symmetry

Article

The Simultaneous Confidence Interval for the Ratios of the Coefficients of Variation of Multiple Inverse Gaussian Distributions and Its Application to $PM_{2.5}$ Data

Wasana Chankham, Sa-Aat Niwitpong * and Suparat Niwitpong

Department of Applied Statistics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand; s6104051910049@email.kmutnb.ac.th (W.C.); suparat.n@sci.kmutnb.ac.th (S.N.)
* Correspondence: sa-aat.n@sci.kmutnb.ac.th

Abstract: Due to slash/burn agricultural activity and frequent forest fires, $PM_{2.5}$ has become a significant air pollution problem in Thailand, especially in the north and north east regions. Since its dispersion differs both spatially and temporally, estimating $PM_{2.5}$ concentrations discretely by area, for which the inverse Gaussian distribution is suitable, can provide valuable information. Herein, we provide derivations of the simultaneous confidence interval for the ratios of the coefficients of variation of multiple inverse Gaussian distributions using the generalized confidence interval, the Bayesian interval based on the Jeffreys' rule prior, the fiducial interval, and the method of variance estimates recovery. The efficacies of these methods were compared by considering the coverage probability and average length obtained from simulation results of daily $PM_{2.5}$ datasets. The findings indicate that in most instances, the fiducial method with the highest posterior density demonstrated a superior performance. However, in certain scenarios, the Bayesian approach using the Jeffreys' rule prior for the highest posterior density yielded favorable results.

Keywords: generalized confidence interval; Bayesian; fiducial; method of variance estimates recovery; Jeffreys' rule prior

Citation: Chankham, W.; Niwitpong, S.-A.; Niwitpong, S. The Simultaneous Confidence Interval for the Ratios of the Coefficients of Variation of Multiple Inverse Gaussian Distributions and Its Application to $PM_{2.5}$ Data. *Symmetry* **2024**, *16*, 331. https://doi.org/10.3390/sym16030331

Academic Editors: Pedro José Fernández de Córdoba Castellá, Juan Carlos Castro-Palacio, Shufei Wu and Miguel Enrique Iglesias Martínez

Received: 31 January 2024
Revised: 29 February 2024
Accepted: 5 March 2024
Published: 8 March 2024

1. Introduction

The inverse Gaussian (IG) distribution (also known as the Wald distribution) is a probability distribution with widespread applications across diverse disciplines. It is characterized by asymmetry and versatility in modeling complex real-world phenomena. Notably, the IG distribution exhibits a skewed nature, featuring a protracted right tail, rendering it particularly suitable for scenarios where events follow a pattern of frequent occurrence followed by a gradual decline. A significant theoretical underpinning of the IG distribution lies in its association with Brownian motion. Chhikara and Folk [1] proposed its application to lifetime modeling, and it has been utilized in various fields such as biology (Hsu et al. [2], Jerves-cobo et al. [3]), pharmacokinetics (Weiss [4]), cardiology (Chaubey [5]), demography (Ewbanks [6]), and finance (Balakrishna [7], Punzo [8]). In addition, it has been applied to particulate matter (PM) data conforming to an IG distribution. For example, Karaca et al. [9] investigated the cyclic patterns in the monthly average concentrations of PM_{10} (PM < 10 µm) and $PM_{2.5}$ (PM < 2.5 µm). Feng et al. [10] investigated the association between daily $PM_{2.5}$ levels and the risk of illness in Beijing by utilizing a generalized additive model. Gavriil et al. [11] examined probability distribution functions applied to PM_{10} and $PM_{2.5}$ concentration data gathered over two years at a central location in Athens; based on goodness-of-fit measures, they identified the most suitable probability density functions as Pearson types VI and V, IG, and lognormal. Confidence intervals (CIs) for functions of the coefficient of variation (CV) of an IG distribution have been proposed. Hsieh [12] analyzed inferences on the CV of an IG distribution by using likelihood ratio testing. Gupta and Akman [13] estimated the square of the CV of a weighted IG distribution. Chaubey et al. [14] investigated the properties of variance stabilizing and symmetrizing transformations for the CV of an IG population. Wasana et al. [15] determined the CIs for the CV of an IG distribution by employing the generalized CI (GCI), adjusted GCI, bootstrap percentile, fiducial CI (FCI), and highest posterior density (HPD) FCI methods.

The simultaneous CI (SCI) is a statistical tool used to estimate the CIs for multiple instances of a distribution function simultaneously to achieve a more comprehensive understanding of data variability. Researchers often need to analyze several parameters simultaneously in various fields, including science, medicine, and economics. For instance, Hannig et al. [16] utilized the notion of fiducial generalized pivotal quantities (GPQs) to provide simultaneous fiducial GCIs for the mean ratios of lognormal distributions. Tian et al. [17] determined the SCI for differences in the medians of multiple independent lognormal distributions by employing the parametric bootstrap, normal approximation, the method of variance estimates recovery (MOVER), and GCI approaches. Abdel-Karim [18] suggested the MOVER method for constructing the SCI for the ratios of the means of multiple lognormal distributions. Yosboonruang et al. [19] provided an SCI for all pairwise differences among the CVs of delta-lognormal distributions by employing the fiducial GCI, Bayesian, and MOVER methods. La-ongkaew et al. [20] constructed the SCI for differences in the means of several Weibull distributions by utilizing the GCI, MOVER, and Bayesian approaches. Kaewprasert et al. [21] calculated the SCI for the mean ratios of multiple zero-inflated gamma populations based on MOVER, fiducial GCI, and Bayesian and HPD interval methods with either the Jeffreys' rule or uniform prior. Zhang [22] investigated the SCI for pairwise comparisons of the means of IG distributions by utilizing fiducial GPQs for the vector parameters.

SCIs have frequently been used to estimate differences in the parameters of various distributions, including lognormal, delta-lognormal, Weibull, delta-gamma, and IG distributions. Moreover, since the SCI for the ratios of the CVs of multiple IG distributions, which is important to measure non-unit data with diverse clusters, has not previously been reported, our aim was to fill this research gap. Herein, we provide methodology involving the GCI, Bayesian, fiducial, and MOVER methods to this end.

2. Methods

For p populations of observations, let $Y_{i1}, ..., Y_{in_i}, i = 1, ..., p$ be random samples from an IG distribution with mean μ_i and variance $\frac{\mu_i^3}{\lambda_i}$. The probability density function is given by

$$f(y_{ij}, \mu_i, \lambda_i) = \left(\frac{\lambda_i}{2\pi y_{ij}^2}\right)^{\frac{1}{2}} exp\left\{-\frac{\lambda_i(y_{ij} - \mu_i)^2}{2\mu_i^2 y_{ij}}\right\}, y_{ij} > 0, \mu_i > 0, \lambda_i > 0. \quad (1)$$

Moreover, the respective maximum likelihood estimators (MLEs) for μ_i and λ_i representing the mean and shape parameters of an IG distribution can be determined as follows:

$$\hat{\mu}_i = \bar{y}_i, \quad \hat{\lambda}_i^{-1} = \frac{1}{n_i}\sum_{j=1}^{n_i}(y_{ij}^{-1} - \bar{y}_i^{-1}). \quad (2)$$

Equation (2) can be rewritten as

$$\bar{Y}_i \sim IG(\mu_i, n_i\lambda_i), \quad n_i\lambda_i\hat{\lambda}^{-1} \sim \chi^2_{n_i-1}, \quad i = 1, ..., p, \quad (3)$$

where $\chi^2_{n_i-1}$ denotes a Chi-square distribution with $n_i - 1$ degrees of freedom and $\hat{\mu}_i$ and $1/\hat{\lambda}_i$ represent comprehensively sufficient and independent statistics.

The CV (a measure of relative variability) is the ratio of the standard deviation to the mean. For an multiple IG distributions with parameters μ and λ, φ denoting the CV can be calculated as

$$\varphi_i = \sqrt{\frac{\mu_i}{\lambda_i}}. \tag{4}$$

The aim of the present study is to construct the SCI for the ratios of the CVs of multiple IG populations as follows:

$$\varphi_{il} = \frac{\varphi_i}{\varphi_l} = \sqrt{\frac{\mu_i}{\lambda_i}} / \sqrt{\frac{\mu_l}{\lambda_l}}, \tag{5}$$

where φ_{il} denotes the ratios of the CVs for $i, l = 1, ..., p$ and $i \neq l$.

By substituting μ_i and λ_i in Equations (2) and (3) with their respective MLEs, one can establish the SCI for the ratios of the CVs of multiple IG populations as follows:

$$\hat{\varphi}_{il} = \frac{\hat{\varphi}_i}{\hat{\varphi}_l} = \sqrt{\frac{\hat{\mu}_i}{\hat{\lambda}_i}} / \sqrt{\frac{\hat{\mu}_l}{\hat{\lambda}_l}}, \tag{6}$$

where $i, l = 1, ..., p$ and $i \neq l$.

2.1. The GCI Approach

Weeranhandi [23] was the pioneer who introduced the GCI, a specific category of the GPQ. Let $Y_i = (Y_{i1}, ...Y_{in_i}), i = 1, ..., p$ be a random sample from an IG distribution with parameters (μ_i, λ_i) across p independent samples and assume that observations $y_i = (y_{i1}, ..., y_{in_i}), i = 1, ..., p$. The corresponding GPQ exists if it satisfies the following two requirements:

1. The distribution conditioned on each y_i is parameter-free.
2. The observed values of $R(Y_i, y_i, \mu_i, \lambda_i)$ comprise the parameter of interest.

Using the MLEs of μ_i and λ_i in Equations (2) and (3) and in accordance with Ye et al. [24], the respective GPQs for μ_i and λ_i become

$$R_{\lambda_i} = \frac{n_i \lambda_i V_i}{n_i v_i} \sim \frac{\chi^2_{n_i-1}}{n_i v_i}, i = 1, ..., p, \tag{7}$$

and

$$R_{\mu_i} = \frac{\bar{y}_i}{|1 + \frac{\sqrt{n_i \lambda_i}(\bar{y}_i - \mu_i)}{\mu_i \sqrt{\bar{y}_i}} \sqrt{\frac{\bar{y}_i}{n_i R_{\lambda_i}}}|} \overset{d}{\sim} \frac{\bar{y}_i}{\left|1 + Z_i \sqrt{\frac{\bar{y}_i}{n_i R_{\lambda_i}}}\right|}, \tag{8}$$

where $\bar{y}_i$ are the observed values of $\bar{Y}_i$ and $\overset{d}{\sim}$ denotes the approximation of the normal distribution $Z_i \sim N(0, 1)$ according to Theorem 2.1 in Chhikara and Folks [25]. Hence, the GPQ for the ratio of two independent CVs can be written as

$$R_{\varphi_{il}} = \frac{R_{\varphi_i}}{R\varphi_l} = \sqrt{\frac{R_{\mu_i}}{R\lambda_i}} / \sqrt{\frac{R_{\mu_l}}{R\lambda_l}}, \tag{9}$$

Therefore, the $100(1-\gamma)\%$ two-sided SCI for φ_{il} based on the GCI approach can be written as $L_{il} \leq \varphi_{il} \leq U_{il}$, where L_{il} and U_{il} are the $\gamma/2$ th and $(1-\gamma/2)$th quantiles of $R_{\varphi_{il}}$, respectively, leading to

$$SCI_{GCI} = \left[R_{\varphi_{il}}(\gamma/2), R_{\varphi_{il}}(1-\gamma/2)\right]. \tag{10}$$

Algorithm 1 details the process of calculating the SCI using the GCI method. Performing 2500 iterations is essential for validating the accuracy of the code and ensuring its stability across different levels of functionality.

Algorithm 1: The GCI method
1. Compute $\hat{\mu}_i$ and $\hat{\lambda}_i$ for a given sample from an IG distribution.
2. Generate $\chi^2_{n_i-1}$ and Z_i from Chi-square and standard normal distributions, respectively.
3. Compute R_{λ_i} and R_{μ_i} using Equations (7) and (8), respectively.
4. Calculate the $R_{\varphi_{il}}$ from Equation (9).
5. Repeat Steps 2–4 2500 times.
6. Complete $R_{\varphi_{ij}}(\gamma/2)$ and $R_{\varphi_{ij}}(1-\gamma/2)$.

2.2. The Bayesian CI (BCI) Approach

Bayesian inference is the process of updating prior beliefs based on new evidence to obtain a posterior probability. For random samples $Y_i, i = 1, ..., p$ from $IG(\mu_i, \lambda_i)$, the joint likelihood function can be written as

$$L(\mu_i, \lambda_i|Y_{ij}) \propto \left(\frac{\lambda_i}{2\pi}\right)^{\frac{n_i}{2}} \prod_{i=1}^{k} Y_{ij}^{\frac{-3}{2}} exp\left(-\lambda_i \sum_{i=1}^{k} \frac{(Y_{ij}-\mu_i)^2}{2\mu_i^2 Y_{ij}}\right). \quad (11)$$

Using Bayes' theorem to estimate the posterior distribution, we obtain

$$\pi(\mu_{i,\lambda_i|Y_{ij}}) \propto L(\mu_{i,\lambda_i|Y_{ij}}) \times \pi(\mu_i) \times \pi(\lambda_i), \quad (12)$$

where $\pi(\mu_i)$ and $\pi(\lambda_i)$ are the prior distributions for μ_i and λ_i, respectively. Through the utilization of the second-order partial derivative of the log-likelihood function concerning the unknown parameters, the Fisher information matrix for said parameters can be formulated as follows:

$$I(\mu_i, \lambda_i) = diag\left(\frac{\lambda_1 n_1}{\mu_1^3} \quad \frac{1}{2\lambda_1^2} \quad .... \quad \frac{\lambda_p n_p}{\mu_p^2} \quad \frac{1}{2\lambda_p^2}\right). \quad (13)$$

The subsequent subsections cover the employment of the Jeffreys' rule prior to construct the SCI and simultaneous HPD intervals. The Bayesian methodology for the IG distribution relies on parameter selection. Instead of using the mean directly, it is more convenient to employ the reciprocal of the mean and consider (δ, λ), where $\delta = \mu^{-1}$ serves for the parametrization. This choice facilitates the derivation of manageable expressions for both the joint and marginal posterior distributions. Utilizing the Jeffreys' rule prior generates proper posteriors when assuming both parameters are unknown. Consequently, this approach enables a flexible comparison with the alternative fiducial approach presented by Amry [26], and eliminates the need for assuming the prior. Although opting for a natural conjugate prior appears to be a viable alternative, this presents challenges in selecting values for its hyperparameters. The choices made in this regard can potentially introduce bias in the inference, thereby favoring the Bayesian perspective over the fiducial one. Using the Jefferys' rule prior, the marginal posterior distributions for both λ_i and δ_i can, respectively, be derived as

$$f(\lambda_i|y_{ij}) \sim Gamma(\frac{n_{ij}}{2}, \beta_i), \quad (14)$$

and

$$f(\delta_i|y_{ij}) = \frac{1}{\Phi(n_i^{\frac{1}{2}} \lambda_i^{\frac{1}{2}} \bar{y}_i^{\frac{-1}{2}})(2n_{ij}\lambda_i \bar{y}_i \pi)^{\frac{1}{2}}} exp\left(-\frac{n_i \lambda_i \bar{y}_i^{-1}}{2}\right) + (\bar{y}_i)^{-1}, \quad (15)$$

where $\beta_i = \frac{1}{2\hat{\mu}_i^2} \sum_{p=1}^{n_i} \left\{\frac{(y_{ij}-\hat{\mu}_i)}{y_{ij}}\right\}$; Φ is the cumulative distribution function for the standard normal distribution; and $\bar{y}_i$ and $\hat{\lambda}_i$ are the MLEs of μ_i and λ_i, respectively, given that all of the observations are considered in Equation (2). In the present work, we assume that both μ_i and λ_i are unknown. Gibbs sampling, which relies on the Monte Carlo Markov Chain (MCMC) method, was used to determine the posterior and fiducial distributions of the parameters [27]. It is commonly used to generate samples from the posterior distribution in Bayesian methodology by sweeping through each variable to sample from its conditional distribution with the remaining variables fixed at their current values. In the Gibbs sampler, convergence of the sampled data is guaranteed using both numerical and graphical summaries. Subsequently, by substituting for μ_i and λ_i, we obtain

$$\varphi_i(BCI) = \sqrt{\frac{\mu_i(BCI)}{\lambda_i(BCI)}}. \tag{16}$$

Therefore, $\varphi_{il}(BCI)$ is given

$$\varphi_{il}(BCI) = \frac{\varphi_i(BCI)}{\varphi_l(BCI)} = \sqrt{\frac{\mu_i(BCI)}{\lambda_i(BCI)}} \Big/ \sqrt{\frac{\mu_l(BCI)}{\lambda_l(BCI)}}. \tag{17}$$

Therefore, the $100(1-\gamma)\%$ SCI and the simultaneous HPD intervals for φ_{il} based on the BCI method are

$$SCI_{BCI} = \left[L_{\varphi_{il}(BCI)}, U_{\varphi_{il}(BCI)}\right], \tag{18}$$

where $L_{il}(BCI)$ and $U_{il}(BCI)$ are the lower and upper bounds of the intervals, respectively. We computed $L_{il}(HPD.BCI)$ and $U_{il}(HPD.BCI)$ using *HPDinterval* in the R software package version 4.2.2 to determine the $100(1-\gamma)\%$ simultaneous HPD intervals for φ_{il}, defined as

$$SCI_{HPD.BCI} = \left[L_{\varphi_{il}(HPD.BCI)}, U_{\varphi_{il}(HPD.BCI)}\right]. \tag{19}$$

The value of φ_{il} can be estimated using the following algorithm.

2.3. The FCI Approach

Fiducial inference was first introduced and studied by Fisher [28]. Under the framework of fiducial inference, parameters are treated as random variables and their distributions (i.e., the fiducial distributions) are produced based on the observed data without assuming prior distributions. Furthermore, according to the fiducial distributions, random samples are generated based on the point and interval estimations of unknown parameters and the MLE. Although challenging, applying the fiducial method to an IG distribution, particularly in conjunction with an MCMC, can be achieved for the parameters of an IG distribution as follows:

$$\mu_i(FCI) \sim IG(\hat{\mu}_i, n_i\hat{\lambda}_i), \tag{20}$$

and

$$\lambda_i(FCI) \sim \left(\frac{\hat{\lambda}_i}{n_i}\right)\chi^2_{n_i-1}, \tag{21}$$

where $\hat{\mu}_i$ and $\hat{\lambda}_i$ are the MLEs of μ_i and λ_i, respectively.

The Gibbs sampler procedure detailed in Algorithm 2 was utilized to sample from the fiducial distribution. Furthermore, a concurrent process for fiducial estimates was carried out by replacing the Bayesian posterior with the fiducial distribution during Step 3 of the Gibbs sampling procedure. After this, the fiducial distribution for $\varphi_i(FCI)$ can be obtained as follows:

$$\varphi_i(FCI) = \sqrt{\frac{\mu_i(FCI)}{\lambda_i(FCI)}}. \tag{22}$$

Following this, the fiducial distribution for $\varphi_{il}(FCI)$ can be defined as

$$\varphi_{il}(FCI) = \frac{\varphi_i(FCI)}{\varphi_l(FCI)} = \sqrt{\frac{\mu_i(FCI)}{\lambda_i(FCI)}} \Big/ \sqrt{\frac{\mu_l(FCI)}{\lambda_l(FCI)}}. \tag{23}$$

Therefore, the $100(1-\gamma)\%$ SCI and simultaneous HPD intervals for φ_{il} based on the FCI method are

$$SCI_{FCI} = \left[L_{\varphi_{il}(FCI)}, U_{\varphi_{il}(FCI)}\right], \tag{24}$$

where $L_{il}(FCI)$ and $U_{il}(FCI)$ are the lower and upper bounds of the intervals, respectively. We computed $L_{il}(HPD.FCI)$ and $U_{il}(HPD.FCI)$ using *HPDinterval* in the R software package to determine the $100(1-\gamma)\%$ simultaneous HPD intervals for φ_{il} using the following relationship:

$$SCI_{HPD.FCI} = \left[L_{\varphi_{il}(HPD.FCI)}, U_{\varphi_{il}(HPD.FCI)}\right]. \tag{25}$$

Algorithm 2: The BCI and HPD.BCI methods

1. Calculate MLEs $\hat{\mu}_{MLE}$ and $\hat{\lambda}_{MLE}$ from the IG distribution and set $\hat{\mu}_{MLE} = \mu_i^0$ and $\hat{\lambda}_{MLE} = \lambda_i^0$ in Equation (2).
2. Generate μ_i^1 and λ_i^1 from their respective posterior distributions given in Equations (14) and (15) with the updated sample observations.
3. Repeat Steps 2 and 3 starting with the current values of μ_i^1 and λ_i^1 for t ($t = 200{,}000$) iterations, where t is the quantity of MCMC replications, and conclude with the results for μ_i^t and λ_i^t.
4. Calculate the desired parameters after burning in 1000 samples.
5. Calculate the 95% SCI using the BCI method in Equation (17).
6. Compute $SCI_{HPD.BCI}$ using *HPDinterval* in the R software package.

2.4. The MOVER Approach

In this section, we briefly describe the concept of the MOVER for constructing confidence intervals. The underlying principle of the MOVER involves initially deriving distinct confidence intervals for two individual parameters, subsequently restoring the variance estimates and finally constructing the confidence interval for the desired function of parameters, such as $\varphi_1 + \varphi_2, \varphi_1/\varphi_2$. This methodology is based on the central limit theorem. Our attention in this paper is specifically directed towards establishing a confidence interval for the parameter related to the ratio function. According to Donner and Zoo [29], the confidence interval for φ_1/φ_2 is formulated as follows:

$$L_{12}(MOVER) = \frac{(\hat{\varphi}_1\hat{\varphi}_2) - \sqrt{(\hat{\varphi}_1\hat{\varphi}_2)^2 - l_1 u_2(2\hat{\varphi}_1 - l_1)(2\hat{\varphi}_2 - u_2)}}{u_2(2\hat{\varphi}_2 - u_2),} \tag{26}$$

and

$$U_{12}(MOVER) = \frac{(\hat{\varphi}_1\hat{\varphi}_2) + \sqrt{(\hat{\varphi}_1\hat{\varphi}_2)^2 - u_1 l_2(2\hat{\varphi}_1 - u_1)(2\hat{\varphi}_2 - l_2)}}{l_2(2\hat{\varphi}_2 - l_2),} \tag{27}$$

where $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are the point parameters and $[l_1, u_1]$ and $[l_2, u_2]$ are the confidence intervals for $\hat{\varphi}_1$ and $\hat{\varphi}_2$. When considering p parameters, the lower and upper bounds of the $100(1-\gamma)\%$ two-sided SCI for φ_{il}, $L_{il}(MOVER)$ and $U_{il}(MOVER)$ can be expressed as

$$L_{il}(MOVER) = \frac{(\hat{\varphi}_i\hat{\varphi}_l) - \sqrt{(\hat{\varphi}_i\hat{\varphi}_l)^2 - l_i u_l(2\hat{\varphi}_i - l_i)(2\hat{\varphi}_l - u_l)}}{u_l(2\hat{\varphi}_l - u_l),} \tag{28}$$

and

$$U_{il}(MOVER) = \frac{(\hat{\varphi}_i \hat{\varphi}_l) + \sqrt{(\hat{\varphi}_i \hat{\varphi}_l)^2 - u_i l_l (2\hat{\varphi}_i - u_i)(2\hat{\varphi}_l - l_l)}}{l_l(2\hat{\varphi}_l - l_l),} \tag{29}$$

for $i, l = 1, ..., p$ and $i \neq l$. The parameters of interest in $\hat{\varphi}_i = \sqrt{\frac{\hat{\mu}_i}{\hat{\lambda}_i}}$ are μ_i and λ_i, for which constructing CIs is achievable. Based on the approach by Gulhar et al. [30], let l_i and u_i be the lower and upper bounds of the CIs of φ_i, respectively, expressed as follows:

$$l_i = \frac{\sqrt{n_i - 1}(\hat{\varphi_i})}{\sqrt{\chi^2_{1-\gamma/2, n_i - 1}}}, \tag{30}$$

and

$$ui = \frac{\sqrt{n_i - 1}(\hat{\varphi_i})}{\sqrt{\chi^2_{\gamma/2, n_i - 1}}}. \tag{31}$$

The $100(1-\gamma)\%$ two-sided MOVER SCI for φ_{il} is

$$SCI_{il}(MOVER) = [L_{il}(MOVER), U_{il}(MOVER)], \tag{32}$$

where $L_{il}(MOVER)$ and $U_{il}(MOVER)$ are defined in Equations (28) and (29), respectively.

Algorithm 3: The MOVER

1. Generate random samples $Y_i, i = 1, 2, ..., p$ with sample size $n_1, n_2, ..., n_p$ from an IG distribution and calculate $\hat{\varphi}_i$
   with sample size $n_1, n_2, ..., n_p$ from an IG distribution and calculate $\hat{\varphi}_i$.
2. Generate $\chi^2_{1-\gamma/2, n_i - 1}$ and $\chi^2_{\gamma/2, n_i - 1}$.
3. Calculate l_i, u_i, l_l, and u_l for $\hat{\varphi}_i$ from Equations (30) and (31).
4. Compute $L_{il}(MOVER)$ and $U_{il}(MOVER)$ by using Equations (28) and (29), and calculate the 95% SCIs for φ_{il}.

2.5. The Simulation Study

We compared the efficacies of the SCI construction approaches via a Monte Carlo simulation study based on 5000 runs. The comparison was made in terms of the coverage probability (CP) and average length (AL). The best-performing method attains a CP equal to or greater than the nominal confidence level of 0.95 together with the shortest AL. In the study, 2500 GPQs were generated for the GCI method and 20,000 iterations with a burn-in of 1000 were utilized for Gibbs sampling in conjunction with the MCMC algorithm for the Bayesian and HPD approaches. In addition, the sample sizes utilized were $n = 30, 50$, or 100; the number of populations (p) was 3 or 5; $\mu_i = 0.5, 1$; and $\lambda_i = 1, 5, 10$.

2.6. Empirical Application of the Approaches to $PM_{2.5}$ Datasets from Northern Thailand

Datasets of the average daily $PM_{2.5}$ concentrations from May to June 2022 in Lampang (N1), Chiang Mai (N2), Mae Hong Son (N3), Chiang Rai (N4), and Nan (N5) in northern Thailand were utilized to assess the effectiveness of the proposed methods in constructing the SCI for the ratios of the CVs of multiple IG distributions, the details for which can be found in Table 1 [31]. As the $PM_{2.5}$ datasets contain positive values, they could be modeled using a lognormal, Cauchy, exponential, Weibull, or IG distribution. Hence, the minimum Akaike information criterion (AIC) and Bayesian information criterion (BIC) were used to identify the best-fitting distribution for these data. The summary statistics for the $PM_{2.5}$ concentration datasets from the five provinces in northern Thailand are reported in Table 2.

Table 1. The daily $PM_{2.5}$ data for May–June 2022 in northern Thailand by province.

Province	Daily $PM_{2.5}$												
	15	15	14	22	24	23	13	8	9	10	9	9	
	12	13	15	14	7	8	9	7	5	13	16	12	
N1	13	20	19	23	18	20	16	12	17	16	7	6	
	6	7	8	10	13	10	10	12	10	10	7	6	
	5	6	5	6	5	7	5	7	9	8	6	6	5
	27	24	20	27	34	30	13	11	12	13	9	12	
	20	18	19	22	15	12	13	12	10	19	22	15	
N2	20	27	24	29	27	27	21	17	22	22	14	12	
	12	11	10	13	17	17	15	18	24	21	18	13	
	12	14	12	11	12	13	12	14	11	12	9	11	12
	17	24	14	25	21	18	13	8	7	6	5	8	
	14	11	16	13	8	6	6	7	4	12	14	8	
N3	12	18	17	20	18	17	15	13	13	11	5	4	
	4	4	6	11	12	11	8	11	12	12	7	5	
	4	4	5	3	4	4	5	5	4	5	3	3	4
	27	16	18	35	57	31	20	12	12	8	8	9	
	12	17	17	19	13	12	13	14	7	9	20	11	
N4	13	18	21	23	28	25	12	12	14	16	12	7	
	8	8	8	9	14	15	12	15	19	12	10	7	
	8	10	8	8	8	11	7	7	11	10	8	8	6
	25	17	22	29	36	30	20	10	12	15	11	10	
	13	15	17	13	10	17	19	16	9	14	21	17	
N5	20	25	28	29	26	28	17	20	27	27	15	12	
	12	14	14	17	18	18	18	18	21	16	11	8	
	10	11	12	11	13	13	12	13	13	14	12	11	6

Table 2. Parameter estimates for the five $PM_{2.5}$ datasets.

Province	n_i	Min	Max	Mean	Variance	CV
N1	61	5	24	11.1148	27.5387	0.4721
N2	61	9	24	16.9672	35.2568	0.3500
N3	61	3	25	9.9016	39.8565	0.6376
N4	61	6	57	14.1803	48.1425	0.4893
N5	61	6	36	16.8525	40.9929	0.3799

3. Results

3.1. The Simulation Study

The results for $p = 3$ and $p = 5$ are provided in Tables 3 and 4, respectively. The CPs for the GCI, FCI, and FCI.HPD methods were above or close to the nominal confidence level of 0.95 under all circumstances, whereas those for MOVER were slightly below it in almost all of them. In most cases, the ALs for FCI and HPD.FCI were shorter than those of the other methods, except when the shape parameter was 5 or 10, for which HPD.BCI provided the shortest ALs. Based on this evidence, we recommend using HPD.FCI and HPD.BCI to construct the SCI for the ratios between the CVs of several IG distributions. Figure 1 displays the CPs of various methods across different sample sizes. It can be observed that the GCI, FCI, and HPD.FCI methods exhibited CPs either above or close to the nominal confidence level. In contrast, the BCI, HPD.BCI, and MOVER methods provided CPs below the nominal confidence level. Figure 2 illustrates the ALs for the various methods across different sample sizes, showing a decrease in the ALs for all methods as the sample size was increased. Consequently, the HPD.FCI method outperformed the others for various parameter shapes displayed in Figures 3 and 4.

Table 3. CPs and ALs for the 95% SCI for the ratios of the CVs of multiple IG distributions in the case of $p = 3$.

n_1, n_2, n_3	μ_1, μ_2, μ_3	$\lambda_1, \lambda_2, \lambda_3$	CPs (ALs)					
			GCI	BCI	$HPD.BCI$	FCI	$HPD.FCI$	$MOVER$
30:30:30	0.5:0.5:0.5	1:1:1	0.9812	0.9180	0.9164	0.9521	**0.9500**	0.9256
			(0.9665)	(0.7615)	(0.7469)	(0.8611)	**(0.8410)**	(0.7825)
		5:5:5	0.9617	0.9437	**0.9504**	0.9536	0.9528	0.9498
			(0.8168)	(0.7626)	**(0.7481)**	(0.7944)	(0.7782)	(0.7835)
		10:10:10	0.9517	0.9517	**0.9550**	0.9503	0.9500	0.9497
			(0.7976)	(0.7639)	**(0.7493)**	(0.7869)	(0.7712)	(0.7849)
		1:5:10	0.9603	0.9277	0.9307	0.9506	**0.9501**	0.9337
			(2.0073)	(1.7257)	(1.6930)	(1.8516)	**(1.8118)**	(1.7736)
	1:1:1	1:1:1	0.9927	0.8970	0.8953	0.9517	**0.9507**	0.9083
			(1.1531)	(0.7662)	(0.7515)	(0.9454)	**(0.9198)**	(0.7875)
		5:5:5	0.9663	0.9320	0.9363	0.9507	**0.9505**	0.9387
			(0.8599)	(0.7652)	(0.7507)	(0.8151)	**(0.7979)**	(0.7864)
		10:10:10	0.9640	0.9517	**0.9526**	0.9540	0.9563	0.9510
			(0.8164)	(0.7609)	**(0.7463)**	(0.7931)	(0.7768)	(0.7818)
		1:5:10	0.9717	0.9150	0.9163	0.9641	**0.9521**	0.9233
			(2.2465)	(1.7354)	(1.7023)	(1.9523)	**(1.9064)**	(1.7834)
30:50:100	0.5:0.5:0.5	1:1:1	0.9793	0.9157	0.9127	0.9503	**0.9501**	0.9207
			(0.7478)	(0.5844)	(0.5733)	(0.6630)	**(0.6493)**	(0.5963)
		5:5:5	0.9580	0.9333	0.9283	0.9501	**0.9504**	0.9400
			(0.6292)	(0.5851)	(0.5739)	(0.6119)	**(0.5997)**	(0.5969)
		10:10:10	0.9633	0.9530	**0.9527**	0.9567	0.9593	0.9563
			(0.6189)	(0.5891)	**(0.5779)**	(0.6093)	(0.5974)	(0.6010)
		1:5:10	0.9760	0.9383	0.9370	0.9590	**0.9570**	0.9430
			(1.6471)	(1.3658)	(1.3384)	(1.4984)	**(1.4665)**	(1.3948)
	1:1:1	1:1:1	0.9923	0.8850	0.8840	0.9521	**0.9501**	0.8903
			(0.8933)	(0.5887)	(0.5775)	(0.7285)	**(0.7120)**	(0.6007)
		5:5:5	0.9667	0.9393	0.9360	0.9523	**0.9512**	0.9423
			(0.6637)	(0.5886)	(0.5774)	(0.6294)	**(0.6169)**	(0.6007)
		10:10:10	0.9633	0.9437	**0.9502**	0.9523	0.9550	0.9483
			(0.6359)	(0.5913)	**(0.5799)**	(0.6186)	(0.6064)	(0.6032)
		1:5:10	0.9860	0.9267	0.9200	0.9560	**0.9500**	0.9307
			(1.8813)	(1.3540)	(1.3270)	(1.5738)	**(1.5386)**	(1.3830)
50:50:50	0.5:0.5:0.5	1:1:1	0.9834	0.9209	0.9181	0.9515	**0.9507**	0.9246
			(0.7187)	(0.5751)	(0.5681)	(0.6472)	**(0.6376)**	(0.5842)
		5:5:5	0.9605	0.9512	0.9503	0.9507	**0.9500**	0.9466
			(0.6104)	(0.5748)	(0.5677)	(0.5947)	**(0.5870)**	(0.5839)
		10:10:10	0.9550	0.9563	**0.9548**	0.9537	0.9510	0.9517
			(0.5939)	(0.5731)	**(0.5660)**	(0.5862)	(0.5786)	(0.5822)
		1:5:10	0.9677	0.9313	0.9257	0.9503	**0.9504**	0.9363
			(1.4696)	(1.2920)	(1.2762)	(1.3776)	**(1.3589)**	(1.3126)
	1:1:1	1:1:1	0.9930	0.8843	0.8843	0.9500	**0.9500**	0.8893
			(0.8478)	(0.5812)	(0.5741)	(0.7146)	**(0.7024)**	(0.5904)
		5:5:5	0.9723	0.9347	0.9370	0.9533	**0.9500**	0.9393
			(0.6375)	(0.5740)	(0.5669)	(0.6072)	**(0.5991)**	(0.5831)
		10:10:10	0.9607	0.9546	**0.9533**	0.9507	0.9507	0.9457
			(0.6163)	(0.5807)	**(0.5737)**	(0.6006)	(0.5928)	(0.5899)
		1:5:10	0.9807	0.9273	0.9177	0.9530	**0.9507**	0.9310
			(1.6256)	(1.2964)	(1.2807)	(1.4504)	**(1.4289)**	(1.3171)

Table 3. *Cont.*

n_1, n_2, n_3	μ_1, μ_2, μ_3	$\lambda_1, \lambda_2, \lambda_3$	CPs(ALs)					
			GCI	*BCI*	*HPD.BCI*	*FCI*	*HPD.FCI*	*MOVER*
100:100:100	0.5:0.5:0.5	1:1:1	0.9873	0.9157	0.9187	0.9520	**0.9509**	0.9173
			(0.4938)	(0.3993)	(0.3964)	(0.4478)	**(0.4441)**	(0.4024)
		5:5:5	0.9630	0.9523	**0.9550**	0.9563	0.9540	0.9533
			(0.4209)	(0.3982)	**(0.3953)**	(0.4099)	(0.4068)	(0.4012)
		10:10:10	0.9553	0.9558	**0.9546**	0.9513	0.9563	0.9480
			(0.4094)	(0.3972)	**(0.3943)**	(0.4041)	(0.4012)	(0.3964)
		1:5:10	0.9643	0.9293	0.9337	0.9516	**0.9510**	0.9307
			(1.0200)	(0.9058)	(0.8992)	(0.9621)	**(0.9546)**	(0.9127)
	1:1:1	1:1:1	0.9933	0.8963	0.8943	0.9577	**0.9570**	0.8983
			(0.5726)	(0.3986)	(0.3957)	(0.4891)	**(0.4843)**	(0.4017)
		5:5:5	0.9743	0.9513	**0.9510**	0.9637	0.9597	0.9537
			(0.4402)	(0.3988)	**(0.3960)**	(0.4200)	(0.4168)	(0.4019)
		10:10:10	0.9597	0.9541	**0.9523**	0.9507	0.9510	0.9467
			(0.4198)	(0.3979)	**(0.3951)**	(0.4097)	(0.4067)	(0.4010)
		1:5:10	0.9783	0.9187	0.9190	0.9561	**0.9531**	0.9213
			(1.1231)	(0.9086)	(0.9022)	(1.0144)	**(1.0057)**	(0.9156)

CPs greater than the nominal confidence level of 0.95 and the shortest ALs are in bold.

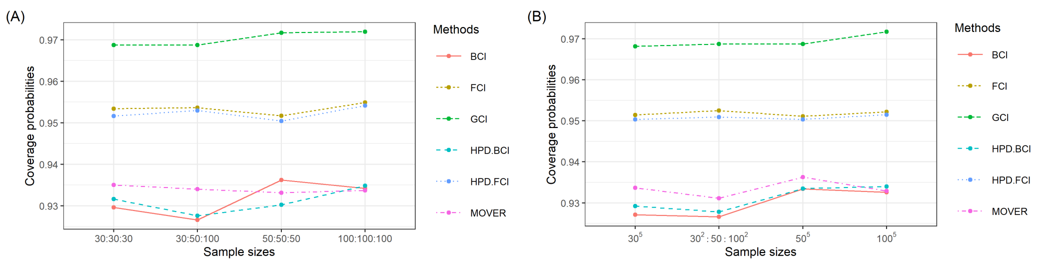


Figure 1. CPs for the 95 % SCI derived using the various methods for various sample sizes in the cases of (**A**) $p = 3$ and (**B**) $p = 5$.

Table 4. CPs and ALs for the 95% SCI for ratios of the CVs of multiple IG distributions in the case of $p = 5$.

n_i^5	μ_i^5	λ_i^5	CPs (ALs)					
			GCI	*BCI*	*HPD.BCI*	*FCI*	*HPD.FCI*	*MOVER*
30^5	0.5^5	1^5	0.9770	0.9155	0.9163	0.9502	**0.9502**	0.9230
			(0.9724)	(0.7639)	(0.7494)	(0.8636)	**(0.8435)**	(0.7850)
		5^5	0.9526	0.9365	0.9351	0.9515	**0.9505**	0.9447
			(0.8160)	(0.7615)	(0.7470)	(0.7935)	**(0.7773)**	(0.7825)
		10^5	0.9535	0.9425	**0.9502**	0.9502	0.9482	0.9426
			(0.7969)	(0.7610)	**(0.7466)**	(0.7842)	(0.7685)	(0.7822)
		1^2:5:10^2	0.9673	0.9317	0.9329	0.9511	**0.9505**	0.9400
			(1.9387)	(1.6644)	(1.6327)	(1.7892)	**(1.7508)**	(1.7103)
	1^5	1^5	0.9915	0.8907	0.8916	0.9529	**0.9511**	0.8992
			(1.1591)	(0.7674)	(0.7528)	(0.9468)	**(0.9212)**	(0.7886)
		5^5	0.9685	0.9393	0.9387	0.9550	**0.9525**	0.9446
			(0.8531)	(0.7594)	(0.7449)	(0.8087)	**(0.7917)**	(0.7804)
		10^5	0.9564	0.9410	**0.9509**	0.9503	0.9489	0.9474
			(0.8103)	(0.7552)	**(0.7408)**	(0.7871)	(0.7711)	(0.7761)
		1^2:5:10^2	0.9781	0.9197	0.9182	0.9501	**0.9506**	0.9281
			(2.1896)	(1.6669)	(1.6349)	(1.8817)	**(1.8373)**	(1.7124)

Table 4. *Cont.*

n_i^5	μ_i^5	λ_i^5	CPs(ALs)					
			GCI	BCI	$HPD.BCI$	FCI	$HPD.FCI$	$MOVER$
30^2:50:100^2	0.5^5	1^5	0.9786	0.9224	0.9192	0.9528	**0.9520**	0.9268
			(0.7519)	(0.5899)	(0.5783)	(0.6693)	**(0.6553)**	(0.6015)
		5^5	0.9584	0.9396	0.9348	0.9514	**0.9509**	0.9444
			(0.6314)	(0.5868)	(0.5756)	(0.6130)	**(0.6010)**	(0.5988)
		10^5	0.9544	0.9460	**0.9512**	0.9524	0.9515	0.9492
			(0.6170)	(0.5884)	**(0.5733)**	(0.6085)	(0.5965)	(0.6006)
		1^2:5:10^2	0.9669	0.9242	0.9266	0.9530	**0.9521**	0.9290
			(2.2674)	(1.8606)	(1.8224)	(2.0485)	**(2.0032)**	(1.9028)
	1^5	1^5	0.9936	0.8938	0.8956	0.9574	**0.9558**	0.8992
			(0.8959)	(0.5923)	(0.5811)	(0.7335)	**(0.7169)**	(0.6047)
		5^5	0.9638	0.9302	0.9262	0.9502	**0.9500**	0.9354
			(0.6613)	(0.5879)	(0.5768)	(0.6280)	**(0.6156)**	(0.5999)
		10^5	0.9546	0.9358	**0.9541**	0.9500	0.9430	0.9388
			(0.6314)	(0.5867)	**(0.5755)**	(0.6133)	(0.6012)	(0.5988)
		1^2:5:10^2	0.9789	0.9207	0.9149	0.9530	**0.9522**	0.9256
			(2.6288)	(1.8612)	(1.8226)	(2.1788)	**(2.1275)**	(1.9034)
50^5	0.5^5	1^5	0.9825	0.9201	0.9188	0.9526	**0.9502**	0.9239
			(0.7187)	(0.5747)	(0.5677)	(0.6467)	**(0.6372)**	(0.5839)
		5^5	0.9592	0.9420	0.9399	0.9504	**0.9503**	0.9465
			(0.6102)	(0.5745)	(0.5675)	(0.5944)	**(0.5867)**	(0.5837)
		10^5	0.9512	0.9486	**0.9502**	0.9435	0.9448	0.9421
			(0.5962)	(0.5752)	**(0.5682)**	(0.5882)	(0.5808)	(0.5843)
		1^2:5:10^2	0.9686	0.9322	0.9319	0.9510	**0.9502**	0.9365
			(1.4379)	(1.2574)	(1.2420)	(1.3447)	**(1.3264)**	(1.2775)
	1^5	1^5	0.9825	0.9201	0.9188	0.9526	**0.9502**	0.9239
			(0.7187)	(0.5747)	(0.5677)	(0.6467)	**(0.6372)**	(0.5839)
		5^5	0.9683	0.9377	0.9390	0.9533	**0.9527**	0.9427
			(0.6407)	(0.5764)	(0.5693)	(0.6098)	**(0.6017)**	(0.5855)
		10^5	0.9624	0.9473	**0.9510**	0.9535	0.9526	0.9505
			(0.6081)	(0.5727)	**(0.5657)**	(0.5927)	(0.5850)	(0.5819)
		1^2:5:10^2	0.9762	0.9194	0.9183	0.9522	**0.9512**	0.9241
			(1.6001)	(1.2607)	(1.2453)	(1.4181)	**(1.3969)**	(1.2807)
100^5	0.5^5	1^5	0.9808	0.9136	0.9144	0.9504	**0.9502**	0.9154
			(0.4936)	(0.3991)	(0.3962)	(0.4479)	**(0.4441)**	(0.4021)
		5^5	0.9601	0.9441	**0.9508**	0.9503	0.9484	0.9454
			(0.4209)	(0.3989)	**(0.3960)**	(0.4107)	(0.4076)	(0.4025)
		10^5	0.9587	0.9518	**0.9514**	0.9548	0.9543	0.9536
			(0.4112)	(0.3986)	**(0.3958)**	(0.4056)	(0.4025)	(0.4017)
		1^2:5:10^2	0.9707	0.9358	0.9353	0.9516	**0.9501**	0.9379
			(0.9934)	(0.8765)	(0.8702)	(0.9339)	**(0.9264)**	(0.8831)
	1^5	1^5	0.9926	0.8935	0.8936	0.9515	**0.9514**	0.8953
			(0.5746)	(0.3996)	(0.3966)	(0.4905)	**(0.4857)**	(0.4026)
		5^5	0.9691	0.9518	**0.9508**	0.9535	0.9545	0.9427
			(0.4412)	(0.3997)	**(0.3968)**	(0.4210)	(0.4177)	(0.4027)
		10^5	0.9618	0.9461	**0.9514**	0.9524	0.9508	0.9479
			(0.4208)	(0.3987)	**(0.3958)**	(0.4103)	(0.4073)	(0.4017)
		1^2:5:10^2	0.9796	0.9238	0.9244	0.9529	**0.9526**	0.9250
			(1.0932)	(0.8754)	(0.8690)	(0.9819)	**(0.9735)**	(0.8821)

CPs greater than the nominal confidence level of 0.95 and the shortest ALs are in bold.

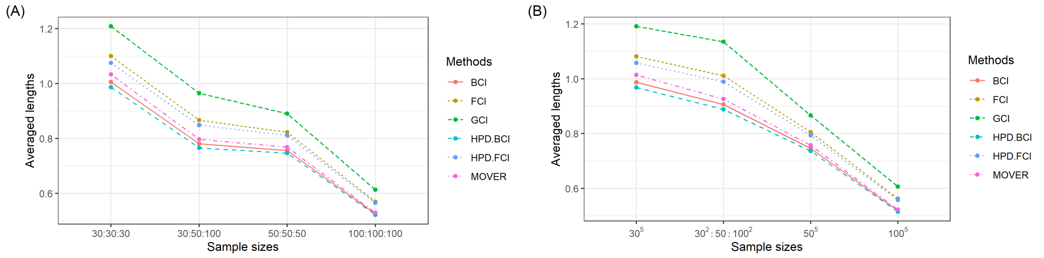


Figure 2. ALs for the 95% SCI derived using the various methods for various sample sizes in the cases of (**A**) $p = 3$ and (**B**) $p = 5$.

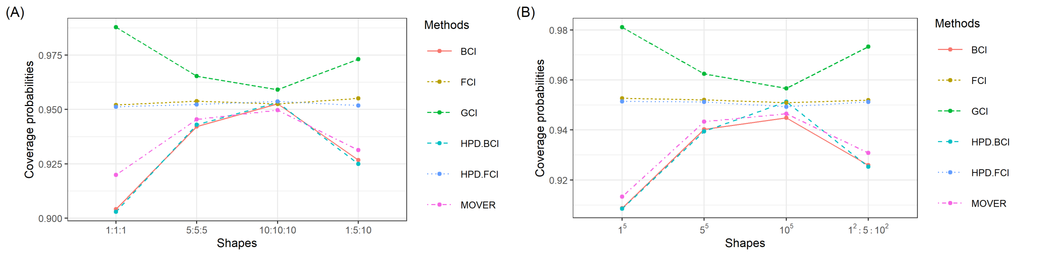


Figure 3. CPs for the 95% SCI derived using the various methods for various parameter shapes in the cases of (**A**) $p = 3$ and (**B**) $p = 5$.

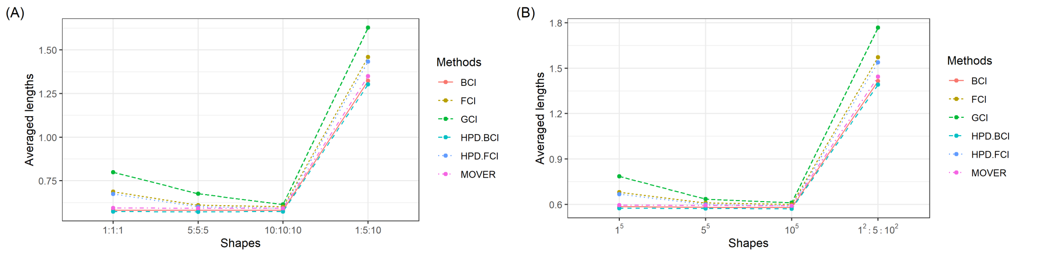


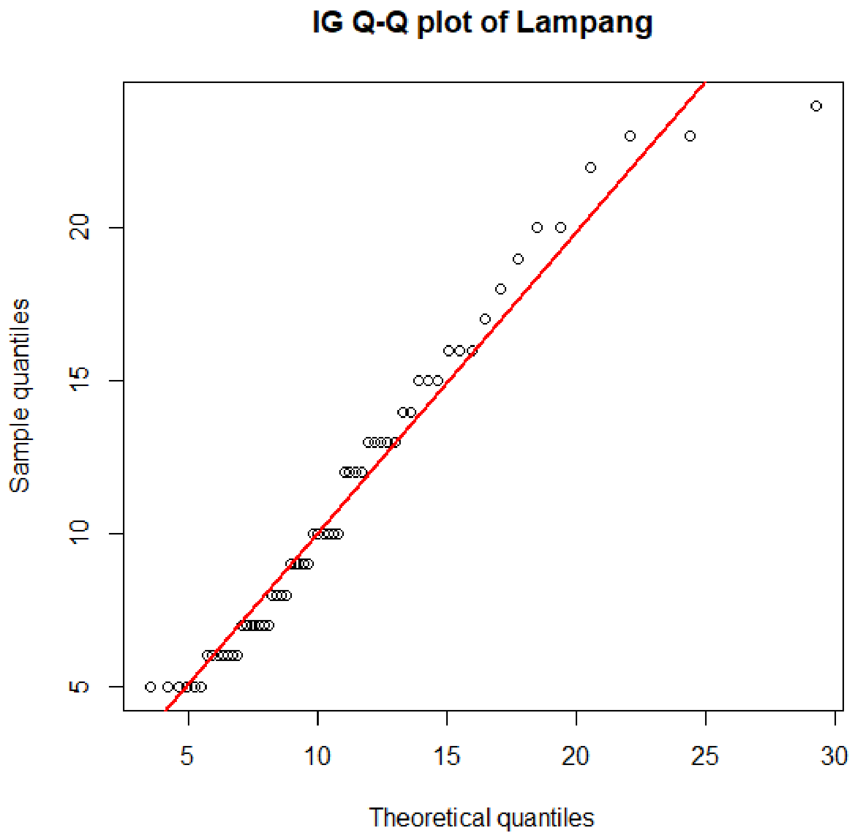
Figure 4. ALs for the 95% SCI using the various methods for various parameter shapes in the cases of (**A**) $p = 3$ and (**B**) $p = 5$.

3.2. Empirical Application of the Methods to $PM_{2.5}$ Datasets from Northern Thailand

The AIC and BIC results in Tables 5 and 6, respectively, indicate that the positive values observed in the $PM_{2.5}$ datasets from the five provinces adhere to the characteristics of an IG distribution. The AIC and BIC values of the inverse Gaussian distribution and the lognormal distribution were not significantly different because both distributions are based on the right-skewed characteristic. In this situation, the model for the inverse Gaussian distribution was considered to be the best due to it providing the lowest AIC and BIC values. Furthermore, the quantile–quantile (Q-Q) plots for the IG distribution confirm this finding in Figure 5.

Table 5. The AIC values for evaluating the distribution for the daily $PM_{2.5}$ data.

Province	Distribution				
	Inverse Gaussian	**Lognormal**	**Cauchy**	**Exponential**	**Weibull**
N1	360.5187	361.6107	398.1535	417.8094	369.3542
N2	383.4496	384.1563	419.3525	469.4165	395.0606
N3	370.8679	372.6417	416.3720	403.7094	375.7245
N4	391.3827	391.526	415.1157	447.5264	415.2512
N5	391.1549	391.3873	418.4176	468.5866	400.4491



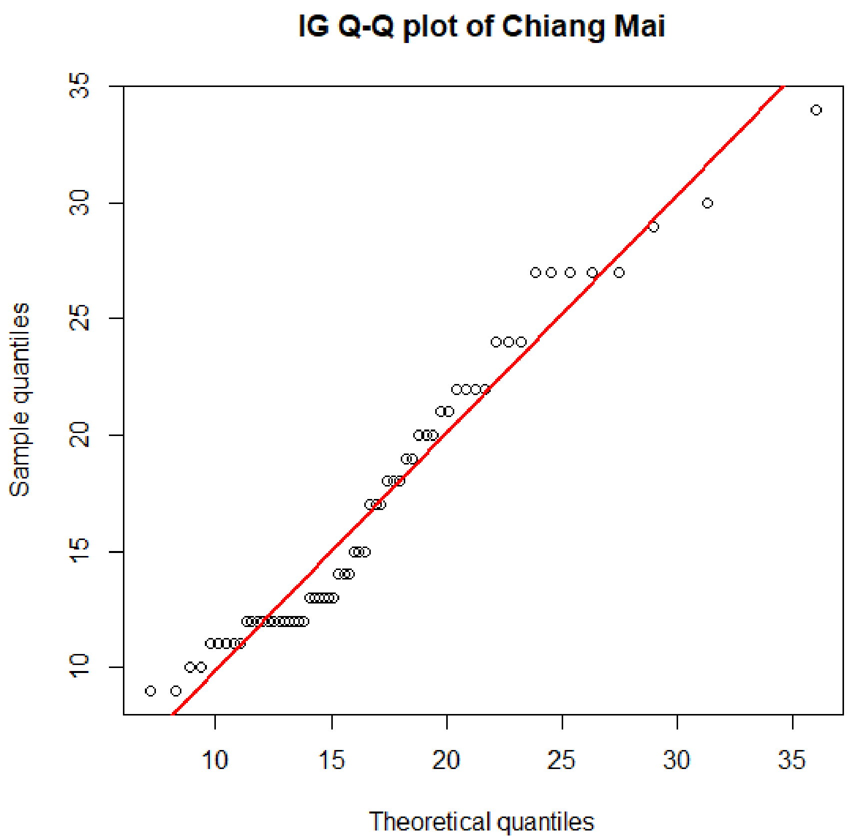
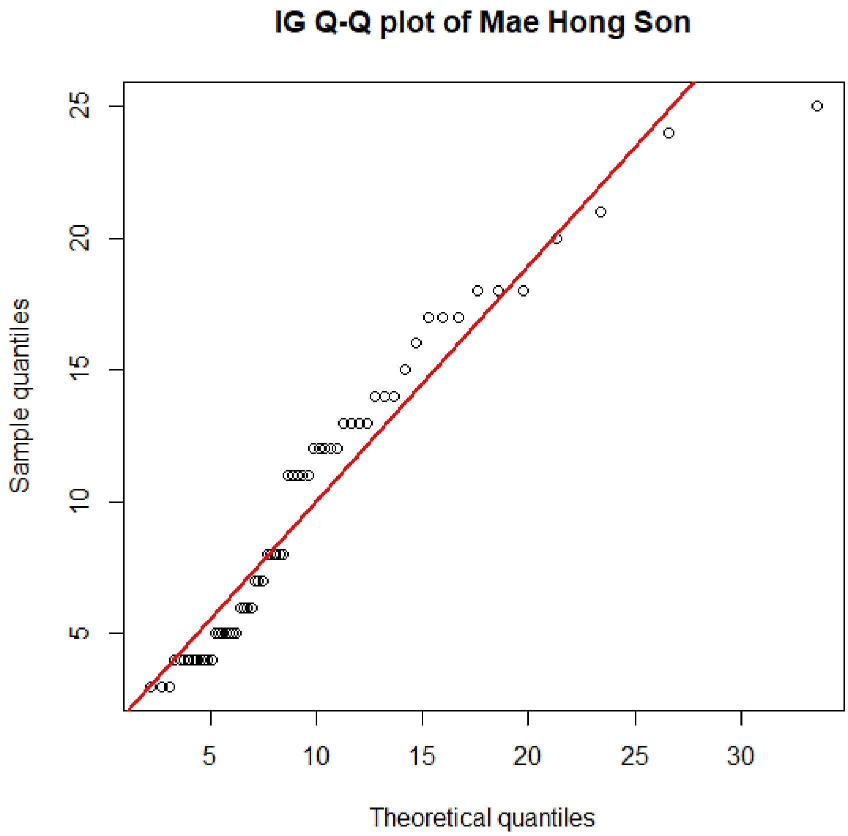


Figure 5. *Cont.*



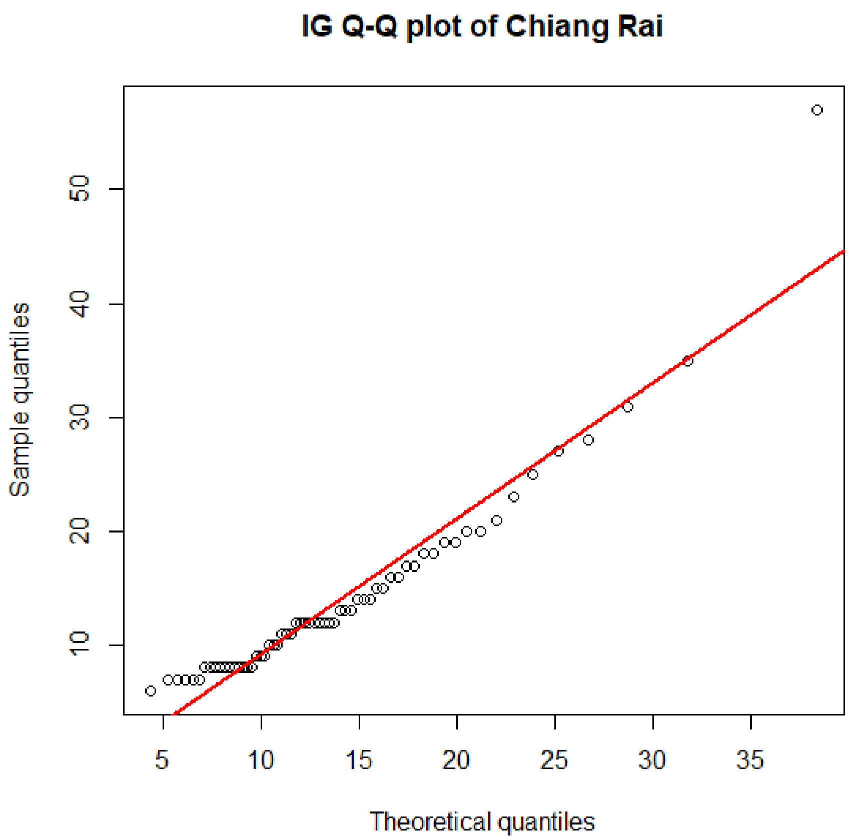


Figure 5. *Cont.*

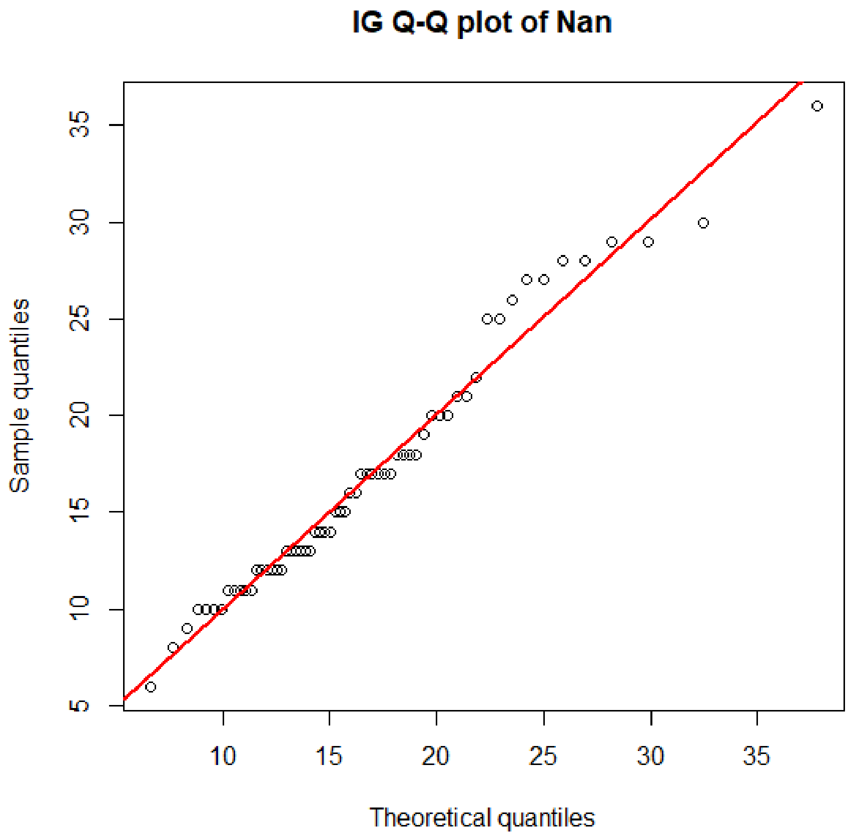


Figure 5. Q-Q plots for fitting the distribution for the $PM_{2.5}$ datasets.

The 95% SCIs for the daily $PM_{2.5}$ datasets from five provinces in northern Thailand are reported in Table 7. The results show that the AL for HPD.BCI was the shortest, which corresponds well with the simulation results. Therefore, it is a good choice for constructing the SCI for the ratios of the CVs of the five $PM_{2.5}$ datasets.

The 95% SCIs for the daily $PM_{2.5}$ dataset from five provinces in northern Thailand in May–June 2022 are reported in Table 7. The results show the AL of the HPD.BCI was the shortest, which corresponds with the simulation results. Therefore, it is a good choice for constructing the SCI for the ratios of the CVs of the $PM_{2.5}$ datasets from the five provinces in northern Thailand.

Table 6. The BIC values for evaluating the distribution for the daily $PM_{2.5}$ data.

Province	Distribution				
	Inverse Gaussian	Lognormal	Cauchy	Exponential	Weibull
N1	364.7405	365.8324	402.3752	419.9202	373.576
N2	387.6714	388.3781	423.5742	471.5274	399.2824
N3	375.0896	376.8635	420.5937	405.8203	379.9462
N4	395.6044	395.7477	419.3375	449.6373	419.473
N5	395.3767	395.6091	422.6394	470.6995	404.6708

Table 7. The ratios of the CV of the daily $PM_{2.5}$ datasets with the nominal 95% SCI.

Provinces	GCI			Bayesian			HPD.Bayesian			FCI			HPD.FCI			MOVER		
	Lower	Upper	Length	Lower	Upper	Length	Lower	Upper	Length	Lower	Upper	Length	Lower	Upper	Length	Lower	Upper	Length
N1/N2	1.0344	1.7718	0.7374	1.0477	1.7314	0.6837	1.0339	1.7101	0.6762	1.0389	1.7479	0.7090	1.0235	1.7284	0.7049	1.0439	1.7437	0.6998
N1/N3	0.5431	0.9937	0.4506	0.5741	0.9534	0.3793	0.5629	0.9387	0.3758	0.5617	0.9681	0.4064	0.5548	0.9575	0.4027	0.5730	0.9570	0.3840
N1/N4	0.7192	1.2710	0.5518	0.7472	1.2426	0.4954	0.7285	1.2181	0.4896	0.7346	1.2608	0.5262	0.7189	1.2410	0.5221	0.7466	1.2471	0.5005
N1/N5	0.9337	1.6365	0.7028	0.9647	1.5999	0.6352	0.9400	1.5693	0.6293	0.9564	1.6256	0.6692	0.9273	1.5863	0.6590	0.9616	1.6061	0.6445
N2/N3	0.4138	0.7229	0.3091	0.4261	0.7083	0.2822	0.4214	0.6996	0.2782	0.4206	0.7202	0.2996	0.4107	0.7062	0.2955	0.4247	0.7094	0.2847
N2/N4	0.5314	0.9443	0.4129	0.5538	0.9202	0.3664	0.5438	0.9054	0.3616	0.5486	0.9317	0.3831	0.5359	0.9127	0.3768	0.5534	0.9244	0.3710
N2/N5	0.7081	1.1962	0.4881	0.7154	1.189	0.4736	0.6987	1.1677	0.4690	0.7106	1.1997	0.4891	0.6972	1.1804	0.4832	0.7127	1.1905	0.4778
N3/N4	0.9636	1.7625	0.7989	1.0110	1.6862	0.6752	0.9839	1.6482	0.6643	0.9905	1.7175	0.7270	0.9722	1.6856	0.7134	1.0082	1.6841	0.6759
N3/N5	1.2584	2.2576	0.9992	1.3064	2.1611	0.8547	1.2777	2.1263	0.8486	1.2804	2.2012	0.9208	1.2420	2.1498	0.9078	1.2985	2.1689	0.8704
N4/N5	0.9729	1.7170	0.7441	0.9977	1.6628	0.6651	0.9750	1.6289	0.6539	0.9884	1.6866	0.6982	0.9539	1.6419	0.6880	0.9965	1.6645	0.6680

4. Discussion

Wasana et al. [32] utilized the FCI and HPD.FCI methods to construct CIs for the ratio of the CVs of two IG distributions. Examination of the efficacies of these methods revealed that HPD.FCI is the most suitable in this scenario. Building on this idea, we developed estimates for the SCI for the ratios of the CVs of multiple IG populations. The results reveal that the CPs and ALs for the 95% SCI for $p = 3$ were similar to those for $p = 5$ across various sample sizes. Notably, for a shape parameter of 10, the HPD.BCI approach performed the best. In contrast, for shape parameter values of 1 or 5, the HPD.FCI approach was the most suitable for all of the situations studied. In addition, the ALs of the approaches decreased with an increasing sample size. The methods were applied in an empirical investigation of the ratios of CVs of $PM_{2.5}$ datasets following IG distributions for five provinces in northern Thailand. The findings aligned with the results of the simulation study, indicating that the HPD.BCI and HPD.FCI methods are the most suitable depending on the scenario. By utilizing our approach for the SCI of the CVs of several $PM_{2.5}$ datasets following IG distributions in a decision-making process, policymakers can enhance the effectiveness and adaptability of measures aimed at mitigating $PM_{2.5}$ pollution, ultimately safeguarding public health and the environment. The proposed approaches could be used in the spatial analysis of $PM_{2.5}$ concentrations to identify areas with high pollution levels. Policymakers can use the information to develop new air pollution prevention and control action plans in key areas.

5. Conclusions

In this research, six approaches (GCI, BCI, HPD.BCI, FCI, HPD.FCI, and MOVER) to constructing the SCI for the ratios of CVs of multiple IG distributions were investigated. The outcomes from a simulation study and an empirical study involving $PM_{2.5}$ datasets in terms of the CP and AL suggest that the HPD.FCI method was the most appropriate in most instances. However, it is noteworthy that HPD.BCI demonstrated effectiveness in certain scenarios involving three or five IG populations. Although the proposed HBD.BCI and HBD.FCI methods have many advantages, they have two limitations. First, the choice of prior distribution in the Bayesian analysis can significantly impact the results. Specifying informative priors may be challenging for areas where prior knowledge is limited or where the environmental conditions are diverse. Second, Bayesian and fiducial methods frequently entail computationally intensive tasks, and the efficiency of these methods can be influenced by factors such as the scale of the data or the complexity of the model, particularly for diverse real-world scenarios. These limitations will be addressed in future studies. Adaptive Bayesian and fiducial methods, which can be adjusted to different environmental conditions by incorporating contextual information, will be developed. Moreover, parallel processing or distributed computing could be carried out to more efficiently handle the computational complexity, along with exploring advancements in Bayesian computation, such as MCMC algorithms, to enhance the speed and scalability of the analysis. The investigation also reveals the limitations of using different priors. Future investigations will also be conducted to refine the choice of prior and to construct the SCI for the ratios of the percentiles of several IG distributions.

Author Contributions: Conceptualization, S.-A.N.; Methodology, W.C., S.-A.N. and S.N.; Software, W.C.; Formal analysis, W.C. and S.N.; Investigation, S.-A.N. and S.N.; Project administration, S.-A.N.; Resources, S.-A.N.; Data curation, W.C.; Writing—original draft, W.C.; Writing—review and editing, S.-A.N. and S.N.; Supervision, S.-A.N. and S.N. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by King Mongkut's University of Technology North Bangkok, contract number: KMUTNB-PHD-63-01.

Data Availability Statement: The $PM_{2.5}$ concentration data were obtained from the Pollution Control Department [31].

Acknowledgments: The authors wish to extend their thanks to King Mongkut's University of Technology, north Bangkok, for supporting their research and offering a space for programming.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

AIC	Akaike information criterion
AL	Average length
BCI	Bayesian confidence interval
BIC	Bayesian information criterion
CI	Confidence interval
CP	Coverage probability
CV	Coefficient of variation
FCI	Fiducial confidence interval
GCI	Generalized confidence interval
GPQ	Generalized pivotal quantity
HPD.BCI	Highest posterior density based on the Bayesian method
HPD.FCI	Highest posterior density based on the fiducial method
IG	Inverse Gaussian
MCMC	Monte Carlo Markov Chain
MLE	Maximum likelihood estimator
MOVER	Method of variance estimates recovery
SCI	Simultaneous confidence interval

References

1. Chhikara, R.S.; Folks, J.L. Optimum test procedures for the mean of first passage time in Brownian motion with positive drift (inverse Gaussian distribution). *Technometrics* **1976**, *18*, 189–193. [CrossRef]
2. Hsu, A.; Ferrage, F.; Palmer, A.G. Analysis of NMR Spin Relaxation Data Using an Inverse Gaussian Distribution Function. *Biophys. J.* **2018**, *115*, 2301–2309. [CrossRef]
3. Jerves-Cobo, R.; Forio, M.A.E.; Lock, K.; Van Butsel, J.; Pauta, G.; Cisneros, F.; Nopens, I.; Goethals, P.L.M. Biological water quality in tropical rivers during dry and rainy seasons: A model-based analysis. *Ecol. Indic.* **2020**, *108*, 15769. [CrossRef]
4. Weiss, M. A note on the role of generalized inverse Gaussian distributions of circulatory transit times in pharmacokinetics *J. Math. Biol.* **1984**, *20*, 95–102. [CrossRef]
5. Chaubey, Y.P. Estimation in inverse Gaussian regression: Comparison of asymptotic and bootstrap distributions. *J. Stat. Plan. Inference* **2002**, *100*, 135–143. [CrossRef]
6. Ewbank, D.C. Mortality differences by APOE genotype estimated from demographic synthesis. *Genet. Epidemiol.* **2002**, *22*, 146–155. [CrossRef]
7. Balakrishna, N.; Rahul, T. Inverse Gaussian Distribution for Modeling Conditional Durations in Finance. *Commun. Stat.-Simul. Comput.* **2002**, *43*, 476–486. [CrossRef]
8. Punzo, A. A new look at the inverse Gaussian distribution with applications to insurance and economic data. *J. Appl. Stat.* **2018**, *46*, 1260–1287. [CrossRef]
9. Karaca, F.; Alagha, O.; Ertürk, F. Statistical characterization of atmospheric PM10 and PM2.5 concentrations at a non-impacted suburban site of Istanbul, Turkey. *Chemosphere* **2005**, *59*, 1183–1190. [CrossRef] [PubMed]
10. Feng, C.; Li, J.; Sun, W.; Zhang, Y.; Wang, Q. Impact of ambient fine particulate matter (PM2.5) exposure on the risk of influenza-like-illness: A time-series analysis in Beijing, China. *Environ. Health* **2016**, *15*, 17. [CrossRef]
11. Gavriil, I.; Grivas, G.; Kassomenos, P.; Chaloulakou, A.; Spyrellis, N. An Application of Theoretical Probability Distributions, to the study of PM10 and PM2.5 time series in Athens, Greece. *Glob. NEST J.* **2006**, *8*, 241–251.
12. Hsieh, H.K. Inferences on the coefficient of variation of an inverse gaussian distribution. *Commun. Stat.-Theory Methods* **1990**, *19*, 1589–1605. [CrossRef]
13. Gupta, R.C.; Akman, O. Estimation of coefficient of variation in a weighted inverse Gaussian model. *Appl. Stoch. Model. Data Anal.* **1996**, *12*, 255–263. [CrossRef]
14. Chaubey, Y.P.; Singh, M.; Sen, D. Symmetrizing and Variance Stabilizing Transformations of Sample Coefficient of Variation from Inverse Gaussian Distribution. *Sankhya B* **2017**, *79*, 217–246. [CrossRef]
15. Chankham, W.; Niwitpong, S.-A.; Niwitpong, S. Measurement of dispersion of PM 2.5 in Thailand using confidence intervals for the coefficient of variation of an inverse Gaussian distribution. *PeerJ* **2022**, *10*, e12988. [CrossRef] [PubMed]
16. Hannig, J.; Iyer, H.; Patterson, P. Fiducial generalized confidence intervals. *J. Am. Stat. Assoc.* **2006**, *101*, 254–269. [CrossRef]

17. Tian, W.; Yang, Y.; Tong, T. Confidence Intervals Based on the Difference of Medians for Independent Log-Normal Distributions. *Mathematics* **2022**, *10*, 2989. [CrossRef]
18. Abdel-Karim, A.H. Construction of Simultaneous Confidence Intervals for Ratios of Means of Lognormal Distributions.*Commun. Stat.-Simul. Comput.* **2014**, *44*, 271–283. [CrossRef]
19. Yosboonruang, N.; Niwitpong, S.-A.; Niwitpong, S. Simultaneous confidence intervals for all pairwise differences between the coefficients of variation of rainfall series in Thailand. *PeerJ* **2021**, *9*, e11651. [CrossRef]
20. La-ongkaew, M.; Niwitpong, S.-A.; Niwitpong, S. Simultaneous Confidence Intervals for All Pairwise Differences between Means of Weibull Distributions. *Symmetry* **2023**, *15*, 2142. [CrossRef]
21. Kaewprasert, T.; Niwitpong, S.-A.; Niwitpong, S. Simultaneous Confidence Intervals for the Ratios of the Means of Zero-Inflated Gamma Distributions and Its Application. *Mathematics* **2022**, *10*, 4724. [CrossRef]
22. Zhang, G. Simultaneous confidence intervals for several inverse Gaussian populations *Stat. Probab. Lett.* **2014**, *92*, 125–131. [CrossRef]
23. Weerahandi, S. Generalized confidence intervals. *J. Am. Stat. Assoc.* **1993**, *88*, 899–905. [CrossRef]
24. Ye, R.D.; Ma, T.F.; Wang, S.G. Inference on the common mean of several inverse Gaussian populations. *Comput. Stat. Data Anal.* **2010**, *54*, 906–915. [CrossRef]
25. Chhikara, R.S.; Folk, J.L. *The Inverse Gaussian Distribution*; Marcel Dekker: New York, NY, USA, 1989.
26. Amry, Z. Bayes Estimator for inverse Gaussian distribution with Jeffrey's Prior. *SCIREA J. Math.* **2021**, *6*, 44–50.
27. Gelfand, A.E.; Smith, A.F.M. Sampling-based approaches to calculating marginal densities. *J. Am. Stat. Assoc.* **1990**, *85*, 398–409. [CrossRef]
28. Fisher, RA. *Statistical Methods and Scientific Inference*; Hafner Publishing Co.: New York, NY, USA, 1973.
29. Donner, A.; Zou, G.Y. Closed-form confidence intervals for functions of the normal mean and standard deviation. *Stat. Methods Med. Res.* **2012**, *21*, 347–359. [CrossRef]
30. Gulhar, M.; Golam Kibria, B.M.; Albatineh, A.N.; Ahmed, N.U. A comparison of some confidence intervals for estimating the population coefficient of variation: A simulation study. *Stat. Oper. Res. Trans.* **2012**, *36*, 45–68.
31. Report on Regional Air Quality and Situation. Available online: http://air4thai.pcd.go.th/webV3/ (accessed on 15 October 2023).
32. Chankham, W.; Niwitpong, S.-A.; Niwitpong, S. Confidence intervals for ratio of coefficients of variation of Inverse Gaussian distribution. In Proceedings of the 2022 International Conference on Big Data, IoT, and Cloud Computing, ICBICC '22, Xicheng, China, 2 December 2022; pp. 1–5.