symmetry

MDPI

Article

Second Hankel Determinant and Fekete–Szegö Problem for a New Class of Bi-Univalent Functions Involving Euler Polynomials

Semh Kadhim Gebur [1] and Waggas Galib Atshan [2,*]

[1] Department of Mathematics, College of Education for Girls, University of Kufa, Najaf 540011, Iraq; semh.alisawi@qu.edu.iq
[2] Department of Mathematics, College of Science, University of Al-Qadisiyah, Diwaniyah 58001, Iraq
* Correspondence: waggas.galib@qu.edu.iq

Abstract: Orthogonal polynomials have been widely employed by renowned authors within the context of geometric function theory. This study is driven by prior research and aims to address the —Fekete-Szegö problem. Additionally, we provide bound estimates for the coefficients and an upper bound estimate for the second Hankel determinant for functions belonging to the category of analytical and bi-univalent functions. This investigation incorporates the utilization of Euler polynomials.

Keywords: Euler polynomials; Fekete–Szegö problem; analytic function; second Hankel determinant; bi-univalent function

MSC: 30C45

check for updates

Citation: Gebur, S.K.; Atshan, W.G. Second Hankel Determinant and Fekete–Szegö Problem for a New Class of Bi-Univalent Functions Involving Euler Polynomials. *Symmetry* **2024**, *16*, 530. https://doi.org/10.3390/sym16050530

Academic Editor: Dongfang Li

Received: 28 March 2024
Revised: 13 April 2024
Accepted: 19 April 2024
Published: 28 April 2024

1. Introduction

The collection of all functions f can be denoted as A and can be represented in the form of a series.

$$f(\xi) = \xi + \sum_{\iota=2}^{\infty} a_\iota \xi^\iota = \xi + a_2\xi^2 + a_3\xi^3 + \ldots + a_\iota \xi^\iota + \ldots, \tag{1}$$

which are analytic in the open unit disk U, where $U = \{\xi \in \mathbb{C} : |\xi| < 1\}$.

A function is considered univalent in U if it never produces the same value twice. Mathematically, $\xi_1 \neq \xi_2$ for all points ξ_1 and ξ_2 in U implies $f(\xi_1) \neq f(\xi_2)$).

Consider the class S, which includes all univalent functions in A, together with the families of starlike and convex functions of order φ. The sets $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$ are notable and extensively studied subclasses of S. Consequently, they have been included in this context, as referenced in [1–3].

$$\mathcal{S}^*(\varphi) = \left\{ f \in S : \mathrm{Re}\left(\frac{zf'(\xi)}{f(\xi)}\right) > \varphi, \xi \in U \right\}, \varphi \in [0,1)$$

$$\mathcal{C}(\varphi) = \left\{ f \in S : \mathrm{Re}\left(1 + \frac{zf''(\xi)}{f'(\xi)}\right) > \varphi, \xi \in U \right\}, \varphi \in [0,1)$$

and

It is readily apparent that

$$\mathcal{S}^*(0) = \mathcal{S}^* \text{ and } \mathcal{C}(0) = \mathcal{C},$$

The function classes $\mathcal{S}^*$ and $\mathcal{C}$ represent the well-established categories of starlike and convex functions, respectively.

If both g and f are analytic functions in U, then f is considered subordinate to g in U and represented as $f \prec g$, if there exists a Schwarz analytic function, w, in U, with $w(0) = 0$, $|w(\xi)| < 1(\xi \in U)$, such that $f(\xi) = g(w(\xi))$, $(\xi \in U)$. Moreover, the existence of the equivalence link is contingent upon the function g being univalent in U. The given expression for ([4,5]) is as follows:

$$f(\xi) \prec g(\xi) \leftrightarrow g(0) = f(0) \text{ and } f(U) \subset g(U), \xi \in U.$$

For further information, please refer to reference [1]. The definition of the inverse function for each $f \in S$ is as follows:

$$f^{-1}(f(\xi)) = \xi,$$

$$f\left(f^{-1}(\omega)\right) = \omega,\ \omega \in D_{\mathfrak{r}_0} = \{\omega \in \mathbb{C},: |\omega| < \mathfrak{r}_0(f)\},\ \mathfrak{r}_0(f) \geq \frac{1}{4},$$

where

$$f^{-1}(\omega) = \omega - a_2\omega^2 + \left(2a_2{}^2 - a_3\right)\omega^3 - \left(5a_2{}^3 - 5a_2a_3 + a_4\right)\omega^4 + \ldots, \omega \in D_{\mathfrak{r}_0}. \tag{2}$$

An analytic function f is considered bi-univalent in the set U if both f and f^{-1} are univalent in U. The classification of all such functions is represented by the symbol Σ. The pioneering research conducted by Srivastava et al. [6] has significantly rejuvenated the investigation of bi-univalent functions in recent decades. Following the study of Srivastava et al. [6], several authors have given and extensively investigated various distinct subclasses of class Σ. The authors Srivastava et al. [7] introduced the function classes $\mathrm{H}_\Sigma(\gamma, \varepsilon, \mu, \varsigma; \alpha)$ and $\mathrm{H}_\Sigma(\gamma, \varepsilon, \mu, \varsigma; \beta)$, and subsequently presented estimates for the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. Srivastava's work has served as a source of inspiration for numerous authors, who have since developed various subclasses of analytic and bi-univalent functions. Several authors have obtained different types of findings for their defined function classes. This paper, inspired by Srivastava's research, introduces novel categories of bi-univalent functions and achieves notable outcomes for these categories. The aforementioned results encompass various aspects, such as the initial coefficient bonds, the Fekete–Szegö problem, and the second Hankel determinant. The study of special functions, which has its roots in their wide range of applications, is a longstanding field of analysis. The enduring fascination with these entities has experienced a recent surge in popularity owing to their novel applications and expanded scope. This theory is currently undergoing extensive development in different unanticipated areas of application. It relies on numerical analysis and computer algebra systems to analyze and display special functions graphically. Moreover, activation functions have a significant role in the domain of computer science, serving as specialized functions. The category of statistical functions known as orthogonal polynomials holds great significance and engagement. A multitude of domains within the domain of natural sciences cover a diverse range of notions, including but not limited to discrete mathematics, theta functions, continuous fractions, Eulerian series, and elliptic functions (see references [2,8–11]).

Moreover, within the realm of pure mathematics, the aforementioned functions possess a multitude of applications. Numerous researchers have commenced their investigations across diverse domains due to the extensive utilization of these functionalities. Contemporary research in geometric function theory places emphasis on the geometric characteristics of specific functions, such as hypergeometric functions, Bessel functions, and other associated functions. In relation to certain geometric properties of these functions, we make reference to [12–18] and any other pertinent sources. This paper introduces a novel category of bi-univalent functions and employs a specific function known as the Euler polynomial. Frequently, the Eulers polynomials $\mathfrak{E}_m$ are defined using the generating function.

$$l(v,t) = \frac{2e^{tv}}{e^t + 1} = \sum_{m=0}^{\infty} \mathfrak{E}_m(v)\frac{t^m}{m!},\ |t| < \pi. \tag{3}$$

A precise equation for $\mathfrak{E}_n(v)$ is provided by

$$\mathfrak{E}_n(v) = \sum_{m=0}^{n} \frac{1}{2^m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (v+k)^n.$$

Now, $\mathfrak{E}_n(v)$ in terms of $\mathfrak{E}_k$ can be obtained from the equation above as

$$\mathfrak{E}_n(v) = \sum_{m=0}^{n} \frac{1}{2^m} \sum_{k=0}^{m} \binom{m}{k} \frac{\mathfrak{E}_k}{2^k} \left(v - \frac{1}{2}\right)^{m-k}. \tag{4}$$

The Euler polynomials in the beginning are:

$$\begin{aligned} \mathfrak{E}_0(v) &= 1, \\ \mathfrak{E}_1(v) &= \tfrac{2v-1}{2}, \\ \mathfrak{E}_2(v) &= v^2 - v, \\ \mathfrak{E}_3(v) &= \tfrac{4v^3-2v^2+1}{4}, \\ \mathfrak{E}_4(v) &= v^4 - 2v^3 + v. \end{aligned} \tag{5}$$

The problem of identifying estimating values of the absolute values of the coefficients remains a persistent challenge in geometric function theory. Their coefficients' magnitude can influence several features of analytic functions, such as univalency, growth rate, and distortion. Efficient issues such as the Fekete–Szegö problem and Hankel determinants involve estimating generic or (l^{th}) coefficient bounds. The aforementioned difficulties regarding coefficients were effectively resolved by multiple scholars employing diverse methodologies. The Fekete–Szegö functional for a function $f(\xi) \in S$ is of considerable importance and is represented by the symbol $l_\beta f = \left|a_3 - a_2^2\right|$. The assumption made by Littlewood and Parley regarding the modulus of coefficients of odd functions $f \in S$ being less than or equal to one was challenged by Fekete and Szegö [19] through the provision of a different functional. Much attention has been paid to the functional, especially in several subfamilies of univalent functions (see [20]).

The a^{th} Hankel determinant of f for $n \geq 1$ and $a \geq 1$, written as $\mathbb{H}_a(n)$ $(a,\ n \in \mathbb{N} = \{1,\ 2,\ 3,\ \ldots\})$, was defined by Pommerenke [21] and Noonan and Thomas [22] for any function $f \in S$ in geometric function theory:

$$\mathbb{H}_a(n) = \begin{vmatrix} t_n & t_{n+1} \ \ldots & t_{n+a-1} \\ t_{n+1} & t_{n+2} \ldots & t_{n+a} \\ \vdots & \vdots & \vdots \\ t_{n+a-1} & t_{n+a} \ldots & t_{n+2a-2} \end{vmatrix},\ (t_1 = 1).$$

For a = 2 and n = 1, we know that the function $\mathbb{H}_2(1) = \begin{vmatrix} t_1 & t_2 \\ t_2 & t_3 \end{vmatrix} = \left|t_3 - t_2^2\right|$ and the second Hankel determinant $\mathbb{H}_2(2)$ is defined as

$$\mathbb{H}_2(2) = \begin{vmatrix} t_2 & t_3 \\ t_3 & t_4 \end{vmatrix} = \left|t_2 t_4 - t_3^2\right|, \tag{6}$$

The bi-starlike and bi-convex classes are referenced in [20,21,23,24]. The second Hankel determinant for specified subsets of bi-univalent functions was investigated by Al-Ameedee et al. [25]. The Hankel determinant of m-fold symmetric bi-univalent functions was examined by Atshan et al. [26] and other researchers [27–30] by the utilization of a novel operator. The Hankel determinant of f was investigated by Fekete and Szegö [19] as $\mathbb{H}_2(1)$. An earlier investigation was conducted to estimate the value of $\left|t_3 - \mu t_2^2\right|$, where $t_1 = 1$ and $\mu \in \mathbb{R}$. Moreover, for instance, in the case of $|\,t_3 - \mu t_2^2\,|$, refer to references [19,31]. Hankel determinants have many applications, such as in random matrix theory and orthogonal polynomials, see, e.g., the recent work of Min and Chen [32].

The paper conducted in this field encompasses the works referenced in [23,24]. The present work introduces and examines a novel subclass, referred to as $\mathcal{R}_\Sigma(v,\delta)$, the set of functions consists of bi-univalent functions that adhere to a specific subordination involving Euler polynomials. The —Fekete-Szegö problem is specifically applied to functions in the class $\mathcal{R}_{\Sigma}(v,\delta)$ and provides bound estimates for the coefficients.

Definition 1. *Assume that the subordination of* $f \in \mathcal{R}_{\Sigma}(v,\delta)$ *is true:*

$$\left[\delta\frac{(\xi f'(\xi))'}{f'(\xi)} + (1-\delta)\frac{f(\xi)}{\xi}\right] \prec l(v,\xi) = \sum_{m=0}^{\infty} \mathfrak{E}_m(v)\frac{\xi^m}{m!} \tag{7}$$

and

$$\left[\delta\frac{(w\mathcal{F}'(w))'}{\mathcal{F}'(w)} + (1-\delta)\frac{\mathcal{F}(w)}{w}\right] \prec l(v,w) = \sum_{m=0}^{\infty} \mathfrak{E}_m(v)\frac{w^m}{m!}\,, \tag{8}$$

where $\delta \geq 0, v \in \left(\frac{1}{2}, 1\right]$*,* ξ*,* $w \in U$*,* $l(v, w)$ *is given by (3), and* $\mathcal{F} = f^{-1}$*is given by (2).*

The univalent of both the functions f and its inverse in the set U leads us to infer that the function f is bi-univalent and belongs to the function class $\mathcal{R}_{\Sigma}(v,\delta)$.

Remark 1. *By assigning a value of 1 to in Definition 1, we obtain a bi-starlike function class* $f \in \mathcal{S}^*(\alpha)$ *that satisfies the following conditions:*

$$\frac{(\xi f'(\xi))'}{f'(\xi)} \prec l(v,\xi) = \sum_{m=0}^{\infty} \mathfrak{E}_m(v)\frac{\xi^m}{m!} \tag{9}$$

and

$$\frac{(w\mathcal{F}'(w))'}{\mathcal{F}'(w)} \prec l(v,w) = \sum_{m=0}^{\infty} \mathfrak{E}_m(v)\frac{w^m}{m!}, \tag{10}$$

where ξ*,* $w \in U$*,* $l(v, w)$ *be defined by Equation (3), and* $\mathcal{F} = f^{-1}$ *be defined by Equation (2).*

Now, let $\mathcal{P}$ *represent the class that includes the analytic functions in U and has the series form as shown below:*

$$p(\xi) = 1 + p_1\xi + p_2\xi^2 + p_2\xi^2 + \ldots = 1 + \textstyle\sum_{l=1}^{\infty} p_l\xi^l, \quad \alpha_1 > 0, \tag{11}$$

with $R\{p(\xi)\} > 0$ $(\forall\xi \in U)$.

Lemma 1 ([33])**.** *If* $\alpha \in \mathcal{P}$*, for each* $l \in \{1,2,3,\ldots\}$*, where* $\mathcal{P}$ *is the family of all* $Re(p(\xi)) > 0$*,* $(z \in U)$*, where*

$$p(\xi) = 1 + c_1\xi + c_2\xi^2 + \ldots, \ (\xi \in U), \tag{12}$$

then

$$|p_l| \leq 2. \tag{13}$$

Lemma 2 ([34])**.** *Letting the function* $p \in \mathcal{P}$ *be denoted by Equation (12). Then*

$$2p_2 = p_1^2 + \chi\left(4 - p_1^2\right) \tag{14}$$

and

$$4p_3 = p_1^3 + 2p_1\chi\left(4 - p_1^2\right) - p_1\chi^2\left(4 - p_1^2\right) + 2\xi\left(4 - p_1^2\right)\left(1 - |\chi|^2\right). \tag{15}$$

2. Coefficient Bounds for the Function Class $\mathcal{R}_{\Sigma}(\mathrm{v},\delta)$

Theorem 1. *Consider the function $f \in \mathcal{R}_{\Sigma}(v,\delta)$, then*

$$|a_2| \leq \sqrt{\mathcal{M}_1(t,v)}\,,$$

$$|a_3| \leq \frac{(2v-1)^2}{4(1+\delta)^2}p_1^2 + \frac{2v-1}{2(1+5\delta)}$$

and

$$|a_4| \leq \frac{(10+10\delta)(2v-1)^3}{2(1+11\delta)(1+\delta)^3}p_1^3 + \frac{(37+23\delta)(3v-1)^2}{4(1+5\delta)(1+\delta)} + \frac{4v^3-6v^2+1}{24(1+11\delta)}\,,$$

where

$$\mathcal{M}_1(t,v) = \frac{(2v-1)^3}{2(1+\delta)(2\delta+2(\delta-1)v^2-2(3\delta+1)v+1)}. \tag{16}$$

Proof. The function $\mathrm{f} \in \Sigma$ as defined by Equation (1) be in the class $\mathcal{R}_{\Sigma}(\mathrm{v},\delta)$. Then

$$\left[\delta\frac{(\xi \mathrm{f}'(\xi))'}{\mathrm{f}'(\xi)} + (1-\delta)\frac{\mathrm{f}(\xi)}{\xi}\right] = \mathrm{l}(\mathrm{v},\mathrm{e}(\xi)) \tag{17}$$

and

$$\left[\delta\frac{(\mathrm{w}\mathcal{F}'(\mathrm{w}))'}{\mathcal{F}'(\mathrm{w})} + (1-\delta)\frac{\mathcal{F}(\mathrm{w})}{\mathrm{w}}\right] = \mathrm{l}(\mathrm{v},\mathrm{h}(\mathrm{w})). \tag{18}$$

Also let the function $\mathrm{p},\mathrm{q} \in \mathcal{P}$ be defined as follows:

$$\mathrm{p}(\xi) = \frac{1+\mathrm{e}(\xi)}{1-\mathrm{e}(\xi)} = 1 + \mathrm{p}_1\xi + \mathrm{p}_2\xi^2 + \mathrm{p}_3\xi^3 + \ldots = 1 + \sum_{\mathrm{l}=1}^{\infty}\mathrm{p}_{\mathrm{l}}\xi^{\mathrm{l}}.$$

Then

$$p(\xi) = \frac{1+e(\xi)}{1-e(\xi)}\,,\ (\xi \in U) \tag{19}$$

and

$$q(\omega) = 1 + q_1\omega + q_2\omega^2 + q_3\omega^3 + \ldots = 1 + \sum_{l=1}^{\infty} q_l\omega^l.$$

Then

$$q(\omega) = \frac{1+h(\omega)}{1-h(\omega)}\ (\omega \in U). \tag{20}$$

It follows that

$$e(\xi) = \frac{p(z)-1}{p(z)+1} = \frac{p_1\xi}{2} + \left(\frac{p_2}{2} - \frac{{p_1}^2}{4}\right)\xi^2 + \left(\frac{p_3}{2} - \frac{p_1p_2}{2} + \frac{p_1^3}{8}\right)\xi^3 + \ldots, \tag{21}$$

and

$$h(\omega) = \frac{q(\omega)-1}{q(\omega)+1} = \frac{q_1}{2}\omega + \left(\frac{q_2}{2} - \frac{{q_1}^2}{4}\right)\omega^2 + \left(\frac{q_3}{2} - \frac{q_1q_2}{2} + \frac{q_1^3}{8}\right)\omega^3 + \ldots. \tag{22}$$

From (21) and (22), we can easily show that

$$l(v,e(\xi)) = \mathfrak{E}_0(\mathrm{v}) + \frac{\mathfrak{E}_1(\mathrm{v})}{2}\mathrm{p}_1\xi + \left[\frac{\mathfrak{E}_1(\mathrm{v})}{2}\left(\mathrm{p}_2 - \frac{{\mathrm{p}_1}^2}{2}\right) + \frac{\mathfrak{E}_2(\mathrm{v})}{8}{\mathrm{p}_1}^2\right]\xi^2 + \left[\frac{\mathfrak{E}_1(\mathrm{v})}{2}\left(\mathrm{p}_3 - \mathrm{p}_1\mathrm{p}_2 + \frac{\mathrm{p}_1^3}{4}\right) + \frac{\mathfrak{E}_2(\mathrm{v})}{4}\mathrm{p}_1\left(\mathrm{p}_2 - \frac{{\mathrm{p}_1}^2}{2}\right) + \frac{\mathfrak{E}_3(\mathrm{v})}{48}{\mathrm{p}_1}^3\right]\xi^3 + \ldots \tag{23}$$

and

$$l(v,e(\omega)) = \mathfrak{E}_0(v) + \tfrac{\mathfrak{E}_1(v)}{2}q_1\omega + \left[\tfrac{\mathfrak{E}_1(v)}{2}\left(q_2 - \tfrac{{q_1}^2}{2}\right) + \tfrac{\mathfrak{E}_2(v)}{8}{q_1}^2\right]\omega^2 \\ + \left[\tfrac{\mathfrak{E}_1(v)}{2}\left(q_3 - q_1q_2 + \tfrac{q_1^3}{4}\right) + \tfrac{\mathfrak{E}_2(v)}{2}q_1\left(q_2 - \tfrac{{q_1}^2}{2}\right) + \tfrac{\mathfrak{E}_3(v)}{48}{q_1}^3\right]\omega^3 + \dots. \tag{24}$$

From (17), (18) and (23), (24), we can easily obtain that

$$(1+\delta)a_2 = \frac{\mathfrak{E}_1(v)}{2}p_1\,, \tag{25}$$

$$(1+5\delta)a_3 - 4a_2^2 = \frac{\mathfrak{E}_1(v)}{2}\left(p_2 - \frac{{p_1}^2}{2}\right) + \frac{\mathfrak{E}_2(v)}{8}{p_1}^2, \tag{26}$$

$$(-18\delta a_2a_3) + (1+11\delta)a_4 + 8\delta a_2^3 = \frac{\mathfrak{E}_1(v)}{2}\left(p_3 - p_1p_2 + \frac{p_1^3}{4}\right) + \frac{\mathfrak{E}_2(v)}{4}p_1\left(p_2 - \frac{{p_1}^2}{2}\right) + \frac{\mathfrak{E}_3(v)}{48}{p_1}^3, \tag{27}$$

and

$$-(1+\delta)a_2 = \frac{\mathfrak{E}_1(v)}{2}q_1\,, \tag{28}$$

$$\left(2(3\delta+1)a_2^2 - (1+5\delta)a_3\right) = \frac{\mathfrak{E}_1(v)}{2}\left(q_2 - \frac{{q_1}^2}{2}\right) + \frac{\mathfrak{E}_2(v)}{8}{q_1}^2, \tag{29}$$

$$(37+5\delta)a_2a_3 - (1+11\delta)a_4 + 8\delta a_2^3 = \frac{\mathfrak{E}_1(v)}{2}\left(q_3 - q_1q_2 + \frac{q_1^3}{4}\right) + \frac{\mathfrak{E}_2(v)}{4}q_1\left(q_2 - \frac{{q_1}^2}{2}\right) + \frac{\mathfrak{E}_3(v)}{48}{q_1}^3. \tag{30}$$

Adding (25) and (28) and further simplification, we have

$$p_1 = -q_1,\ p_1^2 = q_1^2, q_1^3 = -q_1^3. \tag{31}$$

Upon squaring and adding Equations (25) and (28), the resulting value is as follows:

$$2(1+\delta)^2a_2^2 = \frac{\mathfrak{E}_1^2(v)\left(p_1^2 + q_1^2\right)}{4} \tag{32}$$

$$\to a_2^2 = \frac{\mathfrak{E}_1^2(v)\left(p_1^2 + q_1^2\right)}{8(1+\delta)^2}. \tag{33}$$

Additionally, adding (26) and (29) gives

$$2(1+\delta)^2a_2^2 = \tfrac{2\mathfrak{E}_1(v)(p_2+q_2) + p_1^2(\mathfrak{E}_2(v) - 2\mathfrak{E}_1(v))}{4}, \\ 8(1+\delta)^2a_2^2 = 2\mathfrak{E}_1(v)(p_2+q_2) + p_1^2(\mathfrak{E}_2(v) - 2\mathfrak{E}_1(v)). \tag{34}$$

Applying (31) in (32)

$$p_1^2 = \frac{4(1+\delta)^2}{\mathfrak{E}_1^2(v)}a_2^2. \tag{35}$$

In (34), replacing p_1^2 with the following results:

$$|a_2|^2 \le \frac{2\mathfrak{E}_1^3(v)(|p_2| + |q_2|)}{4\left|2(1+\delta)\mathfrak{E}_1^2(v) - (1+\delta)^2[\mathfrak{E}_2(v) - 2\mathfrak{E}_1(v)]\right|}. \tag{36}$$

Applying Lemma 1 and 2, we obtain $|a_2| \leq \sqrt{\mathcal{M}_1(v,\delta)}$, where $\mathcal{M}_1(\mathrm{v},\delta)$ is given by (16). Subtracting (29) and (26) and with some computation, we can easily obtain that

$$a_3 = a_2^2 + \frac{\mathfrak{E}_1(v)}{4(1+5\delta)}(p_2 - q_2), \tag{37}$$

$$a_3 = \frac{\mathfrak{E}_1^2(v)}{4(1+\delta)^2} + \frac{\mathfrak{E}_1(v)}{4(1+5\delta)}(p_2 - q_2). \tag{38}$$

By utilizing Lemma 1 and 3, we derive:

$$|a_3| \leq \frac{(2v-1)^2}{4(1+\delta)^2} + \frac{2v-1}{2(1+5\delta)}\ . \tag{39}$$

By removing (30) from (27), we arrive at:

$$\begin{aligned} a_4 = &\tfrac{(10+10\delta)\mathfrak{E}_1^3(v)}{2(1+11\delta)(1+\delta)^3}p_1^3 + \tfrac{(37+23\delta)\mathfrak{E}_1^2(v)}{8(1+5\delta)(1+\delta)}p_1(p_2-q_2) + \tfrac{\mathfrak{E}_1(v)(p_3-q_3)}{4(1+11\delta)} \\ &+ \tfrac{\mathfrak{E}_2(v)-2\mathfrak{E}_1(v)(p_2+q_2)}{8(1+11\delta)}p_1 + \tfrac{6\mathfrak{E}_1(v)-6\mathfrak{E}_2(v)+\mathfrak{E}_3(v)}{48(1+11\delta)}p_1^3 + \ldots. \end{aligned} \tag{40}$$

By utilizing Lemma 1 and 2, we derive:

$$|a_4| \leq \frac{(10+10\delta)(2v-1)^3}{2(1+11\delta)(1+\delta)^3} + \frac{(37+23\delta)(3v-1)^2}{4(1+5\delta)(1+\delta)} + \frac{4v^3-6v^2+1}{24(1+11\delta)} + \ldots. \square$$

By substituting $\delta = 0$ into Theorem 1, we obtain the subsequent corollary:

Corollary 1. *Let $f \in \mathcal{R}_\Sigma(v,\delta)$. Then,*

$$|a_2| \leq \sqrt{\frac{(2v-1)^3}{2(-2v^2-2v+1)}}\ ,$$

$$|a_3| \leq \frac{(2v-1)^2}{4} + \frac{2v-1}{2}$$

and

$$|a_4| \leq \frac{(10)(2v-1)^3}{2} + \frac{(37)(3v-1)^2}{4} + \frac{4v^3-6v^2+1}{24}. \square$$

For $\delta = 1$, The subsequent corollary of Theorem 1 is obtained.

Corollary 2. *Consider the function $f \in \mathcal{R}_\Sigma(v,\delta)$. Then,*

$$|a_2| \leq \sqrt{\frac{(2v-1)^3}{4(2-8v+1)}}\ ,$$

$$|a_3| \leq \frac{(2v-1)^2}{4(1+\delta)^2} + \frac{2v-1}{2(1+5\delta)}$$

and

$$|a_4| \leq \frac{20(2v-1)^3}{192}p_1^3 + \frac{(37+23)(3v-1)^2}{48} + \frac{4v^3-6v^2+1}{288}. \square$$

3. Fekete–Szegö Inequalities for the Functions of Class $\mathcal{R}_{\Sigma}(\mathbf{v},\boldsymbol{\delta})$

Theorem 2. *If $f \in \mathcal{R}_{\Sigma}(v,\delta)$, then, for some $\mathfrak{T} \in \mathbb{R}$*

$$\left|a_3 - \mathfrak{T}a_2^2\right| \leq \begin{cases} 2|1-\mathfrak{T}|\mathcal{M}_1(t,v) & \left[|1-\mathfrak{T}|\mathcal{M}_1(t,v) \geq \frac{2v-1}{2(1+5\delta)}\right] \\ \frac{2v-1}{(1+5\delta)} & \left[\ |1-\mathfrak{T}|\mathcal{M}_1(t,v) < \frac{2v-1}{2(1+5\delta)}\right] \end{cases},$$

where $\mathcal{M}_1(t,v)$ is given by (16).

Proof. From (37), we obtain

$$a_3 - \mathfrak{T}a_2^2 = a_2^2 + \frac{\mathfrak{E}_1(v)}{4(1+5\delta)}(p_2 - q_2) - \mathfrak{T}a_2^2.$$

By utilizing the well-used triangular inequality, we find:

$$\left|a_3 - \mathfrak{T}a_2^2\right| \leq \frac{2v-1}{2(1+5\delta)} + |1-\mathfrak{T}|\mathcal{M}_1(t,v),$$

if

$$|1-\mathfrak{T}|\mathcal{M}_1(t,v) \geq \frac{2v-1}{2(1+5\delta)}.$$

Furthermore, we obtain

$$\left|a_3 - \mathfrak{T}a_2^2\right| \leq 2|1-\mathfrak{T}|\mathcal{M}_1(t,v),$$

where

$$|1-\mathfrak{T}| \geq \frac{2v-1}{2(1+5\delta)\mathcal{M}_1(t,v)}\ ,$$

and if

$$|1-\mathfrak{T}|\mathcal{M}_1(t,v) \leq \frac{2v-1}{2(1+5\delta)}\ ,$$

then, we obtain

$$\left|a_3 - \mathfrak{T}a_2^2\right| \leq \frac{2v-1}{(1+5\delta)}\ ,$$

where

$$|1-\mathfrak{T}| \leq \frac{2v-1}{2(1+5\delta)\mathcal{M}_1(t,v)}$$

and $\mathcal{M}_1(v,\delta)$ is given by (16). □

Assuming that δ = 0 in the aforementioned Theorem 2, we derive the subsequent outcome.

Corollary 3. *If $f \in \mathcal{R}_{\Sigma}(v,\delta)$, then, for some $\mathfrak{T} \in \mathbb{R}$,*

$$\left|a_3 - \mathfrak{T}a_2^2\right| \leq \begin{cases} 2|1-\mathfrak{T}|\mathcal{M}_1(t,v) & \left[|1-\mathfrak{T}|\mathcal{M}_1(t,v) \geq \frac{2v-1}{2}\right] \\ 2v-1 & \left[\ |1-\mathfrak{T}|\mathcal{M}_1(t,v) < \frac{2v-1}{2}\right] \end{cases},$$

where $\mathcal{M}_1(t,v)$ is given by (16). □

Corollary 4. *If $f \in \mathcal{R}_\Sigma(t,v)$, then, for some $\mathfrak{T} \in \mathbb{R}$,*

$$\left|a_3 - \mathfrak{T}a_2^2\right| \leq \begin{cases} 2|1-\mathfrak{T}|\mathcal{M}_1(t,v) & \left[|1-\mathfrak{T}|\mathcal{M}_1(t,v) \geq \frac{2v-1}{12}\right] \\ \frac{2v-1}{6} & \left[|1-\mathfrak{T}|\mathcal{M}_1(t,v) < \frac{2v-1}{12}\right] \end{cases},$$

where $\mathcal{M}_1(v, \delta)$ is given by (16). □

4. Second Hankel Determinant for the Class $\mathcal{R}_\Sigma(\mathbf{v}, \boldsymbol{\delta})$

Theorem 3. *Consider a function f that belongs to the class $\mathcal{R}_\Sigma(v,\delta)$. Subsequently*

$$\mathbb{H}_2(2) = \left|a_2a_4 - a_3^2\right| \leq \begin{cases} \mathbb{T}(v,2) & (\mathcal{H}_1 \geq 0 \text{ and } \mathcal{H}_2 \geq 0) \\ max\left\{\left(\frac{2v-1}{2(1+5\delta)}\right)^2, \mathbb{T}(v,2)\right\} & (\mathcal{H}_1 > 0 \text{ and } \mathcal{H}_2 > 0) \\ \left(\frac{2v-1}{2(1+5\delta)}\right)^2 & (\mathcal{H}_1 \leq 0 \text{ and } \mathcal{H}_2 \leq 0) \\ max\{\mathbb{T}(\mathfrak{Y}_\circ, t), \mathbb{T}(v,2)\} & (\mathcal{H}_1 < 0 \text{ and } \mathcal{H}_2 < 0) \end{cases},$$

where

$$\mathbb{T}(v,2) := \frac{4(10+10\delta)\mathfrak{E}_1^4(v)}{(1+11\delta)(1+\delta)^4}\mathfrak{Y}^4 + \frac{\mathfrak{E}_1(v)[\mathfrak{E}_3(v)]}{6(1+11\delta)(1+\delta)}\mathfrak{Y}^4 + \frac{\mathfrak{E}_1^4(v)}{(1+\delta)^4},$$

$$\mathbb{T}(\mathfrak{Y}_\circ, t) = \frac{\mathfrak{E}_1^2(v)}{(1+5\delta)^2} + \frac{\mathcal{H}_2^4(1+\delta)^4}{(1+5\delta)^2(1+11\delta)\mathcal{H}_1^3} + \frac{\mathcal{H}_2^3(1+\delta)^2}{(1+5\delta)^2(1+11\delta)\mathcal{H}_1^2},$$

$$\begin{aligned} \mathcal{H}_1 = \; & \mathfrak{E}_1(v)\Big[48(10+10\delta)(1+5\delta)^2\mathfrak{E}_1^3(v) + 2[6\mathfrak{E}_1(v) - 6\mathfrak{E}_2(v) + \mathfrak{E}_3(v)](1+5\delta)^2(1+\delta)^3 \\ & +12\mathfrak{E}_1^3(v)(1+11\delta)(1+5\delta)^2 - 24\mathfrak{E}_1(v)(1+5\delta)^2(1+\delta)^3 \\ & +12\mathfrak{E}_1(v)(1+11\delta)(1+\delta)^4\Big]\mathfrak{Y}^4, \end{aligned}$$

$$\begin{aligned} \mathcal{H}_2 = \mathfrak{E}_1(v)\Big[& 3(1+11\delta)(1+5\delta)\mathfrak{E}_1^2(v) + 6\mathfrak{E}_1(v)(1+5\delta)^2(1+\delta) + 3[\mathfrak{E}_1(v) - 2\mathfrak{E}_2(v)](1+5\delta)^2(1+\delta) \\ & +6\mathfrak{E}_1(v)(1+11\delta)(1+5\delta) - \mathfrak{E}_1(v)(1+11\delta)(1+5\delta)]\mathfrak{Y}^2. \end{aligned}$$

Proof. From (25) and (40), we obtain

$$\begin{aligned} a_2a_4 = & \frac{(10+10\delta)\mathfrak{E}_1^4(v)}{4(1+11\delta)(1+\delta)^4}p_1^4 + \frac{(37+23\delta)\mathfrak{E}_1^3(v)}{16(1+5\delta)(1+\delta)^2}p_1^2(p_2-q_2) + \frac{\mathfrak{E}_1^2(v)(p_3-q_3)}{8(1+11\delta)(1+\delta)}p_1 \\ & + \frac{\mathfrak{E}_1(v)[\mathfrak{E}_2(v)-2\mathfrak{E}_1(v)](p_2+q_2)}{16(1+11\delta)(1+\delta)}p_1^2 + \frac{\mathfrak{E}_1(v)[6\mathfrak{E}_1(v)-6\mathfrak{E}_2(v)+\mathfrak{E}_3(v)]}{96(1+11\delta)(1+\delta)}p_1^4 + \ldots, \end{aligned}$$

with some calculations, we have

$$\begin{aligned} a_2a_4 - a_3^2 = & \frac{(10+10\delta)\mathfrak{E}_1^4(v)}{4(1+11\delta)(1+\delta)^4}p_1^4 + \frac{(37+23\delta)\mathfrak{E}_1^3(v)}{16(1+5\delta)(1+\delta)^2}p_1^2(p_2-q_2) \\ & + \frac{\mathfrak{E}_1^2(v)(p_3-q_3)}{8(1+11\delta)(1+\delta)}p_1 + \frac{\mathfrak{E}_1(v)[\mathfrak{E}_2(v)-2\mathfrak{E}_1(v)](p_2+q_2)}{16(1+11\delta)(1+\delta)}p_1^2 \\ & + \frac{\mathfrak{E}_1(v)[6\mathfrak{E}_1(v)-6\mathfrak{E}_2(v)+\mathfrak{E}_3(v)]}{96(1+11\delta)(1+\delta)}p_1^4 - \frac{\mathfrak{E}_1^4(v)}{16(1+\delta)^2}p_1^4 \\ & - \frac{\mathfrak{E}_1^2(v)}{16(1+5\delta)}(p_2-q_2)^2. \end{aligned}$$

Appling Lemma 2, we obtain

$$p_2 - q_2 = \frac{1}{2}\Big[\Big(4-p_1^2\Big)(\chi - \mathfrak{u})\Big], \tag{41}$$

$$p_2 + q_2 = p_1^2 + \frac{1}{2}\left[\left(4 - p_1^2\right)(\chi + \mathfrak{u})\right], \tag{42}$$

and

$$p_3 - q_3 = \frac{1}{2}\left[p_1^3 + p_1\left(4 - p_1^2\right)(\chi + \mathfrak{u}) - \frac{4 - p_1^2}{2} p_1\left(\chi^2 + \mathfrak{u}^2\right) + \left(4 - p_1^2\right)\left[\left(1 - |\chi|^2\right)\xi - \left(1 - |\mathfrak{u}|^2\right)\sqsupseteq\right]\right], \tag{43}$$

for some $\chi, \mathfrak{u}, \xi, \sqsupseteq$ with $|\chi| \leq 1, |\mathfrak{u}| \leq 1, |p_1| \in [0,2]$ and substituting $(p_2 + q_2), (p_1 + q_1)$ and $(p_3 - q_3)$, then after some straightforward calculations

$$\begin{aligned}
a_2a_4 - a_3^2 \quad &= \frac{(10+10\delta)\mathfrak{E}_1^4(v)}{4(1+11\delta)(1+\delta)^4} p_1^4 + \frac{(37+23\delta)\mathfrak{E}_1^3(v)\left(4-p_1^2\right)(\chi-\mathfrak{u})}{32(1+5\delta)(1+\delta)^2} p_1^2 + \frac{\mathfrak{E}_1^2(v)}{16(1+11\delta)(1+\delta)} p_1^4 \\
&+ \frac{\mathfrak{E}_1^2(v)\left(4-p_1^2\right)(\chi+\mathfrak{u})}{16(1+11\delta)(1+\delta)} p_1^2 - \frac{\mathfrak{E}_1^2(v)\left(4-p_1^2\right)\left(\chi^2+\mathfrak{u}^2\right)}{32(1+11\delta)(1+\delta)} p_1^2 \\
&+ \frac{\mathfrak{E}_1^2(v)\left(1-|\chi|^2\right)\xi-\left(1-|\mathfrak{u}|^2\right)\sqsupseteq)}{16(1+11\delta)(1+\delta)} p_1 + \frac{\mathfrak{E}_1(v)[\mathfrak{E}_2(v)-2\mathfrak{E}_1(v)]}{16(1+11\delta)(1+\delta)} p_1^4 \\
&+ \frac{\mathfrak{E}_1(v)[\mathfrak{E}_2(v)-2\mathfrak{E}_1(v)]\left(4-p_1^2\right)(\chi+\mathfrak{u})}{32(1+11\delta)(1+\delta)} p_1^2 + \frac{\mathfrak{E}_1(v)[6\mathfrak{E}_1(v)-6\mathfrak{E}_2(v)+\mathfrak{E}_3(v)]}{96(1+11\delta)(1+\delta)} p_1^4 \\
&- \frac{\mathfrak{E}_1^4(v)}{16(1+\delta)^2} p_1^4 - \frac{\mathfrak{E}_1^2(v)\left(4-p_1^2\right)^2(\chi-\mathfrak{u})^2}{64(1+5\delta)^2}(p_2 - q_2)^2.
\end{aligned}$$

Let $\mathfrak{Y} = p_1$, assuming that without any restriction $\mathfrak{Y} \in [0,2]$, $T_1 = |\chi| \leq 1, T_2 = |\mathfrak{u}| \leq 1$, and applying triangular inequality, we obtain

$$\begin{aligned}
\left|a_2a_4 - a_3^2\right| \leq \quad & \left\{ \frac{(10+10\delta)\mathfrak{E}_1^4(v)}{4(1+11\delta)(1+\delta)^4}\mathfrak{Y}^4 + \frac{\mathfrak{E}_1^2(v)}{16(1+11\delta)(1+\delta)}\mathfrak{Y}^1 + \frac{\mathfrak{E}_1^2(v)\left(4-\mathfrak{Y}^2\right)}{8(1+11\delta)(1+\delta)}\mathfrak{Y}^4 \right. \\
& \left. + \frac{\mathfrak{E}_1(v)[\mathfrak{E}_2(v)-2\mathfrak{E}_1(v)]}{16(1+11\delta)(1+\delta)}\mathfrak{Y}^4 + \frac{\mathfrak{E}_1(v)[6\mathfrak{E}_1(v)-6\mathfrak{E}_2(v)+\mathfrak{E}_3(v)]}{96(1+11\delta)(1+\delta)}\mathfrak{Y}^4 + \frac{\mathfrak{E}_1^4(v)}{16(1+\delta)^2}\mathfrak{Y}^4 \right\} \\
& + \left\{ \frac{(37+23\delta)\mathfrak{E}_1^3(v)\left(4-p_1^2\right)}{32(1+5\delta)(1+\delta)^2}\mathfrak{Y}^2 + \frac{\mathfrak{E}_1^2(v)\left(4-\mathfrak{Y}^2\right)}{16(1+11\delta)(1+\delta)}\mathfrak{Y}^2 \right. \\
& \left. + \frac{\mathfrak{E}_1(v)[\mathfrak{E}_2(v)-2\mathfrak{E}_1(v)]\left(4-\mathfrak{Y}^2\right)}{32(1+11\delta)(1+\delta)}\mathfrak{Y}^2 \right\}(T_1 + T_2) \\
& + \left\{ \frac{\mathfrak{E}_1^2(v)\left(4-\mathfrak{Y}^2\right)}{32(1+11\delta)(1+\delta)}\mathfrak{Y}^2 - \frac{\mathfrak{E}_1^2(v)}{16(1+11\delta)(1+\delta)}\mathfrak{Y} \right\}\left(T_1^2 + T_2^2\right) \\
& + \frac{\mathfrak{E}_1^2(v)\left(4-\mathfrak{Y}^2\right)^2}{64(1+5\delta)^2}(T_1 + T_2)^2,
\end{aligned}$$

and equivalently, we obtain

$$\left|a_2a_4 - a_3^2\right| \leq \left\{\mathfrak{z}_1(v,\mathfrak{Y}), \mathfrak{z}_2(v,\mathfrak{Y})(T_1 + T_2), \mathfrak{z}_3(v,\mathfrak{Y})\left(T_1^2 + T_2^2\right), \mathfrak{z}_4(v,\mathfrak{Y})(T_1 + T_2)^2\right\} = \mathcal{J}(T_1, T_2), \tag{44}$$

where

$$\begin{aligned}
\mathfrak{z}_1(v,\mathfrak{Y}) = & \left\{ \frac{(10+10\delta)\mathfrak{E}_1^4(v)}{4(1+11\delta)(1+\delta)^4}\mathfrak{Y}^4 + \frac{\mathfrak{E}_1^2(v)}{16(1+11\delta)(1+\delta)}\mathfrak{Y}^4 + \frac{\mathfrak{E}_1^2(v)\left(4-\mathfrak{Y}^2\right)}{8(1+11\delta)(1+\delta)}\mathfrak{Y}^1 + \frac{\mathfrak{E}_1(v)[\mathfrak{E}_2(v)-2\mathfrak{E}_1(v)]}{16(1+11\delta)(1+\delta)}\mathfrak{Y}^4 \right. \\
& \left. + \frac{\mathfrak{E}_1(v)[6\mathfrak{E}_1(v)-6\mathfrak{E}_2(v)+\mathfrak{E}_3(v)]}{96(1+11\delta)(1+\delta)}\mathfrak{Y}^4 + \frac{\mathfrak{E}_1^4(v)}{16(1+\delta)^4}\mathfrak{Y}^4 \right\} \geq 0
\end{aligned}$$

$$\mathfrak{z}_2(v,\mathfrak{Y}) = \left\{ \frac{\mathfrak{E}_1^3(v)\left(4-\mathfrak{Y}^2\right)}{32(1+5\delta)(1+\delta)^2}\mathfrak{Y}^2 + \frac{\mathfrak{E}_1^2(v)\left(4-\mathfrak{Y}^2\right)}{16(1+11\delta)(1+\delta)}\mathfrak{Y}^2 + \frac{\mathfrak{E}_1(v)[\mathfrak{E}_2(v)-2\mathfrak{E}_1(v)]\left(4-\mathfrak{Y}^2\right)}{16(1+11\delta)(1+\delta)}\mathfrak{Y}^2 \right\} \geq 0$$

$$\mathfrak{z}_3(v,\mathfrak{Y}) = \left\{ \frac{\mathfrak{E}_1^2(v)\left(4-\mathfrak{Y}^2\right)}{32(1+11\delta)(1+\delta)}\mathfrak{Y}^2 - \frac{\mathfrak{E}_1^2(v)\left(4-\mathfrak{Y}^2\right)}{16(1+11\delta)(1+\delta)}\mathfrak{Y} \right\} \leq 0$$

$$\mathfrak{z}_4(v,\mathfrak{Y}) = \frac{\mathfrak{E}_1^2(v)\left(4-\mathfrak{Y}^2\right)^2}{64(1+5\delta)^2} \geq 0,\ 0 \ \leq \ \mathfrak{Y} \ \leq \ 2.$$

We now maximize the function $\mathcal{J}(T_1, T_2)$ in the closed square

$$\varnothing = \{(T_1, T_2) : T_1, T_2 \in [0,1]\} \ for \ \mathfrak{Y} \in [0,2].$$

It is necessary to investigate the maximum value of $\mathcal{J}(T_1, T_2)$ with respect to $\mathfrak{Y}$, while considering the scenarios where $\mathfrak{Y}$ = 0, $\mathfrak{Y}$ = 2, and $\mathfrak{Y} \in$ (0, 2). The coefficients of the function $\mathcal{J}(T_1, T_2)$ in Equation (42) are dependent on m, given a constant value of $\mathfrak{Y}$.

The 1st case
where $\mathfrak{Y}$ = 0,

$$\mathcal{J}(T_1, T_2) = \mathfrak{z}_4(v, 0) = \frac{\mathfrak{E}_1^2(v)}{4(1+5\delta)^2}(T_1 + T_2)^2.$$

Clearly the function $\mathcal{J}(T_1, T_2)$ attains its maximum at (T_1, T_2) and

$$max\{\mathcal{J}(T_1, T_2) : T_1, T_2 \in [0,1]\} = \mathcal{J}(1,1) = \frac{\mathfrak{E}_1^2(v)}{(1+5\delta)^2}. \tag{45}$$

The 2nd case

When $\mathfrak{E} = 2$, the function $\mathcal{J}(T_1, T_2)$ is expressed as a constant function with respect to m, resulting in

$$\mathcal{J}(T_1, T_2) = \mathfrak{z}_1(v, 2) = \left\{ \frac{4(10+10\delta)\mathfrak{E}_1^4(v)}{(1+11\delta)(1+\delta)^4}\mathfrak{Y}^4 + \frac{\mathfrak{E}_1(v)[\mathfrak{E}_3(v)]}{6(1+11\delta)(1+\delta)}\mathfrak{Y}^4 + \frac{\mathfrak{E}_1^4(v)}{(1+\delta)^4} \right\}.$$

The 3rd case

When $\mathfrak{Y} \in$ (0, 2), let $T_1 + T_2 = \|$ and $T_1 \cdot T_2 = q$ in this case, and then (48) can be of the form:

$$\mathcal{J}(T_1, T_2) = \mathfrak{z}_1(v, \mathfrak{Y}) + \mathfrak{z}_2(v, \mathfrak{Y}) + (\mathfrak{z}_3(v, \mathfrak{Y}) + \mathfrak{z}_4(v, \mathfrak{Y}) - 2\mathfrak{z}_3(v, \mathfrak{Y})) = \mathbb{Q}(\|, q), \tag{46}$$

where $\| \in [0,\ 2]$ and $q \in [0,\ 1]$. Now, we must examine the upper limit of

$$\mathbb{Q}(\|, q) \in \theta \ \{(\|, q\ ) : \| \in\ [0,\ 2],\ q \in [0,\ 1]\}. \tag{47}$$

By differentiating $\mathfrak{z}(\|, q)$ partially, we have

$$\frac{d\mathbb{Q}}{dc} = \mathfrak{z}_2(v, \mathfrak{Y}) + (\mathfrak{z}_3(v, \mathfrak{Y}) + \mathfrak{z}_4(v, \mathfrak{Y}))d = 0,$$

$$\frac{d\mathbb{Q}}{dc} = -2\mathfrak{z}_3(v, \mathfrak{Y}) = 0.$$

The results indicate that $\mathbb{Q}(\|, q)$ does not have a critical point within the square Ψ. As a result, $\mathcal{J}(T_1, T_2)$ also does not have a critical point within the same region. As a consequence, the function $\mathcal{J}(T_1, T_2)$ is incapable of attaining its maximum value inside the interval Ψ. The examination will focus on the greatest value of $\mathcal{J}(T_1, T_2)$ on the edge of the square. For $T_1 = 0,\ T_2 \in\ [0,\ 1]$ (also, for $T_2 = 0,\ T_1 \in [0,\ 1]$) and

$$\mathcal{J}(0, T_2) = \mathfrak{z}_1(v, \mathfrak{Y}) + \mathfrak{z}_2(v, \mathfrak{Y}) + (\mathfrak{z}_3(v, \mathfrak{Y}) + \mathfrak{z}_4(v, \mathfrak{Y}))T_2^2 = D(T_2). \tag{48}$$

Now, since $(\mathfrak{z}_3(v, \mathfrak{Y}) + \mathfrak{z}_4(v, \mathfrak{Y}))T_2^2 \geq 0$, then we have $D'(T_2) = \mathfrak{z}_2(v, \mathfrak{Y}) + 2(\mathfrak{z}_3(\mathfrak{u}, \mathfrak{Y}) + \mathfrak{z}_4(\mathfrak{u}, \mathfrak{Y}))T_2 > 0$, which implies that $D(T_2)$ is an increasing function. Therefore, for a fixed $\mathfrak{Y} \in [0,2)$ and $\mathfrak{u} \in (1/2, 1]$, the maximum occurs at T_2 = 1. Thus, from (48),

$$max\{\mathcal{J}(T_1, T_2) : T_1, T_2 \in [0,1]\} = \mathcal{J}(0,1) = \mathfrak{z}_1(v, \mathfrak{Y}) + \mathfrak{z}_2(v, \mathfrak{Y}) + \mathfrak{z}_3(v, \mathfrak{Y}) + \mathfrak{z}_4(v, \mathfrak{Y}). \tag{49}$$

For $[T_1 = 1, T_2 \in [0,1]) \ also \ T_2 = 1, T_1 \in [0,1])]$,

$$\mathcal{J}(1,T_2) = \mathfrak{z}_1(v,\mathfrak{Y}) + \mathfrak{z}_2(v,\mathfrak{Y}) + \mathfrak{z}_3(v,\mathfrak{Y}) + \mathfrak{z}_4(v,\mathfrak{Y}) + (\mathfrak{z}_2(v,\mathfrak{Y}) + 2\mathfrak{z}_4(v,\mathfrak{Y}))T_2,$$

$$(\mathfrak{z}_3(v,\mathfrak{Y}) + \mathfrak{z}_4(v,\mathfrak{Y}))T_2^2 = \mathcal{N}(T_2), \tag{50}$$

$$\mathcal{N}'(T_2) = \mathfrak{z}_2(v,\mathfrak{Y}) + 2\mathfrak{z}_4(v,\mathfrak{Y}) + 2(\mathfrak{z}_2(v,\mathfrak{Y}) + \mathfrak{z}_4(v,\mathfrak{Y}))T_2. \tag{51}$$

We know that $\mathfrak{z}_3(v,\mathfrak{Y}) + \mathfrak{z}_4(v,\mathfrak{Y}) \geq 0$, then

$$\mathcal{N}'(T_2) = (\mathfrak{z}_2(\mathfrak{u},\mathfrak{Y}) + 2\mathfrak{z}_4(\mathfrak{u},\mathfrak{Y})) + 2(\mathfrak{z}_2(\mathfrak{u},\mathfrak{Y}) + \mathfrak{z}_4(\mathfrak{u},\mathfrak{Y}))T_2 > 0.$$

Hence, the function $\mathcal{N}(T_2)$ is shown to be an increasing function, with the maximum value observed at $T_2 = 1$. From (50) we have

$$\begin{aligned} &max\{\mathcal{J}(1,T_2) : T_2 \in [0,1]\} \\ &= \mathfrak{z}_1(v,\mathfrak{Y}) + 2(\mathfrak{z}_2(v,\mathfrak{Y}) + \mathfrak{z}_3(v,\mathfrak{Y})) + 4\mathfrak{z}_4(v,\mathfrak{Y}) = \mathcal{J}(1,1)\,. \end{aligned} \tag{52}$$

Therefore, for each $\mathfrak{Y} \in (0,\ 2)$, derived from Equations (52) and (49), we obtain

$$\mathfrak{z}_1(v,\mathfrak{Y}) + 2(\mathfrak{z}_2(v,\mathfrak{Y}) + \mathfrak{z}_3(v,\mathfrak{Y})) + 4\mathfrak{z}_4(v,\mathfrak{Y}) > \mathfrak{z}_1(v,\mathfrak{Y}) + \mathfrak{z}_2(v,\mathfrak{Y}) + \mathfrak{z}_3(v,\mathfrak{Y}) + \mathfrak{z}_4(v,\mathfrak{Y}).$$

Therefore

$$max\{\mathcal{J}(T_1,T_2) : T_1, T_2 \in [0,1]\} = \mathfrak{z}_1(v,\mathfrak{Y}) + 2(\mathfrak{z}_2(v,\mathfrak{Y}) + \mathfrak{z}_3(v,\mathfrak{Y})) + 4\mathfrak{z}_4(v,\mathfrak{Y}).$$

Since

$$D(1) \leq \mathcal{N}(1) \ for \ \mathfrak{Y} \in [0,\ 2],\ v \in [1,\ 1],$$

then

$$max\{\mathcal{J}(T_1,T_2)\} = \mathcal{J}[1,1]\,,$$

and the event takes place in the periphery of square Ψ.

Let $\mathbb{T} : (0,\ 2) \to \mathbb{R}$ defined by

$$\mathbb{T}(v,\mathfrak{Y}) = max\{\mathcal{J}(T_1,T_2)\} = \mathfrak{z}_1(v,\mathfrak{Y}) + 2\mathfrak{z}_2(v,\mathfrak{Y}) + 2\mathfrak{z}_3(v,\mathfrak{Y}) + 4\mathfrak{z}_4(v,\mathfrak{Y}) = \mathcal{J}(1,1). \tag{53}$$

Now, insert the values of $\mathfrak{z}_1(v,\mathfrak{Y}), \mathfrak{z}_2(v,\mathfrak{Y}), \mathfrak{z}_3(v,\mathfrak{Y})$ and $\mathfrak{z}_4(v,\mathfrak{Y})$ into (53) and with some calculations, we obtain

$$\mathbb{T}(v,\mathfrak{Y}) = \frac{\mathfrak{E}_1^2(v)}{(1+5\delta)^2} + \frac{\mathcal{H}_1}{192(1+11\delta)(1+\delta)^4(1+5\delta)^2}\mathfrak{Y}^4 + \frac{\mathcal{H}_2}{12(1+5\delta)^2(1+\delta)^2(1+11\delta)}\mathfrak{Y}^2$$

$$\begin{aligned} \mathcal{H}_1 \quad &= \mathfrak{E}_1(v)\Big[48(10+10\delta)(1+5\delta)^2\mathfrak{E}_1^3(v) + 2[6\mathfrak{E}_1(v) - 6\mathfrak{E}_2(v) + \mathfrak{E}_3(v)](1+5\delta)^2(1+\delta)^3 \\ &\quad +12\mathfrak{E}_1^3(v)(1+11\delta)(1+5\delta)^2 - 24\mathfrak{E}_1(v)(1+5\delta)^2(1+\delta)^3 \\ &\quad +12\mathfrak{E}_1(v)(1+11\delta)(1+\delta)^4\Big]\mathfrak{Y}^4, \end{aligned}$$

$$\begin{aligned} \mathcal{H}_2 = \mathfrak{E}_1(v)\Big[&3(1+11\delta)(1+5\delta)\mathfrak{E}_1^2(v) + 6\mathfrak{E}_1(v)(1+5\delta)^2(1+\delta) + 3[\mathfrak{E}_1(v) - 2\mathfrak{E}_2(v)](1+5\delta)^2(1+\delta) \\ &+6\mathfrak{E}_1(v)(1+11\delta)(1+5\delta) - \mathfrak{E}_1(v)(1+11\delta)(1+5\delta)]\mathfrak{Y}^2. \end{aligned}$$

If $\mathbb{T}(v,\mathfrak{Y})$ attains a maximum value inside the interval $\mathfrak{Y} \in [0,\ 2]$ and by employing fundamental mathematical principles, we can deduce

$$\mathbb{T}'(v,\mathfrak{Y}) = \frac{\mathcal{H}_1}{48(1+11\delta)(1+\delta)^4(1+5\delta)^2}\mathfrak{Y}^3 + \frac{\mathcal{H}_2}{6(1+5\delta)^2(1+\delta)^2(1+11\delta)}\mathfrak{Y}.$$

By virtue of the signs of $\mathcal{H}_1$ and $\mathcal{H}_2$, we need to examine the sign of the function $\mathbb{T}'(v,\mathfrak{Y})$.

First result:

Assuming that $\mathcal{H}_1 \geq 0$ and $\mathcal{H}_2 \geq 0$, then $\mathbb{T}'(v,\mathfrak{Y}) \geq 0$. This demonstrates that $\mathbb{T}(v,\mathfrak{Y})$ is a continuous function on the interval of $\mathfrak{Y} \in [0, 2]$; that is, $\mathfrak{Y} = 2$. Consequently,

$$max\{\mathbb{T}(v,\mathfrak{Y}) : \mathfrak{Y} \in [0,\ 2]\} = \left\{ \frac{4(10+10\delta)\mathfrak{E}_1^4(v)}{(1+11\delta)(1+\delta)^4} + \frac{\mathfrak{E}_1(v)[\mathfrak{E}_3(v)]}{6(1+11\delta)(1+\delta)} + \frac{\mathfrak{E}_1^4(v)}{(1+\delta)^4} \right\}.$$

Second result:

If $\mathcal{H}_1 > 0$ and $\mathcal{H}_2 < 0$, then,

$$\mathbb{T}'(v,\mathfrak{Y}) = \frac{\mathcal{H}_1\mathfrak{Y}^3 + 8\mathcal{H}_2\mathfrak{Y}(1+\delta)^2}{48(1+11\delta)(1+\delta)^4(1+5\delta)^2} = 0. \tag{54}$$

At the critical point,

$$\mathfrak{Y}_\circ = \sqrt{\frac{-8\mathcal{H}_2(1+\delta)^2}{\mathcal{H}_1}} \tag{55}$$

is a critical point of the function. Now,

$$\mathbb{T}''(\mathfrak{Y}_\circ) = -\frac{\mathcal{H}_2}{2(1+11\delta)(1+\delta)^2(1+5\delta)^2}\mathfrak{Y}^3 + \frac{\mathcal{H}_2}{6(1+5\delta)^2(1+\delta)^2(1+11\delta)}\mathfrak{Y} > 0.$$

Therefore, $\mathfrak{Y}_\circ$ is the minimum point of the function $\mathbb{T}(\mathrm{v},\mathfrak{Y})$. Hence, $\mathbb{T}(\mathrm{v},\mathfrak{Y})$, cannot have a maximum.

Third result:

If $\mathcal{H}_1 \leq 0$ and $\mathcal{H}_2 \leq 0$, then $\mathbb{T}(v,\mathfrak{Y}) \leq 0$. Therefore,$\mathbb{T}(v,\mathfrak{Y})$ is a decreasing function on the interval $(0,2)$. Consequently,

$$max\{\mathbb{T}(v_\circ,\mathfrak{Y}) : \mathfrak{Y} \in (0,2)\} = \mathbb{T}(0) = \frac{\mathfrak{E}_1^2(v)}{(1+5\delta)^2}. \tag{56}$$

Fourth result:

If $\mathcal{H}_1 < 0$ and $\mathcal{H}_2 > 0$, then

$$\mathbb{T}''(\mathfrak{Y}_\circ) = \frac{-\mathcal{H}_2}{3(1+11\delta)(1+\delta)^2(1+5\delta)^2} < 0.$$

Therefore, $\mathbb{T}''(\mathfrak{Y}_\circ) < 0$. Hence, $\mathfrak{Y}_\circ$ is the maximum point of the function $\mathbb{T}(v,\mathfrak{Y})$ and $\mathfrak{Y} = \}_0$ is the maximum value. Likewise

$$max\{\mathbb{T}(v,\mathfrak{Y}) : \mathfrak{Y} \in (0,2)\} = \mathbb{T}(\mathfrak{Y}_\circ, s),$$

$$\mathbb{T}(\}_0\,, t) = \frac{\mathfrak{E}_1^2(v)}{(1+5\delta)^2} + \frac{\mathcal{H}_2^4(1+\delta)^4}{(1+5\delta)^2(1+11\delta)\mathcal{H}_1^3} + \frac{\mathcal{H}_2^3(1+\delta)^2}{(1+5\delta)^2(1+11\delta)\mathcal{H}_1^2}. \square$$

Given that $\delta = 0$ in Theorem 3, the subsequent corollary can be derived.

Corollary 5. *Consider a function f to be in the class $\mathcal{R}_{\Sigma}(v,\delta)$. Subsequently:*

$$\mathbb{H}_2(2) = \left|a_2a_4 - a_3^2\right| \leq \begin{cases} \mathbb{T}(v,2) & (\mathcal{H}_1 \geq 0 \text{ and } \mathcal{H}_2 \geq 0) \\ max\left\{\left(\frac{2v-1}{2}\right)^2, \mathbb{T}(v,2)\right\} & (\mathcal{H}_1 > 0 \text{ and } \mathcal{H}_2 > 0) \\ \left(\frac{2v-1}{2}\right)^2 & (\mathcal{H}_1 \leq 0 \text{ and } \mathcal{H}_2 \leq 0) \\ max\{\mathbb{T}(\mathfrak{Y}_\circ, t), \mathbb{T}(v,2)\} & (\mathcal{H}_1 < 0 \text{ and } \mathcal{H}_2 < 0) \end{cases},$$

$$\mathbb{T}(v,2) := 40\mathfrak{E}_1^4(v) + \frac{\mathfrak{E}_1(v)[\mathfrak{E}_3(v)]}{6},$$

$$\mathbb{T}(\mathfrak{Y}_\circ, t) = \mathfrak{E}_1^2(v) + \frac{\mathcal{H}_2^4}{\mathcal{H}_1^3} + \frac{\mathcal{H}_2^3}{\mathcal{H}_1^2},$$

$$\mathcal{H}_1 = \mathfrak{E}_1(v)\left[492\,\mathfrak{E}_1^3(v) + 2[-6\mathfrak{E}_2(v) + \mathfrak{E}_3(v)] - 24\mathfrak{E}_1(v) + 24\mathfrak{E}_1(v)\right]\mathfrak{Y}^4,$$

$$\mathcal{H}_2 = \mathfrak{E}_1(v)\left[3\mathfrak{E}_1^2(v) + 14\mathfrak{E}_1(v) + 3[-2\mathfrak{E}_2(v)]\right]\mathfrak{Y}^2. \square$$

Given that $\delta = 1$ in Theorem 3, we have the next corollary.

Corollary 6. *Consider a function f to be in the class $\mathcal{R}_{\Sigma}(v,\delta)$. Subsequently:*

$$\mathbb{H}_2(2) = \left|a_2a_4 - a_3^2\right| \leq \begin{cases} \mathbb{T}(v,2) & (\mathcal{H}_1 \geq 0 \text{ and } \mathcal{H}_2 \geq 0) \\ max\left\{\left(\frac{2v-1}{12}\right)^2, \mathbb{T}(v,2)\right\} & (\mathcal{H}_1 > 0 \text{ and } \mathcal{H}_2 > 0) \\ \left(\frac{2v-1}{12}\right)^2 & (\mathcal{H}_1 \leq 0 \text{ and } \mathcal{H}_2 \leq 0) \\ max\{\mathbb{T}(\mathfrak{Y}_\circ, t), \mathbb{T}(v,2)\} & (\mathcal{H}_1 < 0 \text{ and } \mathcal{H}_2 < 0) \end{cases},$$

$$\mathbb{T}(v,2) := \frac{80\mathfrak{E}_1^4(v)}{192}\mathfrak{Y}^4 + \frac{\mathfrak{E}_1(v)[\mathfrak{E}_3(v)]}{144}\mathfrak{Y}^4 + \frac{\mathfrak{E}_1^4(v)}{16},$$

$$\mathbb{T}(\mathfrak{Y}_\circ, t) = \frac{\mathfrak{E}_1^2(v)}{36} + \frac{16\mathcal{H}_2^4}{432\mathcal{H}_1^3} + \frac{4\mathcal{H}_2^3}{432\mathcal{H}_1^2},$$

$$\mathcal{H}_1 = \mathfrak{E}_1(v)\left[39744\mathfrak{E}_1^3(v) + 576[-6\mathfrak{E}_2(v) + \mathfrak{E}_3(v)] - 1152\mathfrak{E}_1(v)\right]\mathfrak{Y}^4,$$

$$\mathcal{H}_2 = \mathfrak{E}_1(v)\left[216(v)\mathfrak{E}_1^2(v) + 10008\mathfrak{E}_1(v) - 216[2\mathfrak{E}_2(v)]\right]\mathfrak{Y}^2. \square$$

5. Discussion

We introduce a new subclass $\mathcal{R}_{\Sigma}(v,\delta)$ of bi-univalent functions in the open unit disk U by using subordination conditions and determine estimates of the coefficients $|a_2|$ and $|a_3|$ for functions of this subclass. We obtained some new theorems with new special cases for our new subclass, and these results are different from the previous results for the other authors. Additionally, the present work introduces and examines a novel subclass, referred to as $\mathcal{R}_{\Sigma}(v,\delta)$, the set of functions consisting of bi-univalent functions that adhere to a specific subordination involving Euler polynomials. The Fekete–Szegö problem is specifically applied to functions in the class $\mathcal{R}_{\Sigma}(v,\delta)$ and provides bound estimates for the coefficients. We provide bound estimates for the coefficients and an upper bound estimate for the second Hankel determinant for functions belonging to the category of analytical and bi-univalent functions. This investigation incorporates the utilization of Euler polynomials. The results contained in the paper could inspire ideas for continuing the study, and we opened some windows for authors to generalize our new subclasses to obtain some new results in bi-univalent function theory.

Author Contributions: Conceptualization S.K.G., methodology W.G.A., validation W.G.A., formal analysis W.G.A., investigation S.K.G. and W.G.A., resources W.G.A. and S.K.G., writing—original draft preparation S.K.G., writing—review and editing W.G.A., visualization S.K.G., project administration W.G.A., funding acquisition W.G.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Duren, P.L. Univalent Functions. In *Grundlehren der Mathematischen Wissenschaften*; Band 259; Springer: New York, NY, USA; Berlin/Heidelberg, Germany; Tokyo, Japan, 1983.
2. Kac, V.; Cheung, P. Quantum Calculus. In *Universitext*; Springer: New York, NY, USA, 2002.
3. Srivastava, H.M.; Owa, S. *Current Topics in Analytic Function Theory*; World Scientific: Singapore, 1992.
4. Miller, S.S.; Mocanu, P.T. *Differential Subordinations: Theory and Applications*; Series on Monographs and Text Books in Pure and Applied Mathematics; Marcel Dekker: New York, NY, USA; Basel, Switzerland, 2000.
5. Miller, S.S.; Mocanu, P.T. Subordinants of differential superordinations. *Complex Var. Theory Appl.* **2003**, *48*, 815–826. [CrossRef]
6. Srivastava, H.M.; Mishra, A.K.; Gochhayat, P. Certain subclasses of analytic and bi-univalent functions. *Appl. Math. Lett.* **2010**, *23*, 1188–1192. [CrossRef]
7. Srivastava, H.M.; Gaboury, S.; Ghanim, F. Coefficient estimates for some general subclasses of analytic and bi-univalent functions. *Afr. Mat.* **2017**, *28*, 693–706. [CrossRef]
8. Andrews, G.E. q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra. In *Conference Series in Mathematics*; American Mathematical Society: Providence, RI, USA, 1986; Volume 66.
9. Fine, N.J. *Basic Hypergeometric Series and Applications, Mathematical Surveys and Monographs*; American Mathematical Society: Providence, RI, USA, 1988; Volume 27.
10. Khan, B.; Liu, Z.G.; Srivastava, H.M.; Khan, N.; Darus, M.; Tahir, M. A study of some families of multlivalent q-starlike functions involving higher-order q-derivatives. *Mathematics* **2020**, *8*, 1470. [CrossRef]
11. Riaz, S.; Nisar, U.A.; Xin, Q.; Malik, S.N.; Raheem, A. On Starlike Functions of Negative Order Defined by q-Fractional Derivative. *Fractal Fract.* **2022**, *6*, 30. [CrossRef]
12. Dziok, J.; Srivastava, H.M. Certain subclasses of analytic functions associated with the generalized hypergeometric function. Integral Transform. *Spec. Funct.* **2003**, *14*, 7–18.
13. Hu, Q.; Shaba, T.G.; Younis, J.; Khan, B.; Mashwani, W.K.; Caglar, M. Applications of q-derivative operator to Subclasses of bi-Univalent Functions involving Gegenbauer polynomial. *Appl. Math. Sci. Eng.* **2022**, *30*, 501–520. [CrossRef]
14. Sabir, P.O.; Srivastava, H.M.; Atshan, W.G.; Mohammed, P.O.; Chorfi, N.; Cortez, M.V. A family of holomorphic and m-fold symmetric bi-univalent functions endowed with coefficient estimate problems. *Mathematics* **2023**, *11*, 3970. [CrossRef]
15. Srivastava, H.M. Some families of Mittag-Leffler type functions and associated operators of fractional calculus. *TWMS J. Pure Appl. Math.* **2016**, *7*, 123–145.
16. Srivastava, H.M. Some formulas for the Bernoulli and Euler polynomials at rational arguments. *Math. Proc. Camb. Philos. Soc.* **2000**, *129*, 77–84. [CrossRef]
17. Taj, Y.; Zainab, S.; Xin, Q.; Tawfiq, F.M.O.; Raza, M.; Malik, S.N. Certain Coefficient Problems for q-Starlike Functions Associated with q-Analogue of Sine Function. *Symmetry* **2022**, *14*, 2200. [CrossRef]
18. Vilenkin, N.J.; Klimyk, A.U. *Representations of Lie Groups and Special Functions*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1992; Volume I–III.
19. Fekete, M.; Szego, G. Eine bemerkung uber ungerade schlichte funktionen. *J. Lond. Math. Soc.* **1993**, *8*, 85–89. [CrossRef]
20. Saliu, A.; Al-Shbeil, I.; Gong, J.; Malik, S.N.; Aloraini, N. Properties of q-Symmetric Starlike Functions of Janowski Type. *Symmetry* **2022**, *14*, 1907. [CrossRef]
21. Pommerenke, C. On the coefficients and Hankel determinants of univalent functions. *Proc. Lond. Math. Soc.* **1966**, *41*, 111–122. [CrossRef]
22. Noonan, W.; Thomas, D.K. On the second Hankel determinant of a really mean p-valent functions. *Trans. Am. Math. Soc.* **1976**, *223*, 337–346.
23. Khan, B.; Aldawish, I.; Araci, S.; Khan, M.G. Third Hankel Determinant for the Logarithmic Coefficients of Starlike Functions Associated with Sine Function. *Fractal Fract.* **2022**, *6*, 261. [CrossRef]
24. Zhang, H.Y.; Srivastava, R.; Tang, H. Third-Order Hankel and Toeplitz Determinants for Starlike Functions Connected with the Sine Function. *Mathematics* **2019**, *7*, 404. [CrossRef]

25. Al-Ameedee, S.A.; Atshan, W.G.; Al-Maamori, F.A. Second Hankel determinant for certain subclasses of bi-univalent functions. *J. Phys. Conf. Ser.* **2020**, *1664*, 012044. [CrossRef]
26. Atshan, W.G.; Al-Sajjad, R.A.; Altinkaya, S. On the Hankel determinant of m-fold symmetric bi-univalent functions using a new operator. *Gazi Univ. J. Sci.* **2023**, *36*, 349–360.
27. Al-shbeil, I.; Gong, J.; Khan, S.; Khan, N.; Khan, A.; Khan, M.F.; Goswami, A. Hankel and Symmetric Toeplitz Determinants for a New Subclass of q-Starlike Functions. *Fractal Fract.* **2022**, *6*, 658. [CrossRef]
28. Darweesh, A.M.; Atshan, W.G.; Battor, A.H.; Mahdi, M.S. On the third Hankel determinant of certain subclass of bi-univalent functions. *Math. Model. Eng. Probl.* **2023**, *10*, 1087–1095. [CrossRef]
29. Shakir, Q.A.; Atshan, W.G. On third Hankel determinant for certain subclass of bi-univalent functions. *Symmetry* **2024**, *16*, 239. [CrossRef]
30. Shi, L.; Arif, M.; Iqbal, J.; Ullah, K.; Ghufran, S.M. Sharp Bounds of Hankel Determinant on Logarithmic Coefficients for Functions Starlike with Exponential Function. *Fractal Fract.* **2022**, *6*, 645. [CrossRef]
31. Khan, B.; Liu, Z.-G.; Srivastava, H.M.; Araci, S.; Khan, N.; Ahmad, Z. Higher-order q-derivatives and their applications to subclasses of multivalent Janowski type q-starlike functions. *Adv. Differ. Equ.* **2021**, 440. [CrossRef]
32. Min, C.; Chen, Y. Painleve IV, Chazy II, and asymptotics for recurrence coefficients of semi-classical Laguerre polynomials and their Hankel determinants. *Math. Meth. Appl. Sci.* **2023**, *46*, 15270–15284. [CrossRef]
33. Koornwinder, T.H. Orthogonal polynomials in connection with quantum groups. In *Orthogonal Polynomials, Theory and Practice*; Nevai, P., Ed.; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1990; Volume 294, pp. 257–292.
34. Libera, R.J.; Zlotkiewicz, E.J. Coefficient Bounds for the Inverse of a Function with Derivative. *Proc. Am. Math. Soc.* **1983**, *87*, 251–257. [CrossRef]