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Wave Surface Symmetry and Petrov Types in General Relativity

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Abstract: This paper presents a brief study of (2-dimensional, spacelike) wave surfaces to a null direction l on a space-time (M, g) and studies how certain imposed symmetries on the set of such wave surfaces can be used to describe other geometrical features of l and (M, g). It is mainly a review of known material but contains some novelties. For example, the brief discussion of the nature of wave surfaces (when viewed geometrically as wave fronts to a null ray direction) in Wave Surfaces Section is new in the sense that although it appeared in the author's work by the present author, it has not, to the best of his knowledge, appeared in this form anywhere else. Further, the work on conical symmetry and plane waves are, to the best of the author's knowledge, original with him from earlier papers and are reviewed here while the work on complete wave surface (sectional curvature-) symmetry is believed to be entirely new. Geometrical use of the sectional curvature function is employed in many places. The consequences of the various symmetry conditions imposed on the collection of all wave surfaces to a null direction spanned by a null vector l are described in terms of l spanning a principal null direction of the Weyl tensor (if non-zero) at the point concerned (in the sense of Petrov and Bel).

Keywords: Petrov type; wave surfaces; symmetry sectional curvature

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Citation: Hall, G. Wave Surface Symmetry and Petrov Types in General Relativity. *Symmetry* **2024**, *16*, 230. https://doi.org/10.3390/sym16020230

Academic Editors: Ugur Camci, Ashfaque H. Bokhari, Bobomurat Ahmedov, Fabio Cardone, Calogero Vetro and Sergei D. Odintsov

Received: 19 December 2023
Revised: 5 February 2024
Accepted: 6 February 2024
Published: 14 February 2024

1. Introduction and Notation

In this paper, M is a 4-dimensional, smooth, connected, Hausdorff manifold with smooth Lorentz metric g, and the pair (M, g) is called a *space-time*. Inner products formed from g are denoted using the symbol $\cdot$. For $m \in M$, $T_m M$ is the tangent space to M at m and the set $\Lambda_m M$ is the 6-dimensional vector space of two forms (referred to here as *bivectors*) at m. A tetrad $\{l, n, x, y\}$ of members of $T_m M$ with l and n null, $l \cdot n = 1$, $x \cdot x = y \cdot y = 1$, and with all other inner products between members of this tetrad zero, is a basis for $T_m M$ called a (real) *null tetrad*, while the basis $l, n, m, \bar{m}$, where $m = 2^{-\frac{1}{2}}(x + iy)$ and a bar denotes complex conjugation, is the associated *complex null tetrad*. A 1-dimensional subspace of $T_m M$, spanned by $u \in T_m M$, is called a *direction* and is a *spacelike* (respectively *timelike*, *null*) direction if u is spacelike (respectively, timelike or null). A 2-dimensional subspace $V \subset T_m M$ (referred to as a *2-space* at m) is called *spacelike* if each non-zero member of V is spacelike, *timelike* if V contains exactly two distinct null directions and *null* if V contains exactly one null direction. It is easily shown that any 2-space at m is either spacelike, timelike, or null. A bivector $F \in \Lambda_m M$ necessarily has an even matrix rank and is called *simple* if this rank is 2 and *non-simple* if it is 4. A simple bivector F may be written, in some basis at m, as $F^{ab} = p^a q^b - q^a p^b$ for independent $p, q \in T_m M$, and the 2-space spanned by p and q is uniquely determined by F and called the *blade* of F. Sometimes one writes, symbolically, $F = p \wedge q$ for F or its blade or just for the 2-space spanned by p and q and, for calculations, $p \wedge q$ is the above expression for F^{ab} (or, with an abuse of notation, for F_{ab}). If this blade is spacelike (respectively, timelike or null), F is called *spacelike* (respectively, *timelike* or *null*). The usual (Hodge) dual of a bivector F is denoted by $\overset{*}{F}$ and F is simple if and only if $\overset{*}{F}$ is simple (and their blades are orthogonal). Always F and $\overset{*}{F}$ are independent in $\Lambda_m M$. This latter result is a consequence of Lorentz signature and is false for the other two signatures for a 4-dimensional manifold.

The Levi-Civita connection ∇ of g leads to a curvature tensor denoted *Riem* with components $R^a{}_{bcd}$ and then to the Ricci tensor *Ricc* given in components by $R_{ab} = R^c{}_{acb}$ and to the Ricci scalar $R = R_{ab}g^{ab}$. From this, one obtains the Weyl (conformal) tensor C with components $C^a{}_{bcd}$ given by

$$R_{abcd} = C_{abcd} + E_{abcd} + \frac{R}{6}B_{abcd}, \tag{1}$$

where

$$E_{abcd} = \frac{1}{2}(\widetilde{R}_{ac}g_{db} - \widetilde{R}_{ad}g_{bc} + \widetilde{R}_{bd}g_{ac} - \widetilde{R}_{bc}g_{ad}), \qquad B_{abcd} = \frac{1}{2}(g_{ac}g_{bd} - g_{ad}g_{bc}), \tag{2}$$

B is the bivector metric with components B_{abcd} and where $\widetilde{R}_{ab} = R_{ab} - \frac{R}{4}g_{ab}$ are the components of the tracefree Ricci tensor $\widetilde{R}$. It is easily checked that the Einstein space condition $\widetilde{R}(m) = 0$ at $m \in M$ is equivalent to the condition $E(m) = 0$. (M, g) is called *flat* if $Riem \equiv 0$ on M, *conformally flat* if $C \equiv 0$ on M, and *vacuum* (or *Ricci-flat*) if it is not flat but $Ricc \equiv 0$ on M. Thus, a vacuum space-time is an Einstein space.

Einstein's field equations take the form

$$G_{ab} \equiv R_{ab} - \frac{R}{2}g_{ab} = T_{ab} \tag{3}$$

where G is the Einstein tensor and T is (up to a constant scaling) the energy–momentum tensor. The symmetry of T in its indices means that it may be classified using Jordan form/Segre type theory as, for example, in [1]. The only forms for T required here are the situations $T \equiv 0 (\Leftrightarrow Ricc \equiv 0)$ on M, which is the vacuum condition on M, and the case when T is the outer product of a null vector field l on M with itself (in components, $T_{ab} = \lambda l_a l_b$ on M for some real-valued function λ on M—the "null fluid" condition).

In this paper, the concept of a wave surface (introduced originally in [2]) is studied and which is a spacelike 2-space S at $p \in M$ interpreted as a wave front for a wave disturbance, the latter represented by one of the two independent null directions orthogonal to S at p (Section 3). The idea is to choose one such null direction l at p and to explore the consequences for l of imposing certain symmetry conditions on the collection of all such wave surfaces at p orthogonal to l (the wave fronts of an observer at p) usually in terms of the sectional curvature of the wave surface. On the other hand, at each $p \in M$, certain null directions at p will, in general, be picked out naturally by the geometry of (M, g). The main ones are those arising from the Weyl tensor if it is non-zero (but there are some from the energy–momentum tensor) and which, through the Einstein field equations, may have a significant physical interpretation. Those from the Weyl tensor (the so-called principal null directions of the Weyl tensor) are closely related to the Petrov classification of the Weyl tensor and are described in the next section. The problem is to see if these latter (principal) null directions coincide with those whose wave surfaces satisfy the above imposed symmetry condition. The geometry of the null directions arising from the Weyl tensor will be described in the next section. The first theorem in this direction arises in Section 5, where a type of conical symmetry will be imposed on the wave surfaces. This was first considered by the present author some years ago [3], while a similar symmetry of wave surfaces is described in Section 6 and is believed to be new. Finally, a more restrictive (but perhaps more interesting) Killing type symmetry on wave surfaces, first given in [4], is considered in Section 8.

2. The Petrov Classification And Principal Null Directions

An algebraic classification of the Weyl tensor C at $m \in M$ was given by Petrov [5] and arises by considering the associated complex Weyl tensor $\overset{+}{C} = C + i\overset{*}{C}$, where again $*$ denotes the Hodge dual operator (for the notation, see, e.g., [1]). This is then regarded as a linear map on the complexification of $\Lambda_m M$. This classification may be shown to be equivalent to a similar procedure on the real tensor $C(m)$ but with shorter calculations. The possible Jordan forms arising then lead to the well-known Petrov types **I**, **D**, **II**, **N**, **III**, and **O** at m, where type **O** signifies that $C(m) = 0$ (full details may be found in [1,5]). For Petrov types **N**, **III**, and **O** (and only for these types), each eigenvalue of this linear map is zero. The Petrov types will, in general, vary from point to point in M (subject, of course, to continuity restrictions) but this will not be important for the purposes of this paper and mostly only the Petrov type at a certain $m \in M$ is required. It is thus convenient to assume that (M, g) has the same Petrov type at every point and this is then referred to as *the Petrov type* of (M, g). If (M, g) has Petrov type **I**, it is referred to as *algebraically general*, while for each of the other types, (M, g) is called *algebraically special*.

An alternative view of this classification emerged soon after Petrov's scheme mainly (but not entirely) through the work of Bel [6] (see [1,7,8] for details) and which is, perhaps, more relevant here and often more convenient for calculations. It is noted that for each Petrov type at m, a study of the algebraic structure of $C(m) \neq 0$ (or, alternatively, $\overset{+}{C}(m)$) leads to certain special null directions being "picked out" by $C(m)$. There are finitely many distinct such null directions at m for each (non-zero) Petrov type (in fact, at most four) and each member of this finite set is itself one of two distinct types. To proceed further, let $0 \neq k \in T_m M$ be *real* and consider the following two equations at m for $\overset{+}{C}(m)$, where square brackets (respectively, round brackets) denote the usual skew-symmetrization (respectively, symmetrization) of indices and where q is some complex 1-form at m,

$$k_{[e}\overset{+}{C}_{a]bc[d}k_{f]}k^b k^c = 0, \qquad \overset{+}{C}_{abcd}k^b k^c = k_a q_d + q_a k_d. \tag{4}$$

By using the intermediate (complex) symmetric tensor $T_{ad} \equiv \overset{+}{C}_{abcd}k^b k^c$, it is straightforward to check that the two conditions in Equation (4) are equivalent. Further, Equation (4) in fact implies that k is null, and if $q \neq 0$, it is orthogonal to k. A (null) vector k satisfying Equation (4) is called a *principal null direction* (*pnd*) for C at m. Next, suppose that $0 \neq k \in T_m M$ is *real* and $\alpha \in \mathbb{C}$, and consider the following two equations at m for $\overset{+}{C}(m)$,

$$k_{[e}\overset{+}{C}_{a]bcd}k^b k^c = 0, \qquad \overset{+}{C}_{abcd}k^b k^c = \alpha k_a k_d. \tag{5}$$

Again, these two equations are equivalent, and again, k is necessarily null. A vector such as k is called a *repeated principal null direction* (*repeated pnd*) for C at m. (The reason for this nomenclature can be found in [6,8].)

These conditions for a null direction being a pnd or a repeated pnd can be simplified by noting that, since k is real, the conditions in Equation (4) and Equation (5) can each be replaced by two analogous real conditions on the tensors $C(m)$ and $\overset{*}{C}(m)$ and to which they are obviously equivalent. These are

$$k_{[e}C_{a]bc[d}k_{f]}k^b k^c = 0, \qquad C_{abcd}k^b k^c = k_a q'_d + q'_a k_d, \tag{6}$$

and

$$k_{[e}\overset{*}{C}_{a]bc[d}k_{f]}k^b k^c = 0, \qquad \overset{*}{C}_{abcd}k^b k^c = k_a q''_d + q''_a k_d, \tag{7}$$

where q' and q'' are real 1-forms with $q = q' + iq''$, (and if $q' \neq 0$ (respectively, $q'' \neq 0$) $l \cdot q' = 0$ (respectively, $l \cdot q'' = 0$)) and

$$k_{[e}C_{a]bcd}k^b k^c = 0, \qquad C_{abcd}k^b k^c = \alpha' k_a k_d, \tag{8}$$

and

$$k_{[e}\overset{*}{C}_{a]bcd}k^b k^c = 0, \qquad \overset{*}{C}_{abcd}k^b k^c = \alpha'' k_a k_d, \tag{9}$$

where $\alpha = \alpha' + i\alpha''$ with $\alpha', \alpha'' \in \mathbb{R}$. However, one can go further and show that if $k \in T_m M$ is *assumed to be null*, the (equivalent) conditions in Equation (4) are equivalent to *either of* the two conditions in Equation (6) (these being equivalent) and to either of the two conditions in Equation (7) (these also being equivalent), and thus one achieves the simplification that the condition Equation (4) for a pnd can be replaced by the first equation in Equation (6) if k is assumed to be null. This is often more convenient for calculations. Similarly, the equivalent conditions in Equation (5) are equivalent to *either of* the two conditions in Equation (8), (these being equivalent) and to either of the two conditions in Equation (9) (these also being equivalent), and so Equation (5) can be replaced by the (more convenient) first equation in Equation (8) if k is assumed to be null. These results can be established by considering the general expression (and notation) for $\overset{+}{C}$ in [8] (or Equation (7.67) in [1]) in terms of a complex null tetrad $l, n, m, \bar{m}$ with $k = l$ (since k is assumed to be null) and its associated complex bivectors and computing $k_{[e}\overset{+}{C}_{a]bc[d}k_{f]}k^b k^c$ and separating it into its real and imaginary parts. That the first equations in each of Equations (6) and (7) are equivalent then follows a simple calculation. A similar argument shows the equivalence of the first equations in each of Equations (8) and (9).

It is clear how a repeated pnd is the special case of a pnd when q in Equation (4) is a (complex) multiple of k (and the fact that pnds arise as roots of a certain quadric equation with repeated pnds corresponding to repeated roots justifies the notation). A pnd which is not a repeated pnd is referred to as a general pnd, and general and repeated pnds give the two distinct types of "special" null directions mentioned above. Each individual Petrov type may be characterized by the number of repeated and non-repeated principal null directions it admits at the point in question, the further details of which are not needed here. They are collectively referred to as the *Bel criteria*. The necessity to be careful about whether k is assumed null or not here can be seen from the following example. Consider the general expression (and notation) for $\overset{+}{C}$ in [8] above in terms of a real null tetrad $l, n, m, \bar{m}$ with $C^1 = C^2 = C^4 = C^5 = 0$ and $0 \neq C^3$ pure imaginary. Then $r \equiv l - n$ is real and timelike but satisfies $C_{abcd}r^a r^c = 0$ (but $\overset{+}{C}_{abcd}r^a r^c$ is *not* (complex) proportional to $r_b r_d$).

For later reference, it is remarked that a similar analysis may be undertaken regarding the tensor E in Equations (1) and (2). In fact, there exists a similar set of criteria for E (or for an analogous complex tensor $\overset{+}{E}$ constructed from E in a similar fashion to the progression from C to $\overset{+}{C}$ [1]). This leads to a set of results for E which can be rather useful in calculations involving the tensor *Ricc*. If Equation (6) holds with C replaced by E and for k null, call k a *principal null direction (pnd) for E*, whereas if Equation (8) holds with C replaced by E and for k null, call k a *repeated principal null direction (repeated pnd) for E*. Then it can be shown that a (real) null vector k is a repeated pnd for E if and only if it is a (real) Ricci eigenvector, that is, $R_{ab}k^b = \alpha g_{ab}k^b = \alpha k_a$ holds at m for $\alpha \in \mathbb{R}$, and that k is a pnd for E if and only if $R_{ab}k^a k^b = 0$ at m [1]. It is also noted that, from Equation (2) and for k null, $B_{abcd}k^a k^c = -\frac{1}{2}k_b k_d$.

3. Wave Surfaces

Let $l \in T_m M$ be null. A *wave surface* to l at m is a spacelike 2-space at m, each non-zero member of which is orthogonal to l. There are infinitely many wave surfaces to l at m and these can be described in terms of a null tetrad $\{l, n, x, y\}$ containing l. First, for $p, q \in T_m M$, $p \wedge q$ is a wave surface to l if and only if p and q are spacelike and $p \cdot l = q \cdot l = 0$. Thus, p and q are linear combinations of l, x, and y, $p = ax + by + cl$ and $q = \alpha x + \beta y + \gamma l$ for $a, b, c, \alpha, \beta, \gamma \in \mathbb{R}$. One may assume that $p \cdot p = q \cdot q = 1$ and $p \cdot q = 0$ (as one always can without changing the 2-space $p \wedge q$), which gives $a^2 + b^2 = \alpha^2 + \beta^2 = 1$ and $a\alpha + b\beta = 0$. If $b = \beta = 0$, then one achieves the contradiction that $p \wedge q$, if defined, is the 2-space $l \wedge x$, which is null. Therefore, at least one of b and β is non-zero. By taking linear combinations of p and q (again without changing the 2-space $p \wedge q$), it follows that we may choose $b = 0 \neq \beta$

and then $p = a'x + c'l$ and $q = \alpha'x + \beta'y + \gamma'l$ for $a', c', \alpha', \beta', \gamma' \in \mathbb{R}$. Clearly, $a' \neq 0$ and so one can arrange that $\alpha' = 0$ and then $\beta' \neq 0$. Thus, one may write $p = x + c''l$ and $q = y + \gamma''l$ ($c'', \gamma'' \in \mathbb{R}$). It follows that, for the given null tetrad l, n, x, y, *any* wave surface may be written as $(x + al) \wedge (y + bl)$ for $a, b \in \mathbb{R}$. Further, when taken in this form with the tetrad fixed, the ordered pair (a, b) is uniquely determined by the wave surface and the collection of all such wave surfaces to l at m, denoted by $W_l(m)$, is in a one-to-one correspondence with $\mathbb{R}^2$. (In fact, $W_l(m)$ is a 2-dimensional, connected submanifold of the Grassmann manifold of 2-spaces at m diffeomorphic to $\mathbb{R}^2$ (but not a closed one since it does not contain its null limit points) [1]).

Some properties of this collection can now be described. First, if $u \in T_m M$ is timelike, the 2-space $u \wedge l$ is a timelike 2-space and its orthogonal complement, W, is a wave surface to l uniquely determined by l and u and called the *instantaneous wave surface to l for (an observer represented by) u at m*, that is, u is orthogonal to each non-zero member of W. Another instantaneous wave surface to l at m arising, as above, from a timelike vector u', is the same as W if and only if l, u and u' are coplanar (see e.g., [9]).

A simple geometrical interpretation of wave surfaces can be seen as follows. Consider an orthonormal reference frame ("observer") $I = \{x, y, z, t\}$ at m with x, y, z unit orthogonal spacelike vectors and t a unit timelike vector orthogonal to x, y, and z and a null vector l at m with components $l^a = (1, 0, 0, 1)$ in I. Under a Lorentz transformation to a new orthonormal frame I' related to I by a spatial rotation in the xz plane and then to yet another orthonormal frame I'' obtained from I' by a Lorentz boost in the z' direction, one sees that the rotation and boost may be chosen so that, in the frame I'', the components of l are a multiple of those in I (that is, l appears to be in the same coordinate direction). The observer I clearly sees (in an obvious notation) $y \wedge z$ as its unique instantaneous wave surface to l at m, whereas I'' sees a different instantaneous wave surface of the form $y \wedge (z + cl)$ ($c \in \mathbb{R}$). In fact, one can find transformations which reveal observers with instantaneous wave surfaces of the general form $(y + dl) \wedge (z + el)$ ($d, e \in \mathbb{R}$). However, starting with the frame I, a Lorentz boost along the x direction to a frame I''' reveals that the components of l in I''' are a multiple of those in I and also that the instantaneous wave surface for I''' is the same as that for I, that is, the null direction and the timelike vectors representing these observers are coplanar. Thus, one obtains the *two*-parameter collection of distinct wave surfaces to l described earlier as the 2-dimensional submanifold $W_l(m)$, each of which is an instantaneous wave surface for some (in fact, infinitely many) observers at m. It is thus clear that $W_l(m)$ can be generated by one wave surface to l, say $x \wedge y$, where x, y are unit orthogonal spacelike vectors at m orthogonal to l, and the action on it by the 2-dimensional subgroup of the Lorentz group given on the null tetrad $\{l, n, x, y\}$ by $l \to l$, $x \to x + \alpha l$, $y \to y + \beta l$, $n \to n - \frac{1}{2}(\alpha^2 + \beta^2)l - \alpha x - \beta y$ for $\alpha, \beta \in \mathbb{R}$. Hence, in order to make sense of a claimed property of a *general* wave surface to a null direction l, such a property must be "invariant" under the above Lorentz subgroup. Then, from the first paragraph of this section, these transformations have the *transitivity* property that for any wave surfaces $W, W' \in W_l(m)$ some such transformation maps W to W'.

4. Sectional Curvature

Let $m \in M$, let V be a *non-null* 2-space at m, and let F be a simple bivector at m whose blade equals V. The *sectional curvature* $\sigma_m(V)$ of V is defined as

$$\sigma_m(V) = \frac{R_{abcd}F^{ab}F^{cd}}{2B_{abcd}F^{ab}F^{cd}} = \frac{R_{abcd}F^{ab}F^{cd}}{2F_{ab}F^{ab}} \tag{10}$$

where, since V is non-null, the denominator in Equation (10) is not zero. The term sectional curvature comes from the fact that one can show the existence of an open neighborhood U of m such that the subset N of all points on those geodesics in U starting from m, whose tangent vector at m lies in V is a 2-dimensional submanifold of N and hence of M. One may then choose U so that the tangent plane to N is everywhere non-null and so N admits an induced metric g' from g. The Gauss curvature of (N, g') at m is then equal to $\sigma_m(V)$. (For further details on sectional curvature see, for example, [1,10]).

5. Conical Symmetry

Considering $m \in M$, choose a orthonormal frame $I = \{x, y, z, t\}$ at m and choose a null vector l at m with components $l^a = (1, 0, 0, 1)$ in I. With an abuse of notation, consider the "cone" of vectors $\{r : r^a = Ax^a + \mu y^a + \nu z^a\}$ "about l in I" at m for a fixed $0 \neq A \in \mathbb{R}$ and for $\mu, \nu \in \mathbb{R}$ with $\mu^2 + \nu^2 = 1$. Then consider the timelike 2-space $V = l \wedge r$ at m spanned by l and r. The sectional curvature of V is, thus, from Equation (10), with $F_{ab} = 2l_{[a}r_{b]}$,

$$\sigma_m(V) = \frac{-R_{abcd}l^a l^c r^b r^d}{A^2}. \tag{11}$$

Now suppose the "conical symmetry" condition holds at m, that is, $\sigma_m(V)$ is independent of μ and ν. Thus, $\sigma_m(V)$ is unchanged by "rotating" r about the spatial direction x of l in I. Then, defining the symmetric tensor T at m by the components $T_{bd} = R_{abcd}l^a l^c$, one sees that $T_{ab}l^b = 0$ and that in the frame I, from Equation (11),

$$A^2T_{11} + 2A\mu T_{12} + 2A\nu T_{13} + \mu^2 T_{22} + 2\mu\nu T_{23} + \nu^2 T_{33} \tag{12}$$

(with $\mu^2 + \nu^2 = 1$) is independent of μ and ν and that $T_{a1} + T_{a4} = 0$. It follows that $T_{12} = T_{13} = T_{23} = T_{24} = T_{34} = 0$, $T_{22} = T_{33}$, and that $T_{11} = T_{44} = -T_{14}$. Thus, one obtains at m

$$R_{abcd}l^a l^c (= T_{bd}) = T_{11}l_b l_d + T_{22}(y_b y_d + z_b z_d) \tag{13}$$

and so $T_{22} = \frac{R_{ab}l^a l^b}{2}$. If one constructs a null tetrad $\{l, n, y, z\}$ from the original l, y, and z and uses the completeness relation $g_{ab} = 2l_{(a}n_{b)} + y_a y_b + z_a z_b$, one finally obtains

$$R_{abcd}l^a l^c = T_{11}l_b l_d + \frac{1}{2}R_{cd}l^c l^d [g_{bd} - 2l_{(b}n_{d)}]. \tag{14}$$

Moreover, (see the end of Section 2) one has, for any null vector k, $B_{abcd}k^a k^c = -\frac{1}{2}k_b k_d$. Finally, one obtains from Equation (2)

$$E_{abcd}l^a l^c = \frac{1}{2}(R_{ac}l^a l^c)g_{bd} - \frac{1}{2}(l_d l'_b + l'_d l_b) \tag{15}$$

where $l'_a = \widetilde{R}_{ab}l^b$. Collecting these results together, it then follows from Equation (1) that

$$C_{abcd}l^a l^c = \kappa l_b l_d + l_b q'_d + q'_b l_d = l_b q_d + q_b l_d \tag{16}$$

for $\kappa \in \mathbb{R}$ and for covectors q' and q at m with $l \cdot q = l \cdot q' = 0$. Then Equation (6) shows that l is a pnd for C at m. Thus, one achieves the following result.

Theorem 1. *The conical symmetry assumption on l at m leads to the consequence that l spans a pnd of $C(m)$.*

If, in addition, one has a vacuum or a null fluid energy–momentum tensor at m, l that is a Ricci eigenvector with zero eigenvalue, $R_{ab}l^b = 0$, and so $R_{ab}l^a l^b = 0$, and l', q and q' become multiples of l. Thus, Equation (8) holds for $C(m)$ and l spans a repeated pnd for $C(m)$ [3]. The common sectional curvature of the 2-spaces involved is $-T_{11} - \frac{b}{2A^2}$, where $b = R_{ab}l^a l^b$. In the event where $R_{ab}l^b = 0$ ($\Rightarrow b = 0$), the common sectional curvature is $-T_{11}$.

6. Wave Surface Symmetry

Suppose now that for $m \in M$, l is a null member of $T_m M$. Another possible geometrical symmetry is the statement that the sectional curvatures of each of the wave surfaces to l, that is, of the members of $W_l(m)$, are equal. Thus, for $V \in W_l(m)$, $\sigma_m(V)$ is "invariant" under the null rotations given at the end of Section 3. Choosing a null tetrad $\{l, n, y, z\}$ about l and an "initial" wave surface $V \equiv y \wedge z$, this statement is $\sigma_m(V) = \sigma_m[(y + al) \wedge (z + bl)]$ for each $a, b \in \mathbb{R}$. That is, from Equation (10),

$$R_{abcd}y^a z^b y^c z^d = R_{abcd}(y^a + al^a)(z^b + bl^b)(y^c + al^c)(z^d + bl^d). \tag{17}$$

This can be unraveled by first taking $b = 0$ (but a arbitrary) and expanding to obtain

$$R_{abcd}l^a z^b l^c z^d = R_{abcd}l^a z^b y^c z^d = 0. \tag{18}$$

Similarly, with $a = 0$ and b being arbitrary, one obtains

$$R_{abcd}l^a y^b l^c y^d = R_{abcd}l^a y^b z^c y^d = 0. \tag{19}$$

A back substitution of Equations (18) and (19) into Equation (17) then gives $R_{abcd}l^a y^b l^c z^d = 0$. Next, define the symmetric tensor T by $T_{bd} \equiv R_{abcd}l^a l^c = T_{db}$ which, from the above, satisfies $T_{ab}y^a y^b = T_{ab}z^a z^b = T_{ab}y^a z^b = 0$, and of course, $T_{ab}l^b = 0$. These conditions imply that, at m,

$$R_{abcd}l^a l^c = T_{bd} = \alpha l_b l_d + 2\beta l_{(b}y_{d)} + 2\gamma l_{(b}z_{d)} \tag{20}$$

for $\alpha, \beta, \gamma \in \mathbb{R}$. From Equation (20), one sees that $R_{ab}l^a l^b = 0$. It thus follows that l spans a pnd for E (see the end of Section 2), that is, $l_{[e}E_{a]bc[d}l_{f]}l^b l^c = 0$. In addition, one has $B_{abcd}l^a l^c = -\frac{1}{2}l_b l_d$ and, hence, $l_{[e}B_{a]bc[d}l_{f]}l^b l^c = 0$. But Equation (20) shows that $l_{[e}R_{a]bc[d}l_{f]}l^b l^c = 0$ and so, as Equations (1) and (6) hold for C, show that l spans a pnd for C.

Theorem 2. *A null direction each of whose wave surfaces has the same sectional curvature at $m \in M$ is a pnd for $C(m)$.*

7. A Brief Discussion of Symmetry in General Relativity

In order to proceed further, one requires some brief introduction to mathematical symmetry in Einstein's theory. For the space-time (M, g), a global smooth vector field X is called *Killing* if, using a semi-colon to denote a covariant derivative with respect to the Levi-Civita connection arising from g,

$$X_{a;b} \equiv F_{ab} = -F_{ba}, \tag{21}$$

where F is the *Killing bivector*. The collection of all Killing vector fields on M is denoted by $K(M)$ and is a (finite-dimensional) Lie algebra under the usual Lie bracket operation. If $X \in K(M)$, let ϕ_t be a (necessarily smooth) *local flow* of X (see, e.g., [1]). Then the pullback $\phi_t^* g = g$ and so ϕ_t is a *local isometry* on M. The following results can be deduced from the theory of Killing vector fields (see, e.g., [1]). First, the subset of points $W \subset M$ at which the subspace $\{X(m) : X \in K(M)\}$ of $T_m M$ is trivial is closed and has an empty interior (in the manifold topology of M). Then for a positive integer k, let $X_1, \ldots, X_k \in K(M)$ with local flow maps $\phi_t^1, \ldots, \phi_t^k$. There is a local diffeomorphism between open subsets of M (where defined) given for $m \in M$ by

$$m \to \phi_{t_1}^1(\phi_{t_2}^2(\cdots \phi_{t_k}^k(m)\cdots)) \tag{22}$$

for all choices of k, $X_1, \ldots, X_k$ and $(t_1, \ldots, t_k) \in \mathbb{R}^k$ under the usual rules of composition and inverses. Then define an equivalence relation $\sim$ on M given for $m_1, m_2 \in M$ by $m_1 \sim m_2 \Leftrightarrow$ some map of the form Equation (22) maps m_1 to m_2. The equivalence classes under $\sim$ are called the *(Killing) orbits* in M and each is a submanifold of M. The orbit through $m \in M$ is labeled O_m.

Next, consider the subspace $\overset{*}{K}_m \subset K(M)$ defined by $\overset{*}{K}_m = \{X \in K(M) : X(m) = 0\}$. $\overset{*}{K}_m$ is easily checked to be a subalgebra of $K(M)$ and is called the *isotropy algebra* of $K(M)$ at m. Thus, the orbits connect points which are metrically indistinguishable, whereas if ϕ_t arises from $X \in \overset{*}{K}_m$, $\phi_t(m) = m$ and the pushforward maps ϕ_{t*} give isomorphisms on $T_m M$, linking metrically indistinguishable tangent vectors. Each such ϕ_{t*} is then a member of the *Lorentz group* at m since $g(u,v)(= \phi_t^* g(u,v)) = g(\phi_{t*}u, \phi_{t*}v)$ and $\overset{*}{K}_m$ is a subalgebra of the Lorentz algebra. The tangent space to O_m, $\{X(m) : X \in K(M)\}$, has dimension equal to that of O_m and so, from a consideration of the linear map $K(M) \to T_m M$ given by $X \to X(m)$, elementary linear algebra reveals that $\dim O_m + \dim \overset{*}{K}_m = \dim K(M)$. An orbit O is called *proper* if $1 \leq \dim O \leq 3$ and *dimensionally stable* if, given $m \in O$, there exists an open neighborhood $U \subset M$ of m with the property that the Killing orbit through any point of U has dimension equal to that of O (that is, if "nearby" orbits have the same dimension as O). If $\dim O_m = 4$, for each $m \in M$ there is a single (4-dimensional) orbit and $K(M)$ is called *transitive* and (M, g) is called *homogeneous*.

It is remarked for future use the well-known result that if $\dim \overset{*}{K}_m \geq 3$, the Weyl tensor vanishes at m (see, e.g., [1]).

8. Plane Waves

Let M be $\mathbb{R}^4$ with global coordinates u, v, x, y and consider the metric given by

$$ds^2 = H(x,y,u)du^2 + 2dudv + dx^2 + dy^2 \tag{23}$$

where $H(x,y,u) = a(u)x^2 + b(u)y^2 + c(u)xy$ for functions a, b, and c of u only. It can be checked that the Ricci tensor *Ricc* satisfies the null fluid condition (possibly vacuum form) $R_{ab} = d(u)l_a l_b$ for some function d. Such metrics are usually known as *plane waves* and have been widely discussed (see [1,2,4,7,9] and references contained therein). They are intended to describe a wave motion along the null direction l with local expression $l = \partial/\partial v$ ($l_a = u_{,a}$). The wave surfaces to l at $m \in M$ represent the "plane wave surfaces" to l and contain the instantaneous wave surface to l for each observer at m. The *vacuum* condition $Ricc \equiv 0$ on M (that is, $d \equiv 0$ on M) is equivalent to $a + b \equiv 0$ on M white the conformally flat condition $C \equiv 0$ on M is equivalent to $a \equiv b$ and $c \equiv 0$, on M. Each of these possibilities can occur [7].

Next, suppose that (M, g) is a space-time which is not flat and which admits a Killing algebra $K(M)$ such that at each $m \in M$ there exists a *unique* null direction spanned by a null vector $l \in T_m M$ (referred to as the *wave direction*) and which has the property that the transformations ϕ_{t*} arising from the members of the isotropy algebra $\overset{*}{K}_m$ at m are *transitive* on the set of all wave surfaces $W_l(m)$ to l at m. By this, it is meant that given any two wave surfaces W_1 and W_2 to l at m, some such ϕ_{t*} maps W_1 to W_2. This can be shown to force $\phi_{t*}l$ to be a multiple of l (and that l spans the only direction at m with this property) and expresses the condition that the members of $W_l(m)$ are metrically indistinguishable, and hence, that they have the same sectional curvature (and it then follows from the work of Section 6 that l must span a pnd of the Weyl tensor $C(m)$ at m if this latter is non-zero). In order to remove possibly pathological examples, it will be assumed that all Killing orbits are either 4-dimensional or, if proper, that they are dimensionally stable, and that at no $m \in M$ does *Riem* satisfy the constant curvature condition $Riem(m) = \frac{R(m)}{6}B(m)$ (see Section 1).

Suppose also that $C(m) \neq 0$ so that l is a pnd of $C(m)$. If $X \in \overset{*}{K}_m$, then the local flow maps ϕ_t associated with X satisfy $\phi_t(m) = m$ and $\mathcal{L}_X g = g$, and so $\mathcal{L}_X C = 0$ and $\mathcal{L}_X Ricc = 0$. Since the pnds for $C(m)$ form a finite (discrete) subset of $T_m M$, the condition $\mathcal{L}_X C = 0$ reveals (again) that $\phi_{t*}l$ is a multiple of l. In the case that $C(m) = 0$, it will be seen that a null fluid form for the Ricci/energy–momentum tensor results with null fluid direction l', say, and that this direction is unique. Thus, again, one obtains from the condition $\mathcal{L}_X Ricc = 0$ that $\phi_{t*}l'$ is a multiple of l' (and from this it can be shown that if, in addition, $C(m) \neq 0$ with pnd l, the directions l' and l coincide). One can now, with a little effort and using the information of this and the last paragraph, rule out, systematically, many of the subalgebras of the Lorentz algebra as possibilities for $\overset{*}{K}_m$, and it can be shown that any proper orbit of $K(M)$ is 3-dimensional and null. Thus, any orbit is either 3-dimensional and null, or 4-dimensional. The subalgebra $\overset{*}{K}_m$ is either 2-dimensional or 3-dimensional and $\dim K(M) \geq 5$. The possibilities remaining are that

(i) M admits a single 4-dimensional orbit (and so (M, g) is *homogeneous*) and for $m \in M$ either $\dim\overset{*}{K}_m = 2$ and $\dim K(M) = 6$, or $\dim\overset{*}{K}_m = 3$ and $\dim K(M) = 7$; or
(ii) Each orbit is 3-dimensional and null with normal everywhere parallel to the wave direction and either $\dim\overset{*}{K}_m = 2$ and $\dim K(M) = 5$, or $\dim\overset{*}{K}_m = 3$ and $\dim K(M) = 6$.

The space-times in (ii) above are the *non-homogeneous plane waves* and satisfy $R = 0$ and admit a local, covariantly constant, null vector field l parallel at each point to the wave direction. The energy–momentum tensor is either zero (the vacuum case) or of the null fluid form, and in this latter case the unique null direction of this fluid is parallel to l. In the case $\dim K(M) = 5$ (respectively, $\dim K(M) = 6$), C is either zero or of Petrov type **N** (respectively, $C \equiv 0$ on M). If $C(m)$ is Petrov type **N**, then its repeated pnd is parallel to l.

For the space-times in (i), if the (necessarily constant) R vanishes, either $\dim\overset{*}{K}_m = 2$ and $\dim K(M) = 6$ (with Petrov type **N** or **O**), or $\dim\overset{*}{K}_m = 3$ and $\dim K(M) = 7$ (and Petrov type **O**) and a local covariantly constant, null vector field l is admitted parallel at each point to the wave direction. The space-time is either vacuum or of the null fluid type, and in this latter case, the null fluid direction is proportional to l. If C is type **N**, then the unique pnd is parallel to l. These are the *homogeneous plane waves*. If, however, $R \neq 0$, the homogeneous metrics of Defrise are encountered [11] (see also [7]), which are null fluids with a cosmological constant. They are of Petrov type **N** with $\dim K(M) = 6$, $\dim\overset{*}{K}_m = 2$ for each $m \in M$ and they admit no (local or global) covariantly constant vector fields.

Thus, the assumption of transitivity of ϕ_{t*} on the collection of wave surfaces to l leads, apart from the Defrise metrics, to the plane waves. Conversely, any of the above plane waves has an isotropy algebra $\overset{*}{K}_m$ at any $m \in M$ of dimension 2 or 3 and also gives rise to a locally smooth null vector field on M, either through the (unique repeated) pnd of the Weyl tensor (if this latter is non-zero) or the (unique) null eigendirection of the null fluid energy–momentum tensor (if not vacuum) and which, if each is defined, coincide. These then collectively define the wave direction on M and which is preserved by the associated maps ϕ_{t*} from $\overset{*}{K}_m$. If the Killing orbits are dimensionally stable, 3-dimensional, and null, the normal to them is, by inspection, parallel to the wave direction defined above, and say spanned by a local null vector field l. Then for any $X \in \overset{*}{K}_m$, $X^a l_a = 0$ (since X is tangent to this orbit) and $X(m) = 0$, and a differentiation gives $F_{ab}l^b = 0$ at m for each Killing bivector associated with a member of $\overset{*}{K}_m$. A consideration of the bivector representations of the subalgebras of the Lorentz algebra (see, e.g., the table of such subalgebras in [1]) shows that this leads to $\overset{*}{K}_m$ being either the 2-dimensional subgroup described at the end of Section 3 or to a 3-dimensional subgroup which contains it as a subgroup, and hence to the wave surface transitivity property on $W_l(m)$.

If the orbits are 4-dimensional (the homogeneous case, including the Defrise metric [11]) and if $\dim K(M) = 6$, then $\overset{*}{K}_m$ is the subgroup described at the end of Section 3 (since the other possible subgroups would not allow for a type **N** Weyl tensor or a null fluid type energy–momentum tensor ([1], p. 302)). If $\dim K(M) = 7$, then $\dim\overset{*}{K}_m = 3$ and the Weyl tensor must vanish everywhere and a (non-trivial) null fluid results (to avoid flatness). The isotropy algebra must then contain the subalgebra from Section 3 ([1], p. 302). Again, transitivity holds.

In this sense, the above assumption of transitivity on wave surfaces to a null direction is (roughly speaking and recalling the clauses to remove pathological cases) equivalent to the plane wave assumption (and including the Defrise metrics [11] in the latter). It is remarked here that in [9] it was claimed (see the beginning of Section 6 of that reference) that a certain assumption on a function occurring in the expression for the Weyl tensor was sufficient for the metric discussed there to be a plane wave. This assumption should, in fact, be augmented by an assumption on the Ricci tensor analogous to the condition $d = d(u)$ given at the beginning of this section.

9. Conclusions

In conclusion, it has been shown that certain "continuous" symmetries applied to the wave surfaces of a given null direction at p force that null direction to be a special (that is, a principal) null direction of the Weyl tensor at p. In addition, it is shown that the well-known plane waves are essentially characterized by the extreme (Killing) symmetry condition on wave surfaces described above.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analysed in this study. Data sharing is not applicable to this article.

Acknowledgments: The author wishes to thank Bahar Kırık Rácz for her careful reading of the manuscript.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. Hall, G.S. *Symmetries and Curvature Structure in General Relativity*; World Scientific: Singapore, 2004.
2. Ehlers, J.; Kundt, W. Gravitation. In *An Introduction to Current Research*; Witten, L., Ed.; Wiley: New York, NY, USA, 1962; p. 49.
3. Hall, G.S. On the Petrov classification of gravitational fields. *J. Phys. A.* **1973**, *6*, 619. [CrossRef]
4. Hall, G.S. Celebrating 100 Years of Einstein's Theory of General Relativity. In Proceedings of the International Conference on Relativistic Astrophysics, Lahore, Pakistan, 10–14 February 2015.
5. Petrov, A.Z. *Einstein Spaces*; Pergamon Press: Oxford, UK, 1969. (see also Petrov, A.Z. The classification of spaces defining gravitational fields. *Gen. Rel. Grav.* **2000**, *32*, 1665–1685) [CrossRef]
6. Bel, L. Radiation states and the problem of energy in general relativity. *Cah. Phys.* **1962**, *16*, 59. (See Bel, L. Radiation states and the problem of energy in general relativity. *Gen. Rel Grav.* **2000**, *32*, 2047) [CrossRef]
7. Stephani, H.; Kramer, D.; MacCallum, M.A.H.; Hoenselaers, C.; Herlt, E. *Exact Solutions to Einstein's Field Equations*, 2nd ed.; Cambridge University Press: Cambridge, UK, 2003.
8. Sachs, R.K. Gravitational waves in general relativity. VI. The outgoing radiation condition. *Proc. Roy. Soc.* **1961**, *264*, 309–338.
9. Hall, G.S. Some remarks on wave solutions in general relativity theory. *Arab Journ. Maths.* **2019**, *8*, 171–181. [CrossRef]
10. Hall, G.S. *Four-Dimensional Manifolds and Projective Structure*; Chapman and Hall : London, UK, 2023.
11. Defrise, L. Groupes d'Isotropie et Groupes de Stabilite Conforme dans les Espaces Lorentziens. Ph.D Thesis, Universite Libre de Bruxelles, Brussels, Belgium, 1969.