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Application to Activation Functions through Fixed-Circle Problems with Symmetric Contractions

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Abstract: In this paper, our main aim is to present innovative fixed-point theorems that provide solutions to the fixed-circle problem with symmetric contractions. We accomplish this by employing operator enrichment techniques within the context of Banach spaces. Furthermore, we demonstrate the practical application of these theorems by showcasing their relevance to the rectified linear unit (ReLU) activation function. By exploring the connection between fixed points and activation functions, our work contributes to a deeper understanding of the behavior and properties of these fundamental mathematical concepts.

Keywords: fixed circle; fixed point; symmetric contractions; activation function; neural network

MSC: 92B20; 47H09; 47H10

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1. Introduction

The initial breakthrough in addressing the fixed-circle problem was achieved in a study conducted in [1], which proposed a solution specifically for metric spaces. Since then, subsequent research has focused on exploring new solutions for both metric spaces and generalized metric spaces. For example, Özgür et al. [1] introduced fixed-circle results using the Caristi-type contraction on metric spaces. This approach was further developed by researchers in [2,3], who proved new fixed-circle theorems by employing Wardowski's technique and classical contractive conditions. The fixed-circle problem was also investigated in the context of S-metric spaces in studies conducted in [4,5], where a modified Khan-type contractive condition was utilized to establish a novel fixed-circle theorem in another study [6]. Additionally, generalized fixed-circle results were obtained for Sb-metric spaces and parametric N_b-metric spaces, incorporating a geometric perspective. Furthermore, Mlaiki et al. [7] proposed investigating fixed-circle theorems on extended M_b-metric spaces. In 2020, Taş extended the concept of fixed circles to include Jaggi-type, Dass–Gupta-type-I, and Dass–Gupta-type-II bilateral contractions. For a more comprehensive understanding of this research direction, we recommend referring to the studies conducted in [1–6,8–10] and the references therein.

Activation functions play a crucial role in neural networks as they significantly influence decision-making processes. Therefore, selecting the most suitable activation function is essential for effective network analysis. Several studies, such as [11,12], have provided comprehensive analyses of different activation functions and their real-world applications. It is worth noting that commonly used activation functions, including the Ramp function, ReLU function, and LeakyReLU function, have fixed-point sets consisting of fixed discs and fixed circles [10,13–16]. Fixed-circle theorems have been established and extended to various aspects, including discontinuous activation functions, as well as rectified linear unit activation functions employed in neural networks. Theoretical studies on neural networks often utilize well-known fixed-point theorems like the Banach fixed-point theorem and Brouwer's fixed-point theorem. For instance, Li et al. [17] demonstrated the existence of a fixed point for every recurrent neural network using a geometric approach, with Brouwer's fixed-point theorem ensuring the existence of a fixed point. This study highlights the significance of adopting a geometric viewpoint and leveraging theoretical fixed-point results in practical applications.

Operator enrichment techniques have emerged as a new avenue in fixed-point theory, inspired by Krasnoselskii's fixed-point theorem [18] for nonexpansive operators. These techniques have motivated the exploration of various enriched classes of operators, such as enriched contractions and enriched ϕ-contractions [19], enriched Kannan contractions [20], enriched Chatterjea operators [21], enriched nonexpansive operators in Hilbert spaces [22], enriched multivalued contractions [23], enriched Ćirić–Reich–Rus contractions [24], enriched cyclic contractions [25], enriched modified Kannan pairs [26], and enriched quasi contractions [27]. Notably, Abbas et al. [23] established fixed-point results by imposing the condition that the orbital subset is a complete subset of a normed space (Theorem 3 of [23]), while Gronicki and Bisht [28] considered enriched Ćirić–Reich–Rus contraction operators and proved a fixed-point theorem by imposing the condition that the average operator is an asymptotically regular operator (Theorem 3.1 of [28]).

Motivated by the research mentioned above, this paper presents innovative solutions to the fixed-circle problem by utilizing Caristi-type, general Jaggi-type bilateral, general Dass–Gupta-type-I, and Dass–Gupta-type-II bilateral contractions enriched with operators. In Section 2, a brief survey related to the fixed-circle problem is provided. Section 3 then modifies the known contractive conditions, specifically the Jaggi-type bilateral contraction and the Dass–Gupta-type bilateral contraction, in order to derive new fixed-circle (fixed-disc, common fixed-circle, common fixed-disc) results. Additionally, Section 4 demonstrates the practical effectiveness of our theoretical findings by applying them to rectified linear units activation functions, thus highlighting their reliance on fixed circles.

2. Definition of the Problem

Consider a metric space $(\hat{\mathbb{X}}_h, \wp)$ and a self-operator $\mathscr{T} : \hat{\mathbb{X}}_h \to \hat{\mathbb{X}}_h$. A point $\mathring{g} \in \hat{\mathbb{X}}_h$ that satisfies $\mathscr{T}\mathring{g} = \mathring{g}$ is known as a fixed point of $\mathscr{T}$. We can denote the set of all fixed points of $\mathscr{T}$ as $Fix(\mathscr{T})$.

Self-operators can have either a unique fixed point or multiple fixed points. For example, let us consider the metric space $(\mathbb{R}, \wp)$ with the function $\wp : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ defined as $\wp(\mathring{g}, \vartheta) = |\mathring{g} - \vartheta|$, where $\mathring{g}, \vartheta \in \mathbb{R}$. Now, consider the self-operators $\mathscr{T} : \mathbb{R} \to \mathbb{R}$ and $\mathscr{S} : \mathbb{R} \to \mathbb{R}$ defined as $\mathscr{T}(\mathring{g}) = 1 - \mathring{g}$ and $\mathscr{S}(\mathring{g}) = \mathring{g}^2 - 4\mathring{g} + 6$, respectively, for all $\mathring{g} \in \mathbb{R}$. In this case, $\mathscr{T}$ has a unique fixed point $\mathring{g}_0 = \frac{1}{2}$, while $\mathscr{S}$ has two fixed points $\mathring{g}_1 = 2$ and $\mathring{g}_2 = 3$. When a self-operator has multiple fixed points, it raises the question of the geometric properties of these fixed points.

This question is known as the "fixed-circle problem" and has been extensively studied from a geometric perspective. The problem was initially discussed in [1] and has gained significant importance in both theoretical mathematical studies and various practical applications.

Now, let us define the notion of a fixed circle. Consider the metric space $(\hat{\mathbb{X}}_h, \wp)$, and let $\mathscr{T} : \hat{\mathbb{X}}_h \to \hat{\mathbb{X}}_h$ be an operator. A circle $C_{\mathring{g}_0,r}$ is defined as the set of all points $\mathring{g} \in \hat{\mathbb{X}}_h$ such that $\wp(\mathring{g}, \mathring{g}_0) = r$, where $\mathring{g}_0 \in \hat{\mathbb{X}}_h$ is the center and $r \geq 0$ is the radius of the circle $C_{\mathring{g}_0,r}$.

Now, we can formally state the fixed-circle problem ([1]) in the context of a metric space. Let $\mathscr{T} : \hat{\mathbb{X}}_h \to \hat{\mathbb{X}}_h$ be an operator and $C_{\mathring{g}_0,r}$ be a circle. If $\mathscr{T}\mathring{g} = \mathring{g}$ for every $\mathring{g} \in C_{\mathring{g}_0,r}$, then the circle $C_{\mathring{g}_0,r}$ is referred to as the fixed circle of $\mathscr{T}$.

3. A Survey of Recent Solutions

In 2017, Özgür and Taş initiated the study of fixed circles and proved the results for Caristi-type contractions on a metric space in [1].
The main results in [1] are stated as follows:

Theorem 1 ([1])**.** *Consider a metric space $(\hat{\mathbb{X}}_h, \wp)$ and let $C_{\mathring{g}_0,r}$ be any circle on $\hat{\mathbb{X}}_h$. Let us define the operator $\phi : \hat{\mathbb{X}}_h \to [0, +\infty)$ as $\phi(\mathring{g}) = \wp(\mathring{g}, \mathring{g}_0)$, for all $\mathring{g} \in \hat{\mathbb{X}}_h$. If there exists a self-operator $\mathscr{T}$ on $\hat{\mathbb{X}}_h$ satisfying the following conditions for all $\mathring{g} \in C_{\mathring{g}_0,r}$,*

(C1) $\wp(\mathring{g}, \mathscr{T}\mathring{g}) \leq \phi(\mathring{g}) - \phi(\mathscr{T}\mathring{g})$;
(C2) $\wp(\mathscr{T}\mathring{g}, \mathring{g}_0) \geq r$,

then the circle $C_{\mathring{g}_0,r}$ is a fixed circle for $\mathscr{T}$.

Theorem 2 ([1])**.** *Consider a metric space $(\hat{\mathbb{X}}_h, \wp)$ and let $C_{\mathring{g}_0,r}$ be any circle on $\hat{\mathbb{X}}_h$. Let us define the operator $\phi : \hat{\mathbb{X}}_h \to [0, +\infty)$ as $\phi(\mathring{g}) = \wp(\mathring{g}, \mathring{g}_0)$, for all $\mathring{g} \in \hat{\mathbb{X}}_h$. If there exists a self-operator $\mathscr{T}$ on $\hat{\mathbb{X}}_h$ satisfying the following conditions for all $\mathring{g} \in C_{\mathring{g}_0,r}$,*

(C1)* $\wp(\mathring{g}, \mathscr{T}\mathring{g}) \leq \phi(\mathring{g}) + \phi(\mathscr{T}\mathring{g}) - 2r$;
(C2)* $\wp(\mathscr{T}\mathring{g}, \mathring{g}_0) \leq r$,

then the circle $C_{\mathring{g}_0,r}$ is a fixed circle for $\mathscr{T}$.

Theorem 3 ([1])**.** *Consider a metric space $(\hat{\mathbb{X}}_h, \wp)$ and let $C_{\mathring{g}_0,r}$ be any circle on $\hat{\mathbb{X}}_h$. Let us define the operator $\phi : \hat{\mathbb{X}}_h \to [0, +\infty)$ as $\phi(\mathring{g}) = \wp(\mathring{g}, \mathring{g}_0)$, for all $\mathring{g} \in \hat{\mathbb{X}}_h$. If there exists a self-operator $\mathscr{T}$ on $\hat{\mathbb{X}}_h$ satisfying the following conditions for all $\mathring{g} \in C_{\mathring{g}_0,r}$ and some $h \in [0, 1)$*

(C1)** $\wp(\mathring{g}, \mathscr{T}\mathring{g}) \leq \phi(\mathring{g}) - \phi(\mathscr{T}\mathring{g})$;
(C2)** $h\wp(\mathring{g}, \mathscr{T}\mathring{g}) + \wp(\mathscr{T}\mathring{g}, \mathring{g}_0) \geq r$,

then the circle $C_{\mathring{g}_0,r}$ is a fixed circle for $\mathscr{T}$.

For more results in this direction, we refer the reader to [2–5,8] and the references mentioned therein. These fixed circle results made a significant contribution in fixed-point theory.

In 2020, Taş [10] extended the concept of fixed circles to Jaggi-type, Dass–Gupta-type-I, and Dass–Gupta-type-II bilateral contractions in [10].

The main definitions and results of [10] are the following:

Definition 1 ([10])**.** *If there exist functions $\phi : \hat{\mathbb{X}}_h \to (0, +\infty)$ and $\mathring{g}_0 \in \hat{\mathbb{X}}_h$ such that for all $\mathring{g} \in \hat{\mathbb{X}}_h - \{\mathring{g}_0\}$,*

$$\wp(\mathring{g}, \mathscr{T}\mathring{g}) > 0 \Rightarrow \wp(\mathring{g}, \mathscr{T}\mathring{g}) \leq [\phi(\mathring{g}) - \phi(\mathscr{T}\mathring{g})] R_{\mathscr{T}}(\mathring{g}_0, \mathring{g}), \tag{1}$$

where

$$R_{\mathscr{T}}(\mathring{g}, \mathring{\sigma}) = \max\left(\wp(\mathring{g}, \mathring{\sigma}), \frac{\wp(\mathring{g}, \mathscr{T}\mathring{g})\wp(\mathring{\sigma}, \mathscr{T}\mathring{\sigma})}{\wp(\mathring{g}, \mathring{\sigma})}\right).$$

then $\mathscr{T}$ is called a Jaggi-type bilateral $\mathring{g}_0$-contractive operator.

Theorem 4 ([10])**.** *Let $\mathscr{T} : \hat{\mathbb{X}}_h \to \hat{\mathbb{X}}_h$ be a Jaggi-type bilateral $\mathring{g}_0$-contractive operator with $\mathring{g}_0 \in \hat{\mathbb{X}}_h$ and r defined as*

$$r = \inf\left\{\frac{\wp(\mathring{g}, \mathscr{T}\mathring{g})}{\phi(\mathring{g})} : \mathring{g} \neq \mathscr{T}\mathring{g}, \mathring{g} \in \hat{\mathbb{X}}_h\right\}. \tag{2}$$

If $\mathscr{T}\mathring{g}_0 = \mathring{g}_0$, then $\mathscr{T}$ fixes the circle $C_{\mathring{g}_0,r}$.

Definition 2 ([10]). *If there exists a function $\phi : \hat{\mathbb{X}}_h \to (0, +\infty)$ and $\mathring{g}_0 \in \hat{\mathbb{X}}_h$ such that for all $\mathring{g} \in \hat{\mathbb{X}}_h - \{\mathring{g}_0\}$,*

$$\wp(\mathring{g}, \mathscr{T}\mathring{g}) > 0 \Rightarrow \wp(\mathring{g}, \mathscr{T}\mathring{g}) \leq [\phi(\mathring{g}) - \phi(\mathscr{T}\mathring{g})]Q_{\mathscr{T}}(\mathring{g}, \mathring{g}_0)$$

where,

$$Q_{\mathscr{T}}(\mathring{g}, \mathring{\sigma}) = \max\left(\wp(\mathring{g}, \mathring{\sigma}), \frac{(1 + \wp(\mathring{g}, \mathscr{T}\mathring{g}))\wp(\mathring{\sigma}, \mathscr{T}\mathring{\sigma})}{1 + \wp(\mathring{g}, \mathring{\sigma})}\right).$$

then $\mathscr{T}$ is called a Dass–Gupta-type-II bilateral $\mathring{g}_0$-contractive operator.

Theorem 5 ([10]). *Let $\mathscr{T} : \hat{\mathbb{X}}_h \to \hat{\mathbb{X}}_h$ be a Dass–Gupta-type-II bilateral $\mathring{g}_0$-contractive operator with $\mathring{g}_0 \in \hat{\mathbb{X}}_h$ and r defined as in Equation (2). If $\mathscr{T}\mathring{g}_0 = \mathring{g}_0$, then $\mathscr{T}$ fixes the circle $C_{\mathring{g}_0,r}$.*

4. New Fixed-Circle Theorems

Throughout this paper, we denote $(\hat{\mathbb{X}}_h, \|\cdot\|)$ as the normed space over the field $\mathbb{R}$ (the set of all real numbers). Let $\mathscr{T} : \hat{\mathbb{X}}_h \to \hat{\mathbb{X}}_h$ be a given operator and $\omega \in (0, 1)$. An operator $\mathscr{T}_\omega$ given by

$$\mathscr{T}_\omega(\mathring{g}) = (1 - \omega)\mathring{g} + \omega\mathscr{T}\mathring{g} \quad \text{for all } \mathring{g} \in \hat{\mathbb{X}}_h, \tag{3}$$

is called an averaged operator of $\mathscr{T}$. Note that

$$Fix(\mathscr{T}) = Fix(\mathscr{T}_\omega). \tag{4}$$

4.1. Fixed Circle Results for Caristi-Type Contractions

Throughout this section, we denote $\phi : \hat{\mathbb{X}}_h \to [0, + + \infty)$ as an operator defined by

$$\phi(\mathring{g}) = \|\mathring{g} - \mathring{g}_0\|, \quad \text{for all } \mathring{g} \in \hat{\mathbb{X}}_h. \tag{5}$$

We start with the following result.

Theorem 6. *Let $(\hat{\mathbb{X}}_h, \|\cdot\|)$ be a normed space and $\mathscr{T} : \hat{\mathbb{X}}_h \to \hat{\mathbb{X}}_h$. Assume that there exists $b \in [0, + + \infty)$ such that following conditions are satisfied for each $\mathring{g} \in C_{\mathring{g}_0,r}$;*

(C1) $\frac{1}{b+1}\|\mathring{g} - \mathscr{T}\mathring{g}\| \leq \phi(\mathring{g}) - \phi\left(\frac{b\mathring{g}+\mathscr{T}\mathring{g}}{b+1}\right)$;

(C2) $\frac{1}{b+1}\|(1 + b)\mathring{g}_0 - b\mathring{g} - \mathscr{T}\mathring{g}\| \geq r$.

Then, $C_{\mathring{g}_0,r}$ is the fixed circle of $\mathscr{T}$.

Proof. Let us denote $\omega = \frac{1}{1+b}$. Clearly, $0 < \omega < 1$ for any $\mathring{g} \in C_{\mathring{g}_0,r}$ and condition $(C1)$ becomes

$$\omega\|\mathring{g} - \mathscr{T}\mathring{g}\| \leq \phi(\mathring{g}) - \phi\left(\left(\left(\frac{1}{\omega} - 1\right)\mathring{g} + \mathscr{T}\mathring{g}\right)\omega\right),$$
$$\|\omega\mathring{g} - \omega\mathscr{T}\mathring{g}\| \leq \phi(\mathring{g}) - \phi((1 - \omega)\mathring{g} + \omega\mathscr{T}\mathring{g}),$$
$$\|\mathring{g} - (\mathring{g} - \omega\mathring{g} + \omega\mathscr{T}\mathring{g})\| \leq \phi(\mathring{g}) - \phi(\mathscr{T}_\omega\mathring{g}),$$

which can be written equivalently as

$$\|\mathring{g} - \mathscr{T}_\omega\mathring{g}\| \leq \phi(\mathring{g}) - \phi(\mathscr{T}_\omega\mathring{g}). \tag{6}$$

Based on the same procedure, for any $\mathring{g} \in C_{\mathring{g}_0,r}$, the condition $(\mathrm{C2})$ becomes

$$\|\mathring{g}_0 - \mathscr{T}_\omega\mathring{g}\| \geq r,$$

which can also be written as

$$\phi(\mathcal{T}_\omega \mathring{g}) = \|\mathring{g}_0 - \mathcal{T}_\omega \mathring{g}\| \geq r. \tag{7}$$

By using (7) and (6), we get

$$0 \leq \|\mathring{g} - \mathcal{T}_\omega \mathring{g}\| \leq \phi(\mathring{g}) - r = 0.$$

Thus, we get $\mathring{g} = \mathcal{T}_\omega \mathring{g}$ for all $\mathring{g} \in C_{\mathring{g}_0,r}$. Hence, $C_{\mathring{g}_0,r}$ is a fixed circle for $\mathcal{T}$. □

Example 1. *Let (Y, μ) be a finite measure space. The classical Lebesgue space $\hat{\mathbb{X}}_h = L^2(Y, \mu)$ is defined as the set of all Borel measurable functions $f : Y \to \mathbb{R}$ such that $\int_Y |f(y)|^2 d\mu(y) < +\infty$. It is known that $\hat{\mathbb{X}}_h$, equipped with the norm $\|f\|_{\hat{\mathbb{X}}_h} = \left(\int_Y |f|^2 d\mu\right)^{\frac{1}{2}}$, is a Banach space. Let α be a constant, and consider the Borel measurable function $g(y) = \alpha$ for all $y \in Y$, satisfying*

$$\|g - \mathring{g}_0\| \geq r.$$

Define the operator $\mathcal{T} : L^2(Y, \mu) \to L^2(Y, \mu)$ as follows:

$$\mathcal{T}\mathring{g} = \begin{cases} \mathring{g} & \text{if } \mathring{g} \in C_{\mathring{g}_0,r}, \\ 2g - \mathring{g} & \text{otherwise.} \end{cases}$$

Taking $b = 1$, we have $\omega = \frac{1}{2}$. It follows from the proof of Theorem 6 that the contractive conditions (C1) and (C2) are equivalent to (7) and (6), respectively. On the other hand, for $\omega = \frac{1}{2}$, the averaged operator $\mathcal{T}_{\frac{1}{2}}$ becomes

$$\mathcal{T}_{\frac{1}{2}}\mathring{g} = \begin{cases} \mathring{g} & \text{if } \mathring{g} \in C_{\mathring{g}_0,r}, \\ g & \text{otherwise.} \end{cases}$$

It can be easily seen that conditions (7) and (6) are satisfied. Clearly, $C_{\mathring{g}_0,r}$ is a fixed circle of $\mathcal{T}$.

Theorem 7. *Let $(\hat{\mathbb{X}}_h, \|\cdot\|)$ be a normed space and $\mathcal{T} : \hat{\mathbb{X}}_h \to \hat{\mathbb{X}}_h$. Assume that there exists $b \in [0, ++\infty)$ such that following conditions are satisfied for each $\mathring{g} \in C_{\mathring{g}_0,r}$;*

(C1)* $\frac{1}{b+1}\|\mathring{g} - \mathcal{T}\mathring{g}\| \leq \phi(\mathring{g}) + \phi\left(\frac{b\mathring{g}+\mathcal{T}\mathring{g}}{b+1}\right) - 2r$;

(C2)* $\frac{1}{b+1}\|(b+1)\mathring{g}_0 - (b\mathring{g} + \mathcal{T}\mathring{g})\| \leq r$.

Then, $C_{\mathring{g}_0,r}$ is the fixed circle of $\mathcal{T}$.

Proof. We omit the proof since it follows the same pattern as the proof of Theorem 6. □

Theorem 8. *Let $(\hat{\mathbb{X}}_h, \|\cdot\|)$ be a normed space and $\mathcal{T} : \hat{\mathbb{X}}_h \to \hat{\mathbb{X}}_h$. Assume that there exist $b \in [0, ++\infty)$ and $h \in [0, 1)$ such that following conditions are satisfied for each $\mathring{g} \in C_{\mathring{g}_0,r}$;*

(C1)** $\frac{1}{b+1}\|\mathring{g} - \mathcal{T}\mathring{g}\| \leq \phi(\mathring{g}) - \phi\left(\frac{b\mathring{g}+\mathcal{T}\mathring{g}}{b+1}\right)$;

(C2)** $\frac{h}{b+1}\|\mathring{g} - \mathcal{T}\mathring{g}\| + \frac{1}{1+b}\|(b+1)\mathring{g}_0 - (b\mathring{g} + \mathcal{T}\mathring{g})\| \geq r$.

Then, $C_{\mathring{g}_0,r}$ is the fixed circle of $\mathcal{T}$.

Proof. Let us denote $\omega = \frac{1}{1+b}$. Based on the same procedure as in proof of Theorem 6, the contractive condition (C1)** becomes

$$\|\mathring{g} - \mathcal{T}_\omega \mathring{g}\| \leq \phi(\mathring{g}) - \phi(\mathcal{T}_\omega \mathring{g}), \quad \text{for all } \mathring{g} \in C_{\mathring{g}_0,r}. \tag{8}$$

Similarly, for the value of $\omega = \frac{1}{1+b}$, the condition $(C2)^{**}$ can also be written equivalently in the form of

$$h\|\mathring{g} - \mathscr{T}_\omega \mathring{g}\| + \|\mathring{g}_0 - \mathscr{T}_\omega \mathring{g}\| \geq r, \quad \text{for all } \mathring{g} \in C_{\mathring{g}_0,r}. \tag{9}$$

Using the conditions (8) and (9), we obtain

$$\begin{aligned}
\|\mathring{g} - \mathscr{T}_\omega \mathring{g}\| &\leq \phi(\mathring{g}) - \phi(\mathscr{T}_\omega \mathring{g}) \\
&= \|\mathring{g} - \mathring{g}_0\| - \|\mathring{g}_0 - \mathscr{T}_\omega \mathring{g}\| \\
&= r - \|\mathring{g}_0 - \mathscr{T}_\omega \mathring{g}\| \\
&\leq h\|\mathring{g} - \mathscr{T}_\omega \mathring{g}\| + \|\mathring{g}_0 - \mathscr{T}_\omega \mathring{g}\| - \|\mathring{g}_0 - \mathscr{T}_\omega \mathring{g}\| \\
&= h\|\mathring{g} - \mathscr{T}_\omega \mathring{g}\|,
\end{aligned}$$

which is contradiction to our assumption since $h \in [0,1)$. Therefore, we get $\mathscr{T}_\omega \mathring{g} = \mathring{g}$ and $C_{\mathring{g}_0,r}$ is the fixed circle of $\mathscr{T}$. □

Remark 1. *If we take $b = 0$ in Theorems 6–8, we obtain Theorems 1–3, respectively, of [1] in the setting of normed spaces.*

4.2. Fixed Circle/Disc Results for Bilateral-Type Contractions

We introduce the following idea.

Definition 3. *An operator $\mathscr{T} : \hat{\mathbb{X}}_h \to \hat{\mathbb{X}}_h$ is called a general Jaggi-type bilateral symmetric contraction if there is an operator $\xi : \hat{\mathbb{X}}_h \to (0, +\infty)$ and there exist $b \in [0, +\infty)$ and $\mathring{g}_0 \in \hat{\mathbb{X}}_h$ such that for all $\mathring{g} \in \hat{\mathbb{X}}_h - \{\mathring{g}_0\}$, we have*

$$\frac{1}{b+1}\|\mathring{g} - \mathscr{T}\mathring{g}\| > 0 \Rightarrow \frac{1}{b+1}\|\mathring{g} - \mathscr{T}\mathring{g}\| \leq \left[\xi(\mathring{g}) - \xi\left(\frac{\mathscr{T}\mathring{g} + b\mathring{g}}{1+b}\right)\right] R^*_{\mathscr{T}}(\mathring{g}, \mathring{g}_0), \tag{10}$$

where

$$R^*_{\mathscr{T}}(\mathring{g}, \mathring{g}_0) = \max\left(\|\mathring{g} - \mathring{g}_0\|, \frac{\|\mathring{g} - \mathscr{T}\mathring{g}\|\|\mathring{g}_0 - \mathscr{T}\mathring{g}_0\|}{(b+1)^2\|\mathring{g} - \mathring{g}_0\|}\right),$$

provided that for all $\frac{1}{b+1}\|\mathring{g} - \mathscr{T}\mathring{g}\| > 0$, we have $\xi(\frac{\mathscr{T}\mathring{g} + b\mathring{g}}{b+1}) < \xi(\mathring{g})$.

In order to denote the involvement of parameters in (10), we shall also call $\mathscr{T}$ a $(b, \mathring{g}_0, \xi)$-general Jaggi-type bilateral symmetric contraction.

Theorem 9. *Let $\mathscr{T} : \hat{\mathbb{X}}_h \to \hat{\mathbb{X}}_h$ be a $(b, \mathring{g}_0, \xi)$-general Jaggi-type bilateral symmetric contraction and r be defined as*

$$r = \inf\left\{\frac{\|\mathring{g} - \mathscr{T}\mathring{g}\|}{(b+1)\xi(\mathring{g})} : \mathring{g} \neq \mathscr{T}\mathring{g}, \mathring{g} \in \hat{\mathbb{X}}_h\right\}. \tag{11}$$

Then, $\mathscr{T}$ fixes the circle $C_{\mathring{g}_0,r}$, provided that $\mathscr{T}\mathring{g}_0 = \mathring{g}_0$.

Proof. Let us denote $\omega = \frac{1}{b+1}$. Clearly, $\omega \in (0,1)$. The contractive condition (10) becomes

$$\omega\|\mathring{g} - \mathscr{T}\mathring{g}\| > 0 \Rightarrow \|\mathring{g} - \mathscr{T}_\omega \mathring{g}\| \leq \left[\xi(\mathring{g}) - \xi\left(\frac{\mathscr{T}\mathring{g} + \left(\frac{1}{\omega} - 1\right)\mathring{g}}{1/\omega}\right)\right] R^{**}_{\mathscr{T}}(\mathring{g}, \mathring{g}_0),$$

which can be equivalently written as

$$\|\mathring{g} - \mathscr{T}_\omega \mathring{g}\| > 0 \Rightarrow \|\mathring{g} - \mathscr{T}_\omega \mathring{g}\| \leq \{\xi(\mathring{g}) - \xi(\mathscr{T}_\omega \mathring{g})\} R^{**}_{\mathscr{T}}(\mathring{g}, \mathring{g}_0), \tag{12}$$

where

$$R^{**}_{\mathscr{T}}(\mathring{g}, \mathring{g}_0) = \max\left(\|\mathring{g} - \mathring{g}_0\|, \frac{\|\mathring{g} - \mathscr{T}_\omega \mathring{g}\|\|\mathring{g}_0 - \mathscr{T}_\omega \mathring{g}_0\|}{\|\mathring{g} - \mathring{g}_0\|}\right). \tag{13}$$

Moreover, also notice that for the value of $\omega = \frac{1}{b+1}$, we get

$$r = \inf\left\{ \frac{\|\mathring{g} - \mathscr{T}_\omega \mathring{g}\|}{\xi(\mathring{g})} : \mathring{g} \neq \mathscr{T}\mathring{g}, \mathring{g} \in \hat{\mathbb{X}}_h \right\}. \tag{14}$$

We divide the proof into two following cases.

Case (i): Assume that $r = 0$. Then, in this case, we have $C_{\mathring{g}_0,r} = \{\mathring{g}_0\}$. Clearly, $\mathscr{T}$ fixes $C_{\mathring{g}_0,r}$.

Case (ii): Assume that $r > 0$. Clearly, we have $R^*_{\mathscr{T}}(\mathring{g}, \mathring{g}_0) = ||\mathring{g} - \mathring{g}_0||$. Indeed, $||\mathring{g} - \mathscr{T}_\omega \mathring{g}|| > 0$.

By using (14) and $R^{**}_{\mathscr{T}}(\mathring{g}, \mathring{g}_0) = ||\mathring{g} - \mathring{g}_0||$, we obtain

$$\begin{aligned}
||\mathring{g} - \mathscr{T}_\omega \mathring{g}|| &\leq [\xi(\mathring{g}) - \xi(\mathscr{T}_\omega \mathring{g})]||\mathring{g} - \mathring{g}_0||, \\
&= [\xi(\mathring{g}) - \xi(\mathscr{T}_\omega \mathring{g})]r \\
&[\xi(\mathring{g}) - \xi(\mathscr{T}_\omega \mathring{g})]\frac{||\mathring{g} - \mathscr{T}\mathring{g}||}{(b+1)\xi(\mathring{g})} \\
&= [\xi(\mathring{g}) - \xi(\mathscr{T}_\omega \mathring{g})]\frac{||\mathring{g} - \mathscr{T}_\omega \mathring{g}||}{\xi(\mathring{g})} \\
&= \left[1 - \frac{\xi(\mathscr{T}_\omega \mathring{g})}{\xi(\mathring{g})}\right]||\mathring{g} - \mathscr{T}_\omega \mathring{g}||,
\end{aligned}$$

and therefore, we have

$$||\mathring{g} - \mathscr{T}_\omega \mathring{g}|| \leq \left[1 - \frac{\xi(\mathscr{T}_\omega \mathring{g})}{\xi(\mathring{g})}\right]||\mathring{g} - \mathscr{T}_\omega \mathring{g}||. \tag{15}$$

Note that $\left[1 - \frac{\xi(\mathscr{T}_\omega \mathring{g})}{\xi(\mathring{g})}\right] < 1$. Therefore, condition (15) becomes

$$||\mathring{g} - \mathscr{T}_\omega \mathring{g}|| \leq \left[1 - \frac{\xi(\mathscr{T}_\omega \mathring{g})}{\xi(\mathring{g})}\right]||\mathring{g} - \mathscr{T}_\omega \mathring{g}|| < ||\mathring{g} - \mathscr{T}_\omega \mathring{g}||. \tag{16}$$

which is a contradiction, so our supposition is wrong and we have $\mathscr{T}_\omega \mathring{g} = \mathring{g}$. Hence, $\mathscr{T}\mathring{g} = \mathring{g}$ for all $\mathring{g} \in C_{\mathring{g}_0,r}$; that is, $\mathscr{T}$ fixes the circle $C_{\mathring{g}_0,r}$. □

Corollary 1. *Assume $\mathscr{T}$ satisfies all the assumptions of Theorem 9. If $\mathscr{T}\mathring{g}_0 = \mathring{g}_0$, then $\mathscr{T}$ fixes the disc $D_{\mathring{g}_0,r}$.*

Proof. The proof is obvious and hence omitted. □

Definition 4. *An operator $\mathscr{T} : \hat{\mathbb{X}}_h \to \hat{\mathbb{X}}_h$ is called a general Dass–Gupta-type-I bilateral symmetric contraction if there is an operator $\xi : \hat{\mathbb{X}}_h \to (0,1)$ and there exist $b \in [0, +\infty)$ and $\mathring{g}_0 \in \hat{\mathbb{X}}_h$ such that for all $\mathring{g} \in \hat{\mathbb{X}}_h - \{\mathring{g}_0\}$, we have*

$$\frac{1}{b+1}\|\mathring{g} - \mathscr{T}\mathring{g}\| > 0 \Rightarrow \frac{1}{b+1}\|\mathring{g} - \mathscr{T}\mathring{g}\| \leq \left[\xi(\mathring{g}) - \xi\left(\frac{\mathscr{T}\mathring{g} + b\mathring{g}}{1+b}\right)\right]Q^*_{\mathscr{T}}(\mathring{g}_0, \mathring{g}), \tag{17}$$

where

$$Q^*_{\mathscr{T}}(\mathring{g}_0, \mathring{g}) = \max\left(\|\mathring{g} - \mathring{g}_0\|, \frac{(b+1+\|\mathring{g}_0 - \mathscr{T}\mathring{g}_0\|)\|\mathring{g} - \mathscr{T}\mathring{g}\|}{(b+1)^2(1+\|\mathring{g} - \mathring{g}_0\|)}\right).$$

provided that for all $\frac{1}{b+1}\|\mathring{g} - \mathscr{T}\mathring{g}\| > 0$, we have $\xi(\frac{\mathscr{T}\mathring{g}+b\mathring{g}}{b+1}) < \xi(\mathring{g})$.

In order to denote the involvement of parameters in (17), we shall also call $\mathscr{T}$ a $(b, \mathring{g}_0, \xi)$-Dass–Gupta-type-I bilateral symmetric contraction.

We state the following theorem for the class of $(b, \mathring{g}_0, \xi)$-Dass–Gupta-type-I bilateral symmetric contractions.

Theorem 10. *Let $\mathscr{T} : \hat{\mathbb{X}}_h \to \hat{\mathbb{X}}_h$ be a $(b, \mathring{g}_0, \xi)$-Dass–Gupta-type-I bilateral symmetric contraction and r be defined as (11). Then, $\mathscr{T}$ fixes the circle $C_{\mathring{g}_0,r}$, provided that $\mathscr{T}\mathring{g}_0 = \mathring{g}_0$.*

Proof. Let us denote $\omega = \frac{1}{b+1}$. Clearly, $\omega \in (0,1)$. The contractive condition (17) becomes

$$\|\mathring{g} - \mathscr{T}_\omega\mathring{g}\| > 0 \Rightarrow \|\mathring{g} - \mathscr{T}_\omega\mathring{g}\| \leq [\xi(\mathring{g}) - \xi(\mathscr{T}_\omega\mathring{g})]Q^{**}_{\mathscr{T}}(\mathring{g}_0, \mathring{g}), \tag{18}$$

where

$$Q^{**}_{\mathscr{T}}(\mathring{g}_0, \mathring{g}) = \max\left(||\mathring{g} - \mathring{g}_0||, \frac{(1 + ||\mathring{g}_0 - \mathscr{T}_\omega\mathring{g}_0||)||\mathring{g} - \mathscr{T}_\omega\mathring{g}||}{1 + ||\mathring{g} - \mathring{g}_0||}\right).$$

Moreover,

$$r = \inf\left\{\frac{\|\mathring{g} - \mathscr{T}_\omega\mathring{g}\|}{\xi(\mathring{g})} : \mathring{g} \neq \mathscr{T}\mathring{g}, \mathring{g} \in \hat{\mathbb{X}}_h\right\}. \tag{19}$$

We divide the proof into the following two cases.

Case (i): Assume that $r = 0$. Then, in this case, we have $C_{\mathring{g}_0,r} = \{\mathring{g}_0\}$. Clearly, $\mathscr{T}$ fixes $C_{\mathring{g}_0,r}$.

Case (ii): Assume that $r > 0$. Clearly, we have $Q^{**}_{\mathscr{T}}(\mathring{g}_0, \mathring{g}) = \max\left(r, \frac{||\mathring{g} - \mathscr{T}_\omega\mathring{g}||}{1+r}\right)$. Indeed, $\|\mathring{g} - \mathscr{T}_\omega\mathring{g}\|$.

By using (18) and $Q^{**}_{\mathscr{T}}(\mathring{g}_0, \mathring{g}) = \max\left(r, \frac{||\mathring{g} - \mathscr{T}_\omega\mathring{g}||}{1+r}\right)$, we obtain

$$\begin{aligned}
||\mathring{g} - \mathscr{T}_\omega\mathring{g}|| &\leq [\xi(\mathring{g}) - \xi(\mathscr{T}_\omega\mathring{g})] \max\left(r, \frac{||\mathring{g} - \mathscr{T}_\omega\mathring{g}||}{1 + r}\right), \\
&\leq [\xi(\mathring{g}) - \xi(\mathscr{T}_\omega\mathring{g})] \max\left(\frac{||\mathring{g} - \mathscr{T}_\omega\mathring{g}||}{\xi(\mathring{g})}, \frac{||\mathring{g} - \mathscr{T}_\omega\mathring{g}||}{1 + r}\right), \\
&= \frac{1}{\xi(\mathring{g})}[\xi(\mathring{g}) - \xi(\mathscr{T}_\omega\mathring{g})] \max\left(||\mathring{g} - \mathscr{T}_\omega\mathring{g}||, \frac{||\mathring{g} - \mathscr{T}_\omega\mathring{g}||\xi(\mathring{g})}{1 + r}\right), \\
&= \left[1 - \frac{\xi(\mathring{g})}{\xi(\mathscr{T}_\omega\mathring{g})}\right]||\mathring{g} - \mathscr{T}_\omega\mathring{g}||.
\end{aligned}$$

Since $\frac{\xi(\mathring{g})}{\xi(\mathscr{T}_\omega\mathring{g})} > 0$, we have

$$||\mathring{g} - \mathscr{T}_\omega\mathring{g}|| \leq \left[1 - \frac{\xi(\mathring{g})}{\xi(\mathscr{T}_\omega\mathring{g})}\right]||\mathring{g} - \mathscr{T}_\omega\mathring{g}|| < ||\mathring{g} - \mathscr{T}_\omega\mathring{g}||$$

or

$$||\mathring{g} - \mathscr{T}_\omega\mathring{g}|| < ||\mathring{g} - \mathscr{T}_\omega\mathring{g}||.$$

which is a contradiction, so our supposition is wrong and we have $\mathscr{T}_\omega\mathring{g} = \mathring{g}$. Hence, $\mathscr{T}\mathring{g} = \mathring{g}$; that is, $\mathscr{T}$ fixes the circle $C_{\mathring{g}_0,r}$. □

Remark 2. *If we take $b = 0$ in Theorems 9 and 10, we obtain Theorems 3.2 and 3.10 in [10], respectively, in the setting of normed spaces.*

Corollary 2. *Assume $\mathscr{T}$ satisfies all the assumptions of Theorem 10. If $\mathscr{T}\mathring{g}_0 = \mathring{g}_0$, then $\mathscr{T}$ fixes the disc $D_{\mathring{g}_0,r}$.*

Remark 3. *If we take $b = 0$ in Corollaries 1 and 2, we obtain Corollaries 3.3 and 3.11 in [10], respectively, in the setting of normed spaces.*

Definition 5. *An operator $\mathscr{T}:\hat{\mathbb{X}}_h\to\hat{\mathbb{X}}_h$ is called a general $(b,\mathring{g}_0,\phi)$-Dass–Gupta-type-II bilateral symmetric contraction if there is an operator $\xi:\hat{\mathbb{X}}_h\to(0,+\infty)$ and there exist $b\in[0,+\infty)$ and $\mathring{g}_0\in\hat{\mathbb{X}}_h$ such that for all and $\mathring{g}\in\hat{\mathbb{X}}_h-\{\mathring{g}_0\}$, we have*

$$\frac{1}{b+1}\|\mathring{g}-\mathscr{T}\mathring{g}\|>0\Rightarrow\frac{1}{b+1}\|\mathring{g}-\mathscr{T}\mathring{g}\|\le\left[\xi(\mathring{g})-\xi\left(\frac{\mathscr{T}\mathring{g}+b\mathring{g}}{1+b}\right)\right]Q^*_{\mathscr{T}}(\mathring{g},\mathring{g}_0). \tag{20}$$

where

$$Q^*_{\mathscr{T}}(\mathring{g},\mathring{g}_0)=\max\left(\|\mathring{g}-\mathring{g}_0\|,\frac{(b+1+\|\mathring{g}-\mathscr{T}\mathring{g}\|)\|\mathring{g}_0-\mathscr{T}\mathring{g}_0\|}{(b+1)^2(1+\|\mathring{g}-\mathring{g}_0\|)}\right).$$

provided that for all $\frac{1}{b+1}\|\mathring{g}-\mathscr{T}\mathring{g}\|>0$, we have $\xi(\frac{\mathscr{T}\mathring{g}+b\mathring{g}}{b+1})<\xi(\mathring{g})$.

In order to denote the involvement of parameters in (20), we shall also call $\mathscr{T}$ a $(b,\mathring{g}_0,\xi)$-Dass–Gupta-type-II bilateral symmetric contractions.

We state the following theorem for the class of $(b,\mathring{g}_0,\xi)$-Dass–Gupta-type-II bilateral symmetric contraction.

Theorem 11. *Let $\mathscr{T}:\hat{\mathbb{X}}_h\to\hat{\mathbb{X}}_h$ be a $(b,\mathring{g}_0,\phi)$-Dass–Gupta-type-II bilateral symmetric contraction and r be defined as (11). Then, $\mathscr{T}$ fixes the circle $C_{\mathring{g}_0,r}$, provided that $\mathscr{T}\mathring{g}_0=\mathring{g}_0$.*

Proof. We omit the proof because it follows from similar arguments as used in the proof of Theorem 10. □

Corollary 3. *Assume $\mathscr{T}$ satisfies all the assumptions of Theorem 11. If $\mathscr{T}\mathring{g}_0=\mathring{g}_0$, then $\mathscr{T}$ fixes the disk $D_{\mathring{g}_0,r}$.*

Now, we present an example which supports our result.

Example 2. *Let $\hat{\mathbb{X}}_h=R$ be the usual metric space and $\mathscr{T}:\hat{\mathbb{X}}_h\to\hat{\mathbb{X}}_h$ be defined by*

$$\mathscr{T}\mathring{g}=\begin{cases}\mathring{g}; & \mathring{g}\in[-1,1],\\ -\mathring{g}; & \mathring{g}\in(-\infty,-1)\cup(1,+\infty).\end{cases}$$

Let

$$\xi(\mathring{g})=\begin{cases}1; & \mathring{g}\in[-1,1],\\ 2|\mathring{g}|; & \mathring{g}\in(-\infty,-1)\cup(1,+\infty).\end{cases}$$

If we take $\mathring{g}=2$, then $\mathscr{T}(2)=-2$ and thus we have $|\mathring{g}-\mathscr{T}\mathring{g}|=4>0$. However, $\xi(2)-\xi(-2)=2|2|-2|-2|=0$ and hence for $|\mathring{g}-\mathscr{T}\mathring{g}|>0$, we have $4\nleq 0$. Thus, $\mathscr{T}$ does not satisfy the condition of Definition 1.

If we take $b=1$, then $\omega=\frac{1}{2}$. In this case, we obtain

$$\mathscr{T}_{\frac{1}{2}}\mathring{g}=\begin{cases}\mathring{g}; & \mathring{g}\in[-1,1],\\ 0; & \mathring{g}\in(-\infty,-1)\cup(1,+\infty).\end{cases}$$

Moreover, the contractive condition (10) reduces to

$$\left|\mathring{g}-\mathscr{T}_{\frac{1}{2}}\mathring{g}\right|>0\Rightarrow\left|\mathring{g}-\mathscr{T}_{\frac{1}{2}}\mathring{g}\right|\le\left[\xi(\mathring{g})-\xi\left(\mathscr{T}_{\frac{1}{2}}\mathring{g}\right)\right]R^{**}_{\mathscr{T}}(\mathring{g},\mathring{g}_0), \tag{21}$$

where

$$R^{**}_{\mathscr{T}}(\mathring{g},\mathring{g}_0)=\max\left(|\mathring{g}-\mathring{g}_0|,\frac{\left|\mathring{g}-\mathscr{T}_{\frac{1}{2}}\mathring{g}\right|\left|\mathring{g}_0-\mathscr{T}_{\frac{1}{2}}\mathring{g}_0\right|}{|\mathring{g}-\mathring{g}_0|}\right),$$

provided that for all $\left|\mathring{g}-\mathscr{T}_{\frac{1}{2}}\mathring{g}\right|>0$, we have $\xi(\mathscr{T}_{\frac{1}{2}}\mathring{g})<\xi(\mathring{g})$.

Clearly, $\mathscr{T}$ is a $(1,0,\xi)$-general Jaggi-type bilateral symmetric contraction. Indeed, for $\mathring{g} \in (-\infty,-1) \cup (1,+\infty)$, we have $\frac{1}{2}|\mathring{g} - \mathscr{T}_{\frac{1}{2}}\mathring{g}| = |\mathring{g}| > 0$, and

$$|\mathring{g} - \mathscr{T}_{\frac{1}{2}}\mathring{g}| = |\mathring{g}| \leq (2|\mathring{g}| - 1)|\mathring{g}| = [\xi(\mathring{g}) - \xi(\mathscr{T}_{\frac{1}{2}}\mathring{g})]R^{**}_{\mathscr{T}}(\mathring{g}, 0).$$

We get

$$\begin{aligned} r &= \inf\left\{ \frac{\|\mathring{g} - \mathscr{T}\mathring{g}\|}{2\xi(\mathring{g})} : \mathring{g} \neq \mathscr{T}\mathring{g}, \mathring{g} \in (-\infty,-1) \cup (1,+\infty) \right\} \\ &= \inf\left\{ \frac{\left\|\mathring{g} - \mathscr{T}_{\frac{1}{2}}\mathring{g}\right\|}{\xi(\mathring{g})} : \mathring{g} \neq \mathscr{T}\mathring{g}, \mathring{g} \in (-\infty,-1) \cup (1,+\infty) \right\} \\ &= \inf\left\{ \frac{|\mathring{g}|}{2|\mathring{g}|} : \mathring{g} \neq \mathscr{T}\mathring{g}, \mathring{g} \in (-\infty,-1) \cup (1,+\infty) \right\} = \frac{1}{2} \end{aligned}$$

Hence, $\mathscr{T}$ satisfies the conditions of Theorem 9 and Corollary 1. Consequently, $\mathscr{T}$ fixes $C_{0,\frac{1}{2}} = \{-\frac{1}{2}, \frac{1}{2}\}$ and $D_{0,\frac{1}{2}} = [-\frac{1}{2}, \frac{1}{2}]$.

Example 3. *Let $\hat{\mathbb{X}}_h = R$ be the usual metric space and $\mathscr{T} : \hat{\mathbb{X}}_h \to \hat{\mathbb{X}}_h$ be defined by*

$$\mathscr{T}\mathring{g} = \begin{cases} \mathring{g}; & \mathring{g} \in [-2,+\infty), \\ -\mathring{g}; & \mathring{g} \in (-\infty,-2). \end{cases}$$

Let

$$\xi(\mathring{g}) = \begin{cases} 1/2; & \mathring{g} \in [-2,+\infty), \\ |\mathring{g}|; & \mathring{g} \in (-\infty,-2). \end{cases}$$

If we take $\mathring{g} = -3$, then $\mathscr{T}(-3) = 3$ and thus we have $|\mathring{g} - \mathscr{T}\mathring{g}| = 6 > 0$. However, $\xi(-3) - \xi(3) = |-3| - |3| = 0$ and hence for $|\mathring{g} - \mathscr{T}\mathring{g}| > 0$, we have $6 \nleq 0$. Thus, $\mathscr{T}$ does not satisfy the condition of Definition 2.

If we take $b = 1$, then $\omega = 1/2$. In this case, we obtain

$$\mathscr{T}_{\frac{1}{2}}\mathring{g} = \begin{cases} \mathring{g}; & \mathring{g} \in [-2,+\infty) \\ 0; & \mathring{g} \in (-\infty,-2). \end{cases}$$

Moreover, the contractive condition (20) reduces to

$$|\mathring{g} - \mathscr{T}_{\frac{1}{2}}\mathring{g}| > 0 \Rightarrow |\mathring{g} - \mathscr{T}_{\frac{1}{2}}\mathring{g}| \leq \left[\xi(\mathring{g}) - \xi\left(\mathscr{T}_{\frac{1}{2}}\mathring{g}\right)\right] Q^{**}_{\mathscr{T}}(\mathring{g}, \mathring{g}_0), \tag{22}$$

where

$$Q^{**}_{\mathscr{T}}(\mathring{g}, \mathring{g}_0) = \max\left(|\mathring{g} - \mathring{g}_0|, \frac{\left(1 + \left|\mathring{g} - \mathscr{T}_{\frac{1}{2}}\mathring{g}\right|\right)\left|\mathring{g}_0 - \mathscr{T}_{\frac{1}{2}}\mathring{g}_0\right|}{|\mathring{g} - \mathring{g}_0|} \right),$$

provided that for all $\left|\mathring{g} - \mathscr{T}_{\frac{1}{2}}\mathring{g}\right| > 0$, we have $\xi(\mathscr{T}_{\frac{1}{2}}\mathring{g}) < \xi(\mathring{g})$.

Clearly, $\mathscr{T}$ is a $(1,0,\xi)$-general Dass–Gupta-type-II bilateral symmetric contraction. Indeed, for $\mathring{g} \in (-\infty,-2)$, we have $1/2|\mathring{g} - \mathscr{T}_{\frac{1}{2}}\mathring{g}| = |\mathring{g}| > 0$, and

$$|\mathring{g} - \mathscr{T}_{\frac{1}{2}}\mathring{g}| = |\mathring{g}| \leq [|\mathring{g}| - 1/2]|\mathring{g}| = [\xi(\mathring{g}) - \xi(\mathscr{T}\mathring{g})]Q^{**}_{\mathscr{T}}(\mathring{g}, 0).$$

We get

$$r = \inf\left\{ \frac{\|\mathring{g} - \mathscr{T}\mathring{g}\|}{2\xi(\mathring{g})} : \mathring{g} \neq \mathscr{T}\mathring{g}, \mathring{g} \in (-\infty, -2) \right\}$$
$$= \inf\left\{ \frac{\left\|\mathring{g} - \mathscr{T}_{\frac{1}{2}}\mathring{g}\right\|}{\xi(\mathring{g})} : \mathring{g} \neq \mathscr{T}\mathring{g}, \mathring{g} \in (-\infty, -2) \right\}$$
$$= \inf\left\{ \frac{|\mathring{g}|}{|\mathring{g}|} : \mathring{g} \neq \mathscr{T}\mathring{g}, \mathring{g} \in (-\infty, -2) \right\} = 1.$$

Hence, $\mathscr{T}$ satisfies the conditions of Theorem 11 and Corollary 3. Consequently, $\mathscr{T}$ fixes $C_{0,1} = \{-1, 1\}$ *and disk* $D_{0,1} = [-1, 1]$.

5. Application

Fixed-circle theorems are relevant to activation functions in neural networks because they offer valuable insights into the behavior of iterative processes and their convergence. Activation functions have a vital role in determining the output of a neuron based on its input within a neural network.

By examining fixed-circle theorems, we can analyze the characteristics of activation functions in terms of fixed points or stable cycles. These theorems establish conditions under which iterative processes, such as forward propagation in a neural network, converge to a fixed point or cycle.

Comprehending the properties of fixed-circle theorems can aid in the design and selection of appropriate activation functions for neural networks. It enables us to evaluate the convergence properties of the network and ensure that the learning process attains the desired solution.

In conclusion, the connection between fixed-circle theorems and activation functions lies in their ability to provide insights into the convergence behavior of neural networks, enabling us to choose suitable activation functions for effective and efficient learning.

Activation functions are pivotal in neural networks as they facilitate learning and interpretation. Their primary function is to transform the input signal of a node in the neural network into an output signal. There are numerous examples of activation functions being utilized in neural networks, with "rectified linear units (ReLUs)" [10,29] being one of the most widely used activation functions.

Now, we consider the following activation function

$$NPReLU(\mathring{g}) = \mathscr{T}(\mathring{g}) = \begin{cases} \mathring{g}; & \mathring{g} \geq 0 \\ \frac{\alpha\mathring{g} + \omega\mathring{g} - \mathring{g}}{\omega}; & \mathring{g} < 0, \end{cases}$$

where $\alpha, \omega \in (0, 1)$.

Note that,

$$\mathscr{T}_\omega \mathring{g} = \begin{cases} \mathring{g}; & \mathring{g} \geq 0 \\ \alpha\mathring{g}; & \mathring{g} < 0. \end{cases}$$

The average operator satisfies Theorem 9 on the usual metric space with $\mathring{g}_0 = 1$ and the function $\xi : \hat{\mathbb{X}}_h \to (0, +\infty)$ as

$$\xi(\mathring{g}) = \begin{cases} 1; & \mathring{g} = 0 \\ |\mathring{g}|; & \mathring{g} \neq 0 \end{cases}$$

for all $\mathring{g} \in \mathbb{R}$. Indeed, for $\mathring{g} \in (-+\infty, 0)$, we get

$$||\mathring{g} - \mathscr{T}_\omega \mathring{g}|| = |(1 - \alpha)\mathring{g}| > 0,$$

$$R^{**}_{\mathscr{T}}(1,r) = \max\left\{||1-\mathring{g}||, \frac{||1-\mathscr{T}_\omega 1||||\mathring{g}-\mathscr{T}_\omega \mathring{g}||}{||1-\mathring{g}||}\right\} = |1-\mathring{g}|$$

and

$$\begin{aligned} ||\mathring{g}-\mathscr{T}_\omega \mathring{g}|| &= |\mathring{g}||1-\alpha|, \\ &< |\mathring{g}||1-\alpha||1-\mathring{g}| \\ &= (|\mathring{g}|-|\alpha\mathring{g}|)|1-\mathring{g}|, \\ &= [\xi(\mathring{g})-\xi(\alpha\mathring{g})]|1-\mathring{g}| \\ &= [\xi(\mathring{g})-\xi(\mathscr{T}_{1/3}\mathring{g})]R^{**}_{\mathscr{T}}(1,\mathring{g}). \end{aligned}$$

Hence, $\mathscr{T}_\omega \mathring{g}$ is a Jaggi-type bilateral $\mathring{g}_0$-contractive operator. Furthermore, we obtain

$$r = \inf\left\{\frac{||\mathring{g}-\mathscr{T}_\omega \mathring{g}||}{\xi(\mathring{g})} : \mathring{g} \neq \mathscr{T}_\omega \mathring{g}, \mathring{g} \in \mathbb{R}\right\} = 1-\alpha$$

Hence, the average operator $\mathscr{T}_\omega \mathring{g}$ satisfies the conditions of Theorem 9, so the operator $\mathscr{T}$, which is the activation function $PReLU(\mathring{g})$, fixes the circle $C_{1,1-\alpha} = \alpha, 2-\alpha$ and the disc $D_{1,1-\alpha} = [\alpha, 2-\alpha]$.

6. Conclusions and Future Directions

In this study, we have introduced a comprehensive class of contractive operators called Caristi-type contractions. We have also defined and explored general Jaggi-type bilateral-type contractions, Dass–Gupta-type-I bilateral contractions, and Dass–Gupta-type-II bilateral contractions. These newly defined classes of contractions provide a broader framework for analyzing and studying various types of contractive operators.

We have derived several fixed-circle theorems for different types of contractions. Theorems 6–11 present these results for general Jaggi-type bilateral contractions, Caristi-type contractions, general Dass–Gupta-type-I bilateral contractions, and Dass–Gupta-type-II bilateral contractions, respectively. These theorems demonstrate the versatility and applicability of our approach across different types of symmetric contraction operators.

To support our results, we have provided Examples 2 and 3, which illustrate the effectiveness and applicability of our theorems. These examples highlight the fact that our newly defined contractions are genuine generalizations of other contractions studied in the literature. The obtained results are generalizations of corresponding results in the literature and can be applied to other research areas.

Furthermore, we have showcased the practical application of our main result, Theorem 9, in the context of a novel activation function for neural networks. By demonstrating the effectiveness of this new approach, we contribute to the advancement of neural network research and its potential for improving the performance of various applications.

In terms of future research, it would be interesting to explore the applicability of the contractions defined by Definitions 4 and 5 in the context of partial metric spaces and m-metric spaces. Investigating fixed-circle theorems in these more general settings would further expand our understanding of the geometric properties of fixed points.

Overall, our study provides valuable insights into the theory of contractive operators and opens up new avenues for research and application in various mathematical and applied fields.

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