symmetry

MDPI

Article

Some Properties of a Falling Function and Related Inequalities on Green's Functions

Pshtiwan Othman Mohammed [1,2,*], Ravi P. Agarwal [3], Majeed A. Yousif [4], Eman Al-Sarairah [5,6], Sarkhel Akbar Mahmood [7] and Nejmeddine Chorfi [8]

1 Department of Mathematics, College of Education, University of Sulaimani, Sulaymaniyah 46001, Iraq
2 Research and Development Center, University of Sulaimani, Sulaymaniyah 46001, Iraq
3 Department of Mathematics, Texas A & M University-Kingsville, Kingsville, TX 78363, USA; ravi.agarwal@tamuk.edu
4 Department of Mathematics, College of Education, University of Zakho, Zakho 42002, Iraq
5 Department of Mathematics, Khalifa University, Abu Dhabi P.O. Box 127788, United Arab Emirates
6 Department of Mathematics, Al-Hussein Bin Talal University, P.O. Box 20, Ma'an 71111, Jordan
7 Department of Physics, College of Science, University of Halabja, Halabja 46018, Iraq
8 Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
* Correspondence: pshtiwansangawi@gmail.com

Abstract: Asymmetry plays a significant role in the transmission dynamics in novel discrete fractional calculus. Few studies have mathematically modeled such asymmetry properties, and none have developed discrete models that incorporate different symmetry developmental stages. This paper introduces a Taylor monomial falling function and presents some properties of this function in a delta fractional model with Green's function kernel. In the deterministic case, Green's function will be non-negative, and this shows that the function has an upper bound for its maximum point. More precisely, in this paper, based on the properties of the Taylor monomial falling function, we investigate Lyapunov-type inequalities for a delta fractional boundary value problem of Riemann–Liouville type.

Keywords: Riemann–Liouville operator; Green's functions; falling function; Lyapunov inequalities

MSC: 26A48; 26A51; 33B10; 39A12

check for updates

Citation: Mohammed, P.O.; Agarwal, R.P.; Yousif, M.A.; Al-Sarairah, E.; Mahmood, S.A.; Chorfi, N. Some Properties of a Falling Function and Related Inequalities on Green's Functions. *Symmetry* **2024**, *16*, 337. https://doi.org/10.3390/sym16030337

Academic Editor: Serkan Araci

Received: 14 February 2024
Revised: 29 February 2024
Accepted: 6 March 2024
Published: 11 March 2024

1. Introduction

Discrete fractional calculus in fractional calculus theory is a topic that has motivated a significant number of investigations in the past few decades. Also, many researchers have demonstrated that discrete fractional problems describe natural phenomena in a more systematic way and more precisely than integer-order fraction problems; they are classic with regular time differences. Research has been conducted on practical models appearing in the areas of engineering, physics and computer science (cf. [1–6]).

Taking into account the previous considerations, an important topic in discrete fractional calculus is to achieve computations of boundary and initial value problems whose initial and boundary conditions are of the form of nabla or delta difference operators (cf. [7–13]). In recent years, boundary and initial value problem computations when considering the nabla fractional and the delta fractional with different types of discrete operators bases have been achieved (e.g., [14–21]). In recent years, delta fractional problems when Green's function is deduced from Laplace transformations have been solved (e.g., [22–24]).

Recently, in [25], the following results on the delta BVP have been presented.

Theorem 1 (see [25]). *Let $\alpha \in (1,2)$, let a, b be two real numbers such that $b = a + \mathbb{N}_1$, let $h, A : \mathbb{T}_{a+1} \to \mathbb{R}$, and let $A(t) > 0$. Then, the fractional BVP*

$$\begin{cases} -\left({}_{a+1}^{\mathrm{RL}}\Delta^{\alpha-1}(A\nabla f)\right)(t) = h(t+\alpha-1), \quad t \in \mathbb{T}_{a+1}, \\ f(a) = 0, \quad f(b) = 0, \end{cases}$$

has the unique solution

$$f(t) = \sum_{t=a+1}^{b} G(t,r)\,h(r),$$

where

$$G(t,r) = \begin{cases} \frac{\varphi(b,\rho(r))}{\varphi(b,a)}\varphi(t,a), \qquad t \le r-1, \\ \frac{\varphi(b,\rho(r))}{\varphi(b,a)}\varphi(t,a) - \varphi(t,\rho(r)), \quad t \ge r, \end{cases}$$

and $\varphi(t,\rho(r))$ is the Cauchy function defined by

$$\varphi(t,\rho(r)) = \sum_{\tau=r}^{t} \frac{(\tau+\alpha-\sigma(r))^{\underline{\alpha-1}}}{\Gamma(\alpha)\,A(\tau)}, \quad t \in \mathbb{N}_{a+1}, \tag{1}$$

where $\sigma(r) = r+1$, $\rho(r) = r-1$, and $\mathbb{N}_1$, $\mathbb{T}_a$ and $\tau^{\underline{\alpha}}$ are defined later in Section 2.

Theorem 2 (see [25]). *Let $\alpha \in (1,2)$, $A : \mathbb{T}_{a+1} \to (0,\infty)$, $\delta_1^2 + \delta_2^2 > 0$, and $\delta_3^2 + \delta_4^2 > 0$. Then, the fractional self-adjoint BVP*

$$\begin{cases} -\left({}_{a+1}^{\mathrm{RL}}\Delta^{\alpha-1}(A\nabla f)\right)(t) = 0, \quad t \in \mathbb{T}_{a+2}, \\ \delta_1 f(a+1) - \delta_2\big(\nabla f\big)(a+1) = 0, \\ \delta_3 f(b) + \delta_4\big(\nabla f\big)(b) = 0, \end{cases} \tag{2}$$

only has a trivial solution **if**

$$\xi = \frac{\delta_2\delta_3}{A(a+1)} + \delta_1\delta_3 \sum_{r=a+2}^{b} \frac{(r-a-2+\alpha)^{\underline{\alpha-1}}}{\Gamma(\alpha)A(r)} + \frac{\delta_1\delta_4(b-a-2+\alpha)^{\underline{\alpha-1}}}{\Gamma(\alpha)A(b)} \neq 0.$$

Theorem 3 (see [25]). *The Green's function for the BVP (2) can be expressed as*

$$G(t,r) = \begin{cases} f(t,r), \quad t \le r-1, \\ g(t,r), \quad t \ge r, \end{cases}$$

$$\begin{aligned} f(t,r) = \frac{1}{\xi}\Bigg(&\delta_1\delta_3\varphi(t,a)\varphi(b,\rho(r)) + \delta_1\delta_4\varphi(t,a)\frac{(b+\alpha-r-1)^{\underline{\alpha-1}}}{\Gamma(\alpha)A(b)} \\ &+ \frac{\delta_3(\delta_2-\delta_1)}{A(a+1)}\varphi(b,\rho(r)) + \frac{\delta_4(\delta_2-\delta_1)(b+\alpha-r-1)^{\underline{\alpha-1}}}{\Gamma(\alpha)A(b)}\Bigg), \end{aligned}$$

and

$$g(t,r) = f(t,r) - \varphi(t,\rho(r)).$$

In this paper, based on the above results, the non-negativity of $G(t,r)$ will be proven and we will examine the upper bound for the maximum value of the function. In other words, beyond Green's function $G(t,r)$, in the present paper, we examine and investigate Lyapunov-type inequalities for the delta BVP:

$$\begin{cases} -\left({}^{\mathrm{RL}}_{a+1}\Delta^{\alpha-1}(A\nabla f)\right)(t) = B(t+\alpha-1)f(t+\alpha-1), \quad t \in \mathbb{T}_{a+2}, \\ \delta_1 f(a+1) - \delta_2(\nabla f)(a+1) = 0, \\ \delta_3 f(b) + \delta_4(\nabla f)(b) = 0. \end{cases} \tag{3}$$

The rest of the paper is structured as follows: Section 2 is separated into two parts; in Section 2.1, we review and discuss the literature on delta fractional operators, and we will present and prove some essential lemmas in Section 2.2. Section 3 is devoted to explaining the Taylor monomial falling function and some of its properties (in Section 3.1) and its related results with the implementation of Green's function (in Section 3.2). Section 4 presents an accurate solution obtained when computing a relevant eigenvalue problem corresponding to the BVP (3). Finally, Section 5 includes the most relevant concluding remarks of present and future works.

2. Delta Fractional Operators and the Basic Lemmas

Let $\delta_1 > 0$, and $\mathbb{N}$ be a set of natural numbers. Then, we define the notations $\mathbb{N}_a := a + \mathbb{N}$ and ${}_b\mathbb{N} := b - \mathbb{N}$, for $a, b \in \mathbb{R}$. Furthermore, let $\mathbb{T}_a := \{a, a+1, \ldots, b\}$ such that $b = a + k$, for some $k \in \mathbb{N}_0$.

2.1. Delta Fractional Operators

The $\Delta-$fractional sum operator is defined in [1] (Definition 2.25) as follows:

$$\left({}^{\mathrm{RL}}_{a}\Delta^{-\alpha} f\right)(t) = \sum_{r=a}^{t-\alpha} H_{\alpha-1}(t,\sigma(r))f(r), \quad \text{for } t \text{ in } \mathbb{N}_{a+\alpha}, \tag{4}$$

and the $\Delta-$fractional difference operator is defined in [26] (Theorem 2.2) as follows:

$$\left({}^{\mathrm{RL}}_{a}\Delta^{\alpha} f\right)(t) = \sum_{r=a}^{t+\alpha} H_{-\alpha-1}(t,\sigma(r))f(r), \quad \text{for } t \text{ in } \mathbb{N}_{a+n-\alpha}, \tag{5}$$

for $n-1 < \alpha < n$ and f is defined on $\mathbb{N}_a$. It is important to state that the falling Taylor monomial function is given as follows:

$$H_\alpha(t,r) := \frac{(t-r)^{\underline{\alpha}}}{\Gamma(\alpha+1)}, \tag{6}$$

and the falling function is defined by

$$t^{\underline{\alpha}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}. \tag{7}$$

Lemma 1 (see [1,27]). *If $a, \alpha \in \mathbb{R}^+$, then*

- *For $t \in \mathbb{N}_a$, we have*

$$\begin{aligned} {}^{\mathrm{RL}}_{a+\alpha}\Delta^{-\alpha} H_\alpha(t,a) &= H_{\alpha+\alpha}(t,a), \\ {}^{\mathrm{RL}}_{a-\alpha}\Delta^{\alpha} H_\alpha(t,a) &= H_{\alpha-\alpha}(t,a), \end{aligned}$$

such that $a+\alpha$ and $a-\alpha$ are non-negative integers.

- *For $t \in \mathbb{N}_{a+n}$ as $n-1<\alpha<n$, we have*

$$\left({}_{a+n-\alpha}^{\ \ \mathrm{RL}}\Delta^{-\alpha}\ {}_{a}^{\mathrm{RL}}\Delta^{\alpha} f\right)(t) = f(t). \tag{8}$$

- *$t \in \mathbb{N}_{a+1}$, we have*

$$\sum_{r=a+1}^{t} H_{\alpha}(r+\alpha-1,a) = H_{\alpha+1}(t+\alpha,a),$$
$$\sum_{r=a+1}^{t} H_{\alpha}(r+\alpha+1,\sigma(r)) = H_{\alpha+1}(t+\alpha,a).$$

Lemma 2 (see [28])**.** *Assume that $f,k : S \to \mathbb{R}^{+}$ and attain their max in S. Then, we have*

$$\left|f(t)-k(t)\right| \le \max\{f(t),k(t)\} \le \max\left\{\max_{t\in S} f(t), \max_{t\in S} k(t)\right\},$$

for each fixed $t \in S$.

2.2. Basic Lemmas

In this subsection, we state and prove some necessary lemmas which will be useful later.

Lemma 3. *If $A(t)=1$, then,*

(a) The identity ξ in Theorem 2 can be expressed as follows:

$$\xi = (\delta_2-\delta_1)\delta_3 + \delta_1\delta_3\, H_{\alpha-1}(b+\alpha-2,a) + \delta_1\delta_4\, H_{\alpha-2}(b+\alpha-3,a). \tag{9}$$

(b) The identities $f(t,r)$ and $g(t,r)$ in Theorem 3 can be expressed as follows:

$$\begin{aligned} f(t,r) = \frac{1}{\xi}\Big[&\delta_1\delta_3\, H_{\alpha-1}(t+\alpha-2,a)\, H_{\alpha-1}(b+\alpha,\sigma(r)) \\ &+ \delta_1\delta_4\, H_{\alpha-1}(t+\alpha-2,a)\, H_{\alpha-2}(b+\alpha-1,\sigma(r)) \\ &+ \delta_3(\delta_2-\delta_1)\, H_{\alpha-1}(b+\alpha,\sigma(r)) + \delta_4(\delta_2-\delta_1)\, H_{\alpha-2}(b+\alpha-1,\sigma(r))\Big]. \end{aligned}$$

and

$$g(t,r) = f(t,r) - H_{\alpha-1}(t+\alpha,\sigma(r)). \tag{10}$$

Proof. For the first item, we have

$$\begin{aligned} \sum_{r=a+2}^{b} \frac{(r-a-3+\alpha)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)} &= \sum_{r=a+2}^{b} \frac{\Delta_r (r-a-3+\alpha)^{\underline{\alpha-1}}}{\Gamma(\alpha)} \\ &= \sum_{r=a+2}^{b} \frac{(r-a-2+\alpha)^{\underline{\alpha-1}}}{\Gamma(\alpha)} - \sum_{r=a+2}^{b} \frac{(r-a-3+\alpha)^{\underline{\alpha-1}}}{\Gamma(\alpha)} \\ &= \frac{(b-a+\alpha-2)^{\underline{\alpha-1}}}{\Gamma(\alpha)} - 1. \end{aligned}$$

By putting this result in the equation

$$\xi = \delta_2\delta_3 + \delta_1\delta_3 \sum_{r=a+2}^{b} \frac{(r-a-3+\alpha)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)} + \frac{\delta_1\delta_4 (b-a-3+\alpha)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)},$$

we get the desired result.

The second item can be obtained from the definition of the falling function (6) and (1). Thus, the proof is complete. □

The proofs of the following two lemmas are straightforward and we will omit them.

Lemma 4 (see also [27]). *Assume that $r \in \mathbb{N}_a$. Then, we have*

(i) *If $\alpha > 0$, then*
- *The function $H_\alpha(t+\alpha+1, \sigma(r))$ is decreasing with respect to r, for $t \in \mathbb{N}_{r-1}$.*
- *The function $H_\alpha(t+\alpha+1, \sigma(r))$ is increasing with respect to t, for $t \in \mathbb{N}_r$.*

(ii) *If $\alpha > -1$, then*
- *$H_\alpha(t+\alpha+1, \sigma(r)) \geq 0$, for $t \in \mathbb{N}_{r-1}$.*
- *$H_\alpha(t+\alpha+1, \sigma(r)) > 0$, for $t \in \mathbb{N}_r$.*

(iii) *If $0 > \alpha > -1$, then*
- *The function $H_\alpha(t+\alpha+1, \sigma(r))$ is increasing with respect to r, for $t \in \mathbb{N}_r$.*
- *The function $H_\alpha(t+\alpha+1, \sigma(r))$ is increasing with respect to t, for $t \in \mathbb{N}_{r+1}$.*

(iv) *If $\alpha \geq 0$, then the function $H_\alpha(t+\alpha+1, \sigma(r))$ is non-decreasing with respect to t, for $t \in \mathbb{N}_{r-1}$.*

Lemma 5 (see also [27]). *Assume that $\alpha \geq \alpha > 0$. Then, we have*

$$H_\alpha(t+\alpha-1, a) \geq H_\alpha(t+\alpha-1, a),$$

for each fixed $t \in \mathbb{N}_a$.

3. Taylor Falling Function and Green's Function Results

3.1. Taylor Falling Function

This subsection starts by introducing the Taylor falling function:

$$h_\alpha(t, r) = \frac{H_\alpha(t+\alpha+1, \sigma(r))}{H_\alpha(t+\alpha-1, a)}, \tag{11}$$

for $r \in \mathbb{N}_{a+1}$, $t \in \mathbb{N}_r$ and $\alpha > -1$. Therefore, the following theorem concerns some positivity results on this function.

Theorem 4. *The function defined in (11) has the following properties:*

1. *$h_\alpha(t, r) > 0$.*
2. *$h_\alpha(t, r) \leq 1$, where $\alpha > 0$, and $h_\alpha(t, r) \geq 1$, for $-1 < \alpha < 0$, or, specifically, $h_0(t, r) = 1$.*
3. *The function $h_\alpha(t, r)$ is nonincreasing with respect to t, for $\alpha > 0$.*
4. *The function $h_\alpha(t, r)$ is nonincreasing with respect to t, for $-1 < \alpha < 0$.*

Proof. Proof of (1). By considering the definition, we have

$$h_\alpha(t, r) = \frac{(t-r+\alpha)^{\underline{\alpha}}}{(t-a+\alpha-1)^{\underline{\alpha}}} = \frac{\Gamma(t-a)\Gamma(t-r+\alpha+1)}{\Gamma(t-a+\alpha)\Gamma(t-r+1)}. \tag{12}$$

As it is clear that $\Gamma(t-a), \Gamma(t-r+\alpha+1), \Gamma(t-a+\alpha), \Gamma(t-r+1) > 0$, it follows from (12) that $h_\alpha(t, r) > 0$.

Proof of (2). This follows from the monotonicity of $H_\alpha(t+\alpha+1, \sigma(r))$ with respect to r.

Proof of (3). Let us consider

$$\begin{aligned}\nabla h_\alpha(t,r) &= \nabla\left[\frac{(t-\sigma(r))^{\underline{\alpha}}}{(t-a)^{\underline{\alpha}}}\right] = \frac{(t-r+1)^{\underline{\alpha}}}{(t-a)^{\underline{\alpha}}} - \frac{(t-r)^{\underline{\alpha}}}{(t-a-1)^{\underline{\alpha}}}\\ &= \frac{\Gamma(t-a)\Gamma(t-r+\alpha+1)}{\Gamma(t-a+\alpha)\Gamma(t-r+1)} - \frac{\Gamma(t-a-1)\Gamma(t-r+\alpha)}{\Gamma(t-a+\alpha-1)\Gamma(t-r)}\\ &= \frac{\Gamma(t-a-1)\Gamma(t-r+\alpha)}{\Gamma(t-a+\alpha-1)\Gamma(t-r)}\left[\frac{(t-a-1)(t-r+\alpha)}{(t-a+\alpha-1)(t-r)} - 1\right]\\ &= \alpha(r-a-1)\left[\frac{\Gamma(t-a-1)\Gamma(t-r+\alpha)}{\Gamma(t-a+\alpha)\Gamma(t-r+1)}\right]. \end{aligned} \tag{13}$$

Since $(r-a-1)\geq 0$ and $\Gamma(t-r+\alpha),\Gamma(t-a-1),\Gamma(t-a+\alpha),\ \Gamma(t-r+1)>0$, then we see that $\nabla h_\alpha(t,r)\geq 0$ in (13).

Proof of (4). Rearranging (13), we see that

$$\nabla h_{-\alpha}(t,r) = -\alpha(r-a-1)\left[\frac{\Gamma(t-a-1)\Gamma(t-r+\alpha)}{\Gamma(t-a+\alpha)\Gamma(t-r+1)}\right] \tag{14}$$

From (14), $(r-a-1)\geq 0$ and $\Gamma(t-r+\alpha),\Gamma(t-a-1),\Gamma(t-a+\alpha),\ \Gamma(t-r+1)>0$, then $\nabla h_{-\alpha}(t,r)\leq 0$, as required. Thus, the proof is done. □

3.2. Green's Function Results

In this subsection, we examine some properties of $G(t,r)$. The first lemma shows the positivity of the functions in Theorem 3.

Lemma 6. *Let $\delta_1,\delta_3,\delta_2,\delta_4\geq 0$ and $\delta_1\leq\delta_2$, and let (9) hold. Then, we have*

1. *$\forall t\in\mathbb{T}_a$, and then $\xi>0$.*
2. *$f(t,r)\geq 0,\ \forall (t,r)\in\mathbb{T}_a\times\mathbb{T}_{a+1}$ such that $r-1\geq t$.*
3. *$g(t,r)\geq 0,\ \forall (t,r)\in\mathbb{T}_a\times\mathbb{T}_{a+1}$ such that $r\leq t$.*

Proof. Proof of (1). As we know from Lemma 4 that $H_{\alpha-1}(b+\alpha-2,a), H_{\alpha-2}(b+\alpha-3,a)>0$, we have

$$\xi = (\delta_2-\delta_1)\delta_3 + \delta_1\delta_3 H_{\alpha-1}(b+\alpha-2,a) + \delta_1\delta_4 H_{\alpha-2}(b+\alpha-3,a) > 0.$$

Proof of (2). According to Lemma 4, it is clear that $H_{\alpha-1}(b+\alpha,\sigma(r)), H_{\alpha-2}(b+\alpha-1,\sigma(r))>0,\ \forall\ r\in\mathbb{T}_{a+1}$ and $H_{\alpha-1}(t+\alpha-2,a)\geq 0,\ \forall\ t\in\mathbb{T}_a$. In addition, we know that $\xi>0,\ \forall\ t\in\mathbb{T}_a$ according to (2). Therefore, we have

$$\begin{aligned} f(t,r) = \frac{1}{\xi}\Big[&\delta_1\delta_3\, H_{\alpha-1}(t+\alpha-2,a)\, H_{\alpha-1}(b+\alpha,\sigma(r))\\ &+\delta_1\delta_4\, H_{\alpha-1}(t+\alpha-2,a)\, H_{\alpha-2}(b+\alpha-1,\sigma(r)) + \delta_3(\delta_2-\delta_1)\, H_{\alpha-1}(b+\alpha,\sigma(r))\\ &+\delta_4(\delta_2-\delta_1)\, H_{\alpha-2}(b+\alpha-1,\sigma(r))\Big]\geq 0, \end{aligned}$$

$\forall\ (t,r)\in\mathbb{T}_a\times\mathbb{T}_{a+1}$ then $t\leq r-1$.

Proof of (3). For this property, consider

$$
\begin{aligned}
g(t,r) = \frac{1}{\xi}\Big[&\delta_1\delta_3\, H_{\alpha-1}(t+\alpha-2,a)\, H_{\alpha-1}(b+\alpha,\sigma(r)) \\
&+ \delta_1\delta_4\, H_{\alpha-1}(t+\alpha-2,a)\, H_{\alpha-2}(b+\alpha-1,\sigma(r)) + \delta_3(\delta_2-\delta_1)\, H_{\alpha-1}(b+\alpha,\sigma(r)) \\
&+ \delta_4(\delta_2-\delta_1)\, H_{\alpha-2}(b+\alpha-1,\sigma(r)) - \xi\, H_{\alpha-1}(t+\alpha,\sigma(r))\Big] \\
&= \frac{1}{\xi}\Bigg[\delta_4(\delta_2-\delta_1)\, H_{\alpha-2}(b+\alpha-1,\sigma(r)) + \delta_3(\delta_2-\delta_1)\big(H_{\alpha-1}(b+\alpha,\sigma(r)) \\
&- H_{\alpha-1}(t+\alpha,\sigma(r))\big) + \delta_1\delta_4\Big(H_{\alpha-1}(t+\alpha-2,a)H_{\alpha-2}(b+\alpha-1,\sigma(r)) \\
&- H_{\alpha-1}(t+\alpha,\sigma(r))H_{\alpha-2}(b+\alpha-3,a)\Big) \\
&+ \delta_1\delta_3\Big(H_{\alpha-1}(t+\alpha-2,a)H_{\alpha-1}(b+\alpha,\sigma(r)) \\
&- H_{\alpha-1}(b+\alpha-2,a)H_{\alpha-1}(t+\alpha,\sigma(r))\Big)\Bigg] \\
&= \frac{1}{\xi}[E_1+E_2+E_3+E_4],
\end{aligned}
\tag{15}
$$

where
$E_1 = \delta_4(\delta_2-\delta_1)H_{\alpha-2}(b+\alpha-1,\sigma(r))$;
$E_2 = \delta_3(\delta_2-\delta_1)(H_{\alpha-1}(b,\sigma(r)) - H_{\alpha-1}(t,\sigma(r)))$;
$E_3 = \delta_1\delta_4\Big(H_{\alpha-1}(t+\alpha-2,a)H_{\alpha-2}(b+\alpha-1,\sigma(r)) - H_{\alpha-1}(t+\alpha,\sigma(r))H_{\alpha-2}(b+\alpha-3,a)\Big)$;
$E_4 = \delta_1\delta_3\Big(H_{\alpha-1}(t+\alpha-2,a)H_{\alpha-1}(b+\alpha,\sigma(r)) - H_{\alpha-1}(b+\alpha-2,a)H_{\alpha-1}(t+\alpha,\sigma(r))\Big)$.

In (15), we know that $\xi > 0$, $\forall\, t \in \mathbb{T}_a$. So, it remains to prove that $E_r \geq 0$, for $r = 1,2,3,4$.

- For E_1: According to Lemma 4, $H_{\alpha-2}(b+\alpha-1,\sigma(r)) > 0$, $\forall\, t \in \mathbb{T}_a$. Therefore, $E_1 \geq 0$.
- For E_2: By using Lemma 4, $H_{\alpha-1}(t+\alpha,\sigma(r)) \leq H_{\alpha-1}(b+\alpha,\sigma(r))$, $\forall\,(t,r) \in \mathbb{T}_a \times \mathbb{T}_{a+1}$ such that $t \geq r$. Hence, $E_2 \geq 0$.
- For E_3: Again, by using Lemma 4, $H_{\alpha-1}(t+\alpha,\sigma(r)) \leq H_{\alpha-1}(t+\alpha-2,a) > 0$ and $H_{\alpha-2}(b+\alpha-3,a) \leq H_{\alpha-2}(b+\alpha-1,\sigma(r))$, $\forall\,(t,r) \in \mathbb{T}_a \times \mathbb{T}_{a+1}$ such that $r \leq t$. These lead to $E_3 \geq 0$.
- For E_4: We consider

$$
\begin{aligned}
&H_{\alpha-1}(t+\alpha-2,a)H_{\alpha-1}(b+\alpha,\sigma(r)) - H_{\alpha-1}(b+\alpha-2,a)H_{\alpha-1}(t+\alpha,\sigma(r)) \\
&= H_{\alpha-1}(b+\alpha-2,a)H_{\alpha-1}(t+\alpha,\sigma(r))\left[\frac{H_{\alpha-1}(b+\alpha,\sigma(r))}{H_{\alpha-1}(b+\alpha-2,a)}\cdot\frac{H_{\alpha-1}(t+\alpha-2,a)}{H_{\alpha-1}(t+\alpha,\sigma(r))} - 1\right] \\
&= H_{\alpha-1}(b+\alpha-2,a)H_{\alpha-1}(t+\alpha,\sigma(r))\left[\frac{h_{\alpha-1}(b,r)}{h_{\alpha-1}(t,r)} - 1\right].
\end{aligned}
$$

$H_{\alpha-1}(b+\alpha-2,a), H_{\alpha-1}(t+\alpha,\sigma(r)) > 0$ by Lemma 4, and $h_{\alpha-1}(t,r) \leq h_{\alpha-1}(b,r)$, $\forall\,(t,r) \in \mathbb{T}_a \times \mathbb{T}_{a+1}$ by Theorem 4. Therefore, we get $E_4 \geq 0$ as desired.

As a result, $g(t,r) > 0$, $\forall\, \mathbb{T}_a \times \mathbb{T}_{a+1}$, such that $r \leq t$. Also, $G(t,r) \geq 0$, $\forall\,(t,r) \in \mathbb{T}_a \times \mathbb{T}_{a+1}$. This ends our proof. □

The positivity of Green's function can be deduced from the following theorem.

Theorem 5. *Let $\delta_1,\delta_2,\delta_3,\delta_4 > 0$ and $\delta_1 \leq \delta_2$, and let (9) hold. Then, we have*

$$G(t,r) \geq 0,$$

for $(t,r) \in \mathbb{T}_a \times \mathbb{T}_{a+1}$.

Proof. The proof can be deduced from the next lemma. □

To obtain the above result, we need to show that the functions in Theorem 3 are increasing.

Lemma 7. *Let $\delta_1, \delta_2, \delta_3, \delta_4 > 0$ and $\delta_1 \leq \delta_2$, and let (9) hold. Then, we have that*

1. *$f(t,r)$ is an increasing function with respect to t $\forall$ $(t,r) \in \mathbb{T}_a \times \mathbb{T}_{a+1}$ such that $t \leq r-1$.*
2. *$g(t,r)$ is an increasing function with respect to t $\forall$ $(t,r) \in \mathbb{T}_a \times \mathbb{T}_{a+1}$ such that $r \leq t$.*

Proof. Proof of (1). We consider

$$\nabla_t f(t,r) = \frac{1}{\xi}\Big[\delta_1\delta_3\, H_{\alpha-2}(t+\alpha-3,a)H_{\alpha-1}(b+\alpha,\sigma(r)) \\ + \delta_1\delta_4\, H_{\alpha-2}(t+\alpha-3,a)H_{\alpha-2}(b+\alpha-1,\sigma(r))\Big].$$

From Lemma 4, it is clear that $H_{\alpha-1}(b,\sigma(r)), H_{\alpha-2}(b,\sigma(r)) > 0 \,\forall\, r \in \mathbb{T}_{a+1}$ and $H_{\alpha-2}(t+\alpha-3,a) > 0 \,\forall\, t \in \mathbb{T}_{a+1}$. Moreover, $\xi > 0 \,\forall\, t \in \mathbb{T}_{a+1}$ according to Theorem 2. Thus, we get $\nabla_t f(t,r) > 0$, and this implies that (1) holds true.

Proof of (2). According to (15), we have

$$\begin{aligned}\nabla_t g(t,r) = \frac{1}{\xi}\Bigg[&-\delta_3(\delta_2-\delta_1)\, H_{\alpha-2}(t+\alpha-1,\sigma(r)) \\ &+ \delta_1\delta_4\Big(H_{\alpha-2}(t+\alpha-3,a)H_{\alpha-2}(b+\alpha-1,\sigma(r)) \\ &- H_{\alpha-2}(t+\alpha-1,\sigma(r))H_{\alpha-2}(b+\alpha-3,a)\Big) \\ &+ \delta_1\delta_3\Big(H_{\alpha-2}(t+\alpha-3,a)H_{\alpha-1}(b+\alpha,\sigma(r)) \\ &- H_{\alpha-1}(b+\alpha-2,a)H_{\alpha-2}(t+\alpha-1,\sigma(r))\Big)\Bigg] \\ = &\frac{1}{\xi}\big[E_5+E_6+E_7\big],\end{aligned} \tag{16}$$

where

$$\begin{aligned}E_5 &= -\delta_3(\delta_2-\delta_1)H_{\alpha-2}(t+\alpha-1,\sigma(r)), \\ E_6 &= \delta_1\delta_4\Big(H_{\alpha-2}(t+\alpha-3,a)H_{\alpha-2}(b+\alpha-1,\sigma(r)) \\ &\quad - H_{\alpha-2}(t+\alpha-1,\sigma(r))H_{\alpha-2}(b+\alpha-3,a)\Big), \\ E_7 &= \delta_1\delta_3\Big(H_{\alpha-2}(t+\alpha-3,a)H_{\alpha-1}(b+\alpha,\sigma(r)) \\ &\quad - H_{\alpha-1}(b+\alpha-2,a)H_{\alpha-2}(t+\alpha-1,\sigma(r))\Big).\end{aligned}$$

As it is clear that $\xi > 0 \,\forall\, t \in \mathbb{T}_{a+1}$, we only need to show that $E_i \leq 0,\ \ i = 5,6,7$.

- For E_5: Clearly, from Lemma (4), $H_{\alpha-2}(t,\sigma(r)) > 0 \,\forall\, (t,r) \in \mathbb{T}_a \times \mathbb{T}_{a+1}$ such that $r \geq t$ implies that $E_5 \leq 0$.

- For E_6: We consider

$$\begin{aligned}
&H_{\alpha-2}(t+\alpha-3,a)H_{\alpha-2}(b+\alpha-1,\sigma(r)) - H_{\alpha-2}(b,a)H_{\alpha-2}(t,\sigma(r)) \\
&= H_{\alpha-2}(b+\alpha-3,a)H_{\alpha-2}(t+\alpha-1,\sigma(r))\left[\frac{H_{\alpha-2}(b+\alpha-1,\sigma(r))}{H_{\alpha-2}(b+\alpha-3,a)}\right. \\
&\qquad \left.\cdot\frac{H_{\alpha-2}(t+\alpha-3,a)}{H_{\alpha-2}(t+\alpha-1,\sigma(r))} - 1\right] \\
&= H_{\alpha-2}(b+\alpha-3,a)H_{\alpha-2}(t+\alpha-1,\sigma(r))\left[\frac{h_{\alpha-2}(b,r)}{h_{\alpha-2}(t,r)} - 1\right].
\end{aligned}$$

  By considering Lemma (4), we see that $H_{\alpha-2}(b+\alpha-3,a)H_{\alpha-2}(t+\alpha-1,\sigma(r)) > 0$. Moreover, by considering Theorem 4, we have $h_{\alpha-2}(t,r) \leq h_{\alpha-2}(b,r)$, $\forall\ (t,r) \in \mathbb{T}_a \times \mathbb{T}_{a+1}$. Therefore, $E_6 \leq 0$.
- For E_7: According to Lemma (4), it can be seen that $H_{\alpha-2}(t+\alpha-3,a) \leq H_{\alpha-2}(t+\alpha-1,\sigma(r))$ and $H_{\alpha-1}(b+\alpha,\sigma(r)) \leq H_{\alpha-1}(b+\alpha-2,a)$ $\forall\ (t,r) \in \mathbb{T}_a \times \mathbb{T}_{a+1}$ such that $r \leq t$ implies that $E_7 \leq 0$.

Then, the proof is complete. □

The following theorem demonstrates the boundedness of Green's function.

Theorem 6. *Assume that $\delta_1, \delta_2, \delta_3, \delta_4 \geq 0$ and $\delta_1 \leq \delta_2$, and let (9) hold. Then, for $G(t,r)$, we have*

$$\max_{(t,r)\in\mathbb{T}_a\times\mathbb{T}_{a+1}} G(t,r) < \Upsilon, \tag{17}$$

where

$$\begin{aligned}
\Upsilon = \frac{1}{\xi}\Big[&\delta_1\delta_3\, H_{\alpha-1}(b+\alpha-2,a)\, H_{\alpha-1}(b+\alpha-1,a) \\
&+ \delta_1\delta_4\, H_{\alpha-1}(b+\alpha-2,a)\, H_{\alpha-2}(b+\alpha-2,a) \\
&+ \delta_3(\delta_2-\delta_1)\, H_{\alpha-1}(b+\alpha-1,a) + \delta_4(\delta_2-\delta_1)\Big].
\end{aligned}$$

Proof. According to Lemma 7, we see that

$$\max_{(t,r)\in\mathbb{T}_a\times\mathbb{T}_{a+1}} G(t,r) = \max_{r\in\mathbb{T}_{a+1}} \{f(\rho(r),r), \alpha(r,r)\}.$$

On the other hand,

$$\begin{aligned}
f(\rho(r),r) = \frac{1}{\xi}\Big[&\delta_1\delta_3\, H_{\alpha-1}(\rho(r)+\alpha-2,a)\, H_{\alpha-1}(b+\alpha,\sigma(r)) \\
&+ \delta_1\delta_4\, H_{\alpha-1}(\rho(r)+\alpha-2,a)\, H_{\alpha-2}(b+\alpha-1,\sigma(r)) + \delta_3(\delta_2-\delta_1)\, H_{\alpha-1}(b+\alpha,\sigma(r)) \\
&+ \delta_4(\delta_2-\delta_1)\, H_{\alpha-2}(b+\alpha-1,\sigma(r))\Big],
\end{aligned}$$

and we let

$$\begin{aligned}
k(r) = \frac{1}{\xi}\Big[&\delta_1\delta_3\, H_{\alpha-1}(r+\alpha-2,a)\, H_{\alpha-1}(b+\alpha,\sigma(r)) \\
&+ \delta_1\delta_4\, H_{\alpha-1}(r+\alpha-2,a)\, H_{\alpha-2}(b+\alpha-1,\sigma(r)) \\
&+ \delta_3(\delta_2-\delta_1)\, H_{\alpha-1}(b+\alpha,\sigma(r)) + \delta_4(\delta_2-\delta_1)\, H_{\alpha-2}(b+\alpha-1,\sigma(r))\Big],
\end{aligned}$$

for $r \in \mathbb{T}_{a+1}$. It follows from Lemmas 4 and 6 that

$$0 \le f(\rho(r), r) < k(r) \quad r \in \mathbb{T}_{a+1}. \tag{18}$$

Also, we have

$$\begin{aligned} g(r,r) = \frac{1}{\xi}\Big[&\delta_1\delta_3\, H_{\alpha-1}(r+\alpha-2,a)\, H_{\alpha-1}(b+\alpha,\sigma(r)) \\ &+ \delta_1\delta_4\, H_{\alpha-1}(r+\alpha-2,a)\, H_{\alpha-2}(b+\alpha-1,\sigma(r)) \\ &+ \delta_3(\delta_2-\delta_1)\, H_{\alpha-1}(b+\alpha,\sigma(r)) + \delta_4(\delta_2-\delta_1)\, H_{\alpha-2}(b+\alpha-1,\sigma(r))\Big] - 1 \\ = &\; k(r) - 1, \qquad r \in \mathbb{T}_{a+1}. \end{aligned} \tag{19}$$

Therefore, by considering Lemma 6, it is known that

$$0 \le g(r,r) < k(r), \quad r \in \mathbb{T}_{a+1}. \tag{20}$$

As

$$\begin{aligned} &\max_{r\in\mathbb{T}_{a+1}} H_{\alpha-1}(r+\alpha-2,a) = H_{\alpha-1}(b+\alpha-2,a), \\ &\max_{r\in\mathbb{T}_{a+1}} H_{\alpha-1}(b+\alpha,\sigma(r)) = H_{\alpha-1}(b+\alpha,\sigma(a)), \\ &\max_{r\in\mathbb{T}_{a+1}} H_{\alpha-2}(b+\alpha-1,\sigma(r)) = H_{\alpha-2}(b+\alpha-1,b+1) = 1, \end{aligned}$$

we can deduce that

$$k(r) < \Upsilon, \quad r \in \mathbb{T}_{a+1}. \tag{21}$$

Therefore, according to Lemma 4 and the inequalities (18), (20), (21), we get

$$\begin{aligned} \max_{(t,r)\in\mathbb{T}_a\times\mathbb{T}_{a+1}} G(t,r) &= \max_{r\in\mathbb{T}_{a+1}} \{f(\sigma(r),r), g(r,r)\} \\ &\le \left\{\max_{r\in\mathbb{T}_{a+1}} f(\sigma(r),r),\ \max_{r\in\mathbb{T}_{a+1}} g(r,r)\right\} < \max_{r\in\mathbb{T}_{a+1}} k(r) < \Upsilon, \end{aligned}$$

as desired. □

The next theorem shows the boundedness of Green's function in a limited summation.

Theorem 7. *Let $\delta_1, \delta_2, \delta_3, \delta_4 \ge 0$ and $\delta_1 \le \delta_2$, and let (9) hold. Then, for $G(t,r)$, we have*

$$\sum_{r=a+1}^{b} G(t,r) < \Lambda, \quad \forall\, (t,r) \in \mathbb{T}_a \times \mathbb{T}_{a+1}, \tag{22}$$

where

$$\begin{aligned} \Lambda = \frac{1}{\xi}\Big[&\delta_1\delta_3\, H_{\alpha-1}(b+\alpha-2,a)\, H_{\alpha-1}(b+\alpha-2,a) \\ &+ \delta_1\delta_4\, H_{\alpha-1}(b+\alpha-2,a)\, H_{\alpha-2}(b+\alpha-3,a) + \delta_3(\delta_2-\delta_1)\, H_{\alpha-1}(b+\alpha-2,a) \\ &+ \delta_4(\delta_2-\delta_1)\, H_{\alpha-2}(b+\alpha-3,a)\Big]. \end{aligned}$$

Proof. Consider

$$
\begin{aligned}
\sum_{r=a+1}^{b} G(t,r) &= \sum_{r=a+1}^{t} g(t,r) + \sum_{r=t+1}^{b} f(t,r) \\
&= \sum_{r=a+1}^{b} f(t,r) - \sum_{r=a+1}^{t} H_{\alpha-1}(t+\alpha,\sigma(r)) \\
&= \frac{1}{\xi}\Bigg[\delta_1\delta_3 H_{\alpha-1}(t+\alpha-2,a) \sum_{r=a+1}^{b} H_{\alpha-1}(b+\alpha,\sigma(r)) \\
&\quad + \delta_1\delta_4 H_{\alpha-1}(t+\alpha-2,a) \sum_{r=a+1}^{b} H_{\alpha-2}(b+\alpha-1,\sigma(r)) \\
&\quad + (\delta_2-\delta_1)\delta_3 \sum_{r=a+1}^{b} H_{\alpha-1}(b+\alpha,\sigma(r)) \\
&\quad + (\delta_2-\delta_1)\delta_4 \sum_{r=a+1}^{b} H_{\alpha-2}(b+\alpha-1,\sigma(r))\Bigg] \\
&\qquad - \sum_{r=a+1}^{t} H_{\alpha-1}(t+\alpha,\sigma(r)).
\end{aligned}
$$

Further simplifications lead to

$$
\begin{aligned}
\sum_{r=a+1}^{b} G(t,r) &= \frac{1}{\xi}\Big[(\delta_1\delta_3\, H_{\alpha-1}(t+\alpha-2,a) H_\alpha(b+\alpha-1,a) \\
&\quad + \delta_1\delta_4\, H_{\alpha-1}(t+\alpha-2,a) H_{\alpha-1}(b+\alpha-2,a) \\
&\quad + (\delta_2-\delta_1)\delta_3 H_\alpha(b+\alpha-1,a) + (\delta_2-\delta_1)\delta_4 H_{\alpha-1}(b+\alpha-2,a)\Big] \\
&\quad - H_\alpha(t+\alpha-2,a) \\
&= \frac{1}{\xi}\Big(\delta_1\delta_3\Big(H_{\alpha-1}(t+\alpha-2,a) H_\alpha(b+\alpha-1,a) \\
&\quad - H_{\alpha-1}(b+\alpha-2,a) H_\alpha(t+\alpha-1,a)\Big) \\
&\quad + \delta_1\delta_4\Big(H_{\alpha-1}(t+\alpha-2,a) H_{\alpha-1}(b+\alpha-2,a) \\
&\quad - H_\alpha(t+\alpha-1,a) H_{\alpha-2}(b+\alpha-3,a)\Big) \\
&\quad + (\delta_2-\delta_1)\delta_3\Big(H_\alpha(b+\alpha-1,a) - H_\alpha(t+\alpha-1,a)\Big) \\
&\quad + (\delta_2-\delta_1)\delta_4 H_{\alpha-1}(b,+\alpha-2a)\Big].
\end{aligned}
$$

For $t \in \mathbb{T}_a$, we have $H_\alpha(t,a) \geq 0$ and

$$
\begin{aligned}
&\max_{t\in\mathbb{T}_a} H_\alpha(b+\alpha-1,a) = H_\alpha(b+\alpha-1,a), \\
&\max_{t\in\mathbb{T}_a} H_{\alpha-1}(b+\alpha-2,a) = H_{\alpha-1}(b+\alpha-2,a),
\end{aligned}
$$

which together with the last equation give the required (22). The proof is complete. □

Now, we can formulate a Lyapunov-like inequality for the delta BVP (2) in the following theorem.

Theorem 8. *Let $\delta_1, \delta_2, \delta_3, \delta_4 \geq 0$ and $\delta_1 \leq \delta_2$, and let (9) hold. If the solution to the delta BVP (2) is nontrivial, then*

$$\sum_{r=a+1}^{b} |B(r)| > \frac{1}{\Upsilon}. \tag{23}$$

Proof. Let us set $\mathcal{B}$ as a Banach space with the norm:

$$\|f\| = \max_{t\in\mathbb{T}_a} |f(r)|.$$

By considering Theorems 1 and 3, we see that the solution to (3) satisfies

$$f(t) = \sum_{r=a+1}^{b} G(t,r)B(r)f(r),$$

for $t \in \mathbb{T}_a$. Therefore, according to Theorem 6, we have

$$\begin{aligned}
\|f\| = \max_{t\in\mathbb{T}_a} |f(r)| &= \max_{t\in\mathbb{T}_a} \left| \sum_{r=a+1}^{b} G(t,r)B(r)f(r) \right| \\
&\leq \max_{t\in\mathbb{T}_a} \left[\sum_{r=a+1}^{b} G(t,r)|B(r)||f(r)| \right] \\
&\leq \|f\| \max_{t\in\mathbb{T}_a} \left[\sum_{r=a+1}^{b} G(t,r)|B(r)| \right] \\
&\leq \Upsilon \|f\| \sum_{r=a+1}^{b} |B(r)|.
\end{aligned}$$

This leads to

$$\sum_{r=a+1}^{b} |B(r)| > \frac{1}{\Upsilon}.$$

The proof is complete. □

4. Application

The specific class of operator equations, which appears frequently in quantum mechanics, consists of eigenvalues and eigenfunctions. For this reason, this section has been dedicated to examining the lower bound for the eigenvalues related to the delta BVP (3). If $1 < \alpha < 2$ and f is a nontrivial solution to the delta BVP,

$$\begin{cases}
-\left({}_{a}^{\mathrm{RL}}\Delta^{\alpha-1}(\nabla f)\right)(t) + \lambda f(t) = 0, \quad t \in \mathbb{T}_{a+2}, \\
\delta_1 f(a+1) - \delta_2 (\nabla f)(a+1) = 0, \\
\delta_3 f(b) + \delta_4 (\nabla f)(b) = 0,
\end{cases}$$

whereas $f(t) \neq 0$, for all $t \in \mathbb{T}_{a+1}$, we have

$$|\lambda| = \frac{1}{(b-a)\Upsilon}, \tag{24}$$

according to Theorem 8.

5. Concluding Remarks

Throughout this paper, we have considered a kind of fractional falling function and investigated some of its properties. This occurred in Green's function of a BVP of delta Riemann–Liouville fractional type. Lyapunov-type inequalities have been obtained for the delta fractional BVP under the general boundary conditions. The results show that Green's function is non-negative and this leads to an upper bound for its maximum value. To better understand this point, an example shows the estimation of the lower bound for the eigenvalue of the delta BVP.

Author Contributions: Conceptualization, R.P.A.; Data curation, S.A.M. and N.C.; Formal analysis, E.A.-S.; Funding acquisition, E.A.-S.; Investigation, P.O.M., R.P.A. and E.A.-S.; Methodology, M.A.Y. and N.C.; Project administration, P.O.M.; Software, S.A.M.; Supervision, R.P.A.; Visualization, M.A.Y.; Writing—original draft, P.O.M. and N.C.; Writing—review and editing, M.A.Y. and S.A.M. All of the authors read and approved the final manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Acknowledgments: Researchers Supporting Project number (RSP2024R153), King Saud University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Goodrich, C.S.; Peterson, A.C. *Discrete Fractional Calculus*; Springer: New York, NY, USA, 2015.
2. Wu, G.; Baleanu, D. Discrete chaos in fractional delayed logistic maps. *Nonlinear Dyn.* **2015**, *80*, 1697–1703. [CrossRef]
3. Gholami, Y.; Ghanbari, K. Coupled systems of fractional ∇-difference boundary value problems. *Differ. Equ. Appl.* **2016**, *8*, 459–470. [CrossRef]
4. Atici, F.M.; Eloe, P.W. Initial Value Problems in Discrete Fractional Calculus. *Proc. Am. Math. Soc.* **2009**, *137*, 981–989. [CrossRef]
5. Baleanu, D.; Wu, G.C.; Bai, Y.R.; Chen, F.L. Stability analysis of Caputo-like discrete fractional systems. *Commun. Nonlinear Sci. Numer. Simul.* **2017**, *48*, 520–530. [CrossRef]
6. Mozyrska, D.; Torres, D.F.M.; Wyrwas, M. Solutions of systems with the Caputo-Fabrizio fractional delta derivative on time scales. *Nonlinear Anal. Hybrid Syst.* **2019**, *32*, 168–176. [CrossRef]
7. Goodrich, C.S. On discrete sequential fractional boundary value problems. *J. Math. Anal. Appl.* **2012**, *385*, 111–124. [CrossRef]
8. Wang, Z.; Shiri, B.; Baleanu, D. Discrete fractional watermark technique. *Front. Inf. Technol. Electron. Eng.* **2020**, *21*, 880–883. [CrossRef]
9. Ahrendt, K.; Castle, L.; Holm, M.; Yochman, K. Laplace transforms for the nabla-difference operator and a fractional variation of parameters formula. *Commun. Appl. Anal.* **2012**, *16*, 317–347.
10. Wang, M.; Jia, B.; Chen, C.; Zhu, X.; Du, F. Discrete fractional Bihari inequality and uniqueness theorem of solutions of nabla fractional difference equations with non-Lipschitz nonlinearities. *Appl. Math. Comput.* **2020**, *367*, 125118. [CrossRef]
11. Almusawa, M.Y.; Mohammed, P.O. Approximation of sequential fractional systems of Liouville–Caputo type by discrete delta difference operators. *Chaos Solitons Fract.* **2023**, *176*, 114098. [CrossRef]
12. Baleanu, D.; Mohammed, P.O.; Srivastava, H.M.; Al-Sarairah, E.; Hamed, Y.S. On convexity analysis for discrete delta Riemann–Liouville fractional differences analytically and numerically. *J. Inequalities Appl.* **2023**, *2023*, 4. [CrossRef]
13. Chen, C.R.; Bohner M.; Jia, B.G. Ulam-Hyers stability of Caputo fractional difference equations. *Math. Methods Appl. Sci.* **2019**, *42*, 7461–7470. [CrossRef]
14. Brackins, A. Boundary Value Problems of Nabla Fractional Difference Equations. Ph.D. Thesis, The University of Nebraska–Lincoln, Lincoln, NE, USA, 2014.
15. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier B.V.: Amsterdam, The Netherlands, 2006.
16. Srivastava, H.M. Fractional-order derivatives and integrals: Introductory overview and recent developments. *Kyungpook Math. J.* **2020**, *60*, 73–116.
17. Srivastava, H.M. Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations. *J. Nonlinear Convex Anal.* **2021** *22*, 1501–1520.
18. Cabada, A.; Dimitrov, N.D.; Jonnalagadda, J.M. Non-trivial solutions of non-autonomous nabla fractional difference boundary value problems. *Symmetry* **2021**, *13*, 1101. [CrossRef]
19. Chen, C.; Bohner, M.; Jia, B. Existence and uniqueness of solutions for nonlinear Caputo fractional difference equations. *Turk. J. Math.* **2020**, *44*, 857–869. [CrossRef]

20. Jonnalagadda, J.M. An ordering on Green's function and a Lyapunov-type inequality for a family of nabla fractional boundary value problems. *Fract. Differ. Calc.* **2019**, *9*, 109–124. [CrossRef]
21. Ahrendt, K.; Kissler, C. Cameron Green's function for higher-order boundary value problems involving a nabla Caputo fractional operator. *J. Differ. Equ. Appl.* **2019**, *25*, 788–800. [CrossRef]
22. Atici, F.M.; Eloe, P.W. Linear forward fractional difference equations. *Commun. Appl. Anal.* **2015**, *19*, 31–42.
23. Cabada, A.; Dimitrov, N.; Jagan Mohan, J. Green's functions for fractional difference equations with Dirichlet boundary conditions. *Chaos Solitons Fract.* **2021**, *153*, 111455. [CrossRef]
24. Cabada, A.; Dimitrov, N.; Jagan Mohan, J. Green's functions and existence of solutions of nonlinear fractional implicit difference equations with Dirichlet boundary conditions. *Opusc. Math.* **2024**, *44*, 167–195. [CrossRef]
25. Mohammed, P.O.; Srivastava, H.M.; Muhammad, R.S.; Al-Sarairah, E.; Chorfi, N.; Baleanu, D. On Existence of Certain Delta Fractional Difference Models. *J. King Saud Univ. Sci. accepted*.
26. Guirao, J.L.G.; Mohammed, P.O.; Srivastava, H.M.; Baleanu, D.; Abualrub, M.S. A relationships between the discrete Riemann–Liouville and Liouville–Caputo fractional differences and their associated convexity results. *AIMS Math.* **2022**, *7*, 18127–18141. [CrossRef]
27. Abdeljawad, T. Different type kernel *h*–fractional differences and their fractional *h*–sums. *Chaos Solitons Fract.* **2018**, *116*, 146–156. [CrossRef]
28. Ikram, A. Lyapunov inequalities for nabla Caputo boundary value problems. *J. Differ. Equ. Appl.* **2019**, *25*, 757–775. [CrossRef]