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On Conformable Fractional Milne-Type Inequalities

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Abstract: Building upon previous research in conformable fractional calculus, this study introduces a novel identity. Using this identity as a foundation, we derive a set of conformable fractional Milne-type inequalities specifically designed for differentiable convex functions. The obtained results recover some existing inequalities in the literature by fixing some parameters. These novel contributions aim to enrich the analytical tools available for studying convex functions within the realm of conformable fractional calculus. The derived inequalities reflect an inherent symmetry characteristic of the Milne formula, further illustrating the balanced and harmonious mathematical structure within these frameworks. We provide a thorough example with graphical representations to support our findings, offering both numerical insights and visual confirmation of the established inequalities.

Keywords: conformable fractional integral operator; Milne-type inequalities; convex functions; Hölder inequality; power mean inequality

MSC: 65D32; 26A45; 26D10; 26D15

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1. Introduction

Convexity is a simple and natural notion. A function $\mathcal{K}$ is characterized as convex when it satisfies the condition that, for all $u, v \in I$ and $\gamma \in [0,1]$, the inequality

$$\mathcal{K}(\gamma u + (1-\gamma)v) \leq \gamma\mathcal{K}(u) + (1-\gamma)\mathcal{K}(v)$$

holds, as outlined in [1].

The concept of convexity stands as a cornerstone with profound implications across various disciplines, each underscored by extensive research and practical applications. In mathematics, convexity provides a fundamental framework for analyzing the geometrical properties of sets and functions, shaping the foundation of optimization theory. This mathematical principle extends its influence into economics, where convex optimization plays a crucial role in modeling and solving economic problems, as discussed in works such as [2]. In optimization, convexity is pivotal for the development of efficient algorithms that find optimal solutions, with references to key algorithms available in [3]. Furthermore, convexity is a central concept in game theory, providing insights into strategic interactions among rational decision-makers, as explored in seminal works such as [4]. The versatility of convexity makes it an indispensable notion, contributing significantly to mathematical theories and finding practical applications in decision sciences, resource allocation, and risk management [3].

Integral inequalities, crucial in diverse mathematical and scientific domains, share a pivotal connection with convexity. This relationship has led to the establishment of numerous results related to various quadrature formulas; see [5–7] and the references therein.

The fundamental inequality associated with the concept of convexity is the Hermite–Hadamard inequality [8], formulated as follows. Let $\mathcal{K}$ be a convex function on $[\vartheta_1, \vartheta_2]$; then, we have

$$\mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \leq \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\mathcal{K}(u)du \leq \frac{\mathcal{K}(\vartheta_1)+\mathcal{K}(\vartheta_2)}{2}. \tag{1}$$

Regarding historical considerations about inequality (1), we refer readers to [8–10] and the references therein.

Among the three-point Newton–Cotes quadrature formulas, Milne's formula stands out as a specific expression. It is defined by the following approximation of the integral of a function over a given interval:

$$\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\mathcal{K}(u)du = \frac{1}{3}\left(2\mathcal{K}(\vartheta_1) - \mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + 2\mathcal{K}(\vartheta_2)\right) + \mathcal{R}(\vartheta_1, \vartheta_2, \mathcal{K}),$$

where $\mathcal{R}$ denotes the approximation error; see [11].

In [12], Djenaoui et al. established Milne-type inequalities for differentiable convex functions, as follows:

$$\begin{aligned}&\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1) - \mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + 2\mathcal{K}(\vartheta_2)\right) - \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\mathcal{K}(u)du\right| \\ &\leq \frac{\vartheta_2-\vartheta_1}{24}\left(3\left|\mathcal{K}'(\vartheta_1)\right| + 4\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right| + 3\left|\mathcal{K}'(\vartheta_2)\right|\right)\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1) - \mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + 2\mathcal{K}(\vartheta_2)\right) - \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\mathcal{K}(u)du\right| \\ &\leq \frac{5(\vartheta_2-\vartheta_1)}{24}\left(\left|\mathcal{K}'(\vartheta_1)\right| + \left|\mathcal{K}'(\vartheta_2)\right|\right).\end{aligned}$$

Fractional calculus, an area of mathematical analysis that extends traditional differentiation and integration to non-integer orders, has become increasingly significant across various scientific disciplines. This approach serves as a robust mathematical framework for describing systems with nonlocal or memory-dependent behavior. Its utility ranges from the analysis of anomalous diffusion in physics [13] to modeling intricate biological processes [14]. As researchers delve deeper into its intricacies, fractional calculus continues to prove invaluable in understanding and characterizing phenomena that classical methods may overlook.

Definition 1 ([15]). *Let $\mathcal{K} \in L^1[\vartheta_1, \vartheta_2]$. The Riemann–Liouville fractional integrals $I^{\alpha}_{\vartheta_1^+}\mathcal{K}$ and $I^{\alpha}_{\vartheta_2^-}\mathcal{K}$ of order $\alpha > 0$ with $\vartheta_1 \geq 0$ are defined by*

$$\begin{aligned}I^{\alpha}_{\vartheta_1^+}\mathcal{K}(x) &= \frac{1}{\Gamma(\alpha)}\int_{\vartheta_1}^{x}(x-t)^{\alpha-1}\mathcal{K}(t)dt, \quad x > \vartheta_1, \\ I^{\alpha}_{\vartheta_2^-}\mathcal{K}(x) &= \frac{1}{\Gamma(\alpha)}\int_{x}^{\vartheta_2}(t-x)^{\alpha-1}\mathcal{K}(t)dt, \quad \vartheta_2 > x,\end{aligned}$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$ *is the gamma function and* $I^0_{\vartheta_1^+}\mathcal{K}(x) = I^0_{\vartheta_2^-}\mathcal{K}(x) = \mathcal{K}(x)$.

In the realm of fractional calculus, several scholars have devoted their endeavors to establishing a variety of integral inequalities. Noteworthy contributions in this regard can be found in [5,16–20], along with the additional references contained therein.

In [21], Budak et al. extended the result obtained in [12] to Riemann–Liouville integral operators, as follows:

$$\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1) - \mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + 2\mathcal{K}(\vartheta_2)\right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\vartheta_2-\vartheta_1)^\alpha}\left[I^\alpha_{\vartheta_1^+}\mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + I^\alpha_{\vartheta_2^-}\mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\right|$$
$$\le \frac{\vartheta_2-\vartheta_1}{12}\left(\frac{\alpha+4}{\alpha+1}\right)\left(|\mathcal{K}'(\vartheta_1)| + |\mathcal{K}'(\vartheta_2)|\right),$$

where $|\mathcal{K}'|$ is convex on $[\vartheta_1, \vartheta_2]$.

Other results can be discussed through Hölder's inequality and the power mean inequality.

Theorem 1 (Hölder inequality [22])**.** *Let* $p, q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If* f *and* g *are real functions defined on* $[a, b]$ *and if* $|f|^p$ *and* $|g|^q$ *are integrable functions on* $[a, b]$, *then*

$$\int_a^b |f(u)g(u)|du \le \left(\int_a^b |f(u)|^p du\right)^{\frac{1}{p}} \left(\int_a^b |g(u)|^q du\right)^{\frac{1}{q}}.$$

Theorem 2 (Power Mean Integral inequality [22])**.** *Let* $q \ge 1$. *If* f *and* g *are real functions defined on* $[a, b]$ *and if* $|f|$ *and* $|f||g|^q$ *are integrable functions on* $[a, b]$, *then*

$$\int_a^b |f(u)g(u)|du \le \left(\int_a^b |f(u)|du\right)^{1-\frac{1}{q}} \left(\int_a^b |f(u)||g(u)|^q du\right)^{\frac{1}{q}}.$$

Lemma 1 (Discrete Power Mean inequality [23])**.** *For any* $u, v \ge 0$ *and* $0 \le \varepsilon \le 1$, *we have*

$$u^\varepsilon + v^\varepsilon \le 2^{1-\varepsilon}(u+v)^\varepsilon.$$

To better describe certain phenomena that classical fractional operators fail to model, several new conformable fractional operators have been introduced [24,25]. Among these, the operator proposed by Jarad et al. in [26] stands out as a generalization of various operators, including the Riemann–Liouville and Hadamard operators.

Definition 2 ([26])**.** *Let* $\mathcal{K} \in L[\vartheta_1, \vartheta_2]$. *The left- and right-sided conformable fractional integral operators of order* $\beta \in \mathbb{C}$ *with* $Re(\beta) > 0$ *and* $\alpha \in (0, 1]$ *are expressed as follows:*

$$^\beta\mathcal{J}^\alpha_{\vartheta_1^+}\mathcal{K}(u) = \frac{1}{\Gamma(\beta)}\int_{\vartheta_1}^u \left(\frac{(u-\vartheta_1)^\alpha - (t-\vartheta_1)^\alpha}{\alpha}\right)^{\beta-1}(t-\vartheta_1)^{\alpha-1}\mathcal{K}(t)dt, \quad u > \vartheta_1,$$
$$^\beta\mathcal{J}^\alpha_{\vartheta_2^-}\mathcal{K}(u) = \frac{1}{\Gamma(\beta)}\int_u^{\vartheta_2} \left(\frac{(\vartheta_2-u)^\alpha - (\vartheta_2-t)^\alpha}{\alpha}\right)^{\beta-1}(\vartheta_2-t)^{\alpha-1}\mathcal{K}(t)dt, \quad \vartheta_2 > u.$$

In the wake of the introduction of these novel operators, a plethora of research endeavors has been undertaken to establish inequalities tailored to this class of integrals. Researchers have explored diverse avenues, contributing to the development of a rich landscape of inequalities. Set et al. laid the foundation by presenting Hermite–Hadamard and trapezium inequalities applicable to differentiable convex functions in [27]. Subsequently, Ostrowski-type inequalities were derived in [28], expanding the toolkit for analyzing this class of fractional integrals. Hyder et al. made significant contributions by introducing midpoint-type inequalities in [29]. In [30], Kara et al. extended these efforts by providing midpoint-type and trapezoid-type inequalities specifically tailored for twice-differentiable convex functions.

Further variations and insights into this realm were presented by Hezenci et al. in [31], where Simpson-type inequalities were established, while Ünal et al. contributed Simpson second formula inequalities in [32]. Rashid et al. explored the Minkowski inequality in [33], adding another dimension to the spectrum of inequalities for conformable fractional integrals. Rahman et al. delved into Grüss inequality in [34] and Chebyshev inequality in [35], further enriching the landscape of mathematical tools applicable to this type of integrals. Nisar et al. continued the exploration of the Minkowski inequality in [36], providing additional insights into its applicability.

This research focus was extended to explore specific inequalities, such as the Hermite–Jensen–Mercer inequality investigated by Butt et al. in [37] and the examination of Pachpatte inequality by Akdemir et al. in [38]. These efforts have collectively contributed to a comprehensive understanding of inequalities related to conformable fractional integrals. For those interested in delving further into this topic, additional related works are available for reference in [39–46].

Building upon the insights gleaned from the previously mentioned works, particularly those outlined in [12,21], our current study introduces a novel identity. This identity, rooted in the symmetric principles inherent in the Milne formula, serves as the foundation for establishing a set of new conformable fractional Milne-type inequalities designed for differentiable convex functions. The symmetry in this context not only enhances the mathematical elegance of these inequalities, but also reinforces their theoretical validity. To validate the precision of our findings, we offer a comprehensive example complemented by graphical representations that substantiate the obtained outcomes and visually demonstrate the symmetrical nature of these results.

The rest of this paper is structured as follows. In Section 2, we introduce a new identity, from which we proceed to derive conformable fractional integral inequalities. An illustrative example is presented to demonstrate the accuracy of the obtained results. Section 3 discusses applications to composite quadrature formulas. Finally, the conclusion is presented in Section 4.

2. Main Results

Lemma 2. *Let $\mathcal{K} : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I, $\vartheta_1, \vartheta_2 \in I$ with $\vartheta_1 < \vartheta_2$. If $\mathcal{K}' \in L^1[\vartheta_1, \vartheta_2]$, then the following equality holds for $\beta > 0$ and $\alpha \in (0,1]$:*

$$\begin{aligned}
&\frac{1}{3}\left(2\mathcal{K}(\vartheta_1) - \mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + 2\mathcal{K}(\vartheta_2)\right) - \frac{\alpha^\beta 2^{\beta\alpha-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta\alpha}}\mathcal{W}(\alpha,\beta,\mathcal{K}) \\
&= \frac{\alpha^\beta(\vartheta_2-\vartheta_1)}{4}\left(\int_0^1 \left(\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta - \frac{4}{3\alpha^\beta}\right)\mathcal{K}'\left((1-\gamma)\vartheta_1 + \gamma\frac{\vartheta_1+\vartheta_2}{2}\right)d\gamma\right. \\
&\quad \left. - \int_0^1 \left(\left(\frac{1-\gamma^\alpha}{\alpha}\right)^\beta - \frac{4}{3\alpha^\beta}\right)\mathcal{K}'\left((1-\gamma)\frac{\vartheta_1+\vartheta_2}{2} + \gamma\vartheta_2\right)d\gamma\right), \qquad (2)
\end{aligned}$$

where

$$\mathcal{W}(\alpha,\beta,\mathcal{K}) = {}^\beta\mathcal{J}^\alpha_{\left(\frac{\vartheta_1+\vartheta_2}{2}\right)^-}\mathcal{K}(\vartheta_1) + {}^\beta\mathcal{J}^\alpha_{\left(\frac{\vartheta_1+\vartheta_2}{2}\right)^+}\mathcal{K}(\vartheta_2). \qquad (3)$$

Proof. Let

$$I_1 = \int_0^1 \left(\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta - \frac{4}{3\alpha^\beta}\right)\mathcal{K}'\left((1-\gamma)\vartheta_1 + \gamma\frac{\vartheta_1+\vartheta_2}{2}\right)d\gamma$$

and

$$I_2 = \int_0^1 \left(\left(\frac{1-\gamma^\alpha}{\alpha} \right)^\beta - \frac{4}{3\alpha^\beta} \right) \mathcal{K}'\left((1-\gamma)\frac{\vartheta_1+\vartheta_2}{2} + \gamma\vartheta_2 \right) d\gamma.$$

Integrating by parts I_1, we obtin

$$\begin{aligned}
I_1 &= \frac{2}{\vartheta_2-\vartheta_1} \left(\left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^\beta - \frac{4}{3\alpha^\beta} \right) \mathcal{K}\left((1-\gamma)\vartheta_1 + \gamma\frac{\vartheta_1+\vartheta_2}{2} \right) \Bigg|_0^1 \\
&\quad - \frac{2\beta}{\vartheta_2-\vartheta_1} \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\beta-1} (1-\gamma)^{\alpha-1} \mathcal{K}\left((1-\gamma)\vartheta_1 + \gamma\frac{\vartheta_1+\vartheta_2}{2} \right) d\gamma \\
&= -\frac{2}{3\alpha^\beta(\vartheta_2-\vartheta_1)} \mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2} \right) + \frac{8}{3\alpha^\beta(\vartheta_2-\vartheta_1)} \mathcal{K}(\vartheta_1) \\
&\quad - \frac{2\beta}{\vartheta_2-\vartheta_1} \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\beta-1} (1-\gamma)^{\alpha-1} \mathcal{K}\left((1-\gamma)\vartheta_1 + \gamma\frac{\vartheta_1+\vartheta_2}{2} \right) d\gamma \\
&= -\frac{2}{3\alpha^\beta(\vartheta_2-\vartheta_1)} \mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2} \right) + \frac{8}{3\alpha^\beta(\vartheta_2-\vartheta_1)} \mathcal{K}(\vartheta_1) \\
&\quad - \left(\frac{2}{\vartheta_2-\vartheta_1} \right)^{\beta\alpha+1} \beta \int_{\vartheta_1}^{\frac{3\vartheta_1+\vartheta_2}{4}} \left(\frac{\left(\frac{\vartheta_2-\vartheta_1}{2}\right)^\alpha - \left(\frac{\vartheta_1+\vartheta_2}{2} - u\right)^\alpha}{\alpha} \right)^{\beta-1} \left(\frac{\vartheta_1+\vartheta_2}{2} - u \right)^{\alpha-1} \mathcal{K}(u) du \\
&= \frac{8}{3\alpha^\beta(\vartheta_2-\vartheta_1)} \mathcal{K}(\vartheta_1) - \frac{2}{3\alpha^\beta(\vartheta_2-\vartheta_1)} \mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2} \right) - \frac{2^{\beta\alpha+1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta\alpha+1}} \left({}^{\beta}\mathcal{J}^{\alpha}_{\frac{\vartheta_1+\vartheta_2}{2}} f(\vartheta_1) \right).
\end{aligned} \tag{4}$$

Similarly, we have

$$\begin{aligned}
I_2 &= \frac{2}{\vartheta_2-\vartheta_1} \left(\left(\frac{1-\gamma^\alpha}{\alpha} \right)^\beta - \frac{4}{3\alpha^\beta} \right) \mathcal{K}\left((1-\gamma)\frac{\vartheta_1+\vartheta_2}{2} + \gamma\vartheta_2 \right) \Bigg|_0^1 \\
&\quad + \frac{2\beta}{\vartheta_2-\vartheta_1} \int_0^1 \left(\frac{1-\gamma^\alpha}{\alpha} \right)^{\beta-1} \gamma^{\alpha-1} \mathcal{K}\left((1-\gamma)\frac{\vartheta_1+\vartheta_2}{2} + \gamma\vartheta_2 \right) d\gamma \\
&= -\frac{8}{3\alpha^\beta(\vartheta_2-\vartheta_1)} \mathcal{K}(\vartheta_2) + \frac{2}{3\alpha^\beta(\vartheta_2-\vartheta_1)} \mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2} \right) \\
&\quad + \frac{2^{\alpha\beta+1}\beta}{(\vartheta_2-\vartheta_1)^{\alpha\beta+1}} \int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2} \left(\frac{\left(\frac{\vartheta_2-\vartheta_1}{2}\right)^\alpha - \left(u - \frac{\vartheta_1+\vartheta_2}{2}\right)^\alpha}{\alpha} \right)^{\beta-1} \left(u - \frac{\vartheta_1+\vartheta_2}{2} \right)^{\alpha-1} \mathcal{K}(u) du \\
&= \frac{2}{3\alpha^\beta(\vartheta_2-\vartheta_1)} \mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2} \right) - \frac{8}{3\alpha^\beta(\vartheta_2-\vartheta_1)} \mathcal{K}(\vartheta_2) + \frac{2^{\beta\alpha+1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta\alpha+1}} \left({}^{\beta}_{\frac{\vartheta_1+\vartheta_2}{2}}\mathcal{J}^{\alpha} \mathcal{K}(\vartheta_2) \right).
\end{aligned} \tag{5}$$

By subtracting (5) from (4) and multiplying the resulting equality by $\frac{\alpha^\beta(\vartheta_2-\vartheta_1)}{4}$, we obtain the required outcome. □

Theorem 3. *Let $\mathcal{K}$ be as in Lemma 2. If $|\mathcal{K}'|$ is convex on $[\vartheta_1, \vartheta_2]$, then for $\beta > 0$ and $\alpha \in (0,1]$ we have*

$$\begin{aligned}
&\left| \frac{1}{3}\left(2\mathcal{K}(\vartheta_1) - \mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2} \right) + 2\mathcal{K}(\vartheta_2) \right) - \frac{\alpha^\beta 2^{\beta\alpha-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta\alpha}} \mathcal{W}(\alpha,\beta,\mathcal{K}) \right| \\
&\le \frac{\vartheta_2-\vartheta_1}{4} \left(\frac{2\alpha - 3B\left(\frac{2}{\alpha},\beta+1\right)}{3\alpha} |\mathcal{K}'(\vartheta_1)| + \frac{4\alpha - 6B\left(\frac{1}{\alpha},\beta+1\right) + 6B\left(\frac{2}{\alpha},\beta+1\right)}{3} \left| \mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2} \right) \right| \right. \\
&\quad \left. + \frac{2\alpha - 3B\left(\frac{2}{\alpha},\beta+1\right)}{3\alpha} |\mathcal{K}'(\vartheta_2)| \right),
\end{aligned} \tag{6}$$

where $\mathcal{W}(\alpha,\beta,\mathcal{K})$ is defined by (3) and $B(.,.)$ is the Beta function.

Proof. Taking the absolute value on both sides of (2) and then using the convexity of $|\mathcal{K}'|$, we obtain

$$
\begin{aligned}
&\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1)-\mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+2\mathcal{K}(\vartheta_2)\right)-\frac{\alpha^{\beta}2^{\beta\alpha-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta\alpha}}\mathcal{W}(\alpha,\beta,\mathcal{K})\right| \\
\leq&\frac{\alpha^{\beta}(\vartheta_2-\vartheta_1)}{4}\left(\int_0^1\left(\frac{4}{3\alpha^{\beta}}-\left(\frac{1-(1-\gamma)^{\alpha}}{\alpha}\right)^{\beta}\right)\left|\mathcal{K}'\left((1-\gamma)\vartheta_1+\gamma\frac{\vartheta_1+\vartheta_2}{2}\right)\right|d\gamma\right. \\
&\left.+\int_0^1\left(\frac{4}{3\alpha^{\beta}}-\left(\frac{1-\gamma^{\alpha}}{\alpha}\right)^{\beta}\right)\left|\mathcal{K}'\left((1-\gamma)\frac{\vartheta_1+\vartheta_2}{2}+\gamma\vartheta_2\right)\right|d\gamma\right) \\
\leq&\frac{\alpha^{\beta}(\vartheta_2-\vartheta_1)}{4}\left(\int_0^1\left(\frac{4}{3\alpha^{\beta}}-\left(\frac{1-(1-\gamma)^{\alpha}}{\alpha}\right)^{\beta}\right)\left((1-\gamma)|\mathcal{K}'(\vartheta_1)|+\gamma\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|\right)d\gamma\right. \\
&\left.+\int_0^1\left(\frac{4}{3\alpha^{\beta}}-\left(\frac{1-\gamma^{\alpha}}{\alpha}\right)^{\beta}\right)\left((1-\gamma)\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|+\gamma|\mathcal{K}'(\vartheta_2)|\right)d\gamma\right) \\
=&\frac{\alpha^{\beta}(\vartheta_2-\vartheta_1)}{4}\left(|\mathcal{K}'(\vartheta_1)|\left(\int_0^1\left(\frac{4}{3\alpha^{\beta}}-\left(\frac{1-(1-\gamma)^{\alpha}}{\alpha}\right)^{\beta}\right)(1-\gamma)d\gamma\right)\right. \\
&+\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|\left(\int_0^1\left(\frac{4}{3\alpha^{\beta}}-\left(\frac{1-(1-\gamma)^{\alpha}}{\alpha}\right)^{\beta}\right)\gamma d\gamma+\int_0^1\left(\frac{4}{3\alpha^{\beta}}-\left(\frac{1-\gamma^{\alpha}}{\alpha}\right)^{\beta}\right)(1-\gamma)d\gamma\right) \\
&\left.+|\mathcal{K}'(\vartheta_2)|\left(\int_0^1\left(\frac{4}{3\alpha^{\beta}}-\left(\frac{1-\gamma^{\alpha}}{\alpha}\right)^{\beta}\right)\gamma d\gamma\right)\right) \\
=&\frac{\vartheta_2-\vartheta_1}{4}\left(\frac{2\alpha-3B\left(\frac{2}{\alpha},\beta+1\right)}{3\alpha}|\mathcal{K}'(\vartheta_1)|+\frac{4\alpha-6B\left(\frac{1}{\alpha},\beta+1\right)+6B\left(\frac{2}{\alpha},\beta+1\right)}{3}\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|\right. \\
&\left.+\frac{2\alpha-3B\left(\frac{2}{\alpha},\beta+1\right)}{3\alpha}|\mathcal{K}'(\vartheta_2)|\right),
\end{aligned}
$$

where we have used

$$
\int_0^1\left(\frac{1-(1-\gamma)^{\alpha}}{\alpha}\right)^{\beta}(1-\gamma)d\gamma=\int_0^1\left(\frac{1-\gamma^{\alpha}}{\alpha}\right)^{\beta}\gamma d\gamma=\frac{B\left(\frac{2}{\alpha},\beta+1\right)}{\alpha^{\beta+1}} \tag{7}
$$

and

$$
\int_0^1\left(\frac{1-(1-\gamma)^{\alpha}}{\alpha}\right)^{\beta}\gamma d\gamma=\int_0^1\left(\frac{1-\gamma^{\alpha}}{\alpha}\right)^{\beta}(1-\gamma)d\gamma=\frac{B\left(\frac{1}{\alpha},\beta+1\right)-B\left(\frac{2}{\alpha},\beta+1\right)}{\alpha^{\beta+1}}. \tag{8}
$$

The proof is finished. □

Remark 1. *Taking $\alpha=\beta=1$ results in the reduction of (6) to the first inequality of Corollary 2.4 from [12].*

Corollary 1. *In Theorem 3, using the convexity of $|\mathcal{K}'(\vartheta_1)|$, i.e., $\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|\leq\frac{|\mathcal{K}'(\vartheta_1)|+|\mathcal{K}'(\vartheta_2)|}{2}$, we obtain*

$$
\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1)-\mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+2\mathcal{K}(\vartheta_2)\right)-\frac{\alpha^{\beta}2^{\beta-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta\alpha}}\mathcal{W}(\alpha,\beta,\mathcal{K})\right|
$$

$$\leq \frac{\vartheta_2-\vartheta_1}{4}\left(\frac{4\alpha-3B\left(\frac{1}{\alpha},\beta+1\right)}{3\alpha}\right)\left(\left|\mathcal{K}'(\vartheta_1)\right|+\left|\mathcal{K}'(\vartheta_2)\right|\right). \tag{9}$$

Remark 2. *By taking $\alpha=\beta=1$, Inequality (9) is reduced to the second inequality of Corollary 2.4 from [12]. The same result was obtained by Budak et al. in Remark 1 from [21].*

Corollary 2. *Taking $\alpha=1$ in Theorem 3, we obtain*

$$\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1)-\mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+2\mathcal{K}(\vartheta_2)\right)-\frac{2^{\beta-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta}}\mathcal{R}(\beta,\mathcal{K})\right|$$
$$\leq\frac{\vartheta_2-\vartheta_1}{4(\beta+1)(\beta+2)}\left(\frac{2\beta^2+6\beta+1}{3}\left|\mathcal{K}'(\vartheta_1)\right|+\frac{4\beta^2+6\beta+2}{3}\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|+\frac{2\beta^2+6\beta+1}{3}\left|\mathcal{K}'(\vartheta_2)\right|\right), \tag{10}$$

where

$$\mathcal{R}(\beta,\mathcal{K})=I^{\alpha}_{\left(\frac{\vartheta_1+\vartheta_2}{2}\right)^-}\mathcal{K}(\vartheta_1)+I^{\alpha}_{\left(\frac{\vartheta_1+\vartheta_2}{2}\right)^+}\mathcal{K}(\vartheta_2). \tag{11}$$

Corollary 3. *Taking $\alpha=1$ in Corollary 1, we obtain*

$$\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1)-\mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+2\mathcal{K}(\vartheta_2)\right)-\frac{2^{\beta-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta}}\mathcal{R}(\beta,\mathcal{K})\right|$$
$$\leq\frac{(4\beta+1)(\vartheta_2-\vartheta_1)}{12(\beta+1)}\left(\left|\mathcal{K}'(\vartheta_1)\right|+\left|\mathcal{K}'(\vartheta_2)\right|\right), \tag{12}$$

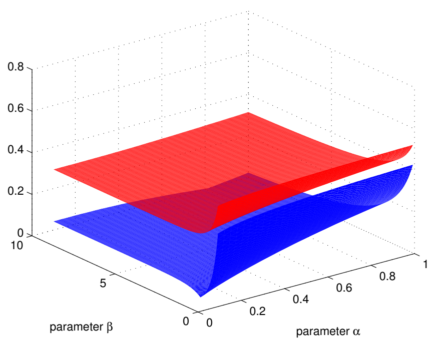
where $\mathcal{R}(\beta,\mathcal{K})$ is defined by (11).

Now, we present an illustrative example supported by graphical representations to substantiate our findings. It is crucial to mention that the figures were created using Matlab, with red representing the right-hand side of the respective inequalities and blue representing the left-hand side.

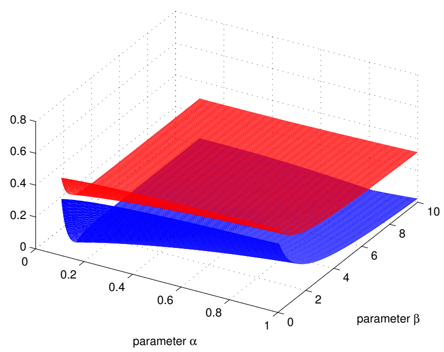
Example 1. *Consider the function $\mathcal{K}$ defined on the interval $[\vartheta_1,\vartheta_2]=[0,1]$ by $\mathcal{K}(u)=u^2$. This function meets the assumptions of our study, as its derivative $\mathcal{K}'(u)=2u$ is convex over the interval $[0,1]$.*

From Theorem 3, we obtain the following inequality, illustrated in Figure 1:

$$\left|\frac{1}{24}-\frac{\beta}{4}\left(B\left(\frac{2}{\alpha}+1,\beta\right)-3B\left(\frac{1}{\alpha}+1,\beta\right)\right)\right|\leq\frac{1}{2\alpha}\left(\frac{4\alpha-B\left(\frac{1}{\alpha},\beta+1\right)}{3}\right).$$



(a) View No. 1.



(b) View No. 2.

Figure 1. Illustration for Theorem 3.

These representations show a consistent trend where the right-hand side is greater than the left-hand side, affirming the accuracy of our results.

Theorem 4. *Let $\mathcal{K}$ be as in Lemma 2. If $|\mathcal{K}'|^q$ is convex on $[\vartheta_1, \vartheta_2]$, then for all $\beta > 0$ and $\alpha \in (0,1]$ we have*

$$
\begin{aligned}
&\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1) - \mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + 2\mathcal{K}(\vartheta_2)\right) - \frac{\alpha^\beta 2^{\beta\alpha-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta\alpha}}\mathcal{W}(\alpha,\beta,\mathcal{K})\right| \\
&\le \frac{\alpha^\beta(\vartheta_2-\vartheta_1)}{4}\left(\int_0^1\left(\frac{4}{3\alpha^\beta} - \left(\frac{1-\gamma^\alpha}{\alpha}\right)^\beta\right)^p d\gamma\right)^{\frac{1}{p}} \\
&\quad\times\left(\left(\frac{|\mathcal{K}'(\vartheta_1)|^q + \left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}} + \left(\frac{\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q + |\mathcal{K}'(\vartheta_2)|^q}{2}\right)^{\frac{1}{q}}\right),
\end{aligned}
\tag{13}
$$

where $\mathcal{W}(\alpha,\beta,\mathcal{K})$ is defined as (3), $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $B(.,.)$ is the Beta function.

Proof. Taking the absolute value on both sides of (2) and then using Hölder's inequality and the fact that $|\mathcal{K}'|^q$ is convex, we deduce

$$
\begin{aligned}
&\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1) - \mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + 2\mathcal{K}(\vartheta_2)\right) - \frac{\alpha^\beta 2^{\beta\alpha-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta\alpha}}\mathcal{W}(\alpha,\beta,\mathcal{K})\right| \\
&\le \frac{\alpha^\beta(\vartheta_2-\vartheta_1)}{4}\left(\int_0^1\left(\frac{4}{3\alpha^\beta} - \left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta\right)\left|\mathcal{K}'\left((1-\gamma)\vartheta_1 + \gamma\frac{\vartheta_1+\vartheta_2}{2}\right)\right|d\gamma\right. \\
&\quad\left.+\int_0^1\left(\frac{4}{3\alpha^\beta} - \left(\frac{1-\gamma^\alpha}{\alpha}\right)^\beta\right)\left|\mathcal{K}'\left((1-\gamma)\frac{\vartheta_1+\vartheta_2}{2} + \gamma\vartheta_2\right)\right|d\gamma\right) \\
&\le \frac{\alpha^\beta(\vartheta_2-\vartheta_1)}{4}\left(\left(\int_0^1\left(\frac{4}{3\alpha^\beta} - \left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta\right)^p d\gamma\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{K}'\left((1-\gamma)\vartheta_1 + \gamma\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_0^1\left(\frac{4}{3\alpha^\beta} - \left(\frac{1-\gamma^\alpha}{\alpha}\right)^\beta\right)^p d\gamma\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{K}'\left((1-\gamma)\frac{\vartheta_1+\vartheta_2}{2} + \gamma\vartheta_2\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right) \\
&\le \frac{\alpha^\beta(\vartheta_2-\vartheta_1)}{4}\left(\int_0^1\left(\frac{4}{3\alpha^\beta} - \left(\frac{1-\gamma^\alpha}{\alpha}\right)^\beta\right)^p d\gamma\right)^{\frac{1}{p}}\left(\left(\int_0^1\left((1-\gamma)|\mathcal{K}'(\vartheta_1)|^q + \gamma\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q\right)d\gamma\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_0^1\left((1-\gamma)\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q + \gamma|\mathcal{K}'(\vartheta_2)|^q\right)d\gamma\right)^{\frac{1}{q}}\right) \\
&= \frac{\alpha^\beta(\vartheta_2-\vartheta_1)}{4}\left(\int_0^1\left(\frac{4}{3\alpha^\beta} - \left(\frac{1-\gamma^\alpha}{\alpha}\right)^\beta\right)^p d\gamma\right)^{\frac{1}{p}} \\
&\quad\times\left(\left(\frac{|\mathcal{K}'(\vartheta_1)|^q + \left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}} + \left(\frac{\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q + |\mathcal{K}'(\vartheta_2)|^q}{2}\right)^{\frac{1}{q}}\right).
\end{aligned}
$$

The proof is finished. □

Remark 3. *By setting $\alpha = \beta = 1$, (13) is simplified to the first inequality of Corollary 2.8 in [12].*

Corollary 4. *In Theorem 4, using the convexity of $|\mathcal{K}'|$, we obtain*

$$
\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1) - \mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + 2\mathcal{K}(\vartheta_2)\right) - \frac{\alpha^\beta 2^{\beta\alpha-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta\alpha}}\mathcal{W}(\alpha,\beta,\mathcal{K})\right|
$$

$$\leq \frac{\alpha^{\beta}(\vartheta_2-\vartheta_1)}{4}\left(\int_0^1\left(\frac{4}{3\alpha^{\beta}}-\left(\frac{1-\gamma^{\alpha}}{\alpha}\right)^{\beta}\right)^p d\gamma\right)^{\frac{1}{p}}$$
$$\times\left(\left(\frac{3|\mathcal{K}'(\vartheta_1)|^q+|\mathcal{K}'(\vartheta_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{K}'(\vartheta_1)|^q+3|\mathcal{K}'(\vartheta_2)|^q}{4}\right)^{\frac{1}{q}}\right). \tag{14}$$

Remark 4. *By taking $\alpha=\beta=1$, (14) is reduced to the second inequality of Corollary 2.8 from [12].*

Corollary 5. *In Corollary 4, using the discrete power mean inequality, we obtain*

$$\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1)-\mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+2\mathcal{K}(\vartheta_2)\right)-\frac{\alpha^{\beta}2^{\beta\alpha-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta\alpha}}\mathcal{W}(\alpha,\beta,\mathcal{K})\right|$$
$$\leq \frac{\alpha^{\beta}(\vartheta_2-\vartheta_1)}{2}\left(\int_0^1\left(\frac{4}{3\alpha^{\beta}}-\left(\frac{1-\gamma^{\alpha}}{\alpha}\right)^{\beta}\right)^p d\gamma\right)^{\frac{1}{p}}\left(\frac{|\mathcal{K}'(\vartheta_1)|^q+|\mathcal{K}'(\vartheta_2)|^q}{2}\right)^{\frac{1}{q}}. \tag{15}$$

Corollary 6. *In Corollary 5, taking $\alpha=\beta=1$, we obtain*

$$\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1)-\mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+2\mathcal{K}(\vartheta_2)\right)-\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\mathcal{K}(u)du\right|$$
$$\leq \frac{\vartheta_2-\vartheta_1}{6}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{4^{p+1}-1}{3}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{K}'(\vartheta_1)|^q+|\mathcal{K}'(\vartheta_2)|^q}{2}\right)^{\frac{1}{q}}. \tag{16}$$

Corollary 7. *Taking $\alpha=1$ in Theorem 4, we obtain*

$$\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1)-\mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+2\mathcal{K}(\vartheta_2)\right)-\frac{2^{\beta-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta}}\mathcal{R}(\beta,\mathcal{K})\right|$$
$$\leq \frac{\vartheta_2-\vartheta_1}{4}\left(\frac{2^{1+2p-\frac{2}{\beta}}\sqrt{\pi}\,\Gamma\left(\frac{1}{\beta}\right)}{3^p\Gamma\left(\frac{1}{2}+\frac{1}{\beta}\right)}\,{}_2F_1\left(-p,\frac{1}{\beta},\frac{1}{\beta}+1;\frac{3}{4}\right)\right)^{\frac{1}{p}}$$
$$\times\left(\left(\frac{|\mathcal{K}'(\vartheta_1)|^q+\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q+|\mathcal{K}'(\vartheta_2)|^q}{2}\right)^{\frac{1}{q}}\right), \tag{17}$$

where $\mathcal{R}(\beta,\mathcal{K})$ is defined by (11) and ${}_2F_1$ is the hypergeometric function.

Corollary 8. *In Corollary 7, using the convexity of $|\mathcal{K}'|^q$, we obtain*

$$\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1)-\mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+2\mathcal{K}(\vartheta_2)\right)-\frac{2^{\beta-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta}}\mathcal{R}(\beta,\mathcal{K})\right|$$
$$\leq \frac{\vartheta_2-\vartheta_1}{4}\left(\frac{2^{1+2p-\frac{2}{\beta}}\sqrt{\pi}\,\Gamma\left(\frac{1}{\beta}\right)}{3^p\Gamma\left(\frac{1}{2}+\frac{1}{\beta}\right)}\,{}_2F_1\left(-p,\frac{1}{\beta},\frac{1}{\beta}+1;\frac{3}{4}\right)\right)^{\frac{1}{p}}$$
$$\times\left(\left(\frac{3|\mathcal{K}'(\vartheta_1)|^q+|\mathcal{K}'(\vartheta_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{K}'(\vartheta_1)|^q+3|\mathcal{K}'(\vartheta_2)|^q}{4}\right)^{\frac{1}{q}}\right), \tag{18}$$

where $\mathcal{R}(\beta,\mathcal{K})$ is defined by (11) and ${}_2F_1$ is the hypergeometric function.

Corollary 9. *In Corollary 8, using the discrete power mean inequality, we obtain*

$$\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1)-\mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+2\mathcal{K}(\vartheta_2)\right)-\frac{2^{\beta-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta}}\mathcal{R}(\beta,\mathcal{K})\right|$$

$$\leq \frac{\vartheta_2-\vartheta_1}{2}\left(\frac{2^{1+2p-\frac{2}{\beta}}\sqrt{\pi}\Gamma\left(\frac{1}{\beta}\right)}{3^p\Gamma\left(\frac{1}{2}+\frac{1}{\beta}\right)}\,{}_2F_1\left(-p,\frac{1}{\beta},\frac{1}{\beta}+1;\frac{3}{4}\right)\right)^{\frac{1}{p}}\left(\frac{|\mathcal{K}'(\vartheta_1)|^q+|\mathcal{K}'(\vartheta_2)|^q}{2}\right)^{\frac{1}{q}}, \tag{19}$$

where $\mathcal{R}(\beta,\mathcal{K})$ is defined by (11) and ${}_2F_1$ is the hypergeometric function.

Theorem 5. *Let $\mathcal{K}$ be as in Lemma 2. If $|\mathcal{K}'|^q$ is convex on $[\vartheta_1,\vartheta_2]$, then for all $\beta>0$ and $\alpha\in(0,1]$ we have*

$$\begin{aligned}
&\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1)-\mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+2\mathcal{K}(\vartheta_2)\right)-\frac{\alpha^\beta 2^{\beta\alpha-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta\alpha}}\mathcal{W}(\alpha,\beta,\mathcal{K})\right|\\
&\leq \frac{\vartheta_2-\vartheta_1}{4\alpha}\left(\frac{4\alpha-3B\left(\frac{1}{\alpha},\beta+1\right)}{3}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{2\alpha-3B\left(\frac{2}{\alpha},\beta+1\right)}{3}|\mathcal{K}'(\vartheta_1)|^q+\frac{2\alpha-3B\left(\frac{1}{\alpha},\beta+1\right)+3B\left(\frac{2}{\alpha},\beta+1\right)}{3}\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\\
&+\left(\frac{2\alpha-3B\left(\frac{1}{\alpha},\beta+1\right)+3B\left(\frac{2}{\alpha},\beta+1\right)}{3}\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q+\frac{2\alpha-3B\left(\frac{2}{\alpha},\beta+1\right)}{3}|\mathcal{K}'(\vartheta_2)|^q\right)^{\frac{1}{q}}\Bigg),
\end{aligned}\tag{20}$$

where $\mathcal{W}(\alpha,\beta,\mathcal{K})$ is defined by (3), $q\geq 1$ and $B(.,.)$ is the Beta function.

Proof. Taking the absolute value on both sides of (2) and then using power mean inequality and convexity of $|\mathcal{K}'|^q$, we deduce

$$\begin{aligned}
&\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1)-\mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+2\mathcal{K}(\vartheta_2)\right)-\frac{\alpha^\beta 2^{\beta\alpha-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta\alpha}}\mathcal{W}(\alpha,\beta,\mathcal{K})\right|\\
&\leq\frac{\alpha^\beta(\vartheta_2-\vartheta_1)}{4}\left(\left(\int_0^1\left(\frac{4}{3\alpha^\beta}-\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta\right)d\gamma\right)^{1-\frac{1}{q}}\right.\\
&\left(\int_0^1\left(\frac{4}{3\alpha^\beta}-\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta\right)\left|\mathcal{K}'\left((1-\gamma)\vartheta_1+\gamma\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q d\gamma\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left(\frac{4}{3\alpha^\beta}-\left(\frac{1-\gamma^\alpha}{\alpha}\right)^\beta\right)d\gamma\right)^{1-\frac{1}{q}}\\
&\left.\times\left(\int_0^1\left(\frac{4}{3\alpha^\beta}-\left(\frac{1-\gamma^\alpha}{\alpha}\right)^\beta\right)\left|\mathcal{K}'\left((1-\gamma)\frac{\vartheta_1+\vartheta_2}{2}+\gamma\vartheta_2\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right)\\
&\leq\frac{\alpha^\beta(\vartheta_2-\vartheta_1)}{4}\left(\left(\int_0^1\left(\frac{4}{3\alpha^\beta}-\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta\right)d\gamma\right)^{1-\frac{1}{q}}\right.\\
&\times\left(\int_0^1\left(\frac{4}{3\alpha^\beta}-\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta\right)\left((1-\gamma)|\mathcal{K}'(\vartheta_1)|^q+\gamma\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q\right)d\gamma\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left(\frac{4}{3\alpha^\beta}-\left(\frac{1-\gamma^\alpha}{\alpha}\right)^\beta\right)d\gamma\right)^{1-\frac{1}{q}}
\end{aligned}$$

$$\times\left(\left(\int_0^1\left(\frac{4}{3\alpha^\beta}-\left(\frac{1-\gamma^\alpha}{\alpha}\right)^\beta\right)\left((1-\gamma)\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q+\gamma|\mathcal{K}'(\vartheta_2)|^q\right)d\gamma\right)^{\frac{1}{q}}\right)$$

$$=\frac{\vartheta_2-\vartheta_1}{4\alpha}\left(\frac{4\alpha-3B\left(\frac{1}{\alpha},\beta+1\right)}{3}\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{2\alpha-3B\left(\frac{2}{\alpha},\beta+1\right)}{3}|\mathcal{K}'(\vartheta_1)|^q+\frac{2\alpha-3B\left(\frac{1}{\alpha},\beta+1\right)+3B\left(\frac{2}{\alpha},\beta+1\right)}{3}\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{2\alpha-3B\left(\frac{1}{\alpha},\beta+1\right)+3B\left(\frac{2}{\alpha},\beta+1\right)}{3}\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q+\frac{2\alpha-3B\left(\frac{2}{\alpha},\beta+1\right)}{3}|\mathcal{K}'(\vartheta_2)|^q\right)^{\frac{1}{q}}\Bigg),$$

where we have used (7), (8), and

$$\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta d\gamma=\int_0^1\left(\frac{1-\gamma^\alpha}{\alpha}\right)^\beta d\gamma\frac{1}{\alpha^{\beta+1}}\int_0^1(1-z)^\beta z^{\frac{1}{\alpha}-1}dz=\frac{1}{\alpha^{\beta+1}}B\left(\frac{1}{\alpha},\beta+1\right).$$

The proof is finished. □

Remark 5. *Taking $\alpha=\beta=1$ results in the reduction of* (20) *to the first inequality of Corollary 2.11 from* [12].

Corollary 10. *In Theorem 5, using the convexity of $|\mathcal{K}'|^q$, we obtain*

$$\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1)-\mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+2\mathcal{K}(\vartheta_2)\right)-\frac{\alpha^\beta 2^{\beta\alpha-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta\alpha}}\mathcal{W}(\alpha,\beta,\mathcal{K})\right|$$

$$\le\frac{\vartheta_2-\vartheta_1}{4\alpha}\left(\frac{4\alpha-3B\left(\frac{1}{\alpha},\beta+1\right)}{3}\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{6\alpha-3B\left(\frac{1}{\alpha},\beta+1\right)-3B\left(\frac{2}{\alpha},\beta+1\right)}{6}|\mathcal{K}'(\vartheta_1)|^q+\frac{2\alpha-3B\left(\frac{1}{\alpha},\beta+1\right)+3B\left(\frac{2}{\alpha},\beta+1\right)}{6}|\mathcal{K}'(\vartheta_2)|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{2\alpha-3B\left(\frac{1}{\alpha},\beta+1\right)+3B\left(\frac{2}{\alpha},\beta+1\right)}{6}|\mathcal{K}'(\vartheta_1)|^q+\frac{6\alpha-3B\left(\frac{1}{\alpha},\beta+1\right)-3B\left(\frac{2}{\alpha},\beta+1\right)}{6}|\mathcal{K}'(\vartheta_2)|^q\right)^{\frac{1}{q}}\Bigg).\quad(21)$$

Remark 6. *Taking $\alpha=\beta=1$, inequality* (21) *is reduced to the second inequality of Corollary 2.11 from* [12].

Corollary 11. *In Corollary 10, using the discrete power mean inequality, we obtain*

$$\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1)-\mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+2\mathcal{K}(\vartheta_2)\right)-\frac{\alpha^\beta 2^{\beta\alpha-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta\alpha}}\mathcal{W}(\alpha,\beta,\mathcal{K})\right|$$

$$\le\frac{\left(4\alpha-3B\left(\frac{1}{\alpha},\beta+1\right)\right)(\vartheta_2-\vartheta_1)}{6\alpha}\left(\frac{|\mathcal{K}'(\vartheta_1)|^q+|\mathcal{K}'(\vartheta_2)|^q}{2}\right)^{\frac{1}{q}}.\quad(22)$$

Corollary 12. *Taking $\alpha=1$ in Theorem 5, we obtain*

$$\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1)-\mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+2\mathcal{K}(\vartheta_2)\right)-\frac{2^{\beta-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta}}\mathcal{R}(\beta,\mathcal{K})\right|$$

$$\le\frac{(4\beta+1)(\vartheta_2-\vartheta_1)}{12(\beta+1)}\left(\frac{2\beta^2+6\beta+1}{(4\beta+1)(\beta+2)}|\mathcal{K}'(\vartheta_1)|^q+\frac{2\beta^2+3\beta+1}{(4\beta+1)(\beta+2)}\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{2\beta^2+3\beta+1}{(4\beta+1)(\beta+2)}\left|\mathcal{K}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q+\frac{2\beta^2+6\beta+1}{(4\beta+1)(\beta+2)}|\mathcal{K}'(\vartheta_2)|^q\right)^{\frac{1}{q}}\Bigg),\quad(23)$$

where $\mathcal{R}(\beta,\mathcal{K})$ *is defined as* (11).

Corollary 13. *In Corollary 12, using the convexity of* $|\mathcal{K}'|^q$*, we obtain*

$$\begin{aligned}&\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1)-\mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+2\mathcal{K}(\vartheta_2)\right)-\frac{2^{\beta-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^\beta}\mathcal{R}(\beta,\mathcal{K})\right|\\&\leq\frac{(4\beta+1)(\vartheta_2-\vartheta_1)}{12(\beta+1)}\left(\frac{6\beta^2+15\beta+3}{2(4\beta+1)(\beta+2)}|\mathcal{K}'(\vartheta_1)|^q+\frac{2\beta^2+3\beta+1}{2(4\beta+1)(\beta+2)}|\mathcal{K}'(\vartheta_2)|^q\right)^{\frac{1}{q}}\\&+\left(\frac{2\beta^2+3\beta+1}{2(4\beta+1)(\beta+2)}|\mathcal{K}'(\vartheta_1)|^q+\frac{6\beta^2+15\beta+3}{2(4\beta+1)(\beta+2)}|\mathcal{K}'(\vartheta_2)|^q\right)^{\frac{1}{q}}\Bigg),\end{aligned}\tag{24}$$

where $\mathcal{R}(\beta,\mathcal{K})$ *is defined as* (11).

Corollary 14. *In Corollary 13, using the discrete power mean inequality, we obtain*

$$\begin{aligned}&\left|\frac{1}{3}\left(2\mathcal{K}(\vartheta_1)-\mathcal{K}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+2\mathcal{K}(\vartheta_2)\right)-\frac{2^{\beta-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^\beta}\mathcal{R}(\beta,\mathcal{K})\right|\\&\leq\frac{(4\beta+1)(\vartheta_2-\vartheta_1)}{6(\beta+1)}\left(\frac{|\mathcal{K}'(\vartheta_1)|^q+|\mathcal{K}'(\vartheta_2)|^q}{2}\right)^{\frac{1}{q}},\end{aligned}\tag{25}$$

where $\mathcal{R}(\beta,\mathcal{K})$ *is defined as* (11).

3. Applications

Application to Composite Quadrature Formula

Let $\mathcal{P}$ be the partition of the interval $[\vartheta_1,\vartheta_2]$ such that $\vartheta_1=u_0<u_1<\ldots<u_n=\vartheta_2$, taking the quadrature formula into consideration; then,

$$\int_{\vartheta_2}^{\vartheta_2}\mathcal{K}(u)du=\mathcal{Q}(\mathcal{K},\mathcal{P})+\mathcal{E}(\mathcal{K},\mathcal{P}),$$

where

$$\mathcal{Q}(\mathcal{K},\mathcal{P})=\sum_{i=0}^{n-1}\frac{u_{i+1}-u_i}{3}\left(2\mathcal{K}(u_i)-\mathcal{K}\left(\frac{u_i+u_{i+1}}{2}\right)+2\mathcal{K}(u_{i+1})\right),$$

with $\mathcal{E}(\mathcal{K},\mathcal{P})$ denoting the associated approximation error.

Proposition 1. *Let* $\mathcal{K}$ *be as in Theorem 3; then, we have*

$$|\mathcal{E}(\mathcal{K},\mathcal{P})|\leq\sum_{i=0}^{n-1}\frac{5(u_{i+1}-u_i)^2}{24}\left(|\mathcal{K}'(u_i)|+|\mathcal{K}'(u_{i+1})|\right).$$

Proof. By applying inequality Equation (12) with $\beta=1$ to the partition $\mathcal{P}$ of subintervals $[u_i,u_{i+1}]$ $(i=0,1,\ldots,n-1)$, we obtain

$$\begin{aligned}&\left|\frac{1}{3}\left(2\mathcal{K}(u_i)-\mathcal{K}\left(\frac{u_i+u_{i+1}}{2}\right)+2\mathcal{K}(u_{i+1})\right)-\frac{1}{u_{i+1}-u_i}\int_{u_i}^{u_{i+1}}\mathcal{K}(u)du\right|\\&\leq\frac{5(u_{i+1}-u_i)}{24}\left(|\mathcal{K}'(u_i)|+|\mathcal{K}'(u_{i+1})|\right).\end{aligned}$$

We reach the necessary result by multiplying both sides of the aforementioned inequality by $(u_{i+1}-u_i)$, summing the generated inequalities for all $i=0,1,\ldots,n-1$, and applying the triangular inequality. □

Proposition 2. *Let $\mathcal{K}$ be as in Theorem 4; then, we have*

$$|\mathcal{E}(\mathcal{K},\mathcal{P})| \leq \sum_{i=0}^{n-1} \frac{(u_{i+1}-u_i)^2}{6} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{4^{p+1}-1}{3}\right)^{\frac{1}{p}} \left(\frac{|\mathcal{K}'(u_i)|^q+|\mathcal{K}'(u_i)|^q}{2}\right)^{\frac{1}{q}}.$$

Proof. By applying inequality (19) with $\beta = 1$ to the partition $\mathcal{P}$ of subintervals $[u_i, u_{i+1}]$ $(i = 0, 1, \ldots, n-1)$, we obtain

$$\left|\frac{1}{3}\left(2\mathcal{K}(u_i) - \mathcal{K}\left(\frac{u_i+u_{i+1}}{2}\right) + 2\mathcal{K}(u_{i+1})\right) - \frac{1}{u_{i+1}-u_i}\int_{u_i}^{u_{i+1}} \mathcal{K}(u)du\right|$$
$$\leq \frac{u_{i+1}-u_i}{6} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{4^{p+1}-1}{3}\right)^{\frac{1}{p}} \left(\frac{|\mathcal{K}'(u_i)|^q+|\mathcal{K}'(u_i)|^q}{2}\right)^{\frac{1}{q}}.$$

We reach the necessary result by multiplying both sides of the aforementioned inequality by $(u_{i+1} - u_i)$, summing the generated inequalities for all $i = 0, 1, \ldots, n-1$, and applying the triangular inequality. □

Proposition 3. *Let $\mathcal{K}$ be as in Theorem 5; then, we have*

$$|\mathcal{E}(\mathcal{K},\mathcal{P})| \leq \sum_{i=0}^{n-1} \frac{5(u_{i+1}-u_i)^2}{12} \left(\frac{|\mathcal{K}'(u_i)|^q+|\mathcal{K}'(u_i)|^q}{2}\right)^{\frac{1}{q}}.$$

Proof. By applying inequality (25) with $\beta = 1$ to the partition $\mathcal{P}$ of subintervals $[u_i, u_{i+1}]$ $(i = 0, 1, \ldots, n-1)$, we obtain

$$\left|\frac{1}{3}\left(2\mathcal{K}(u_i) - \mathcal{K}\left(\frac{u_i+u_{i+1}}{2}\right) + 2\mathcal{K}(u_{i+1})\right) - \frac{1}{u_{i+1}-u_i}\int_{u_i}^{u_{i+1}} \mathcal{K}(u)du\right|$$
$$\leq \frac{5(u_{i+1}-u_i)}{12} \left(\frac{|\mathcal{K}'(u_i)|^q+|\mathcal{K}'(u_i)|^q}{2}\right)^{\frac{1}{q}}.$$

We reach the necessary result by multiplying both sides of the aforementioned inequality by $(u_{i+1} - u_i)$, summing the generated inequalities for all $i = 0, 1, \ldots, n-1$, and applying the triangular inequality. □

4. Conclusions

In summary, our efforts have made a significant contribution to the field of conformable fractional calculus. The introduction of a novel symmetrical identity followed by the derivation of multiple Milne-type inequalities tailored for differentiable convex functions highlights the innovative nature of our work. The illustrative example serves to visually elucidate the underlying concepts of the derived inequalities through graphical representations. This research adds valuable insights to the advancing field of conformable fractional calculus, demonstrating its potential applications in mathematical analysis and modeling. The presented results not only contribute to the current understanding, but can pave the way for future investigations in this promising area of study.

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