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Contact CR-Warped Product Submanifold of a Sasakian Space Form with a Semi-Symmetric Metric Connection

Meraj Ali Khan [1,*], Ibrahim Al-Dayel [1], Foued Aloui [1] and Shyamal Kumar Hui [2]

[1] Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), P.O. Box 65892, Riyadh 11566, Saudi Arabia; iaaldayel@imamu.edu.sa (I.A.-D.); fyaloui@imamu.edu.sa (F.A.)

[2] Department of Mathematics, The University of Burdwan, Golapbag, Burdwan 713104, West Bengal, India; skhui@math.buruniv.ac.in

* Correspondence: mskhan@imamu.edu.sa

Abstract: The main goal of this research paper is to investigate contact CR-warped product submanifolds within Sasakian space forms, utilizing a semi-symmetric metric connection. We conduct a comprehensive analysis of these submanifolds and establish several significant results. Additionally, we formulate an inequality that establishes a relationship between the squared norm of the second fundamental form and the warping function. Lastly, we present a number of geometric applications derived from our findings.

Keywords: contact CR; warped product manifolds; semi-symmetric; Sasakian manifolds

MSC: primary 53B50; secondary 53C20; 53C40

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1. Introduction

The concept of a semi-symmetric linear connection on a Riemannian manifold was initially introduced in the publication mentioned as [1]. In a subsequent work, the author of [2] provided a precise definition for a semi-symmetric connection within the context of a linear connection, denoted as ∇, on an n-dimensional Riemannian manifold (M, g). This definition states that the torsion tensor, denoted as T, satisfies the condition $T(\Lambda_1, \Lambda_2) = \pi(\Lambda_2)\Lambda_1 - \pi(\Lambda_1)\Lambda_2$, where π represents a 1-form and $\Lambda_1, \Lambda_2 \in TM$.

The properties of semi-symmetric metric connections were further explored by K. Yano in [3]. In this work, it was demonstrated that a conformally flat Riemannian manifold equipped with a semi-symmetric connection exhibits a curvature tensor that vanishes. This observation highlights a notable characteristic of such manifolds and contributes to our understanding of the interplay between conformal flatness, semi-symmetric connections, and the curvature properties of Riemannian manifolds. Overall, the developments discussed in these papers provide a foundation for the study of semi-symmetric connections on Riemannian manifolds. They introduce the concept, define its properties, and explore the implications of such connections in relation to the curvature and conformal flatness of the manifold.

On the other hand, warped product manifolds present an intriguing geometric framework for studying the behavior of spacetime near black holes and objects with powerful gravitational fields. Bishop and O'Neill initially introduced these manifolds to explore spaces with negative curvature, but they have since evolved to incorporate warping functions, expanding on the concept of Riemannian product manifolds. By combining two pseudo-Riemannian manifolds—a base manifold (B, g_B) and a fiber (F, g_F)—using a smooth function, b, defined on the base manifold, warped product manifolds are constructed. The resulting metric tensor, denoted as $g = g_B \times b^2 g_F$, reflects the amalgamation of the two manifolds. In this construction, the base manifold represents the underlying space, while the fiber represents an additional space that is warped or scaled by the warping function, b. Each point in the base manifold is assigned a positive value by the warping function, which influences the manifold's geometry. When a conformal Killing vector is present in a warped product manifold, it has been extensively studied in the context of Einstein–Weyl geometry, where the warping function acts as a conformal factor, altering the manifold's geometry. The geometry of these manifolds is determined by a conformal class of metrics, capturing the fundamental geometric properties shared by metrics related through conformal transformations. Leistner and Nurowski's works [4,5] provide comprehensive information and insights on this topic. Furthermore, the use of warped product manifolds has facilitated the exploration of various instances of Ricci solitons, which represent self-similar solutions to the equation governing Ricci flow. These solitons offer valuable insights into the dynamics and evolution of Riemannian manifolds. Notably, extensive research and analysis have focused on investigating "cigar solitons" within Euclidean space [6,7].

The investigation of warped products in submanifold theory initially originated from the pioneering work of B. Y. Chen, as referenced in [8]. Chen's research primarily focused on CR-warped product submanifolds within the framework of almost Hermitian manifolds. In his seminal work, Chen introduced an estimation for the norm of the second fundamental form by incorporating a warping function. This concept played a crucial role in understanding the geometry of CR-warped product submanifolds.

Expanding upon Chen's ideas, Hesigawa and Mihai, as mentioned in [9], delved further into the contact form of these submanifolds. They explored the properties of contact CR-warped product submanifolds embedded within Sasakian space forms. In their study, Hesigawa and Mihai derived a similar approximation for the second fundamental form of such submanifolds. This approximation provided valuable insights into the geometric characteristics and behavior of contact CR-warped product submanifolds within the context of Sasakian space forms.

In general, the contributions of Chen, Hesigawa, and Mihai have significantly advanced our understanding of warped product submanifolds. Their works introduced important concepts, such as the estimation of the norm of the second fundamental form and the exploration of the contact form, which have paved the way for further research and developments in the field of submanifold theory.

The investigation of Einstein warped product manifolds equipped with a semi-symmetric metric connection within the context of warped product manifolds was undertaken by Sular and Ozgur, as referenced in [10]. Their research primarily focused on exploring the properties and behavior of such manifolds, shedding light on this specific geometric framework.

In a subsequent publication, cited as [11], Sular and Ozgur expanded on their previous work and obtained additional results concerning warped product manifolds endowed with a semi-symmetric metric connection. These new findings further enhanced the understanding of this geometric framework and contributed to its development.

Moreover, theoretical research and development on submanifold theory, soliton theory, and related topics were carried out by researchers like Li et al., as evidenced by a series of referenced papers [12–24]. The works of Li et al. have made significant contributions to the advancement of these research areas, providing valuable insights and motivation for further exploration.

Building upon the prior investigations carried out by Friedmann, Schouten, Hayden, K. Yano, Sular, Ozgur, Li, and others, our research is motivated by their significant contributions. In particular, we are intrigued by the impact of a semi-symmetric metric connection on contact CR-warped product submanifolds within a Sasakian space form. Our objective is to delve into the geometry and properties of these submanifolds when influenced by a semi-symmetric metric connection. By doing so, we aim to make advancements in the understanding of this mathematical topic, contributing to the broader knowledge in this field of study.

2. Definitions and Basic Results

Consider an odd-dimensional Riemannian manifold $(\bar{\Omega}, g)$. We define $\bar{\Omega}$ as an almost contact metric manifold if it possesses a $(1,1)$ tensor field ϕ and a global vector field χ that satisfy the following conditions:

$$\phi^2\Lambda_1 = -\Lambda_1 + \eta(\Lambda_1)\chi, \quad g(\Lambda_1, \chi) = \eta(\Lambda_1) \tag{1}$$

$$g(\phi\Lambda_1, \phi\Lambda_2) = g(\Lambda_1, \Lambda_2) - \eta(\Lambda_1)\eta(\Lambda_2). \tag{2}$$

Let η denote the dual 1-form of the vector field χ. It is well-known that an almost contact metric manifold can be classified as a Sasakian manifold if it satisfies the following conditions

$$(\bar{\bar{\nabla}}_{\Lambda_1}\phi)\Lambda_2 = g(\Lambda_1, \Lambda_2)\chi - \eta(\Lambda_2)\Lambda_1. \tag{3}$$

On For a Sasakian manifold $\bar{\Omega}$, the following observations can be readily made

$$\bar{\bar{\nabla}}_{\Lambda_1}\chi = -\phi\Lambda_1. \tag{4}$$

In this context, the symbols Λ_1 and Λ_2 represent elements of the tangent space of $\bar{\Omega}$, while $\bar{\bar{\nabla}}$ denotes the Riemannian connection corresponding to the metric g on $\bar{\Omega}$.

Now, let us define a connection $\bar{\nabla}$ as follows:

$$\bar{\nabla}_{\Lambda_1}\Lambda_2 = \bar{\bar{\nabla}}_{\Lambda_1}\Lambda_2 + \eta(\Lambda_2)\Lambda_1 - g(\Lambda_1, \Lambda_2)\chi \tag{5}$$

such that $\bar{\nabla}g = 0$ for any $\Lambda_1, \Lambda_2 \in T\Omega$, The Riemannian connection with respect to the metric g is denoted as $\bar{\bar{\nabla}}$. The connection $\bar{\nabla}$ is classified as semi-symmetric due to the property $T(\Lambda_1, \Lambda_2) = \eta(\Lambda_2)\Lambda_1 - \eta(\Lambda_1)\Lambda_2$, where η represents the dual 1-form. Using (5) in (3), we have

$$(\bar{\nabla}_{\Lambda_1}\phi)\Lambda_2 = g(\Lambda_1, \Lambda_2)\chi - g(\Lambda_1, \phi\Lambda_2)\chi - \eta(\Lambda_2)\Lambda_1 - \eta(\Lambda_2)\phi\Lambda_1 \tag{6}$$

and

$$\bar{\nabla}_{\Lambda_1}\chi = \Lambda_1 - \eta(\Lambda_1)\chi - \phi\Lambda_1. \tag{7}$$

When a Sasakian manifold $\bar{\Omega}$ possesses a constant ϕ-holomorphic sectional curvature c, it is referred to as a Sasakian space form and is denoted as $\bar{\Omega}(c)$.

The curvature tensor $\bar{R}$, which corresponds to the semi-symmetric metric connection $\bar{\nabla}$, is expressed as follows:

$$\bar{R}(\Lambda_1, \Lambda_2)\Lambda_3 = \bar{\nabla}_{\Lambda_1}\bar{\nabla}_{\Lambda_2}\Lambda_3 - \bar{\nabla}_{\Lambda_2}\bar{\nabla}_{\Lambda_1}\Lambda_3 - \bar{\nabla}_{[\Lambda_1,\Lambda_2]}\Lambda_3. \tag{8}$$

In a similar manner, the curvature tensor $\bar{\bar{R}}$ can be defined for the Riemannian connection $\bar{\bar{\nabla}}$.

Let

$$\beta(\Lambda_1, \Lambda_2) = (\bar{\nabla}_{\Lambda_1}\eta)\Lambda_2 - \eta(\Lambda_1)\eta(\Lambda_2) + \frac{1}{2}g(\Lambda_1, \Lambda_2). \tag{9}$$

By applying Equations (5), (8), and (9), we obtain

$$\begin{aligned} \bar{R}(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = {} & \bar{\bar{R}}(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) + \beta(\Lambda_1, \Lambda_3)g(\Lambda_2, \Lambda_4) \\ & - \beta(\Lambda_2, \Lambda_3)g(\Lambda_1, \Lambda_4) + \beta(\Lambda_2, \Lambda_4)g(\Lambda_1, \Lambda_3) \\ & - \beta(\Lambda_1, \Lambda_4)g(\Lambda_2, \Lambda_3). \end{aligned} \tag{10}$$

By utilizing the computed value of $\bar{\bar{R}}(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4)$, as described in [25], we can compute the following expression for the curvature tensor, $\bar{R}$, of a Sasakian space form:

$$\begin{aligned}
\bar{R}(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) =& \frac{c+3}{4}\{g(\Lambda_2, \Lambda_3)g(\Lambda_1, \Lambda_4) - g(\Lambda_1, \Lambda_3)g(\Lambda_2, \Lambda_4)\} \\
&+ \frac{c-1}{4}\{\eta(\Lambda_1)\eta(\Lambda_3)g(\Lambda_2, \Lambda_4) - \eta(\Lambda_2)\eta(\Lambda_3)g(\Lambda_1, \Lambda_4) \\
&+ g(\Lambda_1, \Lambda_3)\eta(\Lambda_2)\eta(\Lambda_4) - g(\Lambda_2, \Lambda_3)\eta(\Lambda_1)\eta(\Lambda_4) \\
&+ g(\phi\Lambda_2, \Lambda_3)g(\phi\Lambda_1, \Lambda_4) + g(\phi\Lambda_3, \Lambda_1)g(\phi\Lambda_2, \Lambda_4) \\
&- 2g(\phi\Lambda_1, \Lambda_2)g(\phi\Lambda_3, \Lambda_4)\} + \beta(\Lambda_1, \Lambda_3)g(\Lambda_2, \Lambda_4) \\
&- \beta(\Lambda_2, \Lambda_3)g(\Lambda_1, \Lambda_4) + \beta(\Lambda_2, \Lambda_4)g(\Lambda_1, \Lambda_3) \\
&- \beta(\Lambda_1, \Lambda_4)g(\Lambda_2, \Lambda_3),
\end{aligned} \tag{11}$$

for all $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4 \in T\bar{\Omega}$.

For a submanifold, $\Omega, \hookrightarrow \bar{\Omega}$, with SSM connection, then it is easy to observe that the Gauss and Weingarten formulae are given by, respectively,

$$h(\Lambda_1, \Lambda_2) = \bar{\nabla}_{\Lambda_1}\Lambda_2 - \nabla_{\Lambda_1}\Lambda_2$$

and

$$\bar{\nabla}_{\Lambda_1}N = -A_N\Lambda_1 + \nabla^{\perp}_{\Lambda_1}N + \eta(N)\Lambda_1.$$

Here, $\bar{\nabla}$ represents the covariant derivative with respect to the S-S-M connection on $\bar{\Omega}$, ∇ denotes the induced SSM connection on Ω, and Λ_1 and Λ_2 are tangent vectors on Ω. Moreover, N denotes a normal vector to the submanifold Ω, $\nabla^{\perp}$ represents the covariant derivative along the normal bundle $T^{\perp}\Omega$, and $\eta(N)$ is a scalar function.

The relation between the shape operator A_N and the second fundamental form h can be described by the following mathematical expression.

$$g(h(\Lambda_1, \Lambda_2), N) = g(A_N\Lambda_1, \Lambda_2).$$

Let Λ_1 and Λ_3 be vector fields, where Λ_1 belongs to TM and Λ_3 belongs to $T^{\perp}M$. The expression can be decomposed as follows:

$$\phi\Lambda_1 = P\Lambda_1 + F\Lambda_1 \tag{12}$$

and

$$\phi\Lambda_3 = t\Lambda_3 + f\Lambda_3, \tag{13}$$

where $P\Lambda_1(t\Lambda_3)$ and $F\Lambda_1(f\Lambda_3)$ are the tangential and normal components of $\phi\Lambda_1(\phi\Lambda_3)$ respectively.

The equation of Gauss, which relates to a SSM connection, and involves the Riemannian curvature tensor R, can be expressed as follows according to reference [25]:

$$\bar{R}(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = R(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) - g(h(\Lambda_1, \Lambda_4), h(\Lambda_2, \Lambda_3)) + g(h(\Lambda_2, \Lambda_4), h(\Lambda_1, \Lambda_3)) \tag{14}$$

for $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4 \in T\Omega$.

In their publication [10], Sular and Özgür conducted a study on warped product manifolds denoted as $\Omega_1 \times_f \Omega_2$. These manifolds were equipped with a semi-symmetric metric connection and were associated with a vector field P. The components of this construction include Riemannian manifolds Ω_1 and Ω_2, as well as a positive differentiable function f defined on Ω_1 serving as the warping function. The authors made significant discoveries and observations, which are summarized in a lemma presented here. These findings will be relevant and useful for our subsequent analysis.

Lemma 1. *Consider the warped product manifold $\Omega_1 \times_f \Omega_2$ with a SSM connection $\bar{\nabla}$. In this setting, we have the following*

(i) *if $P \in TM_1$, then*

$$\bar{\nabla}_{\Lambda_1}\Lambda_3 = \frac{\Lambda_1 f}{f}\Lambda_3 \quad \text{and} \quad \bar{\nabla}_{\Lambda_3}\Lambda_1 = \frac{\Lambda_1 f}{f}\Lambda_3 + \pi(\Lambda_1)\Lambda_3,$$

(ii) *if $P \in T\Omega_2$, then*

$$\bar{\nabla}_{\Lambda_1}\Lambda_3 = \frac{\Lambda_1 f}{f}\Lambda_3 \quad \text{and} \quad \nabla_{\Lambda_3}\Lambda_1 = \frac{\Lambda_1 f}{f}\Lambda_3.$$

Here, $\Lambda_1 \in T\Omega_1,\ \Lambda_3 \in T\Omega_2$, and π is the 1-form correspondng to the vector field, P.

Consider the warped product submanifold $\Omega = \Omega_1 \times_f \Omega_2$ within a smooth manifold $\bar{\Omega}$. In this context, we can define the curvature tensors R and $\tilde{R}$ for the submanifold Ω, which are associated with its induced semi-symmetric metric (SSM) connection ∇ and induced Riemannian connection $\tilde{\nabla}$, respectively. Considering this, we can express the interconnection between these tensors as follows:

$$\begin{aligned} R(\Lambda_1,\Lambda_2)\Lambda_3 =& \tilde{R}(\Lambda_1,\Lambda_2)\Lambda_3 + g(\Lambda_3,\nabla_{\Lambda_1}P)\Lambda_2 - g(\Lambda_3,\nabla_{\Lambda_2}P)\Lambda_1 \\ &+ g(\Lambda_1,\Lambda_3)\nabla_{\Lambda_2}P - g(\Lambda_2,\Lambda_3)\nabla_{\Lambda_1}P \\ &+ \eta(P)[g(\Lambda_1,\Lambda_3)\Lambda_2 - g(\Lambda_2,\Lambda_3)\Lambda_1] \\ &+ [g(\Lambda_2,\Lambda_3)\eta(\Lambda_1) - g(\Lambda_1,\Lambda_3)\eta(\Lambda_2)]P \\ &+ \eta(\Lambda_3)[\eta(\Lambda_2)\Lambda_1 - \eta(\Lambda_1)\Lambda_2], \end{aligned} \tag{15}$$

for any vector field $\Lambda_1, \Lambda_2, \Lambda_3$ on Ω [10].

As mentioned in part (ii) of Lemma 3.2 in reference [10], for the warped product submanifold $\Omega = \Omega_1 \times_f \Omega_2$, the following relationship is established:

$$\tilde{R}(\Lambda_1,\Lambda_2)\Lambda_3 = \frac{H^f(\Lambda_1,\Lambda_2)}{f}\Lambda_3. \tag{16}$$

In the given equation, Λ_1 and Λ_2 belong to $T\Omega_1$, while Λ_3 belongs to $T\Omega_2$. The term H^f denotes the Hessian of the warping function.

By considering Equations (15) and (16), we can deduce the following:

$$\begin{aligned} R(\Lambda_1,\Lambda_3)\Lambda_2 = \frac{H^f(\Lambda_1,\Lambda_2)}{f} + \frac{Pf}{f}g(\Lambda_1,\Lambda_2)\Lambda_3 + \eta(P)g(\Lambda_1,\Lambda_2)\Lambda_3 + g(\Lambda_2,\nabla_{\Lambda_1}P)\Lambda_3 \\ - \eta(\Lambda_1)\eta(\Lambda_2)\Lambda_3, \end{aligned} \tag{17}$$

for the vector fields $\Lambda_1, \Lambda_2 \in T\Omega_1$, $\Lambda_3 \in T\Omega_2$, and $P \in T\Omega_1$.

By assuming $P = \chi$ into Equation (5), we introduce the SSM connection. Hence, utilizing part (i) of Lemma 2.1, we can establish the following relation for a WP submanifold $\Omega = \Omega_1 \times_f \Omega_2$ within the Riemannian manifold $\bar{\Omega}$.

$$\nabla_{\Lambda_1}\Lambda_3 = \Lambda_1(\ln f)\Lambda_3 \tag{18}$$

and

$$\nabla_{\Lambda_3}\Lambda_1 = \Lambda_1(\ln f)\Lambda_3 + \eta(\Lambda_1)\Lambda_3. \tag{19}$$

Furthermore, the combination of Equation (19) with (7) yields

$$R(\Lambda_1,\Lambda_3)\Lambda_2 = \frac{H^f(\Lambda_1,\Lambda_2)}{f}\Lambda_3 + \frac{\chi f}{f}g(\Lambda_1,\Lambda_2)\Lambda_3 + 2g(\Lambda_1,\Lambda_2)\Lambda_3 - 2\eta(\Lambda_1)\eta(\Lambda_2)\Lambda_3 - g(\Lambda_2,\phi\Lambda_1)\Lambda_3, \tag{20}$$

for $\chi,\Lambda_1,\Lambda_2 \in T\Omega_1$, and $\Lambda_3 \in T\Omega_2$.

The expression for the Laplacian Δf of the warping function can be observed as follows:

$$\frac{\Delta f}{f} = \Delta(\ln f) - \|\nabla lnf\|^2. \tag{21}$$

3. Contact CR-W-P Submanifolds

In his work [26], Bejancu introduced the notion of semi-invariant submanifolds within the context of almost contact metric manifolds. A smooth manifold $\bar{\Omega}$ containing an m-dimensional Riemannian submanifold Ω is considered a semi-invariant submanifold if the vector field χ is tangent to Ω and if there exists a differentiable distribution $D : x \mapsto D_x \subset T_x\Omega$ on Ω. This distribution D_x is invariant by the structure vector field ϕ. Additionally, the orthogonal complementary distribution $D_x^{\perp}$ to D_x on Ω is anti-invariant, meaning that $\phi D^{\perp} \subseteq T_x^{\perp}\Omega$, where $T_x\Omega$ and $T_x^{\perp}\Omega$ refer to the tangent space and normal space at point $x \in \Omega$, respectively.

Hesigawa and Mihai further explored the topic by examining a specific class of submanifolds called warped product submanifolds in a Sasakian manifold $\bar{\Omega}$ [9]. These submanifolds are of the form $\Omega_T \times_f \Omega_{\perp}$, where Ω_T represents an invariant submanifold, $\Omega_{\perp}$ represents an anti-invariant submanifold, and χ belongs to $T\Omega_T$. The authors labeled these submanifolds as contact CR-submanifolds and presented significant findings concerning their properties and characteristics.

To initiate our investigation, we delve into the study of a specific category of submanifolds known as contact CRWP submanifolds within a smooth manifold equipped with a SSM connection. These submanifolds exhibit the characteristic structure $\Omega_{\perp} \times_f \Omega_T$, where $\Omega_{\perp}$ represents an anti-invariant submanifold, and Ω_T denotes an invariant submanifold that fulfills the condition $\chi \in T\Omega_T$.

Theorem 2. *Let us consider $(\bar{\Omega},\phi,\chi,\eta,g)$ as an SM with an SSM connection. In this scenario, we can deduce that there is no existence of a WP submanifold in the form of $\Omega_{\perp} \times_f \Omega_T$ that fulfills the condition $\chi \in T\Omega_T$.*

Proof. For any $\Lambda_1,\Lambda_2 \in T\Omega_T$, and $\Lambda_3 \in T\Omega_{\perp}$, we utilize equation (19), the Gauss formula, along with equation (2), to obtain the following result

$$\begin{aligned}\Lambda_3(\ln f)g(\Lambda_1,\Lambda_2) &= g(\bar{\nabla}_{\Lambda_1}\phi\Lambda_3,\phi\Lambda_2) - g((\nabla_{\Lambda_1}\phi)\Lambda_3,\phi\Lambda_3) - \Lambda_3(\ln f)\eta(\Lambda_1)\eta(\Lambda_2)\\ &= g(\nabla_{\Lambda_1}\phi\Lambda_3,\phi\Lambda_2) - \Lambda_3(\ln f)\eta(\Lambda_1)\eta(\Lambda_2).\end{aligned} \tag{22}$$

Equivalently

$$\Lambda_3(\ln f)g(\Lambda_1,\Lambda_2) = \phi\Lambda_3(\ln f)g(\Lambda_1,\phi\Lambda_2) - \Lambda_3(\ln f)\eta(\Lambda_1)\eta(\Lambda_2). \tag{23}$$

Upon replacing both Λ_1 and Λ_2 with χ in the previously mentioned equation, we arrive at the conclusion that $\Lambda_3 \ln f = 0$. This deduction implies that in order for the equation to hold, the function f must be a constant. This finding ultimately validates the desired result. □

The primary objective of this investigation is to examine the characteristics of WP submanifolds, denoted as $\Omega = \Omega_T \times_f \Omega_{\perp}$, within the context of a Sasakian manifold, $\bar{\Omega}$. These submanifolds possess a SSM connection, and the vector field χ is an element of $T\Omega_T$.

To be more precise, we categorize these submanifolds as contact CRWP submanifolds. Furthermore, we designate the invariant subspace of the normal bundle, $T^{\perp}\Omega$, as μ.

With this background in mind, let us proceed by presenting the initial findings of our analysis.

Lemma 3. *Consider $\Omega = \Omega_T \times_f \Omega_{\perp}$ as a contact CRWP submanifold within the context of a SM $\bar{\Omega}$ equipped with a SSM connection. In this case, we have the following*

(i) $g(h(\phi\Lambda_1, \Lambda_3), \phi\Lambda_4) = \Lambda_1(\ln f)g(\Lambda_3, \Lambda_4) + \eta(\Lambda_1)g(\Lambda_3, \Lambda_4)$;
(ii) $g(h(\Lambda_1, \Lambda_2), \phi\Lambda_3) = 0$;
(iii) $\chi(\ln f) = 0$.

$\forall\ \Lambda_1, \Lambda_2 \in T\Omega_T$ *and* $\Lambda_3, \Lambda_4 \in T\Omega_{\perp}$, $\chi \in T\Omega_T$.

Proof. By applying the Gauss formula and utilizing Equation (6), we derive the following:

$$g(h(\phi\Lambda_1, \Lambda_3), \phi\Lambda_4) = g(\bar{\nabla}_{\Lambda_3}\phi\Lambda_1, \phi\Lambda_4) = g(\bar{\nabla}_{\Lambda_3}\Lambda_1, \Lambda_4).$$

By employing formula (19), we arrive at the following result

$$g(h(\phi\Lambda_1, \Lambda_3), \phi\Lambda_4) = g(\nabla_{\Lambda_3}\Lambda_1, \Lambda_4) = \Lambda_1(\ln f)g(\Lambda_3, \Lambda_4) + \eta(\Lambda_1)g(\Lambda_3, \Lambda_4),$$

which is part (i). Again, using (6), (18), and the Gauss formula, part (ii) is proven straightforwardly. Now, using the formula $\nabla_{\Lambda_3}\chi = \Lambda_3 - \eta(\Lambda_3) - P\Lambda_3$ and applying Equation (19), we have $\chi(\ln f) + \eta(\chi)\Lambda_3 = \Lambda_3$ or $\chi(\ln f) = 0$, which is part (iii). □

Lemma 4. *Let $\Omega = \Omega_T \times_f \Omega_{\perp}$ be a contact CRWP submanifold of a SM $\bar{\Omega}$ admitting a SSM connection, then*

$$g(h(\Lambda_1, \Lambda_1), V) = -g(h(\phi\Lambda_1, \phi\Lambda_1), V), \tag{24}$$

for all $\Lambda_1 \in T\Omega$ and $V \in \mu$.

Proof. Through the utilization of the Gauss formula and Equation (6), we derive the following result

$$\nabla_{\Lambda_1}\phi\Lambda_1 + h(\Lambda_1, \phi\Lambda_1) - \phi\nabla_{\Lambda_1}\Lambda_1 - \phi h(\Lambda_1, \Lambda_1) = g(\Lambda_1, \Lambda_1)\chi - \Lambda_1 - \phi\Lambda_1, \tag{25}$$

When we take the Riemannian product with $\phi V \in \mu$, we get

$$g(h(\Lambda_1, \phi\Lambda_1), \phi V) = g(h(\Lambda_1, \Lambda_1), V). \tag{26}$$

By substituting Λ_1 with $\phi\Lambda_1$ and utilizing Equation (1), we arrive at:

$$-g(h(\phi\Lambda_1, \Lambda_1), \phi V) = g(h(\phi\Lambda_1, \phi\Lambda_1), V. \tag{27}$$

By considering Equations (26) and (27), we can deduce the following

$$g(h(\Lambda_1, \Lambda_1), V) = -g(h(\phi\Lambda_1, \phi\Lambda_1), V). \tag{28}$$

This establishes the claim. □

4. Inequality for Second Fundamental Form

In this section, we examine a submanifold, Ω, that is a contact CRWP of dimensions n. It can be expressed as $\Omega = \Omega_T^{n_1} \times_f \Omega_{\perp}^{n_2}$, where $\Omega_T^{n_1}$ and $\Omega_{\perp}^{n_2}$ are submanifolds of dimensions n_1 and n_2, respectively. The ambient manifold, $\bar{\Omega}$, is a SM of dimension $2m + 1$ with a SSM connection. Additionally, χ belongs to the tangent space $T\Omega_T$. Consider the orthonormal frame $\{\xi_1, \ldots, \xi_s, \xi_{s+1} = \phi\xi_1, \ldots, \xi_{n_1-1} = \phi\xi_s, \xi_n = \chi, \xi_{n_1+1}, \ldots, \xi_n = \xi_{n_1+n_2}\}$ for the submanifold Ω, where the vector fields $\{\xi_1, \ldots, \xi_s, \xi_{s+1} = \phi\xi_1, \ldots, \xi_{n_1-1} = \phi\xi_s, \xi_n = \chi\}$ are tangent to $T\Omega_T^{n_1}$ and $\{\xi_{n_1+1}, \ldots, \xi_n = \xi_{n_1+n_2}\}$ are tangent to $\Omega_\perp^{n_2}$.

Definition 5. Consider $\Omega = \Omega_1 \times_f \Omega_2$, a WP submanifold of an SM. In this context, if the partial second fundamental form, h_i, vanishes identically, we refer to Ω as being Ω_i-totally geodesic. Similarly, if the partial mean curvature vector, H_i, vanishes for $i = 1, 2$, we classify Ω as Ω_i-minimal.

Presented below is the theorem at hand

Theorem 6. *Let $\Omega = \Omega_T^{n_1} \times_f \Omega_\perp^{n_2}$ be a contact CRWP submanifold of a Sasakian manifold, $\bar{\Omega}$, admitting a SSM connection. Therefore, the squared mean curvature of Ω can be expressed as follows*

$$||H||^2 = \frac{1}{n^2} \sum_{r=n+1}^{2m+1} (h^r_{n_1+1 n_1+1} + \cdots + h^r_{nn})^2. \tag{29}$$

Proof. The squared norm of mean curvature vector for the submanifold Ω is given by

$$\begin{aligned} ||H||^2 = \frac{1}{n^2} \sum_{r=n+1}^{2m+1} (h^r_{11} + \ldots h^r_{ss} + h^r_{s+1s+1} + \cdots + h^r_{2s2s} \\ + h^r_{n_1+1n_1+1} + \cdots + h^r_{nn})^2, \end{aligned} \tag{30}$$

where $h^r_{ij} = g(h(\xi_i, \xi_j), \xi_r)$, and $1 \le i,\ j \le n, \quad n+1 \le r \le 2m+1$. Applying Lemma 4, we oget the required result. $\square$

For the contact CRWP submanifold $\Omega = \Omega_T^{n_1} \times_f \Omega_\perp^{n_2}$, from the relation (20), we deduce the following formula

$$\sum_{i=1}^{n_1} \sum_{j=n_1+1}^{n} K(\xi_i \wedge \xi_j) = n_2 \frac{\Delta f}{f} + 2n_1 n_2 - 2n_2. \tag{31}$$

Theorem 7. *Let $\Omega = \Omega_T^{n_1} \times_f \Omega_\perp^{n_2}$ be a CRWP submanifold of an $2m+1-$dimensional SM $\bar{\Omega}$, with an SSM connection, then*

(i) *The second fundamental form fulfills the following condition*

$$||h||^2 \ge 2\bar{\tau}(T\Omega) - 2\bar{\tau}(T\Omega_T^{n_1}) - 2\bar{\tau}(T\Omega_\perp^{n_2}) - 2n_2 \frac{\Delta f}{f} - 4n_1 n_2 + 4n_2.$$

(ii) *In the case where the equality in (i) holds, it follows that $\Omega_T^{n_1}$ and $\Omega_\perp^{n_2}$ are completely geodesic and completely umbilical submanifolds, respectively, within $\bar{\Omega}$.*

Proof. By substituting $\Lambda_1 = \Lambda_4 = \xi_i$ and $\Lambda_2 = \Lambda_3 = \xi_j$ into the Gauss Equation (14) and summing over $1 \le i,\ j \le n$ with $i \ne j$, we derive the following expression

$$||h||^2 = -2\tau(T\Omega) + 2\bar{\tau}(T\Omega) + n^2||H||^2. \tag{32}$$

As the submanifold Ω is a contact CR-warped product submanifold, the above equation yields

$$||h||^2 = -2\sum_{i=1}^{n_1} \sum_{j=n_1+1}^{n} K(\xi_i, \xi_j) - 2\tau(T\Omega_T) - 2\tau(T\Omega_\perp) + 2\bar{\tau}(T\Omega) + n^2||H||^2. \tag{33}$$

On further using the Gauss equation, we obtain

$$
\begin{aligned}
||h||^2 = & -2\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} K(\xi_i,\xi_j) - 2\bar{\tau}(T\Omega_T) - 2\bar{\tau}(T\Omega_\perp) + 2\bar{\tau}(T\Omega) \\
& -2\sum_{r=n+1}^{2m+1}\sum_{1\le i<k\le n_1}(h^r_{ii}h^r_{kk} - (h^r_{ik})^2) + n^2||H||^2 \\
& -2\sum_{r=n+1}^{2m+1}\sum_{n_1+1\le j<t\le n}(h^r_{jj}h^r_{tt} - (h^r_{jt})^2).
\end{aligned} \tag{34}
$$

The equivalent version of the above equation can be written as

$$
\begin{aligned}
||h||^2 = & -2\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} K(\xi_i,\xi_j) - 2\bar{\tau}(T\Omega_T) - 2\bar{\tau}(T\Omega_\perp) + 2\bar{\tau}(T\Omega) \\
& -2\sum_{r=n+1}^{2m+1}\sum_{1\le i\neq k\le n_1}(h^r_{ii}h^r_{kk} - (h^r_{ik})^2) + n^2||H||^2 \\
& -2\sum_{r=n+1}^{2m+1}\sum_{n_1+1\le j\neq \le n}(h^r_{jj}h^r_{tt} - (h^r_{jt})^2).
\end{aligned} \tag{35}
$$

Since the submanifold $\Omega_T^{n_1} \times_f \Omega_\perp^{n_2}$ is Ω_T-minimal, we have

$$
\sum_{r=n+1}^{2m+1}\sum_{1\le i\neq k\le n_1}(h^r_{ii}h^r kk - (h^r_{ik})^2) = -\sum_{r=n+1}^{2m+1}\sum_{r=n+1}^{2m+1}\sum_{i,k=1}^{n_1}(h^r_{ik})^2. \tag{36}
$$

Thus, Equation (35) can be written as follows

$$
\begin{aligned}
||h||^2 = & -2\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} K(\xi_i,\xi_j) - 2\bar{\tau}(T\Omega_T) - 2\bar{\tau}(T\Omega_\perp) + 2\bar{\tau}(T\Omega) \\
& +\sum_{r=n+1}^{2m+1}\sum_{1\le i\neq k\le n_1}(h^r_{ik})^2 + n^2||H||^2 - n^2||H||^2 \\
& +\sum_{r=n+1}^{2m+1}\sum_{n_1+1\le j\neq t\le n}(h^r_{jt})^2.
\end{aligned} \tag{37}
$$

From (37) and (31), we obtain

$$
||h||^2 \ge 2\bar{\tau}(T\Omega) - 2\bar{\tau}(T\Omega_T) - 2\bar{\tau}(T\Omega_\perp) - 2n_2\frac{\Delta f}{f} - 4n_1n_2 + 4n_2.
$$

This establishes the validity of part (i) of the theorem. If the equality in (i) is satisfied, it implies that $h(\Lambda_1,\Lambda_2) = 0$ for any Λ_1 and Λ_2 belonging to the tangent space $T\Omega_T$. Consequently, the submanifold $\Omega_T^{n_1}$ can be classified as totally geodesic, while $\Omega_\perp^{n_2}$ can be classified as totally umbilical. □

If the ambient manifold is an unit sphere, S^{2m+1} of odd dimension, then we have the following theorem

Theorem 8. *Let $\Omega = \Omega_T^{n_1} \times_f \Omega_\perp^{n_2}$ be a contact CRWP submanifold of an unit sphere $S^{2m+1}(1)$ of odd dimension, with SSM connection, then*

(i) *The following inequality holds*

$$||h||^2 \geq -2n_2\frac{\Delta f}{f} - 2n_1n_2 + 4n_2 - 2n_1\sum_{i=n_1+1}^{n_2}\beta(\xi_i,\xi_i) - 2n_2\sum_{i=1}^{n_1}\beta(\xi_i,\xi_i).$$

(ii) *If the equality in (i) is fulfilled, it signifies that $\Omega_T^{n_1}$ and $\Omega_\perp^{n_2}$ are totally geodesic and totally umbilical submanifolds in $\bar{\Omega}$, respectively.*

Proof. If the Sasakian manifold is an unit sphere of dimension $2m+1$, then by Equation (11)

$$2\bar{\tau}(T\Omega_T) = n_1(n_1-1) - 2n_1\sum_{i=1}^{n_1}\beta(\xi_i,\xi_i). \tag{38}$$

Similarly, on summing over the vector field on $TN_\perp$, we obtain

$$2\bar{\tau}(T\Omega_\perp) = n_2(n_2-1) - 2n_2\sum_{i=n_1+1}^{n}\beta(\xi_i,\xi_i). \tag{39}$$

Now, summing up over basis vector fields of $T\Omega$, such that $1 \leq i \neq j \leq n$, we have

$$2\bar{\tau}(T\Omega) = n(n-1) - 2n\ trace\beta. \tag{40}$$

Therefore,

$$\begin{aligned} 2\bar{\tau}(T\Omega) - 2\bar{\tau}(T\Omega_T) - 2\bar{\tau}(T\Omega_\perp) = {}& n(n-1) - 2n\ trace\beta - n_1(n_1-1) - n_2(n_2-1) \\ & + 2n_1\sum_{i=1}^{n_1}\beta(\xi_i,\xi_i) + 2n_2\sum_{i=n_1+1}^{n}\beta(\xi_i,\xi_i). \end{aligned} \tag{41}$$

After, simplification, we obtain

$$2\bar{\tau}(T\Omega) - 2\bar{\tau}(T\Omega_T) - 2\bar{\tau}(T\Omega_\perp) = 2n_1n_2 - 2n_1\sum_{i=n_1+1}^{n}\beta(\xi_i,\xi_i) - 2n_2\sum_{i=1}^{n_1}\beta(\xi_i,\xi_i). \tag{42}$$

Hence, from part (i), we obtain the desired inequality. □

In view of Equation (21), we deduce the following

Theorem 9. *Let $\Omega = \Omega_T^{n_1} \times_f \Omega_\perp^{n_2}$ be a contact CRWP of an unit sphere, $S^{2m+1}(1)$ of odd dimension, with SSM connection, then*

(i) *the following inequality holds*

$$||h||^2 \geq 2n_2(||\nabla(\ln f)||^2 - \Delta(\ln f)) - 2n_1n_2 + 4n_2 - 2n_1\sum_{i=n_1+1}^{n}\beta(\xi_i,\xi_i) - 2n_2\sum_{i=1}^{n_1}\beta(\xi_i,\xi_i).$$

(ii) *If the equality in (i) is fulfilled, it signifies that $\Omega_T^{n_1}$ and $\Omega_\perp^{n_2}$ are totally geodesic and totally umbilical submanifolds in $\bar{\Omega}$, respectively.*

5. Some Applications

In this section, we will explore various applications utilizing the outcomes of our findings.

Theorem 10. *Let $\Omega = \Omega_T^{n_1} \times_f \Omega_\perp^{n_2}$ be a contact CRWP submanifold an unit sphere, $S^{2m+1}(1)$ of odd dimension, with SSM connection. Then, Ω is a Riemannian product if*

$$||h||^2 + 2n_2\sum_{i=1}^{n_1}\beta(\xi_i,\xi_i) + 2n_1\sum_{i=n_1+1}^{n}\beta(\xi_i,\xi_i) \geq 4n_2 - 2n_1n_2. \tag{43}$$

Proof. From Theorem 9, we obtain

$$2n_2||\nabla(\ln f)||^2 - 2n_1n_2 + 4n_2 - 2n_1 \sum_{i=n_1+1}^{n} \beta(\xi_i, \xi_i) - 2n_2 \sum_{i=1}^{n_1} \beta(\xi_i, \xi_i) \leq 2n_2\Delta(\ln f). \quad (44)$$

On integrating

$$\int_{N_T \times n_2} 2n_2||\nabla(\ln f)||^2 - 2n_1n_2 + 4n_2 - 2n_1 \sum_{i=n_1+1}^{n} \beta(\xi_i, \xi_i) - 2n_2 \sum_{i=1}^{n_1} \beta(\xi_i, \xi_i) \leq 2n_2 \int_{\Omega_T \times n_2} \Delta(\ln f) = 0. \quad (45)$$

Now, if $||h||^2 + 2n_2 \sum_{i=1}^{n_1} \beta(\xi_i, \xi_i) + 2n_1 \sum_{i=n_1+1}^{n} \beta(\xi_i, \xi_i) \geq 4n_2 - 2n_1n_2$, then, from (45), we find $\int_{N_T \times n_2} 2n_2||\nabla(\ln f)||^2 \leq 0$, which is not possible for a positive function. Therefore, $\nabla(\ln f) = 0$, which means that f is constant and Ω is simply a Riemannian product. □

Suppose λ is a nonzero eigenvalue of the Laplacian on Ω_T. Then, by the property of minimum principle, we obtain

$$\int_{\Omega_T} ||\nabla(\ln f)||^2 dV_T \geq \lambda \int_{\Omega_T} ((\ln f))^2 dV_T. \quad (46)$$

Using this fact in Theorem 9, we obtain the following result.

Corollary 11. *Let $\Omega = \Omega_T^{n_1} \times_f \Omega_\perp^{n_2}$ be a contact CRWP submanifold of an unit sphere $S^{2m+1}(1)$ of odd dimension, with SSM connection. Let Ω_T be a compact invariant submanifold and λ be a nonzero eigenvalue of the Laplacian on Ω_T. Then*

$$\int_{\Omega_T \times n_2} ||h||^2 \geq \int_{\Omega_T \times n_2} \{4n_2 - 2n_1n_2 - 2n_1 \sum_{i=n_1+1}^{n} \beta(\xi_i, \xi_i) - 2n_2 \sum_{i=1}^{n_1} \beta(\xi_i, \xi_i)\} Vol(\Omega_T) + 2n_2\lambda \int_{\Omega_T} ((\ln f))^2 dV_T. \quad (47)$$

6. Conclusions

In conclusion, this research paper delved into the investigation of contact CRWP submanifolds within SSF, utilizing a SSM connection. The comprehensive examination of these submanifolds has made significant contributions to the field. Firstly, important results have been established regarding the properties and characteristics of contact CRWP submanifolds. This analysis has provided valuable insights into their geometric structure and shed light on their behavior within Sasakian space forms.

Additionally, a notable inequality has been derived, establishing a relationship between the squared norm of the second fundamental form and the warping function. This inequality serves as a fundamental tool for understanding the interplay between the geometric properties of contact CRWP submanifolds and the underlying warping function. It quantitatively measures the connection between these two aspects, enabling further investigations in this area.

The findings presented in this paper contribute to a broader understanding of contact CRWP submanifolds and their relevance to SSF. The results deepen our knowledge of these geometric structures and provide new avenues for future research in related fields. In summary, this study successfully explores contact CRWP submanifolds within the framework of SSF equipped with a SSM connection. The established results and derived inequality offer valuable insights and opportunities for further investigation. It is hoped that these findings will inspire and guide future research in this exciting area of differential geometry.

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Abbreviations

The following abbreviations are used in this manuscript:

Semi-symmetric metric	S-S-M
Sasakian manifold	S-M
Sasakian space form	S-S-F

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