2$ , in which the edges connecting vertices between  $V_1$  and  $V_2$  are chosen independently with probability  $p \in (0, 1)$ . We assume  $|V_i| = n_i$  ( $i = 1, 2$ ),  $n_1 + n_2 = n$  and  $\lim_{n \rightarrow \infty} n_2/n_1 := y \in (0, 1]$ .

For brevity, let  $A_n = A(G_{n_1, n_2}(p))$  be the adjacency matrix, and denote by  $Q_n = (q_{ij}) \in \mathbb{R}^{n \times n}$  a quasi-unit matrix, where  $q_{ij} = q_{ji} = 1$  if  $i, j \in V_1$  or  $i, j \in V_2$ , and  $q_{ij} = q_{ji} = 0$  otherwise. Let



Combining (10) with (11) we derive

$$\begin{aligned}\lim_{n \rightarrow \infty} \int_1^{\infty} e^x d\Phi_{\frac{1}{n_1} XX^T}(x) &= \lim_{n \rightarrow \infty} \left( \int_0^{\infty} e^x d\Phi_{\frac{1}{n_1} XX^T}(x) - \int_0^1 e^x d\Phi_{\frac{1}{n_1} XX^T}(x) \right) \\ &= \int_0^{\infty} e^x dF_y(x) - \int_0^1 e^x dF_y(x) \\ &= \int_1^{\infty} e^x dF_y(x) \quad a.s.\end{aligned}$$

□

**Proposition 2.**

$$EE \left( \frac{\bar{A}_n}{\sqrt{n_1}} \right) = n_1 - n_2 + n_2 \left( \frac{\int_{\sqrt{a}}^{\sqrt{b}} 2 \cosh(x) \sqrt{\left(\frac{b}{x^2} - 1\right) \left(1 - \frac{a}{x^2}\right)} dx}{\pi p(1-p)y} + o(1) \right) \quad a.s.,$$

where  $a$  and  $b$  are given as in Lemma 1.

**Proof.** Define

$$\Psi(x) = \frac{1}{n_2} \sum_{i=1}^{n_2} 1_{\{\sqrt{\lambda_i(\frac{1}{n_1} XX^T)} \leq x\}}.$$

Then we have

$$e^{\bar{\lambda}_1} + \dots + e^{\bar{\lambda}_{n_2}} = n_2 \int_0^{\infty} e^x d\Psi(x) = n_2 \int_0^{\infty} e^{\sqrt{x}} d\Phi_{\frac{1}{n_1} XX^T}(x). \quad (12)$$

Analogous to the proof of (10) we derive

$$\lim_{n \rightarrow \infty} \int_0^1 e^{\sqrt{x}} d\Phi_{\frac{1}{n_1} XX^T}(x) = \int_0^1 e^{\sqrt{x}} dF_y(x) \quad a.s.$$

For any  $x \in [1, \infty)$ , we have  $0 \leq e^{\sqrt{x}} \leq e^x$ . By Lemma 2 and Proposition 1 we deduce that

$$\lim_{n \rightarrow \infty} \int_1^{\infty} e^{\sqrt{x}} d\Phi_{\frac{1}{n_1} XX^T}(x) = \int_1^{\infty} e^{\sqrt{x}} dF_y(x) \quad a.s.$$

Accordingly, we have

$$\lim_{n \rightarrow \infty} \int_0^{\infty} e^{\sqrt{x}} d\Phi_{\frac{1}{n_1} XX^T}(x) = \int_0^{\infty} e^{\sqrt{x}} dF_y(x) \quad a.s. \quad (13)$$

It follows from (12) and (13) that

$$\begin{aligned}& e^{\bar{\lambda}_1} + \dots + e^{\bar{\lambda}_{n_2}} \\ \rightarrow & n_2 \int_0^{\infty} e^{\sqrt{x}} dF_y(x) \\ = & \int_a^b \frac{e^{\sqrt{x}}}{2\pi p(1-p)xy} \sqrt{(b-x)(x-a)} dx \\ = & \frac{1}{\pi p(1-p)y} \int_{\sqrt{a}}^{\sqrt{b}} e^x \sqrt{\left(\frac{b}{x^2} - 1\right) \left(1 - \frac{a}{x^2}\right)} dx \quad a.s.\end{aligned} \quad (14)$$

Next, we calculate the sum of the exponentials of the smallest  $n_2$  eigenvalues of  $\bar{A}_n/\sqrt{n_1}$ . Similarly as in (12), we obtain

$$e^{\bar{\lambda}_{n_1+1}} + \dots + e^{\bar{\lambda}_n} = n_2 \int_0^\infty e^{-x} d\Psi(x) = n_2 \int_0^\infty e^{-\sqrt{x}} d\Phi_{\frac{1}{n_1}XX^T}(x). \tag{15}$$

Noting that  $0 \leq e^{-\sqrt{x}} \leq e^x$  for  $x \in [1, \infty)$ , we likewise have

$$\lim_{n \rightarrow \infty} \int_0^\infty e^{-\sqrt{x}} d\Phi_{\frac{1}{n_1}XX^T}(x) = \int_0^\infty e^{-\sqrt{x}} dF_y(x) \quad a.s. \tag{16}$$

by employing Lemma 2 and Proposition 1. It then follows from (15) and (16) that

$$\begin{aligned} & e^{\bar{\lambda}_{n_1+1}} + \dots + e^{\bar{\lambda}_n} \\ \rightarrow & n_2 \int_0^\infty e^{-\sqrt{x}} dF_y(x) \\ = & \frac{1}{\pi p(1-p)y} \int_{\sqrt{a}}^{\sqrt{b}} e^{-x} \sqrt{\left(\frac{b}{x^2} - 1\right) \left(1 - \frac{a}{x^2}\right)} dx \quad a.s. \end{aligned} \tag{17}$$

Finally, combining (14), (17) and the fact that  $\bar{\lambda}_{n_2+1} = \dots = \bar{\lambda}_{n_1} = 0$ , we readily deduce the assertion of Proposition 2.  $\square$

**Proposition 3.**

$$\begin{aligned} n_1 - n_2 + n_2 \left( \frac{(e^{-\sqrt{n_1}b} + e^{\sqrt{n_1}a})C}{\pi p(1-p)y} + o(1) \right) \\ \leq EE(\bar{A}_n) \leq n_1 - n_2 + n_2 \left( \frac{(e^{-\sqrt{n_1}a} + e^{\sqrt{n_1}b})C}{\pi p(1-p)y} + o(1) \right) \quad a.s., \end{aligned}$$

where  $a, b$  are given as in Lemma 1, and  $C = \int_{\sqrt{a}}^{\sqrt{b}} \sqrt{\left(\frac{b}{x^2} - 1\right) \left(1 - \frac{a}{x^2}\right)} dx$ .

**Proof.** Define

$$\tilde{\Psi}(x) = \frac{1}{n_2} \sum_{i=1}^{n_2} 1_{\{\sqrt{n_1 \lambda_i(\frac{1}{n_1}XX^T)} \leq x\}}.$$

Then we have  $\tilde{\Psi}(x) = \Psi(x/\sqrt{n_1})$ . Therefore, the sum of the exponentials of the largest  $n_2$  eigenvalues of  $\bar{A}_n$  is

$$\begin{aligned} e^{\lambda_1(\bar{A}_n)} + \dots + e^{\lambda_{n_2}(\bar{A}_n)} &= n_2 \int_0^\infty e^x d\tilde{\Psi}(x) = n_2 \int_0^\infty e^{\sqrt{n_1}x} d\Psi(x) \\ &= n_2 \int_0^\infty e^{\sqrt{n_1}x} d\Phi_{\frac{1}{n_1}XX^T}(x). \end{aligned}$$

Analogous to the proof of (14) we obtain

$$\begin{aligned} & \lim_{n \rightarrow \infty} e^{\lambda_1(\bar{A}_n)} + \dots + e^{\lambda_{n_2}(\bar{A}_n)} \\ = & \frac{1}{\pi p(1-p)y} \int_{\sqrt{a}}^{\sqrt{b}} e^{\sqrt{n_1}x} \sqrt{\left(\frac{b}{x^2} - 1\right) \left(1 - \frac{a}{x^2}\right)} dx \\ \in & \left[ \frac{Ce^{\sqrt{n_1}a}}{\pi p(1-p)y}, \frac{Ce^{\sqrt{n_1}b}}{\pi p(1-p)y} \right] \quad a.s. \end{aligned} \tag{18}$$

where  $C = \int_{\sqrt{a}}^{\sqrt{b}} \sqrt{\left(\frac{b}{x^2} - 1\right) \left(1 - \frac{a}{x^2}\right)} dx$ . Similarly, the sum of the exponentials of the smallest  $n_2$  eigenvalues of  $\bar{A}_n$  satisfies

$$\begin{aligned} & \lim_{n \rightarrow \infty} e^{\lambda_{n_1+1}(\bar{A}_n)} + \dots + e^{\lambda_n(\bar{A}_n)} \\ &= \frac{1}{\pi p(1-p)y} \int_{\sqrt{a}}^{\sqrt{b}} e^{-\sqrt{n_1}x} \sqrt{\left(\frac{b}{x^2} - 1\right) \left(1 - \frac{a}{x^2}\right)} dx \\ &\in \left[ \frac{C e^{-\sqrt{n_1}b}}{\pi p(1-p)y}, \frac{C e^{-\sqrt{n_1}a}}{\pi p(1-p)y} \right] \quad a.s. \end{aligned} \quad (19)$$

Combining (18), (19) and the fact that  $\lambda_{n_2+1}(\bar{A}_n) = \dots = \lambda_{n_1}(\bar{A}_n) = 0$ , we complete the proof of Proposition 3.  $\square$

#### Proposition 4.

$$e^{-pn_1}(e^{O(\sqrt{n_1})} + O(1)) \leq EE(A_n) \leq e^{pn_1}(e^{O(\sqrt{n_1})} + O(1)) \quad a.s.$$

**Proof.** Since  $n_1 \geq n_2$  for large  $n$ , by the Geršgorin circle theorem we deduce

$$-pn_1 \leq \lambda_n(p(J_n - Q_n)) \leq \dots \leq \lambda_1(p(J_n - Q_n)) \leq pn_1.$$

In view of Lemma 3 and  $A_n = \bar{A}_n + p(J_n - Q_n)$ , we get

$$\lambda_i(\bar{A}_n) + \lambda_n(p(J_n - Q_n)) \leq \lambda_i(A_n) \leq \lambda_i(\bar{A}_n) + \lambda_1(p(J_n - Q_n))$$

for all  $i$ . Consequently,  $e^{-pn_1} EE(\bar{A}_n) \leq EE(A_n) \leq e^{pn_1} EE(\bar{A}_n)$ . It then follows from Proposition 3 that

$$\begin{aligned} & e^{-pn_1}(n_1 - n_2) + e^{-pn_1} n_2 \left( \frac{(e^{-\sqrt{n_1}b} + e^{\sqrt{n_1}a})C}{\pi p(1-p)y} + o(1) \right) \\ & \leq EE(A_n) \leq e^{pn_1}(n_1 - n_2) + e^{pn_1} n_2 \left( \frac{(e^{-\sqrt{n_1}a} + e^{\sqrt{n_1}b})C}{\pi p(1-p)y} + o(1) \right) \quad a.s. \end{aligned} \quad (20)$$

Note that the rate of convergence in (18) as well as (19) can be bounded by  $O(n^{-1})$  using the moment approach (for instance, see Theorem 4.5.5 in [29]) and the estimates in [26] (pp. 621–623). Hence, the infinitesimal quantity  $o(1)$  on both sides of (20) is equivalent to  $O(n^{-1})$ . Inserting  $n_1 = n/(1+y)$  and  $n_2 = yn_1$  into (20), we have

$$e^{-pn_1}(e^{O(\sqrt{n_1})} + O(1)) \leq EE(A_n) \leq e^{pn_1}(e^{O(\sqrt{n_1})} + O(1)) \quad a.s.$$

as desired.  $\square$

With Proposition 4 in our hands, we quickly get the proof of our main result.

**Proof of Theorem 1.** Since  $n_1 = O(n)$ , the upper bound in Proposition 4 yields  $EE(G_{n_1, n_2}(p)) \leq e^{n_1 p}(e^{O(\sqrt{n})} + O(1))$  a.s. On the other hand, the lower bound  $EE(G_{n_1, n_2}(p)) \geq e^{n_2 p}(e^{O(\sqrt{n})} + o(1))$  a.s. can be easily read out from Theorem 3.1 of [20].  $\square$

We mention that it is possible that our method can be pushed further to obtain sharper bounds, for example, a more fine-grained analysis in Proposition 2 could give better estimates for the second order terms in the expansion for  $\ln(EE(A_n))$ .

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### Conflicts of Interest

The author declares no conflict of interest.

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