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Two-Dimensional Quaternion Fourier Transform Method in Probability Modeling

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Abstract: The Fourier transform plays a crucial role in statistics, applied mathematics, and engineering sciences. In this study, we give a definition of the two-dimensional quaternion Fourier transform, which is an extension of the two-dimensional Fourier transform. We present a new convolution theorem including this transformation. We study the characteristic function in the setting of quaternion algebra and obtain the essential properties. Based on this, we seek the expected value, variance, covariance, and their basic relations to the two-dimensional quaternion Fourier transform. We illustrate the results by giving examples to see how the obtained results differ from the classical case.

Keywords: two-dimensional quaternion Fourier transform; quaternion characteristic function; quaternion probability density function; quaternion covariance

1. Introduction

The Fourier transformation is an important mathematical tool that has been widely utilized in many fields of scientific study, including signal analysis and image processing (see, for example, [1–3]). As a natural generalization of the two-dimensional Fourier transform (2DFT), the two-dimensional quaternion Fourier transform (2DQFT) has attracted a significant amount of attention from many scholars in applied and theoretic aspects (see, for example, [4–13]). Various properties of the 2DQFT have been investigated in detail, such as linearity, modulation, the convolution theorem, partial derivatives, energy conservation, and uncertainty principles. These properties are amendments of the corresponding properties of the 2DFT.

On the other hand, the use of the classical Fourier transform in probability modeling is quite widespread. It is related to the characteristic function of any real-valued random variable to compute the moment, variance, covariance, distribution function, etc. In [14], the authors studied the representation of the characteristic function using the vector and the classical Fourier transform and then provided some examples related to the proposed characteristic function. A more general probability theory was investigated in [15] using the one-dimensional Clifford–Fourier transform. Distinct from the one-dimensional quaternion Fourier transform method in probability modeling [16], the two-dimensional quaternion Fourier transform method in quaternion probability modeling is more complicated. This work deals with an application of the two-dimensional quaternion Fourier transform method in quaternion probability modeling. It may be regarded as an extension and continuation of our previous work [16]. To arrive at the results, we first introduce the two-dimensional quaternion Fourier transform (2DQFT) and state some of its useful properties. We explore the application of this considered transformation in probability modeling. In particular, we study the characteristic function in the setting of quaternion algebra and present its relation to the two-dimensional quaternion Fourier transformation. Several properties of the quaternion characteristic function are also investigated in detail. We utilize the results to calculate the moments, variance, and covariance in the context of quaternion

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algebra. We provide several examples to verify how the obtained results differ from the classical case.

The material presented in this paper is organized in the following way. In Section 2, we shortly review the basic knowledge of quaternion algebra needed for the next section. In Section 3, we provide useful properties of the 2DQFT such as the inversion theorem and Parseval's formula. We also establish a new form of the convolution theorem, including the 2DQFT in this part. In Section 4, we study the utility of the 2DQFT in probability modeling. Some conclusions are presented in Section 5.

2. Notations

A quaternions is an expansion of real and complex numbers to higher dimensions. The set of quaternions is denoted by $\mathbb{H}$. A quaternion q can be written in the following form [17]

$$\mathbb{H} = \{q = q^a + \boldsymbol{i}q^b + \boldsymbol{j}q^c + \boldsymbol{k}q^d : q^a, q^b, q^c, q^d \in \mathbb{R}\}, \tag{1}$$

where imaginary numbers $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ satisfy:

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1. \tag{2}$$

This equation will lead to

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k},\ \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i},\ \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}. \tag{3}$$

The scalar part and vector part of quaternion $q = q^a + \boldsymbol{i}q^b + \boldsymbol{j}q^c + \boldsymbol{k}q^d$ are denoted by $S_c(q) = q^a$ and $V_c(q) = \boldsymbol{q} = \boldsymbol{i}q^b + \boldsymbol{j}q^c + \boldsymbol{k}q^d$, respectively. Using Equations (2) and (3), we obtain the quaternion product qp as

$$qp = q^a p^a - \boldsymbol{q} \cdot \boldsymbol{p} + q^a \boldsymbol{p} + p^a \boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{p}, \tag{4}$$

where

$$\boldsymbol{q} \cdot \boldsymbol{p} = q^b p^b + q^c p^c + q^d p^d,$$

and

$$\boldsymbol{q} \times \boldsymbol{p} = \boldsymbol{i}(q^c p^d - q^d p^c) + \boldsymbol{j}(q^d p^b - q^b p^d) + \boldsymbol{k}(q^b p^c - q^c p^d). \tag{5}$$

Similar to the complex case, the conjugate of quaternion q is defined by

$$\overline{q} = q^a - \boldsymbol{i}q^b - \boldsymbol{j}q^c - \boldsymbol{k}q^d, \tag{6}$$

which satisfies

$$\overline{qp} = \overline{p}\overline{q}, \quad \forall p, q \in \mathbb{H}. \tag{7}$$

It is obvious to see that from Equation (7), the quaternion conjugate changes the order of the quaternion product. Due to relation (6), we get the modulus as

$$|q| = \sqrt{q\overline{q}} = \sqrt{(q^a)^2 + (q^b)^2 + (q^c)^2 + (q^d)^2}. \tag{8}$$

It should be observed that the scalar part of any quaternion satisfies the symmetry property; that is,

$$S_c(rpq) = S_c(prq) = S_c(qrp), \quad \forall p, q, r \in \mathbb{H}. \tag{9}$$

This formula plays a key role in deriving some properties of the quaternion Fourier transform. Also, one can easily verify that

$$|S_c(q)| \leq |q|, \quad |q^2| = |q|^2, \quad |qp| = |q||p|, \quad \text{and} \quad |q + p| \leq |q| + |p|, \quad \forall p, q \in \mathbb{H}. \tag{10}$$

Due to Equations (6), (8) and (10), we obtain the inverse of a non-zero quaternion $q \in \mathbb{H}$ by

$$q^{-1} = \frac{\overline{q}}{|q^2|}. \tag{11}$$

When $|q| = 1, q$ is called a unit quaternion and when $q^a = 0, q$ is called a pure quaternion. We introduce the inner product for two functions $f, g : \mathbb{R}^2 \longrightarrow \mathbb{H}$ as

$$(f, g)_{L^2(\mathbb{R}^2;\mathbb{H})} = \int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}d\boldsymbol{x}, \quad d\boldsymbol{x} = dx_1 dx_2. \tag{12}$$

For $f = g$, we get

$$\|f\|_{L^2(\mathbb{R}^2;\mathbb{H})} = \left(\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 d\boldsymbol{x}\right)^{\frac{1}{2}}. \tag{13}$$

3. Two-Dimensional Quaternion Fourier Transform with Properties

In this section, we provide the definition of the two-dimensional quaternion Fourier transform (2DQFT). We present useful facts such as Parseval's formula and the convolution theorem, which will be used later on. More details regarding the properties of the 2DQFT have been presented in [6,7,11].

Definition 1. *The two-dimensional quaternion Fourier transform of the function $f \in L^1(\mathbb{R}^2;\mathbb{H}) \cap L^2(\mathbb{R}^2;\mathbb{H})$ is defined by*

$$\mathcal{F}_{\boldsymbol{H}}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 x_1} f(\boldsymbol{x}) e^{\boldsymbol{j}\omega_2 x_2} d\boldsymbol{x}, \tag{14}$$

for $\boldsymbol{x}, \boldsymbol{\omega} \in \mathbb{R}^2$.

Observe that

$$\begin{aligned} |\mathcal{F}_{\boldsymbol{H}}\{f\}(\boldsymbol{\omega})| &= \left|\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 x_1} f(\boldsymbol{x}) e^{\boldsymbol{j}\omega_2 x_2} d\boldsymbol{x}\right| \\ &\leq \int_{\mathbb{R}^2} \left|e^{\boldsymbol{i}\omega_1 x_1} f(\boldsymbol{x}) e^{\boldsymbol{j}\omega_2 x_2}\right| d\boldsymbol{x} \\ &= \|f\|_{L^1(\mathbb{R}^2;\mathbb{H})}, \end{aligned} \tag{15}$$

which shows that $|\mathcal{F}_{\boldsymbol{H}}\{f\}(\boldsymbol{\omega})|$ is bounded by the quaternion constant $\|f\|_{L^1(\mathbb{R}^2;\mathbb{H})}$.

Further, the reconstruction formula related to the 2DQFT is calculated by the following.

Definition 2. *Let $f \in L^1(\mathbb{R}^2;\mathbb{H})$ and $\mathcal{F}_{\boldsymbol{H}}\{f\} \in L^1(\mathbb{R}^2;\mathbb{H})$. The inverse transform of the 2DQFT is obtained through*

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 x_1} \mathcal{F}_{\boldsymbol{H}}\{f\}(\boldsymbol{\omega}) e^{-\boldsymbol{j}\omega_2 x_2} d\boldsymbol{\omega}. \tag{16}$$

Based on Definition 2, one may easily obtain the following theorem.

Theorem 1. *Let the quaternion function $f \in L^2(\mathbb{R}^2;\mathbb{H})$. One obtains*

$$\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 d\boldsymbol{x} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |\mathcal{F}_{\boldsymbol{H}}\{f\}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega}, \tag{17}$$

which is often called the Plancherel formula for the 2DQFT.

One of the fundamental results of the 2DQFT is the convolution theorem. It is known that the form of the convolution theorem in the 2DQFT is more complicated compared with the convolution theorem of the FT (see [8] for more details). Let us introduce the convolution and correlation definitions and then establish a new version of the convolution theorem associated with the 2DQFT.

Definition 3. *Let two functions $f, g \in L^1(\mathbb{R}^2; \mathbb{H})$. The convolution operator of f and g is described by*

$$(f \star g)(\boldsymbol{x}) = \int_{\mathbb{R}^2} f(\boldsymbol{y}) g(\boldsymbol{x} - \boldsymbol{y}) d\boldsymbol{y}. \tag{18}$$

With Definition 3 above, we obtain the following result.

Theorem 2. *For f, g in $L^2(\mathbb{R}^2; \mathbb{H})$, it holds that*

$$\mathcal{F}_{\boldsymbol{H}}\{f \star g\}(\boldsymbol{\omega}) = \mathcal{F}_{\boldsymbol{H}}\Big\{\mathcal{F}_{\boldsymbol{H}}\{f\}(\boldsymbol{\omega})(g^a + \boldsymbol{j}g^c)\Big\}(\boldsymbol{\omega}) + \mathcal{F}_{\boldsymbol{H}}\Big\{\mathcal{F}_{\boldsymbol{H}}\{f\}(\omega_1, -\omega_2)(\boldsymbol{i}g^b + \boldsymbol{k}g^d)\Big\}(\boldsymbol{\omega}). \tag{19}$$

In particular, we get

$$\mathcal{F}_{\boldsymbol{H}}\{f \star g\}(\boldsymbol{\omega}) = \mathcal{F}_{\boldsymbol{H}}\{\mathcal{F}_{\boldsymbol{H}}\{f\}g\}(\boldsymbol{\omega}), \tag{20}$$

where, in this case, g is a real function.

Proof. According to Definition 3, we have

$$\mathcal{F}_{\boldsymbol{H}}\{f \star g\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 x_1} f(\boldsymbol{y}) g(\boldsymbol{x} - \boldsymbol{y}) e^{\boldsymbol{j}\omega_2 x_2} d\boldsymbol{y} d\boldsymbol{x}. \tag{21}$$

We change the variables to $\boldsymbol{x} - \boldsymbol{y} = \boldsymbol{z}$ and get

$$\begin{aligned}
\mathcal{F}_{\boldsymbol{H}}\{f \star g\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1(y_1+z_1)} f(\boldsymbol{y}) g(\boldsymbol{z}) e^{\boldsymbol{j}\omega_2(y_2+z_2)} d\boldsymbol{y} d\boldsymbol{z} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1} e^{\boldsymbol{i}\omega_1 z_1} f(\boldsymbol{y}) g(\boldsymbol{z}) e^{\boldsymbol{j}\omega_2 y_2} e^{\boldsymbol{j}\omega_2 z_2} d\boldsymbol{y} d\boldsymbol{z} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1} e^{\boldsymbol{i}\omega_1 z_1} f(\boldsymbol{y}) (g^a(\boldsymbol{z}) + \boldsymbol{i}g^b(z_1, z_2) + \boldsymbol{j}g^c(\boldsymbol{z}) + \boldsymbol{k}g^d(z_1, z_2)) \\
&\quad e^{\boldsymbol{j}\omega_2 y_2} e^{\boldsymbol{j}\omega_2 z_2} d\boldsymbol{y} d\boldsymbol{z} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1} e^{\boldsymbol{i}\omega_1 z_1} f(\boldsymbol{y}) e^{\boldsymbol{j}\omega_2 y_2} (g^a(\boldsymbol{z}) + \boldsymbol{j}g^c(\boldsymbol{z})) e^{\boldsymbol{j}\omega_2 z_2} d\boldsymbol{y} d\boldsymbol{z} \\
&\quad + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1} e^{\boldsymbol{i}\omega_1 z_1} f(\boldsymbol{y}) e^{-\boldsymbol{j}\omega_2 y_2} (\boldsymbol{i}g^b(\boldsymbol{z}) + \boldsymbol{k}g^d(\boldsymbol{z})) e^{\boldsymbol{j}\omega_2 z_2} d\boldsymbol{y} d\boldsymbol{z} \\
&= \int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 z_1} \mathcal{F}_{\boldsymbol{H}}\{f\}(\boldsymbol{\omega}) (g^a(\boldsymbol{z}) + \boldsymbol{j}g^c(\boldsymbol{z})) e^{\boldsymbol{j}\omega_2 z_2} d\boldsymbol{z} \\
&\quad + \int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 z_1} \mathcal{F}_{\boldsymbol{H}}\{f\}(\omega_1, -\omega_2) (\boldsymbol{i}g^b(\boldsymbol{z}) + \boldsymbol{k}g^d(\boldsymbol{z})) e^{\boldsymbol{j}\omega_2 z_2} d\boldsymbol{z} \\
&= \mathcal{F}_{\boldsymbol{H}}\Big\{\mathcal{F}_{\boldsymbol{H}}\{f\}(\boldsymbol{\omega})(g^a + \boldsymbol{j}g^c)\Big\}(\boldsymbol{\omega}) + \mathcal{F}_{\boldsymbol{H}}\Big\{\mathcal{F}_{\boldsymbol{H}}\{f\}(\omega_1, -\omega_2)(\boldsymbol{i}g^b + \boldsymbol{k}g^d)\Big\}(\boldsymbol{\omega}),
\end{aligned}$$

which is a new version of convolution theorem for the 2DQFT. □

Definition 4. *The correlation operator related to the 2DQFT for two quaternion functions $f, g \in L^1(\mathbb{R}^2; \mathbb{H})$ is given by the integral*

$$(f \circ g)(\boldsymbol{x}) = \int_{\mathbb{R}^2} f(\boldsymbol{x} + \boldsymbol{y}) \overline{g(\boldsymbol{y})} d\boldsymbol{y}. \tag{22}$$

4. Two-Dimensional Quaternion Fourier Transform in Quaternion Probability

In [14,18], the authors studied the characteristic function in the setting of complex numbers. The authors of [16] presented the quaternion characteristic function using the one-dimensional quaternion Fourier transform. In this part, we study the use of the two-dimensional quaternion Fourier transform in quaternion probability, see Table 1. For this aim, we start with the following definition.

Definition 5. *Let $\boldsymbol{X} = (X_1, X_2)$ be real two random variables. A quaternion-valued function $f_{\boldsymbol{X}}(\boldsymbol{x}) = f^a_{\boldsymbol{X}}(\boldsymbol{x}) + \boldsymbol{i} f^b_{\boldsymbol{X}}(\boldsymbol{x}) + \boldsymbol{j} f^c_{\boldsymbol{X}}(\boldsymbol{x}) + \boldsymbol{k} f^d_{\boldsymbol{X}}(\boldsymbol{x})$ is called the quaternion probability density function of $\boldsymbol{X}$ (compare to [19,20]) if*

$$\int_{\mathbb{R}^2} f^l_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} = 1, \quad f^l_{\boldsymbol{X}}(\boldsymbol{x}) \geq 0, \quad \forall \boldsymbol{x} \in \mathbb{R}^2,\ l = a, b, c, d. \tag{23}$$

In this case, $f^l_{\boldsymbol{X}}(\boldsymbol{x})$ is a real probability density function. The quaternion cumulative distribution function is expressed as (compare to [18])

$$f_{\boldsymbol{X}}(x_1, x_2) = \frac{\partial}{\partial u} \frac{\partial}{\partial v} F_{\boldsymbol{X}}(u, v) \bigg|_{u = x_1, v = x_2}. \tag{24}$$

Here, the quaternion probability P is related to $F_{\boldsymbol{X}}$ of X_1 and X_2 given by

$$F_{\boldsymbol{X}}(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2). \tag{25}$$

Table 1. Comparison of one-dimensional quaternion probability and two-dimensional quaternion probability.

Property	One-Dimensional Quaternion Probability	Two-Dimensional Quaternion Probability
Characteristic Function	$\phi_X(t) = \phi^a_X(t) + \boldsymbol{i}\phi^b_X(t) + \boldsymbol{j}\phi^c_X(t) + \boldsymbol{k}\phi^d_X(t)$	$\phi_{\boldsymbol{X}}(\boldsymbol{t}) = \phi^a_{\boldsymbol{X}}(\boldsymbol{t}) + \boldsymbol{i}\phi^b_{\boldsymbol{X}}(\boldsymbol{t}) + \boldsymbol{j}\phi^c_{\boldsymbol{X}}(-t_1, t_2) + \boldsymbol{k}\phi^d_{\boldsymbol{X}}(-t_1, t_2)$
Moments	$E[X^k] = \frac{d^k}{dt^k}\phi_X(0)(-\boldsymbol{i})^k$	$E[X_1^n X_2^m] = \boldsymbol{i}^{-m} \frac{\partial^{m+n}}{\partial t_1^m \partial t_2^n} \phi_{\boldsymbol{X}}(\boldsymbol{0}) \boldsymbol{j}^{-n}$
Variance, Covariance	$var(X) = \frac{d^2}{dt^2}\phi_X(0)(-\boldsymbol{i})^2 - \left(\frac{d^2}{dt^2}\phi_X(0)(-\boldsymbol{i})^2\right)$	$Cov(X_1, X_2) = \boldsymbol{i} \frac{\partial^2}{\partial t_1 \partial t_2} \phi_{\boldsymbol{X}}(\boldsymbol{0}) \boldsymbol{j} - \left(\boldsymbol{i} \frac{\partial}{\partial t_1} \phi_{\boldsymbol{X}}(\boldsymbol{0}) \frac{\partial}{\partial t_1} \phi_{\boldsymbol{X}}(\boldsymbol{0}) \boldsymbol{j}\right)^2$

Definition 6 (Expected value)**.** *Let $\boldsymbol{X} = (X_1, X_2)$ be any two real random variables with the quaternion probability density function $f_{\boldsymbol{X}}(\boldsymbol{x})$. The expected value of X_1 and X_2 is defined as*

$$E[X_1 X_2] = \int_{\mathbb{R}^2} x_1 x_2 f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x}. \tag{26}$$

The expected value of X_1 is given by

$$E[X_1] = \int_{\mathbb{R}^2} x_1 f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x}, \tag{27}$$

and the expected value of X_2 is defined as

$$E[X_2] = \int_{\mathbb{R}^2} x_2 f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x}. \tag{28}$$

Now, write Equation (26) above as

$$\begin{aligned} E[X_1 X_2] &= \int_{\mathbb{R}^2} x_1 x_2 \big(f^a_{\boldsymbol{X}}(\boldsymbol{x}) + \boldsymbol{i} f^b_{\boldsymbol{X}}(\boldsymbol{x}) + \boldsymbol{j} f^c_{\boldsymbol{X}}(\boldsymbol{x}) + \boldsymbol{k} f^d_{\boldsymbol{X}}(\boldsymbol{x})\big) d\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} x_1 x_2 f^a_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} + \boldsymbol{i} \int_{\mathbb{R}^2} x_1 x_2 f^b_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} + \boldsymbol{j} \int_{\mathbb{R}^2} x_1 x_2 f^c_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \\ &\quad + \boldsymbol{k} \int_{\mathbb{R}^2} x_1 x_2 f^d_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \\ &= E_a[X_1 X_2] + \boldsymbol{i} E_b[X_1 X_2] + \boldsymbol{j} E_c[X_1 X_2] + \boldsymbol{k} E_d[X_1 X_2], \end{aligned} \tag{29}$$

where

$$E_l[X_1X_2] = \int_{\mathbb{R}^2} x_1x_2 f^l_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x}, \quad l = a,b,c,d \tag{30}$$

Let us illustrate the above results in the following example.

Example 1. *Let X_1 and X_2 have the quaternion probability density function in the form of*

$$f(x_1,x_2) = \begin{cases} 4x_1x_2 + 3\boldsymbol{i}x_1^2 + 3\boldsymbol{j}x_2^2 + \boldsymbol{k}(x_1+x_2), & \text{if} \quad 0 < x_1,x_2 < 1 \\ 0, & \text{elsewhere.} \end{cases} \tag{31}$$

Find the expected value of X_1 and X_2.

Solution. It is easy to check that

$$\int_{\mathbb{R}^2} f^l_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x} = 1, \quad l = a,b,c,d. \tag{32}$$

Now, the expected value of X_1 is

$$\begin{aligned} E[X_1] &= \int_0^1\int_0^1 x_1\big(4x_1x_2 + 3\boldsymbol{i}x_1^2 + 3\boldsymbol{j}x_2^2 + \boldsymbol{k}(x_1+x_2)\big)dx_1dx_2 \\ &= \tfrac{2}{3} + \tfrac{3}{4}\boldsymbol{i} + \tfrac{1}{2}\boldsymbol{j} + \tfrac{7}{12}\boldsymbol{k}. \end{aligned} \tag{33}$$

The expected value of X_2 is

$$\begin{aligned} E[X_2] &= \int_0^1\int_0^1 x_2\big(4x_1x_2 + 3\boldsymbol{i}x_1^2 + 3\boldsymbol{j}x_2^2 + \boldsymbol{k}(x_1+x_2)\big)dx_1dx_2 \\ &= \tfrac{2}{3} + \tfrac{1}{2}\boldsymbol{i} + \tfrac{3}{4}\boldsymbol{j} + \tfrac{7}{12}\boldsymbol{k}. \end{aligned} \tag{34}$$

The expected value of X_1X_2 is

$$\begin{aligned} E[X_1X_2] &= \int_0^1\int_0^1 x_1x_2\big(4x_1x_2 + 3\boldsymbol{i}x_1^2 + 3\boldsymbol{j}x_2^2 + \boldsymbol{k}(x_1+x_2)\big)dx_1dx_2 \\ &= \tfrac{4}{9} + \tfrac{3}{8}\boldsymbol{i} + \tfrac{3}{8}\boldsymbol{j} + \tfrac{1}{3}\boldsymbol{k}. \end{aligned} \tag{35}$$

From Equations (33)–(35), it is straightforward to see that

$$E_i[X_1X_2] = E_i[X_1]E_i[X_2], \quad i = a,b,c. \tag{36}$$

The product of $E[X_1]E[X_2]$ is

$$\begin{aligned} E[X_1]E[X_2] &= \left(\tfrac{2}{3} + \tfrac{3}{4}\boldsymbol{i} + \tfrac{1}{2}\boldsymbol{j} + \tfrac{7}{12}\boldsymbol{k}\right)\left(\tfrac{2}{3} + \tfrac{1}{2}\boldsymbol{i} + \tfrac{3}{4}\boldsymbol{j} + \tfrac{7}{12}\boldsymbol{k}\right) \\ &= \tfrac{4}{9} - \tfrac{3}{8} - \tfrac{1}{2} - \tfrac{49}{144} + \left(\tfrac{2}{6} + \tfrac{6}{12} + \tfrac{7}{24} - \tfrac{7}{12}\right)\boldsymbol{i} + \left(\tfrac{2}{3} - \tfrac{21}{48} - \tfrac{2}{6} + \tfrac{7}{24}\right)\boldsymbol{j} \\ &\quad + \left(\tfrac{14}{36} - \tfrac{3}{4} - \tfrac{1}{4} + \tfrac{14}{36}\right)\boldsymbol{k} \\ &= \tfrac{-31}{48} + \tfrac{11}{16}\boldsymbol{i} + \tfrac{11}{16}\boldsymbol{j} + 1\tfrac{13}{144}\boldsymbol{k}. \end{aligned} \tag{37}$$

Definition 7. *Let $\boldsymbol{X} = (X_1,X_2)$ be real random variables with the quaternion probability density function $f_{\boldsymbol{X}}(\boldsymbol{x})$. The characteristic function of X_1 and X_2, $\phi_{\boldsymbol{X}} : \mathbb{R}^2 \longrightarrow \mathbb{H}$, is defined by the formula*

$$\begin{aligned} \phi_{\boldsymbol{X}}(\boldsymbol{t}) &= E[e^{\boldsymbol{i}t_1X_1}e^{\boldsymbol{j}t_2X_2}] \\ &= \int_{\mathbb{R}^2} e^{\boldsymbol{i}t_1x_1} f_{\boldsymbol{X}}(\boldsymbol{x}) e^{\boldsymbol{j}t_2x_2}d\boldsymbol{x}. \end{aligned} \tag{38}$$

Based on Equation (38), the generalization of the characteristic function in higher dimensions is given by the following.

Remark 1. *Assume $f \in L^2(\mathbb{R}^{2n};\mathbb{H})$, $n \in \mathbb{N}$. The definition of the n-dimensional quaternion Fourier transform of the function f is given by*

$$\mathcal{F}_{\boldsymbol{H}}\{f\}(\boldsymbol{u},\boldsymbol{v}) = \int_{\mathbb{R}^{2n}} e^{\boldsymbol{i}\boldsymbol{u}\cdot\boldsymbol{x}} f(\boldsymbol{x},\boldsymbol{y}) e^{\boldsymbol{j}\boldsymbol{v}\cdot\boldsymbol{y}} d\boldsymbol{x}d\boldsymbol{y}$$

for $\boldsymbol{u},\boldsymbol{v},\boldsymbol{x},\boldsymbol{y},\in \mathbb{R}^n$. Moreover, the generalization of the characteristic function of $\boldsymbol{X} = (X_1, X_2, \cdots X_n)$, $\phi_{\boldsymbol{X}} : \mathbb{R}^{2n} \longrightarrow \mathbb{H}$, is given by the formula

$$\phi_{\boldsymbol{X}}(\boldsymbol{u},\boldsymbol{v}) = \int_{\mathbb{R}^{2n}} e^{\boldsymbol{i}\boldsymbol{u}\cdot\boldsymbol{x}} f_{\boldsymbol{X}}(\boldsymbol{x},\boldsymbol{y}) e^{\boldsymbol{j}\boldsymbol{v}\cdot\boldsymbol{y}} d\boldsymbol{x}d\boldsymbol{y}.$$

Theorem 3. *Let $\boldsymbol{X} = (X_1, X_2)$ be real random variables. If the quaternion probability density function related to the quaternion characteristic function $\phi_{\boldsymbol{X}}$ satisfies*

$$\int_{\mathbb{R}^2} |f_{\boldsymbol{X}}(\boldsymbol{x})| < \infty,$$

then $\phi_{\boldsymbol{X}}$ is uniformly continuous.

Proof. Simple computations yield

$$\begin{aligned}
|\phi_{\boldsymbol{X}}(\boldsymbol{t}+\boldsymbol{h}) - \phi_{\boldsymbol{X}}(\boldsymbol{t})| &= \left| \int_{\mathbb{R}^2} e^{\boldsymbol{i}x_1(\boldsymbol{t_1}+\boldsymbol{h_1})} f_{\boldsymbol{X}}(\boldsymbol{x}) e^{\boldsymbol{j}x_2(\boldsymbol{t_2}+\boldsymbol{h_2})} - e^{\boldsymbol{i}x_1\boldsymbol{t_1}} f_{\boldsymbol{X}}(\boldsymbol{x}) e^{\boldsymbol{j}x_2\boldsymbol{t_2}} d\boldsymbol{x} \right| \\
&= \left| \int_{\mathbb{R}^2} e^{\boldsymbol{i}x_1\boldsymbol{t_1}} \left(e^{\boldsymbol{i}x_1\boldsymbol{h_1}} f_{\boldsymbol{X}}(\boldsymbol{x}) e^{\boldsymbol{j}x_2\boldsymbol{h_2}} - f_{\boldsymbol{X}}(\boldsymbol{x}) \right) e^{\boldsymbol{j}x_2\boldsymbol{t_2}} d\boldsymbol{x} \right| \\
&\leq \int_{\mathbb{R}^2} \left| \left(e^{\boldsymbol{i}x_1\boldsymbol{h_1}} f_{\boldsymbol{X}}(\boldsymbol{x}) e^{\boldsymbol{j}x_2\boldsymbol{h_2}} - f_{\boldsymbol{X}}(\boldsymbol{x}) \right) \right| d\boldsymbol{x}.
\end{aligned} \tag{39}$$

Invoking the triangle inequality for quaternion (10) gives

$$\begin{aligned}
|\phi_{\boldsymbol{X}}(\boldsymbol{t}+\boldsymbol{h}) - \phi_{\boldsymbol{X}}(\boldsymbol{t})| &\leq \int_{\mathbb{R}^2} \left(\left| e^{\boldsymbol{i}x_1\boldsymbol{h_1}} f_{\boldsymbol{X}}(\boldsymbol{x}) e^{\boldsymbol{j}x_2\boldsymbol{h_2}} \right| + \left| f_{\boldsymbol{X}}(\boldsymbol{x}) \right| \right) d\boldsymbol{x} \\
&= 2 \int_{\mathbb{R}^2} |f_{\boldsymbol{X}}(\boldsymbol{x})| d\boldsymbol{x}.
\end{aligned}$$

Using the dominated convergence theorem implies that $|\phi_{\boldsymbol{X}}(\boldsymbol{t}+\boldsymbol{h}) - \phi_{\boldsymbol{X}}(\boldsymbol{t})| \to 0$ when $\boldsymbol{h} \to 0$ and so, $\phi_{\boldsymbol{X}}$ is uniformly continuous. □

Theorem 4. *Let $\boldsymbol{X} = (X_1, X_2)$ be real random variables. If X_1 and X_2 are independent and the quaternion probability density function $f_{\boldsymbol{X}}(\boldsymbol{x})$ factorizes as*

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = f_{X_1}(x_1) f_{X_2}(x_2), \tag{40}$$

then

$$\phi_{\boldsymbol{X}}(\boldsymbol{t}) = \phi_{X_1}(t_1)\phi_{X_2}(t_2) \tag{41}$$

Proof. In fact, we have

$$\begin{aligned}
\phi_{\boldsymbol{X}}(\boldsymbol{t}) &= E[e^{\boldsymbol{i}t_1X_1} e^{\boldsymbol{j}t_2X_2}] \\
&= \int_{\mathbb{R}^2} e^{\boldsymbol{i}t_1x_1} f_{\boldsymbol{X}}(\boldsymbol{x}) e^{\boldsymbol{j}t_2x_2} d\boldsymbol{x} \\
&= \int_{\mathbb{R}^2} e^{\boldsymbol{i}t_1x_1} f_{X_1}(x_1) f_{X_2}(x_2) e^{\boldsymbol{j}t_2x_2} d\boldsymbol{x}.
\end{aligned} \tag{42}$$

If we assume that f_{X_2} is a real-valued function, the above relation changes to

$$\begin{aligned}\phi_{\boldsymbol{X}}(\boldsymbol{t}) &= \int_{\mathbb{R}^2} e^{\boldsymbol{i}t_1x_1} f_{X_1}(x_1) f_{X_2}(x_2) e^{\boldsymbol{j}t_2x_2} d\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} e^{\boldsymbol{i}t_1x_1} f_{X_1}(x_1) e^{\boldsymbol{j}t_2x_2} f_{X_2}(x_2) d\boldsymbol{x},\end{aligned} \tag{43}$$

and the proof is complete. □

Relation (38) above may be expressed in the form

$$\begin{aligned}\phi_{\boldsymbol{X}}(t) &= \int_{\mathbb{R}^2} e^{\boldsymbol{i}t_1x_1} \left(f_{\boldsymbol{X}}^a(\boldsymbol{x}) + \boldsymbol{i} f_{\boldsymbol{X}}^b(\boldsymbol{x}) + \boldsymbol{j} f_{\boldsymbol{X}}^c(\boldsymbol{x}) + \boldsymbol{k} f_{\boldsymbol{X}}^d(\boldsymbol{x})\right) e^{\boldsymbol{j}t_2x_2} d\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} e^{\boldsymbol{i}t_1x_1} f_{\boldsymbol{X}}^a(\boldsymbol{x}) e^{\boldsymbol{j}t_2x_2} d\boldsymbol{x} + \boldsymbol{i} \int_{\mathbb{R}^2} e^{\boldsymbol{i}t_1x_1} f_{\boldsymbol{X}}^b(\boldsymbol{x}) e^{\boldsymbol{j}t_2x_2} d\boldsymbol{x} \\ &\quad + \boldsymbol{j} \int_{\mathbb{R}^2} e^{-\boldsymbol{i}t_1x_1} f_{\boldsymbol{X}}^c(\boldsymbol{x}) e^{\boldsymbol{j}t_2x_2} d\boldsymbol{x} + \boldsymbol{k} \int_{\mathbb{R}^2} e^{-\boldsymbol{i}t_1x_1} f_{\boldsymbol{X}}^d(\boldsymbol{x}) e^{\boldsymbol{j}t_2x_2} d\boldsymbol{x} \\ &= \phi_{\boldsymbol{X}}^a(\boldsymbol{t}) + \boldsymbol{i}\phi_{\boldsymbol{X}}^b(\boldsymbol{t}) + \boldsymbol{j}\phi_{\boldsymbol{X}}^c(-t_1, t_2) + \boldsymbol{k}\phi_{\boldsymbol{X}}^d(-t_1, t_2),\end{aligned}$$

where

$$\phi_{\boldsymbol{X}}^l(\boldsymbol{t}) = \int_{\mathbb{R}^2} e^{\boldsymbol{i}t_1x_1} f_{\boldsymbol{X}}^l(\boldsymbol{x}) e^{\boldsymbol{j}t_2x_2} d\boldsymbol{x}, \quad l = a, b, c, d. \tag{44}$$

Proposition 1. *Let* $\boldsymbol{X} = (X_1, X_2)$ *be real random variables. Then, one gets*

1. $\phi_{\boldsymbol{X}}^l(\boldsymbol{0}) = 1, \quad \phi_{\boldsymbol{X}}(\boldsymbol{0}) = \boldsymbol{1}$.
2. $|\phi_{\boldsymbol{X}}^l(\boldsymbol{t})| \leq 1, \quad |\phi_{\boldsymbol{X}}(\boldsymbol{t})| \leq |\boldsymbol{1}| = 4, \quad l = a, b, c, d$

where $\boldsymbol{1} = (1, 1, 1, 1) = 1 + \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$.

Proof. According to relations (23) and (44), it is straightforward to see that

$$\phi_{\boldsymbol{X}}^l(0) = \int_{\mathbb{R}^2} f_{\boldsymbol{X}}^l(\boldsymbol{x}) d\boldsymbol{x} = 1, \tag{45}$$

and

$$\begin{aligned}\phi_{\boldsymbol{X}}(0) &= \int_{\mathbb{R}^2} f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} \left(f_{\boldsymbol{X}}^a + \boldsymbol{i} f_{\boldsymbol{X}}^b + \boldsymbol{j} f_{\boldsymbol{X}}^c + \boldsymbol{k} f_{\boldsymbol{X}}^d\right) d\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} f_{\boldsymbol{X}}^a d\boldsymbol{x} + \boldsymbol{i} \int_{\mathbb{R}^2} f_{\boldsymbol{X}}^b d\boldsymbol{x} + \boldsymbol{j} \int_{\mathbb{R}^2} f_{\boldsymbol{X}}^c d\boldsymbol{x} + \boldsymbol{k} \int_{\mathbb{R}^2} f_{\boldsymbol{X}}^d d\boldsymbol{x} \\ &= \boldsymbol{1}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}|\phi_{\boldsymbol{X}}^l(\boldsymbol{t})| &= \left| \int_{\mathbb{R}^2} e^{\boldsymbol{i}t_1x_1} f_{\boldsymbol{X}}^l(\boldsymbol{x}) e^{\boldsymbol{j}t_2x_2} d\boldsymbol{x} \right| \\ &\leq \int_{\mathbb{R}^2} |e^{\boldsymbol{i}t_1x_1} f_{\boldsymbol{X}}^l(\boldsymbol{x}) e^{\boldsymbol{j}t_2x_2}| d\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} |f_{\boldsymbol{X}}^l(\boldsymbol{x})| d\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} f_{\boldsymbol{X}}^l(\boldsymbol{x}) d\boldsymbol{x} \\ &= 1.\end{aligned}$$

We also find

$$
\begin{aligned}
|\phi_{\boldsymbol{X}}(\boldsymbol{t})| &= \left| \int_{\mathbb{R}^2} e^{\boldsymbol{i}t_1x_1} f_{\boldsymbol{X}}(\boldsymbol{x}) e^{\boldsymbol{j}t_2x_2} d\boldsymbol{x} \right| \\
&\leq \int_{\mathbb{R}^2} |f_{\boldsymbol{X}}(\boldsymbol{x})| d\boldsymbol{x}.
\end{aligned}
$$

Applying (10) and (23) results in

$$
\begin{aligned}
|\phi_{\boldsymbol{X}}(\boldsymbol{t})| &= \left| \int_{\mathbb{R}^2} e^{\boldsymbol{i}t_1x_1} (f^a_{\boldsymbol{X}}(\boldsymbol{x}) + \boldsymbol{i}f^b_{\boldsymbol{X}}(\boldsymbol{x}) + \boldsymbol{j}f^c_{\boldsymbol{X}}(\boldsymbol{x}) + \boldsymbol{k}f^d_{\boldsymbol{X}}(\boldsymbol{x})) e^{\boldsymbol{j}t_2x_2} d\boldsymbol{x} \right| \\
&\leq \int_{\mathbb{R}^2} |f^a_{\boldsymbol{X}}(\boldsymbol{x})| d\boldsymbol{x} + \int_{\mathbb{R}^2} |\boldsymbol{i}f^b_{\boldsymbol{X}}(\boldsymbol{x})| d\boldsymbol{x} + \int_{\mathbb{R}^2} |\boldsymbol{j}f^c_{\boldsymbol{X}}(\boldsymbol{x})| d\boldsymbol{x} + \int_{\mathbb{R}^2} |\boldsymbol{k}f^d_{\boldsymbol{X}}(\boldsymbol{x})| d\boldsymbol{x} \\
&= \int_{\mathbb{R}^2} |f^a_{\boldsymbol{X}}(\boldsymbol{x})| d\boldsymbol{x} + \int_{\mathbb{R}^2} |f^b_{\boldsymbol{X}}(\boldsymbol{x})| d\boldsymbol{x} + \int_{\mathbb{R}^2} |f^c_{\boldsymbol{X}}(\boldsymbol{x})| d\boldsymbol{x} + \int_{\mathbb{R}^2} |f^d_{\boldsymbol{X}}(\boldsymbol{x})| d\boldsymbol{x} \\
&= \int_{\mathbb{R}^2} f^a_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} + \int_{\mathbb{R}^2} f^b_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} + \int_{\mathbb{R}^2} f^c_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} + \int_{\mathbb{R}^2} f^d_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \\
&= 1 + 1 + 1 + 1 \\
&= 4,
\end{aligned}
$$

and the proof is complete. □

Definition 8. *The quaternion probability density function $f_{\boldsymbol{X}}(\boldsymbol{x})$ can be recovered from the characteristic function using the formula*

$$
f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-\boldsymbol{i}t_1x_1} \phi_{\boldsymbol{X}}(\boldsymbol{t}) e^{-\boldsymbol{j}t_2x_2} d\boldsymbol{t}. \tag{46}
$$

Let us now compute the quaternion characteristic function of Example 1. It follows from Equation (38) that

$$
\begin{aligned}
&\phi_{X_1,X_2}(t_1,t_2) \\
&= \int_0^1 \int_0^1 e^{\boldsymbol{i}t_1x_1} (4x_1x_2 + 3\boldsymbol{i}x_1^2 + 3\boldsymbol{j}x_2^2 + \boldsymbol{k}(x_1+x_2)) e^{\boldsymbol{j}t_2x_2} dx_1dx_2 \\
&= \int_0^1 \int_0^1 e^{\boldsymbol{i}t_1x_1} (4x_1x_2) e^{\boldsymbol{j}t_2x_2} dx_1dx_2 + \int_0^1 \int_0^1 e^{\boldsymbol{i}t_1x_1} (3\boldsymbol{i}x_1^2) e^{\boldsymbol{j}t_2x_2} dx_1dx_2 \\
&\quad + \int_0^1 \int_0^1 e^{\boldsymbol{i}t_1x_1} (3\boldsymbol{j}x_2^2) e^{\boldsymbol{j}t_2x_2} dx_1dx_2 + \int_0^1 \int_0^1 e^{\boldsymbol{i}t_1x_1} (\boldsymbol{k}(x_1+x_2)) e^{\boldsymbol{j}t_2x_2} dx_1dx_2 \\
&= 4\int_0^1 x_1 e^{\boldsymbol{i}t_1x_1} dx_1 \int_0^1 x_2 e^{\boldsymbol{j}t_2x_2} dx_2 + 3\boldsymbol{i} \int_0^1 e^{\boldsymbol{i}t_1x_1} x_1^2 dx_1 \int_0^1 e^{\boldsymbol{j}t_2x_2} dx_2 \\
&\quad + 3\int_0^1 e^{\boldsymbol{i}t_1x_1} dx_1 \boldsymbol{j} \int_0^1 x_2^2 e^{\boldsymbol{j}t_2x_2} dx_2 + \int_0^1 e^{\boldsymbol{i}t_1x_1} x_1 dx_1 \boldsymbol{k} \int_0^1 e^{\boldsymbol{j}t_2x_2} dx_2 \\
&\quad + \int_0^1 e^{\boldsymbol{i}t_1x_1} dx_1 \boldsymbol{k} \int_0^1 x_2 e^{\boldsymbol{j}t_2x_2} dx_2.
\end{aligned} \tag{47}
$$

Further, we get

$$
\begin{aligned}
&\phi_{X_1,X_2}(t_1,t_2) \\
&= 4\left(\tfrac{e^{\boldsymbol{i}t_1}}{\boldsymbol{i}t_1} + \tfrac{e^{\boldsymbol{i}t_1}-1}{t_1^2}\right)\left(\tfrac{e^{\boldsymbol{j}t_2}}{\boldsymbol{j}t_2} + \tfrac{e^{\boldsymbol{j}t_2}-1}{t_2^2}\right) + 3\boldsymbol{i}\left(\tfrac{e^{\boldsymbol{i}t_1}}{\boldsymbol{i}t_1} + \tfrac{2e^{\boldsymbol{i}t_1}}{t_1^2} - \tfrac{2e^{\boldsymbol{i}t_1}-2}{\boldsymbol{i}t_1^3}\right)\left(\tfrac{e^{\boldsymbol{j}t_2}-1}{\boldsymbol{j}t_2}\right) \\
&\quad + 3\boldsymbol{j}\left(\tfrac{e^{\boldsymbol{i}t_1}-1}{\boldsymbol{i}t_1}\right)\left(\tfrac{e^{\boldsymbol{j}t_2}}{\boldsymbol{j}t_2} + \tfrac{2e^{\boldsymbol{j}t_2}}{\boldsymbol{j}t_2} - \tfrac{2e^{\boldsymbol{j}t_2}-2}{\boldsymbol{j}t_2}\right) + \boldsymbol{k}\left(\tfrac{e^{\boldsymbol{i}t_1}}{\boldsymbol{i}t_1} + \tfrac{e^{\boldsymbol{i}t_1}-1}{t_1^2}\right)\left(\tfrac{e^{\boldsymbol{j}t_2}-1}{\boldsymbol{j}t_2}\right) \\
&= 4\left(\tfrac{t_1e^{\boldsymbol{i}t_1} + \boldsymbol{i}(e^{\boldsymbol{i}t_1}-1)}{\boldsymbol{i}t_1^2}\right)\left(\tfrac{t_2e^{\boldsymbol{j}t_2} + \boldsymbol{j}(e^{\boldsymbol{j}t_2}-1)}{\boldsymbol{j}t_1^2}\right) \\
&\quad + 3\boldsymbol{i}\left(\tfrac{t_1^2e^{\boldsymbol{i}t_1} + \boldsymbol{i}t_1 2e^{\boldsymbol{i}t_1} - 2e^{\boldsymbol{i}t_1} + 2}{\boldsymbol{i}t_1^3}\right)\left(\tfrac{e^{\boldsymbol{j}t_2}-1}{\boldsymbol{j}t_2}\right) \\
&\quad + 3\boldsymbol{j}\left(\tfrac{e^{\boldsymbol{i}t_1}-1}{\boldsymbol{i}t_1}\right)\left(\tfrac{t_2^2e^{\boldsymbol{j}t_2} + \boldsymbol{j}t_2 2e^{\boldsymbol{j}t_2} - 2e^{\boldsymbol{j}t_2} + 2}{\boldsymbol{j}t_2^3}\right) \\
&\quad + \boldsymbol{k}\left(\tfrac{t_1e^{\boldsymbol{i}t_1} + \boldsymbol{i}(e^{\boldsymbol{i}t_1}-1)}{\boldsymbol{i}t_1^2}\right)\left(\tfrac{e^{\boldsymbol{j}t_2}-1}{\boldsymbol{j}t_2}\right) \\
&\quad + \boldsymbol{k}\left(\tfrac{e^{\boldsymbol{i}t_1}-1}{\boldsymbol{i}t_1}\right)\left(\tfrac{t_2e^{\boldsymbol{j}t_2} + \boldsymbol{j}(e^{\boldsymbol{j}t_2}-1)}{\boldsymbol{j}t_2^2}\right).
\end{aligned} \tag{48}
$$

Equation (48) may be further simplified to

$$\begin{aligned}&\phi_{X_1,X_2}(t_1,t_2)\\&=4\left(\frac{\left(t_1e^{it_1}+i(e^{it_1}-1)\right)\left(t_2e^{jt_2}+j(e^{jt_2}-1)\right)}{kt_1^2t_2^2}\right)\\&\quad+3i\left(\frac{\left(t_1^2e^{it_1}+it_12e^{it_1}-2e^{it_1}+2\right)\left(e^{jt_2}-1\right)}{kt_1^3t_2}\right)\\&\quad+3j\left(\frac{\left(e^{it_1}-1\right)\left(t_2^2e^{jt_2}+2jt_2^2e^{jt_2}-2e^{jt_2}+2\right)}{kt_1t_2^3}\right)\\&\quad+k\left(\frac{\left(t_1e^{it_1}+i(e^{it_1}-1)\right)\left(e^{jt_2}-1\right)}{kt_1^2t_2}\right)+k\left(\frac{\left(e^{it_1}-1\right)\left(t_2e^{jt_2}+j(e^{jt_2}-1)\right)}{kt_1t_2^2}\right)\end{aligned}\tag{49}$$

We finally arrive at

$$\begin{aligned}\phi_{X_1,X_2}(t_1,t_2)=&\frac{4}{kt_1^2t_2^2}\Big(\left(t_1e^{it_1}+i(e^{it_1}-1)\right)\left(t_2e^{jt_2}+j(e^{jt_2}-1)\right)\Big)\\&+\frac{1}{kt_1^3t_2}\Big(3i\left(2+e^{it_1}(t_1^2+2it_1-2)\right)\left(e^{jt_2}-1\right)\Big)\\&+\frac{1}{kt_1t_2^3}\Big(3j\left(2+e^{jt_2}(t_2^2+2jt_2-2)\right)\left(e^{it_1}-1\right)\Big)\\&+\frac{1}{t_1^2t_2}\Big(\left(t_1e^{it_1}+i(e^{it_1}-1)(e^{jt_2}-1)\right)\Big)\\&+\frac{1}{t_1t_2^2}\Big(\left(e^{it_1}-1\right)\left(t_2e^{jt_2}+j(e^{jt_2}-1)\right)\Big).\end{aligned}$$

As an immediate consequence of Definition 8, the following result can be found.

Theorem 5. *Assume that the quaternion characteristic functions $\phi_{\boldsymbol{X}}$ and $\psi_{\boldsymbol{X}}$ of the random variables $\boldsymbol{X}$ are defined by*

$$\phi_{\boldsymbol{X}}(\boldsymbol{t})=\int_{\mathbb{R}^2}e^{\boldsymbol{i}t_1x_1}f_{\boldsymbol{X}}(\boldsymbol{x})e^{\boldsymbol{j}t_2x_2}d\boldsymbol{x},\quad \psi_{\boldsymbol{X}}(\boldsymbol{x})=\int_{\mathbb{R}^2}e^{\boldsymbol{i}t_1x_1}g_{\boldsymbol{X}}(\boldsymbol{t})e^{\boldsymbol{j}t_2x_2}d\boldsymbol{t}.\tag{50}$$

Assume that $f_{\boldsymbol{X}}$ is a real probability density function; then, one gets

$$\begin{aligned}\int_{\mathbb{R}^2}g_{\boldsymbol{X}}(\boldsymbol{t})e^{-\boldsymbol{i}t_1y_1}\phi_{\boldsymbol{X}}(\boldsymbol{t})e^{-\boldsymbol{j}t_2y_2}d\boldsymbol{t}=\;&\int_{\mathbb{R}^2}f_{\boldsymbol{X}}(\boldsymbol{x})\psi^a_{\boldsymbol{X}}(\boldsymbol{x}-\boldsymbol{y})\,d\boldsymbol{x}+\boldsymbol{i}\int_{\mathbb{R}^2}f_{\boldsymbol{X}}(\boldsymbol{x})\psi^b_{\boldsymbol{X}}(\boldsymbol{x}-\boldsymbol{y})d\boldsymbol{x}\\&+\boldsymbol{j}\int_{\mathbb{R}^2}f_{\boldsymbol{X}}(\boldsymbol{x})\psi^c_{\boldsymbol{X}}(\boldsymbol{y}-\boldsymbol{x})d\boldsymbol{x}+\boldsymbol{k}\int_{\mathbb{R}^2}f_{\boldsymbol{X}}(\boldsymbol{x})\psi^d_{\boldsymbol{X}}(\boldsymbol{y}-\boldsymbol{x})d\boldsymbol{x}.\end{aligned}\tag{51}$$

Proof. By virtue of the quaternion characteristic function defined in (38), we obtain

$$\begin{aligned}e^{-\boldsymbol{i}t_1y_1}\phi_{\boldsymbol{X}}(\boldsymbol{t})e^{-\boldsymbol{j}t_2y_2}&=e^{-\mathbf{i}t_1y_1}\int_{\mathbb{R}^2}e^{\boldsymbol{i}t_1x_1}f_{\boldsymbol{X}}(\boldsymbol{x})e^{\boldsymbol{j}t_2x_2}d\boldsymbol{x}e^{-\mathbf{j}t_2y_2}\\&=\int_{\mathbb{R}^2}e^{\boldsymbol{i}t_1(x_1-y_1)}f_{\boldsymbol{X}}(\boldsymbol{x})e^{\boldsymbol{j}t_2(x_2-y_2)}d\boldsymbol{x}.\end{aligned}\tag{52}$$

Multiplying both sides of relation (52) by $g_{\boldsymbol{X}}(\boldsymbol{t})$ and then integrating with respect to $d\boldsymbol{t}$, we see that

$$\begin{aligned}\int_{\mathbb{R}^2}g_{\boldsymbol{X}}(\boldsymbol{t})e^{-\boldsymbol{i}t_1y_1}\phi_{\boldsymbol{X}}(\boldsymbol{t})e^{-\mathbf{j}t_2y_2}d\boldsymbol{t}&=\int_{\mathbb{R}^2}g_{\boldsymbol{X}}(\boldsymbol{t})\left(\int_{\mathbb{R}^2}e^{\boldsymbol{i}t_1(x_1-y_1)}f_{\boldsymbol{X}}(\boldsymbol{x})e^{\boldsymbol{j}t_2(x_2-y_2)}d\boldsymbol{x}\right)d\boldsymbol{t}\\&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(g^a_{\boldsymbol{X}}(\boldsymbol{t})+\boldsymbol{i}g^b_{\boldsymbol{X}}(\boldsymbol{t})+\boldsymbol{j}g^c_{\boldsymbol{X}}(\boldsymbol{t})+\boldsymbol{k}g^d_{\boldsymbol{X}}(\boldsymbol{t})\right)\\&\quad e^{\boldsymbol{i}t_1(x_1-y_1)}f_{\boldsymbol{X}}(\boldsymbol{x})e^{\boldsymbol{j}t_2(x_2-y_2)}d\boldsymbol{x}d\boldsymbol{t}\\&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(g^a_{\boldsymbol{X}}(\boldsymbol{t})+\boldsymbol{i}g^b_{\boldsymbol{X}}(\boldsymbol{t})+\boldsymbol{j}g^c_{\boldsymbol{X}}(\boldsymbol{t})+\boldsymbol{k}g^d_{\boldsymbol{X}}(\boldsymbol{t})\right)\\&\quad e^{\boldsymbol{i}t_1(x_1-y_1)}f_{\boldsymbol{X}}(\boldsymbol{x})e^{\boldsymbol{j}t_2(x_2-y_2)}d\boldsymbol{x}d\boldsymbol{t}.\end{aligned}$$

Fubini's theorem allows us to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} g_{\boldsymbol{X}}(\boldsymbol{t})e^{-\boldsymbol{i}t_1y_1}\phi_{\boldsymbol{X}}(\boldsymbol{t})e^{-\boldsymbol{j}t_2y_2}d\boldsymbol{t}\\
&=\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} f_{\boldsymbol{X}}(\boldsymbol{x})e^{\boldsymbol{i}t_1(x_1-y_1)}g^a_{\boldsymbol{X}}(\boldsymbol{t})e^{\boldsymbol{j}t_2(x_2-y_2)}d\boldsymbol{t}+\boldsymbol{i}f_{\boldsymbol{X}}(\boldsymbol{x})\int_{\mathbb{R}^2} e^{\boldsymbol{i}t_1(x_1-y_1)}g^b_{\boldsymbol{X}}(\boldsymbol{t})e^{\boldsymbol{j}t_2(x_2-y_2)}d\boldsymbol{t}\right)d\boldsymbol{x}\\
&\quad+\int_{\mathbb{R}^2}\left(\boldsymbol{j}f_{\boldsymbol{X}}(\boldsymbol{x})\int_{\mathbb{R}^2} e^{\boldsymbol{i}t_1(x_1-y_1)}g^c_{\boldsymbol{X}}(\boldsymbol{t})e^{\boldsymbol{j}t_2(x_2-y_2)}d\boldsymbol{t}+\boldsymbol{k}f_{\boldsymbol{X}}(\boldsymbol{x})\int_{\mathbb{R}^2} e^{\boldsymbol{i}t_1(x_1-y_1)}g^d_{\boldsymbol{X}}(\boldsymbol{t})e^{\boldsymbol{j}t_2(x_2-y_2)}d\boldsymbol{t}\right)d\boldsymbol{x}\\
&=\int_{\mathbb{R}^2}\left(f_{\boldsymbol{X}}(\boldsymbol{x})\psi^a_{\boldsymbol{X}}(\boldsymbol{x}-\boldsymbol{y})+\boldsymbol{i}f_{\boldsymbol{X}}(\boldsymbol{x})\psi^b_{\boldsymbol{X}}(\boldsymbol{x}-\boldsymbol{y})\right)d\boldsymbol{x}\\
&\quad+\int_{\mathbb{R}^2}\left(\boldsymbol{j}f_{\boldsymbol{X}}(\boldsymbol{x})\psi^c_{\boldsymbol{X}}(\boldsymbol{y}-\boldsymbol{x})+\boldsymbol{k}f_{\boldsymbol{X}}(\boldsymbol{x})\psi^d_{\boldsymbol{X}}(\boldsymbol{y}-\boldsymbol{x})\right)d\boldsymbol{x}\\
&=\int_{\mathbb{R}^2} f_{\boldsymbol{X}}(\boldsymbol{x})\psi^a_{\boldsymbol{X}}(\boldsymbol{x}-\boldsymbol{y})d\boldsymbol{x}+\boldsymbol{i}\int_{\mathbb{R}^2} f_{\boldsymbol{X}}(\boldsymbol{x})\psi^b_{\boldsymbol{X}}(\boldsymbol{x}-\boldsymbol{y})d\boldsymbol{x}\\
&\quad+\boldsymbol{j}\int_{\mathbb{R}^2} f_{\boldsymbol{X}}(d\boldsymbol{x})\ \psi^c_{\boldsymbol{X}}(\boldsymbol{y}-\boldsymbol{x})d\boldsymbol{x}+\boldsymbol{k}\int_{\mathbb{R}^2} f_{\boldsymbol{X}}(d\boldsymbol{x})\psi^d_{\boldsymbol{X}}(\boldsymbol{y}-\boldsymbol{x})d\boldsymbol{x}.
\end{aligned}$$

This is the desired result. □

Based on Equation (26), we may define the $(n+m)$th moment of real random variables X_1 and X_2 as

$$E[X_1^m X_2^n]=\int_{\mathbb{R}^2} x_1^n x_2^m f_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x},\quad n,m=1,2,3,\cdots, \tag{53}$$

provided the integral exists. It is obvious that for $n=1, m=0$ and $n=0, m=1$ in (53), we obtain

$$\mu_1=E[X_1]=\int_{\mathbb{R}^2} x_1 f_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x}, \tag{54}$$

and

$$\mu_2=E[X_2]=\int_{\mathbb{R}^2} x_2 f_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x}. \tag{55}$$

Theorem 6. *If $\boldsymbol{X}=(X_1,X_2)$ are real random variables, then there exist $(n+m)$th continuous derivatives for the quaternion characteristic function $\phi_{\boldsymbol{X}}(\boldsymbol{t})$ described by the formula*

$$\frac{\partial^{m+n}}{\partial t_1^m\partial t_2^n}\phi_{\boldsymbol{X}}(\boldsymbol{t})=\boldsymbol{i}^m\int_{\mathbb{R}^2} x_1^m e^{\boldsymbol{i}t_1x_1}f_{\boldsymbol{X}}(\boldsymbol{x})\ x_2^n e^{\boldsymbol{j}t_2x_2}d\boldsymbol{x}\ \boldsymbol{j}^n. \tag{56}$$

In addition,

$$E[X_1^n X_2^m]=\boldsymbol{i}^{-m}\frac{\partial^{m+n}}{\partial t_1^m\partial t_2^n}\phi_{\boldsymbol{X}}(\boldsymbol{0})\boldsymbol{j}^{-n}. \tag{57}$$

Proof. For $m=1$ and $n=0$, direct calculations reveal that

$$\begin{aligned}
\tfrac{\partial}{\partial t_1}\phi_{\boldsymbol{X}}(\boldsymbol{t}) &= \tfrac{\partial}{\partial t_1}\int_{\mathbb{R}^2} e^{\boldsymbol{i}t_1x_1}f_{\boldsymbol{X}}(\boldsymbol{x})e^{\boldsymbol{j}t_2x_2}d\boldsymbol{x}\\
&=\int_{\mathbb{R}^2}\left(\tfrac{\partial}{\partial t_1}e^{\boldsymbol{i}t_1x_1}\right)f_{\boldsymbol{X}}(\boldsymbol{x})e^{\boldsymbol{j}t_2x_2}d\boldsymbol{x}\\
&=\int_{\mathbb{R}^2}(\boldsymbol{i}x_1)e^{\boldsymbol{i}t_1x_1}f_{\boldsymbol{X}}(\boldsymbol{x})\ e^{\boldsymbol{j}t_2x_2}d\boldsymbol{x}.
\end{aligned} \tag{58}$$

In view of relation (58), we obtain

$$\begin{aligned}
\tfrac{\partial^2}{\partial t_1^2}\phi_{\boldsymbol{X}}(\boldsymbol{t}) &= \tfrac{\partial}{\partial t_1}\left(\int_{\mathbb{R}^2}(\boldsymbol{i}x_1)e^{\boldsymbol{i}t_1x_1}f_{\boldsymbol{X}}(\boldsymbol{x})\ e^{\boldsymbol{j}t_2x_2}\ d\boldsymbol{x}\right)\\
&=\boldsymbol{i}^2\int_{\mathbb{R}^2} x_1^2 e^{\boldsymbol{i}t_1x_1}f_{\boldsymbol{X}}(\boldsymbol{x})\ e^{\boldsymbol{j}t_2x_2}\ d\boldsymbol{x}\ .
\end{aligned} \tag{59}$$

This implies that

$$\frac{\partial^{m+n}}{\partial t_1^m\partial t_2^n}\phi_{\boldsymbol{X}}(\boldsymbol{t})=\boldsymbol{i}^m\int_{\mathbb{R}^2} x_1^m e^{\boldsymbol{i}t_1x_1}f_{\boldsymbol{X}}(\boldsymbol{x})\ x_2^n e^{\boldsymbol{j}t_2x_2}d\boldsymbol{x}\ \boldsymbol{j}^n. \tag{60}$$

Applying Equation (53) gives

$$\frac{\partial^{m+n}}{\partial t_1^m\partial t_2^n}\phi_{\boldsymbol{X}}(\boldsymbol{0})=\boldsymbol{i}^m\int_{\mathbb{R}^2} x_1^m f_{\boldsymbol{X}}(\boldsymbol{x})\ x_2^n d\boldsymbol{x}\ \boldsymbol{j}^n. \tag{61}$$

Hence,

$$\boldsymbol{i}^{-m}\frac{\partial^{m+n}}{\partial t_1^m \partial t_2^n}\phi_{\boldsymbol{X}}(\mathbf{0})\boldsymbol{j}^{-n} = \int_{\mathbb{R}^2} x_1^m f_{\boldsymbol{X}}(\boldsymbol{x})\, x_2^n d\boldsymbol{x},$$

thus, the proof of the theorem is completed. □

Let us now introduce the definition of the quaternion covariance of real random variables X_1 and X_2.

Definition 9. *Let $\boldsymbol{X} = (X_1, X_2)$ be any real random variables. The covariance of X_1 and X_2 in the quaternion setting is given by*

$$\begin{aligned} Cov(X_1, X_2) &= E\big[(X_1 - E[X_1])(\overline{X_2 - E[X_2]})\big] \\ &= E\big[(X_1 - E[X_1])(X_2 - \overline{E[X_2]})\big] \\ &= E\big[(X_1X_2 - X_1\overline{E[X_2]} - E[X_1]X_2 + E[X_1]\overline{E[X_2]}\big] \\ &= E\big[X_1X_2\big] - E[X_1]\overline{E[X_2]} - E[X_1]E[X_2] + E[X_1]\overline{E[X_2]} \\ &= E\big[X_1X_2\big] - E[X_1]E[X_2]. \end{aligned} \tag{62}$$

According to (62), we obtain

$$\begin{aligned} Cov(X_2, X_1) &= E[X_2X_1] - E[X_2]E[X_1] \\ &= E[X_1X_2] - E[X_2]E[X_1] \end{aligned} \tag{63}$$

In general, since $E[X_1]E[X_2] \neq E[X_2]E[X_1]$, then $Cov(X_1, X_2) \neq Cov(X_2, X_1)$. Due to (62), we get

$$\begin{aligned} Var(X_1) = \sigma_1 &= E\big[X_1X_1\big] - E[X_1]E[X_1] \\ &= E\big[X_1^2\big] - E[X_1]^2. \end{aligned} \tag{64}$$

which is the quaternion variance of real random variable X_1.

From Example 1, we get

$$\begin{aligned} Cov(X_1, X_2) &= E\big[X_1X_2\big] - E[X_1]E[X_2] \\ &= \Big(\tfrac{4}{9} + \tfrac{3}{8}\boldsymbol{i} + \tfrac{3}{8}\boldsymbol{j} + \tfrac{1}{3}\boldsymbol{k}\Big) - \Big(\tfrac{-31}{48} + \tfrac{11}{16}\boldsymbol{i} + \tfrac{11}{16}\boldsymbol{j} + 1\tfrac{13}{144}\boldsymbol{k}\Big) \\ &= 1\tfrac{13}{144} - \tfrac{5}{16}\boldsymbol{i} + \tfrac{5}{16}\boldsymbol{j} - \tfrac{109}{144}\boldsymbol{k}, \end{aligned} \tag{65}$$

and

$$\begin{aligned} |Cov(X_1, X_2)| &= \sqrt{\Big(\tfrac{157}{144}\Big)^2 + \Big(-\tfrac{5}{16}\Big)^2 + \Big(\tfrac{5}{16}\Big)^2 + \Big(-\tfrac{109}{144}\Big)^2} \\ &= \sqrt{\tfrac{10145}{5184}} = \tfrac{1}{72}\sqrt{10145}. \end{aligned} \tag{66}$$

It can be obtained that the quaternion covariance $Cov(X_1, X_2)$ of real random variables X_1 and X_2 in terms of the quaternion characteristic function is

$$\begin{aligned} Cov(X_1, X_2) &= \boldsymbol{i}^{-1}\tfrac{\partial^2}{\partial t_1 \partial t_2}\phi_{\boldsymbol{X}}(\mathbf{0})\boldsymbol{j}^{-1} - \Big(\boldsymbol{i}\tfrac{\partial}{\partial t_1}\phi_{\boldsymbol{X}}(0)\tfrac{\partial}{\partial t_2}\phi_{\boldsymbol{X}}(0)\boldsymbol{j}\Big) \\ &= \boldsymbol{i}\tfrac{\partial^2}{\partial t_1 \partial t_2}\phi_{\boldsymbol{X}}(\mathbf{0})\boldsymbol{j} - \Big(\boldsymbol{i}\tfrac{\partial}{\partial t_1}\phi_{\boldsymbol{X}}(0)\tfrac{\partial}{\partial t_1}\phi_{\boldsymbol{X}}(0)\boldsymbol{j}\Big). \end{aligned} \tag{67}$$

Remark 2. *Due to Equation (63), one gets*

$$\begin{aligned} Cov(X_2, X_1) &= \boldsymbol{i}^{-1}\tfrac{\partial^2}{\partial t_1 \partial t_2}\phi_{\boldsymbol{X}}(\mathbf{0})\boldsymbol{j}^{-1} - \Big(\tfrac{\partial}{\partial t_2}\phi_{\boldsymbol{X}}(0)\boldsymbol{j}^{-1}\boldsymbol{i}^{-1}\tfrac{\partial}{\partial t_1}\phi_{\boldsymbol{X}}(0)\Big) \\ &= \boldsymbol{i}\tfrac{\partial^2}{\partial t_1 \partial t_2}\phi_{\boldsymbol{X}}(\mathbf{0})\boldsymbol{j} - \Big(\tfrac{\partial}{\partial t_2}\phi_{\boldsymbol{X}}(0)\boldsymbol{k}\tfrac{\partial}{\partial t_1}\phi_{\boldsymbol{X}}(0)\Big). \end{aligned} \tag{68}$$

The following example illustrates the use of the results mentioned above.

Example 2. *The random variables X_1 and X_2 have a probability density function of the form*

$$f(x_1, x_2) = \frac{1}{2\pi|\sigma_1||\sigma_2|} e^{-\frac{(x_1-|\mu_1|)^2}{2|\sigma_1|^2}} e^{-\frac{(x_2-|\mu_2|)^2}{2|\sigma_2|^2}}. \tag{69}$$

where μ_1 and μ_2 are defined by Equations (54) and (55). Find the quaternion variance of X_1 and X_2.

Solution. It follows from Equation (38) that

$$\phi_X(t_1, t_2) = \frac{1}{2\pi|\sigma_1||\sigma_2|} \int_{\mathbb{R}^2} e^{\boldsymbol{i}t_1 x_1} e^{-\frac{(x_1-|\mu_1|)^2}{2|\sigma_1|^2}} e^{-\frac{(x_2-|\mu_2|)^2}{2|\sigma_2|^2}} e^{\boldsymbol{j}t_2 x_2} d\boldsymbol{x}.$$

Changing the variables to $x_1 - |\mu_1| = y_1$ and $x_2 - |\mu_2| = y_2$, it is easily seen that

$$\begin{aligned}
&\phi_X(t_1, t_2) \\
&= \frac{1}{2\pi|\sigma_1||\sigma_2|} \int_{\mathbb{R}^2} e^{\boldsymbol{i}t_1(|\mu_1|+y_1)} e^{-\frac{y_1^2}{2|\sigma_1|^2}} e^{-\frac{y_2^2}{2|\sigma_2|^2}} e^{\boldsymbol{j}t_2(|\mu_2|+y_2)} dy_1 dy_2 \\
&= \frac{e^{\boldsymbol{i}t_1|\mu_1|} e^{\boldsymbol{j}t_2|\mu_2|}}{2\pi|\sigma_1||\sigma_2|} \int_{\mathbb{R}} e^{-\frac{y_1^2}{2|\sigma_1|^2} + \boldsymbol{i}t_1 y_1} dy_1 \int_{\mathbb{R}} e^{-\frac{y_2^2}{2|\sigma_2|^2} + \boldsymbol{j}t_2 y_2} dy_2.
\end{aligned} \tag{70}$$

We further obtain

$$\begin{aligned}
&\phi_X(t_1, t_2) \\
&= \frac{e^{\boldsymbol{i}t_1|\mu_1|} e^{\boldsymbol{j}t_2|\mu_2|}}{2\pi|\sigma_1||\sigma_2|} \int_{\mathbb{R}} e^{-\frac{1}{2|\sigma_1|^2}(y_1^2 - 2|\sigma_1|^2 \boldsymbol{i}t_1 y_1)} dy_1 \int_{\mathbb{R}} e^{-\frac{1}{2|\sigma_2|^2}(y_2^2 - 2|\sigma_2|^2 \boldsymbol{j}t_2 y_2)} dy_2 \\
&= \frac{e^{\boldsymbol{i}t_1|\mu_1|} e^{\boldsymbol{j}t_2|\mu_2|}}{2\pi|\sigma_1||\sigma_2|} \int_{\mathbb{R}} e^{-\frac{1}{2|\sigma_1|^2}\left((y_1 - |\sigma_1|^2 \boldsymbol{i}t_1)^2 - (|\sigma_1|^2 \boldsymbol{i}t_1)^2\right)} dy_1 \int_{\mathbb{R}} e^{-\frac{1}{2\sigma_2^2}\left((y_2 - |\sigma_2|^2 \boldsymbol{j}t_2)^2 - (|\sigma_2|^2 \boldsymbol{j}t_2)^2\right)} dy_2 \\
&= \frac{e^{\boldsymbol{i}t_1|\mu_1|} e^{\boldsymbol{j}t_2|\mu_2|}}{2\pi|\sigma_1||\sigma_2|} \int_{\mathbb{R}} e^{\frac{(|\sigma_1|^2 \boldsymbol{i}t_1)^2}{2|\sigma_1|^2}} e^{-\frac{1}{2\sigma_1^2}(y_1 - |\sigma_1|^2 \boldsymbol{i}t_1)^2} dy_1 \int_{\mathbb{R}} e^{\frac{(|\sigma_2|^2 \boldsymbol{j}t_2)^2}{2|\sigma_2|^2}} e^{-\frac{1}{2|\sigma_2|^2}(y_2 - |\sigma_2|^2 \boldsymbol{j}t_2)^2} dy_2 \\
&= \frac{e^{\boldsymbol{i}t_1|\mu_1|} e^{\boldsymbol{j}t_2|\mu_2|}}{2\pi|\sigma_1||\sigma_2|} e^{\frac{(|\sigma_1|^2 \boldsymbol{i}t_1)^2}{2|\sigma_1|^2}} \int_{\mathbb{R}} e^{-\frac{1}{2|\sigma_1|^2}(y_1 - |\sigma_1|^2 \boldsymbol{i}t_1)^2} dy_1 \int_{\mathbb{R}} e^{-\frac{1}{2|\sigma_2|^2}(y_2 - |\sigma_2|^2 \boldsymbol{j}t_2)^2} dy_2\, e^{\frac{(|\sigma_2|^2 \boldsymbol{j}t_2)^2}{2|\sigma_2|^2}}.
\end{aligned} \tag{71}$$

Substituting $u_1 = y_1 - |\sigma_1|^2 \boldsymbol{i}t_1$ and $u_2 = y_2 - |\sigma_2|^2 \boldsymbol{j}t_2$ yields

$$\begin{aligned}
\phi_X(t_1, t_2) &= \frac{e^{\boldsymbol{i}t_1|\mu_1|} e^{\boldsymbol{j}t_2|\mu_2|}}{2\pi|\sigma_1||\sigma_2|} e^{-\frac{|\sigma_1|^2 t_1^2}{2}} e^{-\frac{|\sigma_2|^2 t_2^2}{2}} \int_{\mathbb{R}} e^{-\frac{1}{2|\sigma_1|^2} u_1^2} du_1 \int_{\mathbb{R}} e^{-\frac{1}{2|\sigma_2|^2} u_2^2} du_2 \\
&= \frac{e^{\boldsymbol{i}t_1|\mu_1|} e^{\boldsymbol{j}t_2|\mu_2|}}{2\pi|\sigma_1||\sigma_2|} e^{\frac{-|\sigma_1|^2 t_1^2}{2}} e^{-\frac{|\sigma_2|^2 t_2^2}{2}} \sqrt{2\pi|\sigma_1|^2} \sqrt{2\pi|\sigma_2|^2} \\
&= e^{\boldsymbol{i}t_1|\mu_1| - \frac{|\sigma_1|^2 t_1^2}{2}} e^{\boldsymbol{j}t_2|\mu_2| - \frac{|\sigma_2|^2 t_2^2}{2}}.
\end{aligned} \tag{72}$$

Therefore,

$$\begin{aligned}
&\frac{\partial}{\partial t_1} \phi_X(t_1, t_2) = (\boldsymbol{i}|\mu_1| - t_1|\sigma_1|^2) e^{\boldsymbol{i}t_1|\mu_1| - \frac{|\sigma_1|^2 t_1^2}{2}} e^{\boldsymbol{j}t_2|\mu_2| - \frac{|\sigma_2|^2 t_2^2}{2}} \\
&\frac{\partial}{\partial t_2} \phi_X(t_1, t_2) = e^{\boldsymbol{i}t_1|\mu_1| - \frac{|\sigma_1|^2 t_1^2}{2}} (\boldsymbol{j}|\mu_2| - t_2|\sigma_2|^2) e^{\boldsymbol{j}t_2|\mu_2| - \frac{|\sigma_2|^2 t_2^2}{2}} \\
&\frac{\partial^2}{\partial t_1 \partial t_2} \phi_X(t_1, t_2) = (\boldsymbol{i}|\mu_1| - t_1|\sigma_1|^2) e^{\boldsymbol{i}t_1|\mu_1| - \frac{|\sigma_1|^2 t_1^2}{2}} (\boldsymbol{j}|\mu_2| - t_2|\sigma_2|^2) e^{\boldsymbol{j}t_2|\mu_2| - \frac{|\sigma_2|^2 t_2^2}{2}} \\
&\frac{\partial^2}{\partial t_1^2} \phi_X(t_1, t_2) = \left((\boldsymbol{i}|\mu_1| - t_1|\sigma_1|^2)^2 - |\sigma_1|^2\right) e^{\boldsymbol{i}t_1|\mu_1| - \frac{|\sigma_1|^2 t_1^2}{2}} e^{\boldsymbol{j}t_2|\mu_2| - \frac{|\sigma_2|^2 t_2^2}{2}} \\
&\frac{\partial^2}{\partial t_2^2} \phi_X(t_1, t_2) = \left(\boldsymbol{i}|m| - (t|\sigma|^2)^3 - 3|\sigma|^2(\boldsymbol{i}|m| - t|\sigma|^2)\right) e^{\boldsymbol{i}t|m| - \frac{|\sigma|^2 t^2}{2}}.
\end{aligned} \tag{73}$$

Due to Equation (67), we obtain

$$\begin{aligned} Cov(X_1,X_2) &= \boldsymbol{i}\frac{\partial^2}{\partial t_1 \partial t_2}\phi_{X}(\mathbf{0})\boldsymbol{j} - \left(\boldsymbol{i}\frac{\partial}{\partial t_1}\phi_{X}(0)\frac{\partial}{\partial t_2}\phi_{X}(0)\boldsymbol{j}\right) \\ &= \boldsymbol{i}(\boldsymbol{i}|\mu_1|\boldsymbol{j}|\mu_2|)\boldsymbol{j} - \left(\boldsymbol{i}^2|\mu_1|\boldsymbol{j}|\mu_2|\boldsymbol{j}\right) \\ &= \boldsymbol{i}^2\boldsymbol{j}^2|\mu_1||\mu_2| - |\mu_2||\mu_1| \\ &= |\mu_1||\mu_2| - |\mu_2||\mu_1| \\ &= 0. \end{aligned}$$

Remark 3. *According to Equation (68), one has*

$$\begin{aligned} Cov(X_2,X_1) &= \boldsymbol{i}\frac{\partial^2}{\partial t_1 \partial t_2}\phi_{X}(\mathbf{0})\boldsymbol{j} - \left(\frac{\partial}{\partial t_2}\phi_{X}(0)\boldsymbol{k}\frac{\partial}{\partial t_1}\phi_{X}(0)\right) \\ &= \boldsymbol{i}^2\boldsymbol{j}^2|\mu_1||\mu_2| - \boldsymbol{j}|\mu_2|\boldsymbol{k}\boldsymbol{i}|\mu_1| \\ &= |\mu_1||\mu_2| + |\mu_2||\mu_1| \\ &= 2|\mu_1||\mu_2|. \end{aligned}$$

This example shows that $Cov(X_1, X_2)$ in Equation (62) is different from $Cov(X_2, X_1)$ in Equation (63). The above results are summarized in Table 2.

Table 2. Comparison of quaternion probability and classical probability.

Quaternion Probability	Classical Probability
$Cov(x_1,x_2) \neq Cov(x_2,x_1)$	$Cov(X_1,X_2) = Cov(X_2,X_1)$
$E[X_1]E[X_2] \neq E[X_2]E[X_1]$	$E[X_1]E[X_2] = E[X_2]E[X_1]$
$\int_{\mathbb{R}^2} f(\boldsymbol{x})d\boldsymbol{x} = \mathbf{1}$	$\int_{\mathbb{R}^2} f(\boldsymbol{x})d\boldsymbol{x} = 1$
$\lvert\phi_{X}(\boldsymbol{t})\rvert \leq \lvert\mathbf{1}\rvert = 4$	$\lvert\phi_{X}(\boldsymbol{t})\rvert \leq 1$

5. Conclusions and Future Works

In this paper, we have introduced the two-dimensional quaternion Fourier transform (2DQFT) and investigated a new convolution theorem related to this transformation. We demonstrated its utility in quaternion probability modeling and obtained some results that differ from classical probability modeling.

All works reported in this paper are only preliminary results. There are many issues that can be studied through further research. For instance, instead of the continuous quaternion Fourier transform, the discrete quaternion Fourier transform [21,22] may be implemented in the construction of the discrete characteristic function, discrete expectation, and so on. In addition, the generalization of the n-dimensional quaternion Fourier transform allows us to define the characteristic function, expectation, etc., in higher dimensions. Also, in a forthcoming paper, we will investigate the use of two-dimensional quaternion differential equations to solve quaternion-valued differential equations like the quaternion Laplace transform [23].

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