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Geometry of Warped Product Hemi-Slant Submanifolds of an S-Manifold

Ahlam Al-Mutairi, Reem Al-Ghefari and Awatif Al-Jedani *

Department of Mathematics and Statistics, Faculty of Science, University of Jeddah, Jeddah 23890, Saudi Arabia; aalmutairi0565.stu@uj.edu.sa (A.A.-M.); ralghefari@uj.edu.sa (R.A.-G.)
* Correspondence: amaljedani@uj.edu.sa

Abstract: The purpose of this paper is to investigate a warped product of hemi-slant submanifolds on an S-manifold. We prove many interesting results for the existence of warped product hemi-slant submanifold of the type $M_\theta \times_f M_\perp$ with $\xi_\alpha \in M_\theta$ of an S-manifold. For such submanifolds, a characterization theorem is proven. In addition, we form an inequality for the squared norm of the second fundamental form in terms of the warping function and the slant angle. We also provide some examples, and the equality case is also considered.

Keywords: warpedproduct; slant submanifold; hemi-slant submanifold; warped product hemi-slant submanifold; S-manifold

1. Introduction

In 1963, the concept of a φ-structure on a smooth manifold $\overline{M}$ of dimension $(2n+s)$ was introduced by Yano [1] as a non-vanishing tensor field of type $(1,1)$ on $\overline{M}$, which satisfies $\varphi^3+\varphi=0$ and has a constant rank $r=2n$. φ-structures are almost complex if $(s=0)$, and almost contact if $(s=1)$. In 1970, Goldberg and Yano [2] defined globally framed φ-structures for which the sub-bundle kerφ is parallelizable. Then there exists a global frame $\{\xi_1,\xi_2,\ldots,\xi_s\}$ for the sub-bundle kerφ, the vector fields $\xi_1,\xi_2,\ldots,\xi_s$ are called the structure vector fields with dual 1-forms $\eta_1,\eta_2,\ldots,\eta_s$ such that $g(\varphi X,\varphi Y)=g(X,Y)-\sum_{\alpha=1}^{s}\eta_\alpha(X)\eta_\alpha(Y)$ for any vector fields X,Y in $\overline{M}$, and then the structure is called a metric φ-structure. In [3], a wider class of a globally framed φ-manifold was introduced by the following definition: a metric φ-structure is said to be a K-structure if the fundamental 2-form Φ given by $\Phi(X,Y)=g(X,\varphi Y)$ for any vector fields X and Y on $\overline{M}$ is closed and the normality condition holds, that is, $[\varphi,\varphi]+2\sum_{\alpha=1}^{s}\xi_\alpha\otimes d\eta_\alpha=0$, where $[\varphi,\varphi]$ denotes the Nijenhuis tensor of φ. A K-manifold is called an S-manifold if $d\eta_\alpha=\Phi$ for all $\alpha=1,\ldots,s$. An S-manifold is a Sasakian manifold if $s=1$. For $s\geq 2$, examples of an S-manifold are presented in [3–6]. Furthermore, an S-manifold has been studied by several authors (see, for example, [2,7–10]).

The geometry of slant submanifolds has been extensively investigated since Chen defined and studied slant immersions in complex geometry as a natural generalization of both holomorphic and totally real immersions [11,12]. Later, this study for almost contact metric manifolds was expanded by Lotta [13].After that, Cabrerizo et al. [14] studied these submanifolds in the case of K-contact and Sasakian manifolds. To generalize these submanifolds, Papaghiuc [15] studied a new class of submanifolds known as semi-slant submanifolds, which were then expanded by Cabrerizo et al. for contact metric manifolds [16]. Recently, Carriazo [17] introduced the notion of anti-slant submanifolds, which were later renamed pseudo-slant submanifolds because the name anti-slant appears to refer to the fact that they lack a slant factor. However, in [18], Sahin refers to these submanifolds as hemi-slant submanifolds. Several geometers have studied hemi-slant submanifolds in various structures since then (see, for example, [19–21]).

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On the other hand, Bishop and O'Neill [22] initiated the concept of a warped product in 1969 as a natural generalization of Riemannian product manifolds. The warped product manifolds have their applications in general relativity. Many spacetime models are warped product manifolds, including Robertson–Walker spacetime, asymptotically flat spacetime, Schwarzschild spacetime and Reissner-Nordström spacetime. For more information, see [23].

At the turn of this century, the idea of warped product submanifolds was introduced by Chen in his series of papers [24,25]. He proved that the warped product CR-submanifolds of the type $M_\perp \times_f M_T$ does not exist, where M_T and $M_\perp$ are holomorphic and totally real submanifolds of a Kaehler manifold $\overline{M}$, respectively. Then, he considered CR-warped products in a Kaehler manifold, which are warped products of the form $M_T \times_f M_\perp$. He showed several fundamental results on the existence of CR-warped products in Kaehler manifolds, such as optimal inequalities and characterizatios in [24–26]. Many geometers researched warped product submanifolds for the various structures on Riemannian manifolds, as inspired by Chen's work [27–32]. Some researchers have also extended this approach to warped product semi-slant and pseudo-slant submanifolds (see [32–36]). In [34], Sahin showed that there are no warped product semi-slant submanifolds other than CR-warped products in a Kaehler manifold introduced by Chen in [24,25]. Recently, Sahin studied the warped product pseudo-slant submanifolds of a Keahler manifold under the name of the hemi-slant warped product in [18]. He provided many interesting results, including a characterization and an inequality by the mixed totally geodesic condition. In the context of an S-manifold, we have seen no warped product semi-slant submanifolds other than a contact CR-warped product submanifold [37].

In this paper, we investigate the warped product submanifold where one of the factors is a slant and another is an anti-invariant, and we call such submanifolds warped product hemi-slant submanifolds of an S-manifold.

This paper is organized as follows: Section 2 goes over some fundamental formulas and definitions for an S-manifold and its submanifolds. We review the definitions of slant and hemi-slant submanifolds in Section 3. In addition, we will study the integrability conditions of distributions and some basic properties related to the totally geodesicness of distributions involved in the definition of the hemi-slant submanifold. In Section 4, we investigate a warped product hemi-slant submanifold. We obtain a characterization result and then construct an example of such warped product immersions. In Section 5, we form an inequality for the squared norm of the second fundamental form in terms of the warping function and the slant angle.

2. Basic Concepts

An S-manifold is a $(2n+s)$-dimensional differentiable manifold $\overline{M}$ which carries a $(1,1)$-tensor field φ (φ—structure and has a constant rank $2n$), s global vector fields ξ_α (structure vector fields), and s 1-forms η_α satisfying [3]

$$\varphi^2 = -I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha, \quad \varphi\xi_\alpha = 0, \quad \eta_\alpha \circ \varphi = 0, \quad \eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}, \tag{1}$$

where $I : T\overline{M} \to T\overline{M}$ is the identity mapping. In addition, $\overline{M}$ admits a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X,Y) - \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y), \tag{2}$$

for any $X, Y \in T\overline{M}$, the Lie algebra of vector fields on $\overline{M}$. As an immediate consequence of (2),

$$\eta_\alpha(X) = g(X, \xi_\alpha), \tag{3}$$

$$g(\varphi X, Y) = -g(X, \varphi Y) \tag{4}$$

Moreover, the S-structure $\left(\overline{M}, \varphi, \xi_\alpha, \eta_\alpha, g\right)$ is *normal*, that is

$$[\varphi, \varphi] + 2\sum_{\alpha=1}^{s} \xi_\alpha \otimes d\eta_\alpha = 0$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. Furthermore, in an S-manifold, we have

$$\eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_s \wedge (d\eta_\alpha)^n \neq 0 \qquad \text{and} \qquad \eta_\alpha = \Phi, \quad \alpha = 1, \ldots, s$$

where Φ is the fundamental 2-form defined by $\Phi(X, Y) = g(X, \varphi Y)$.

For the Levi–Civita connection $\overline{\nabla}$ of g on an S-manifold, $\overline{M}$ can be expressed by

$$(\overline{\nabla}_X \varphi)Y = \sum_{\alpha=1}^{s} \Big[g(\varphi X, \varphi Y)\xi_\alpha + \eta_\alpha(Y)\varphi^2 X \Big], \tag{5}$$

$$\overline{\nabla}_X \xi_\alpha = -\varphi X. \tag{6}$$

for all $X, Y \in T\overline{M}$.

Let $\mathcal{L}$ denote the distribution determined by $-\varphi^2$ and μ. The complementary distribution is determined by $\varphi^2 + I$ and spanned by $\xi_\alpha, \alpha = 1, \ldots, s$. If $X \in \mathcal{L}$, then $\eta_\alpha(X) = 0$ for all α, and if $X \in \mu$, then $\varphi X = 0$.

The covariant derivative of φ is defined by

$$\left(\overline{\nabla}_X \varphi\right)Y = \overline{\nabla}_X \varphi Y - \varphi \overline{\nabla}_X Y \tag{7}$$

for all $X, Y \in T\overline{M}$.

Now, let M be an isometrically immersed submanifold in $\overline{M}$ with induced metric g. Let TM be the Lie algebra of vector fields on M, and $T^\perp M$ the set of all vector fields normal to M. If we denote the Levi–Civita connection induced on the tangent bundle TM by ∇ and $\nabla^\perp$ is the normal connection in the normal bundle $T^\perp M$ of M, then the Gauss and Weingarten formulas are, respectively, given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{8}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^\perp_X V, \tag{9}$$

for any vector field $X, Y \in TM$ and $V \in T^\perp M$, where h and A_V are the second fundamental form and A_V the shape operator (corresponding to the normal vector field V), respectively, for the immersion of M into $\overline{M}$. They are related by

$$g(h(X, Y), V) = g(A_V X, Y) \tag{10}$$

For any $X \in TM$ and $V \in T^\perp M$, we write

$$\varphi X = TX + FX, \tag{11}$$

$$\varphi V = tX + fX, \tag{12}$$

where TX and tX are the tangential components, and FX and fX are the normal components of φX and φV, respectively. The covariant derivatives ∇T and ∇F are defined by

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y$$

$$(\nabla_X F)Y = \nabla^\perp_X FY - F\nabla_X Y$$

for all $X, Y \in TM$. For a submanifold M of an S-manifold $\overline{M}$ by Equations (6), (8) and (11),

$$\nabla_X \xi_\alpha = -TX, \tag{13}$$

$$h(X, \xi_\alpha) = -FX. \tag{14}$$

Let $p \in M$ and $\{e_1, \ldots, e_m, \ldots, e_{2n+s}\}$ be an orthonormal basis of the tangent space $T_p\overline{M}$, then for a smooth function f on M such that $e_1, \ldots, e_m$ are tangent to M at p. Then, the *mean curvature vector* is $H(p) = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_i)$. Furthermore, the squared norm of the second fundamental form h is defined by

$$\|h\|^2 = \sum_{j=1}^{m}\sum_{i=1}^{m} g\big(h(e_i, e_j), h(e_i, e_j)\big).$$

and

$$h_{ij}^r = g\big(h(e_i, e_j), e_r\big), \quad i, j \in \{1, 2, \ldots, m\}, \quad r \in \{m+1, \ldots, 2n+s\}.$$

The gradient of a smooth function f on a manifold M, denoted as $\vec{\nabla} f$, is defined by

$$g\left(\vec{\nabla} f, X\right) = Xf,$$

for any $X \in T\overline{M}$.

For the submanifold tangent to the structure vector field ξ_α, the submanifold M is said to be an *invariant* submanifold if $\varphi\big(T_pM\big) \subset T_pM$, for every $p \in M$. Otherwise, M is said to be an *anti-invariant* submanifold if $\varphi\big(T_pM\big) \subset T_p^{\perp}M$, for every $p \in M$.

A submanifold M tangent to ξ_α is said to be a *contact CR-submanifold* if there exists a pair of orthogonal distributions $D : p \to D_p$ and $D^{\perp} : p \to D_p^{\perp}, \forall p \in M$ such that

$$TM = D \oplus D^{\perp} \oplus \langle \xi_\alpha \rangle,$$

where $\langle \xi_\alpha \rangle$ is the S-dimensional distribution spanned by the structure vector field ξ_α, D is invariant, i.e., $\varphi D = D$ and $D^{\perp}$ is anti-invariant, i.e., $\varphi D^{\perp} \subseteq T^{\perp}M$.

3. Slant and Hemi-Slant Submanifold

In this section, we discuss the other classes of submanifolds tangent to ξ_α. If X and ξ_α are linearly independent for each nonzero vector X tangent to M at p and the angle between φX and T_pM is constant $\theta(X) \in \left[0, \frac{\pi}{2}\right]$ for all nonzero $X \in T_pM - \langle \xi_\alpha \rangle, \forall p \in M$, then M is said to be a *slant* submanifold and the angle $\theta(X)$ is called *slant angle* of M. Obviously, if $\theta = 0$ or $\theta = \frac{\pi}{2}$, then M is an *invariant* or *anti-invariant* submanifold, respectively. A slant submanifold which is not invariant nor anti-invariant is called a *proper slant* submanifold.

We recall the following result for the slant submanifold of an S-manifold [38].

Theorem 1. *Let M be a submanifold of an S-manifold $\overline{M}$ such that $\xi_\alpha \in TM$. Then, M is a slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = \lambda\left(-I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha\right) \tag{15}$$

Furthermore, in such case, if θ is a slant angle, then $\lambda = \cos^2\theta$.

The following relations are a straightforward consequence of (15):

$$g(TX, TY) = \cos^2\theta\left[g(X, Y) - \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y)\right], \tag{16}$$

$$g(FX, FY) = \sin^2\theta\left[g(X, Y) - \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y)\right], \tag{17}$$

for any $X, Y \in TM$.

Now, for a slant submanifold M, it is easy to show the following result for subsequent use:

Theorem 2. *Let M be a proper slant submanifold of an S-manifold $\overline{M}$, such that $\xi_\alpha \in TM$. Then, for any $X \in TM$*

$$(a) \quad tFX = \sin^2\theta\left(-X + \sum_{\alpha=1}^{s} \eta_\alpha(X)\xi_\alpha\right), \tag{18}$$

$$(b) \quad fFX = -FTX, \tag{19}$$

Proof. From the relation (11), we have

$$\varphi^2 X = \varphi TX + \varphi FX,$$

for any $X \in TM$. Applying the Equations (1) and (12) and again by (11), we derive

$$-X + \sum_{\alpha=1}^{s} \eta_\alpha(X)\xi_\alpha = T^2X + FTX + tFX + fFX.$$

Then, using Theorem 1, we arrive at the desired result by equaling the tangential and normal components. $\square$

A submanifold M tangent to ξ_α is said to be a *hemi-slant submanifold* if there exists a pair of orthogonal distributions $D^\perp$ and D_θ on M such that $TM = D^\perp \oplus D_\theta \oplus \langle\xi_\alpha\rangle$, $D^\perp$ is anti-invariant, i.e., $\varphi D^\perp \subseteq T^\perp M$, D_θ is a proper slant with a slant angle $\theta \neq \frac{\pi}{2}$.

If we denote the dimensions of $D^\perp$ and D_θ by n_1 and n_2, respectively, then M is an invariant (resp. an anti-invariant) submanifold if $n_1 = 0$ and $\theta = 0$ (resp. $n_2 = 0$). Also, the contact CR-submanifold and slant submanifold are special cases of a hemi-slant submanifold with a slant angle $\theta = 0$ and $n_1 = 0$, respectively. A hemi-slant submanifold M is a *proper* hemi-slant if neither $n_1 = 0$ nor $\theta = 0$ or $\frac{\pi}{2}$.

A hemi-slant submanifold M of an S-manifold $\overline{M}$ is said to be *mixed geodesic* if $h(X, Z) = 0$, for any $X \in D^\perp$ and $Z \in D_\theta$.

Now, we will discuss the integrability of distributions involved in the definition of a hemi-slant submanifold of an S-manifold $\overline{M}$, and we also investigate some basic properties related to the totally geodesicness of the distributions.

For a hemi-slant submanifold M of an S-manifold $\overline{M}$, we have

$$TM = D^\perp \oplus D_\theta \oplus \langle\xi_\alpha\rangle,$$

where $\langle\xi_\alpha\rangle$ is the S-dimensional distribution spanned by the structure vector field ξ_α. Then, for any $X \in TM$, put

$$X = P_1X + P_2X + \sum_{\alpha=1}^{s} \eta_\alpha(X)\xi_\alpha \tag{20}$$

where P_i $(i = 1, 2)$ are projection maps on the distributions $D^\perp$ and D_θ. Now, operating φ on both sides of Equation (20)

$$\varphi X = FP_1X + TP_2X + FP_2X.$$

It easy to see that

$$TX = TP_2X, \quad FX = FP_1X + FP_2X$$

and

$$\varphi P_1X = FP_1X, \quad TP_1X = 0; \quad TP_2X \in D_\theta.$$

Then, the normal bundle $T^\perp M$ can be decomposed as

$$T^\perp M = \varphi D^\perp \oplus FD_\theta \oplus \nu, \tag{21}$$

where ν is the normal invariant sub-bundle under φ.

As $D^{\perp}$ and D_{θ} are orthogonal distributions on M, $g(X,Z)=0$ for each $X\in D^{\perp}$ and $Z\in D_{\theta}$, then, by Equations (4) and (11), we may write

$$g(FX,FZ)=g(\varphi X,\varphi Z)=g(X,Z)=0$$

That means the distributions $\varphi D^{\perp}$ and FD_{θ} are mutually perpendicular. In fact, the decomposition (21) is an orthogonal direct decomposition.

Now, the following lemmas play an important role in working out the integrability conditions of distributions involved in this setting.

Lemma 1. *Let M be a hemi-slant submanifold of an S-manifold $\overline{M}$. Then,*

$$A_{\varphi Y}X=A_{\varphi X}Y,$$

for all $X,Y\in D^{\perp}$.

Proof. For any $X,Y\in D^{\perp}$ and $Z\in TM$, using Equations (4) and (10), we find

$$g(A_{\varphi Y}X,Z)=-g(\varphi h(X,Z),Y)$$

Then, by the Gauss Formula (8), and since $\varphi D^{\perp}\subset T^{\perp}M$, we arrive at

$$g(A_{\varphi Y}X,Z)=-g(\varphi\overline{\nabla}_{Z}X,Y)$$

Now, applying the Equations (3), (5), (7) and (9), we have the following

$$g(A_{\varphi Y}X,Z)=g(A_{\varphi X}Z,Y)+g(\varphi Z,\varphi X)g(\xi_{\alpha},Y)+g(X,\xi_{\alpha})g(\varphi^{2}Z,Y)$$

Since $D^{\perp}$ and $\langle\xi_{\alpha}\rangle$ are orthogonal, we derive

$$g(A_{\varphi Y}X,Z)=g(A_{\varphi X}Z,Y)$$

The result follows from the above equation and by the symmetry of the shape operator. This proves the lemma completely. □

Lemma 2. *Let M be a hemi-slant submanifold of an S-manifold $\overline{M}$. Then,*

$$[X,\xi_{\alpha}]\in D^{\perp},$$

for all $X\in D^{\perp}$.

Proof. For any $X\in D^{\perp}$ and $Z\in D_{\theta}$,

$$g([X,\xi_{\alpha}],TZ)=g\big(\nabla_{X}\xi_{\alpha}-\nabla_{\xi_{\alpha}}X,TZ\big)=g\big(\overline{\nabla}_{X}\xi_{\alpha}-\overline{\nabla}_{\xi_{\alpha}}X,TZ\big)$$

From the relations (6), (7) and (11), we obtain

$$g([X,\xi_{\alpha}],TZ)=g\big(\overline{\nabla}_{\xi_{\alpha}}TZ,X\big)=g\big((\overline{\nabla}_{\xi_{\alpha}}\varphi)Z,X\big)+g\big(\varphi\overline{\nabla}_{\xi_{\alpha}}Z,X\big)-g\big(\overline{\nabla}_{\xi_{\alpha}}FZ,X\big)$$

Using the Equations (1), (5), (8) and (9), we get

$$g([X,\xi_{\alpha}],TZ)=-g\big(\nabla_{\xi_{\alpha}}Z,\varphi X\big)-g(h(\xi_{\alpha},Z),\varphi X)+g(h(\xi_{\alpha},X),FZ)$$

Since $X\in D^{\perp}$, so $\varphi X\in T^{\perp}M$ and $\varphi X=FX$. Thus, from (14), we obtain

$$g([X,\xi_{\alpha}],TZ)=0$$

This proves the lemma completely. □

Proposition 1. *Let M be a proper hemi-slant submanifold of an S-manifold $\overline{M}$. Then, the anti-invariant distribution $D^{\perp}$ is always integrable.*

Proof. From the Gauss Formula (8), for any $X, Y \in D^{\perp}$ and $Z \in D_{\theta}$, we get

$$g([X,Y],Z) = g(\nabla_X Y, Z) - g(\nabla_Y X, Z) = g(\overline{\nabla}_X Y, Z) - g(\overline{\nabla}_Y X, Z)$$

By (2) and (3), we find

$$g([X,Y],Z) = g(\varphi\overline{\nabla}_X Y, \varphi Z) + \eta_\alpha(Z) g(\overline{\nabla}_X Y, \xi_\alpha) - g(\varphi\overline{\nabla}_Y X, \varphi Z) - \eta_\alpha(Z) g(\overline{\nabla}_Y X, \xi_\alpha)$$

Then, using (4) and (6), we derive

$$g([X,Y],Z) = g(\varphi\overline{\nabla}_X Y, \varphi Z) - g(\varphi\overline{\nabla}_Y X, \varphi Z)$$

From (1), (4), (5) and (7), we have

$$\begin{aligned} g([X,Y],Z) &= g(\overline{\nabla}_X \varphi Y, \varphi Z) - g(\overline{\nabla}_Y \varphi X, \varphi Z) \\ &= g(\overline{\nabla}_X \varphi Y, TZ) + g(\overline{\nabla}_X \varphi Y, FZ) - g(\overline{\nabla}_Y \varphi X, TZ) - g(\overline{\nabla}_Y \varphi X, FZ) \end{aligned}$$

From the Equations (4) and (9), and using the orthogonality of vector fields, we obtain

$$g([X,Y],Z) = g(A_{\varphi X} Y - A_{\varphi Y} X, TZ) - g(\overline{\nabla}_X FZ, \varphi Y) + g(\overline{\nabla}_Y FZ, \varphi X)$$

By Lemma 1, Equations (4) and (7), we get

$$\begin{aligned} g([X,Y],Z) &= g(\varphi\overline{\nabla}_X FZ, Y) - g(\varphi\overline{\nabla}_Y FZ, X) \\ &= g(\overline{\nabla}_X \varphi FZ, Y) - g((\overline{\nabla}_X \varphi) FZ, Y) - g(\overline{\nabla}_Y \varphi FZ, X) + g((\overline{\nabla}_Y \varphi) FZ, X) \end{aligned}$$

Using (1), (5), (12), and the fact that $D^{\perp}$ and $\langle \xi_\alpha \rangle$ are orthogonal, we arrive at

$$g([X,Y],Z) = g(\overline{\nabla}_X tFZ, Y) + g(\overline{\nabla}_X fFZ, Y) - g(\overline{\nabla}_Y tFZ, X) - g(\overline{\nabla}_Y fFZ, X)$$

Then, by the relations (18) and (19), we get

$$\begin{aligned} g([X,Y],Z) = &- \sin^2\theta g(\overline{\nabla}_X Z, Y) + \sin^2\theta \eta_\alpha(Z) g(\overline{\nabla}_X \xi_\alpha, Y) - g(\overline{\nabla}_X FTZ, Y) \\ &+ \sin^2\theta g(\overline{\nabla}_Y Z, X) - \sin^2\theta \eta_\alpha(Z) g(\overline{\nabla}_Y \xi_\alpha, X) + g(\overline{\nabla}_Y FTZ, X) \end{aligned}$$

Applying (6), (9) and by the orthogonality of vector fields, we derive

$$g([X,Y],Z) = \sin^2\theta g(\overline{\nabla}_X Y, Z) + g(A_{FTZ} X, Y) - \sin^2\theta g(\overline{\nabla}_Y X, Z) - g(A_{FTZ} Y, X)$$

Using the fact that the shape operator is symmetric, we arrive at

$$g([X,Y],Z) = \sin^2\theta g([X,Y],Z)$$

which means that

$$\cos^2\theta g([X,Y],Z) = 0$$

Since M is a proper hemi-slant submanifold, then $\cos^2\theta \neq 0$, and hence we conclude that $g([X,Y],Z) = 0$. Therefore, $[X,Y] \in D^{\perp}$, for any $X, Y \in D^{\perp}$, i.e., the anti-invariant distribution $D^{\perp}$ is integrable. The proof is complete. □

From Proposition 1 and Lemma 2, we have the following corollary:

Corollary 1. *On a hemi-slant submanifold M of an S-manifold $\overline{M}$, the distribution $D^{\perp} \oplus \langle \xi_\alpha \rangle$ is integrable.*

Lemma 3. *Let M be a hemi-slant submanifold of an S-manifold $\overline{M}$. Then,*

$$g([X,Y],\xi_\alpha) = 2g(X,TY),$$

for any $X, Y \in D^{\perp} \oplus D_\theta$.

Proof. For any $X, Y \in D^{\perp} \oplus D_\theta$, we have

$$g([X,Y],\xi_\alpha) = g(\nabla_X Y, \xi_\alpha) - g(\nabla_Y X, \xi_\alpha)$$

Applying Equations (4) and (13), the result follows. □

From the above Lemma 3, we have the following:

Corollary 2. *In an S-manifold the distribution $D^{\perp} \oplus D_\theta$ is not integrable.*

Lemma 4. *Let M be a hemi-slant submanifold of an S-manifold $\overline{M}$. Then, the slant distribution D_θ is not integrable.*

Proof. By Lemma 3, for any $Z, W \in D_\theta$,

$$\eta_\alpha([Z,W]) = g([Z,W],\xi_\alpha) = 2g(Z,TW)$$

By the definition of a hemi-slant submanifold the result follows. □

Proposition 2. *Let M be a proper hemi-slant submanifold of an S-manifold $\overline{M}$. Then, the distribution $D_\theta \oplus \langle \xi_\alpha \rangle$ is integrable if and only if*

$$h(Z,TW) - h(W,TZ) + \nabla^{\perp}_Z FW - \nabla^{\perp}_W FZ$$

lies in FD_θ, for each $Z, W \in D_\theta \oplus \langle \xi_\alpha \rangle$.

Proof. By the relation (2), for any $Z, W \in D_\theta \oplus \langle \xi_\alpha \rangle$ and $X \in D^{\perp}$, we obtain

$$g([Z,W],X) = g(\varphi[Z,W],\varphi X) + \eta_\alpha([Z,W])\eta_\alpha(X)$$

From the Equation (3) and (11), and the facts that $TX = 0$, and $D^{\perp}$ and $\langle \xi_\alpha \rangle$ are orthogonal, we find

$$g([Z,W],X) = g(\varphi[Z,W],FX) = g\big(\varphi\overline{\nabla}_Z W, FX\big) - g\big(\varphi\overline{\nabla}_W Z, FX\big)$$

Then, by the Equations (1), (4), (5) and (7), we have

$$\begin{aligned} g([Z,W],X) &= g\big(\overline{\nabla}_Z \varphi W, FX\big) - g\big(\overline{\nabla}_W \varphi Z, FX\big) \\ &= g\big(\overline{\nabla}_Z TW, FX\big) + g\big(\overline{\nabla}_Z FW, FX\big) - g\big(\overline{\nabla}_W TZ, FX\big) - g\big(\overline{\nabla}_W FZ, FX\big) \end{aligned}$$

Applying the Formulas (8) and (9) gives

$$g([Z,W],X) = g\Big(h(Z,TW) - h(W,TZ) + \nabla^{\perp}_Z FW - \nabla^{\perp}_W FZ, FX\Big)$$

By the fact that $FD^{\perp}$ and FD_θ are mutually perpendicular, the result follows. □

Now, we have the following results for a hemi-slant submanifold of an S-manifold.

Lemma 5. *On a hemi-slant submanifold M of an S-manifold $\overline{M}$, we have*

$$g(\nabla_X Z, Y) = \sec^2\theta\{g(h(X, TZ), \varphi Y) - g(h(X, Y), FTZ)\}$$

for any $X, Y \in D^{\perp}$ and $Z \in D_\theta \oplus \langle\xi_\alpha\rangle$.

Proof. By the Gauss Formula (2), (3) and (8), for any $X, Y \in D^{\perp}$ and $Z \in D_\theta \oplus \langle\xi_\alpha\rangle$, we get

$$g(\nabla_X Y, Z) = g(\overline{\nabla}_X Y, Z) = g(\varphi\overline{\nabla}_X Y, \varphi Z)$$

Then, using the Equations (1), (5), (7), and the fact that $D^{\perp}$ and $\langle\xi_\alpha\rangle$ are orthogonal, we get

$$g(\nabla_X Y, Z) = g(\overline{\nabla}_X \varphi Y, \varphi Z)$$

Applying (3), (5), (7), and (9)–(11), we find

$$g(\nabla_X Y, Z) = -g(h(X, TZ), \varphi Y) - g(\overline{\nabla}_X Y, \varphi FZ)$$

From the formulas (6), (12), (18) and (19), thus

$$g(\nabla_X Y, Z) = -g(h(X, TZ), \varphi Y) + \sin^2\theta g(\overline{\nabla}_X Y, Z) - g(\overline{\nabla}_X FTZ, Y)$$

By the relations (8)–(10), we have

$$\cos^2\theta g(\nabla_X Y, Z) = g(h(X, Y), FTZ) - g(h(X, TZ), \varphi Y)$$

Finally,

$$g(\nabla_X Z, Y) = \sec^2\theta\{g(h(X, TZ), \varphi Y) - g(h(X, Y), FTZ)\}$$

This proves the lemma completely. □

Lemma 6. *On a hemi-slant submanifold M of an S-manifold $\overline{M}$, we have*

$$g(\nabla_Z X, W) = \sec^2\theta\{g(h(Z, X), FTW) - g(h(Z, TW), \varphi X)\}$$

for any $Z, W \in D_\theta \oplus \langle\xi_\alpha\rangle$ and $X \in D^{\perp}$.

Proof. Using the Formulas (1)–(3), (5), (7) and (8), for any $Z, W \in D_\theta \oplus \langle\xi_\alpha\rangle$ and $X \in D^{\perp}$, we get

$$g(\nabla_Z W, X) = g(\overline{\nabla}_Z \varphi W, \varphi X) = g(\overline{\nabla}_Z TW, \varphi X) + g(\overline{\nabla}_Z FW, \varphi X)$$

Applying (4) and (8), we find that

$$g(\nabla_Z W, X) = g(h(Z, TW), \varphi X) - g(\overline{\nabla}_Z \varphi X, FW)$$

From the relations (1), (4), (5) and (7), thus

$$g(\nabla_Z W, X) = g(h(Z, TW), \varphi X) + g(\overline{\nabla}_Z X, \varphi FW)$$

Using the Equations (4), (6), (12), (18) and (19), we arrive at

$$g(\nabla_Z W, X) = g(h(Z, TW), \varphi X) + \sin^2\theta g(\overline{\nabla}_Z W, X) + g(\overline{\nabla}_Z FTW, X)$$

Then, by the relations (8)–(10),

$$\cos^2\theta g(\nabla_Z W, X) = g(h(Z, TW), \varphi X) - g(h(Z, X), FTW)$$

Finally, we get

$$g(\nabla_Z X, W) = \sec^2\theta\{g(h(Z,X), FTW) - g(h(Z,TW), \varphi X)\}$$

This proves the lemma completely. □

Theorem 3. *Let M be a hemi-slant submanifold of an S-manifold $\overline{M}$. Then, the leaves of the distribution $D^{\perp}$ are totally geadesic if and only if*

$$g\big(A_{FTZ}Y - A_{\varphi Y}TZ, X\big) = 0,$$

for any $X, Y \in D^{\perp}$ and $Z \in D_\theta \oplus \langle\xi_\alpha\rangle$.

Proof. For any $X, Y \in D^{\perp}$ and $Z \in D_\theta \oplus \langle\xi_\alpha\rangle$, by Lemma 5 and relation (10), we have

$$g(\nabla_X Z, Y) = \sec^2\theta g\big(A_{\varphi Y}TZ - A_{FTZ}Y, X\big)$$

From (4), we get

$$g(\nabla_X Y, Z) = \sec^2\theta g\big(A_{FTZ}Y - A_{\varphi Y}TZ, X\big)$$

of which the assertion follows immediately. □

Theorem 4. *Let M be a hemi-slant submanifold of an S-manifold $\overline{M}$. Then, the leaves of the distribution $D_\theta \oplus \langle\xi_\alpha\rangle$ are totally geadesic if and only if*

$$g\big(A_{\varphi X}TW - A_{FTW}X, Z\big) = 0,$$

for $Z, W \in D_\theta \oplus \langle\xi_\alpha\rangle$ and $X \in D^{\perp}$.

Proof. For any $Z, W \in D_\theta \oplus \langle\xi_\alpha\rangle$ and $X \in D^{\perp}$, by Lemma 6 and relation (10), we have

$$g(\nabla_Z X, W) = \sec^2\theta g\big(A_{FTW}X - A_{\varphi X}TW, Z\big)$$

From (4), we get

$$g(\nabla_Z W, X) = \sec^2\theta g\big(A_{\varphi X}TW - A_{FTW}X, Z\big)$$

of which the assertion follows immediately. □

Thus, from Theorems 3 and 4 we can state the following theorem:

Theorem 5. *Let M be a proper hemi-slant submanifold of an S-manifold $\overline{M}$. Then, M is a locally Riemannian product manifold of $M_\perp$ and M_θ if and only if*

$$A_{\varphi X}TZ = A_{FTZ}X,$$

for any $X \in D^{\perp}$ and $Z \in D_\theta \oplus \langle\xi_\alpha\rangle$, where $M_\perp$ is an anti-invariant submanifold and M_θ is a proper slant submanifold tangent to the structure vector fields ξ_α of $\overline{M}$.

4. Warped Product Hemi-Slant Submanifold

A hemi-slant submanifold M is said to be a hemi-slant product if the distributions $D^{\perp}$ and D_θ are involutive and parallel on M, i.e., $D^{\perp}$ and D_θ are integrable on M. In this case, M is foliated by the leaves of these distributions. As a generalization of this product manifold, we can consider the warped product manifold, which is defined as follows:

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds with Riemannian metrics and a positive differentiable function f on M_1. Consider the product manifold $M_1 \times M_2$ with its projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$. Then, their warped product manifold $M = M_1 \times_f M_2$ is the Riemannian manifold $M_1 \times M_2 = (M_1 \times M_2, g)$ equipped with the Riemannian $M = M_1 \times_f M_2$ being the structure such that

$$g(X,Y) = g_1(\pi_{1_*}X, \pi_{1_*}Y) + (f \circ \pi_1)^2 g_2(\pi_{2_*}X, \pi_{2_*}Y),$$

for any vector field X, Y tangent to M, where $*$ is the symbol for the tangent maps. A warped product manifold $M = M_1 \times_f M_2$ is said to be *trivial*, or simply a *Riemannian product manifold*, if the warping function f is constant.

We recall the following result for warped product manifolds.

Lemma 7 ([22])**.** *On a warped product manifold $M = M_1 \times_f M_2$. If $X, Y \in TM_1$ and $Z, W \in TM_2$, then*

(i) $\nabla_X Y \in TM_1$,
(ii) $\nabla_X Z = \nabla_Z X = (X \ln f)Z$,
(iii) $nor(\nabla_Z W) = -g(Z,W)\vec{\nabla} \ln f$,

where ∇ is the Levi–Civita connection on M and $nor(\nabla_Z W)$ is the normal component of $\nabla_Z W$ in TM_2.

As a consequence, we have

$$\|\vec{\nabla} f\|^2 = \sum_{i=1}^{m} (e_i(f))^2. \tag{22}$$

for an orthonormal frame $\{e_1, \ldots, e_m\}$ on M_1. Furthermore, M_1 is a totally geodesic submanifold and M_2 is a totally umbilical submanifold of M.

In this section, we shall discuss the warped product of an S-manifold, in particular of a hemi-slant submanifold. Let $M_\perp$ and M_θ be an anti-invariant and a proper slant submanifolds of an S-manifold $\overline{M}$, respectively. Then, we consider the warped product hemi-slant submanifold of the form $M_\theta \times_f M_\perp$ such that the structure vector fields ξ_α tangent to the base. Firstly, we have the following results for later use.

Lemma 8. *Let $M = M_\theta \times_f M_\perp$ be a warped product submanifold of an S-manifold $\overline{M}$ such that $\xi_\alpha \in M_\theta$, where M_θ and $M_\perp$ are proper slant and anti-invariant submanifolds of $\overline{M}$, respectively. Then,*

(i) $g(h(X,Y), \varphi Z) = g(h(X,Z), FY)$;
(ii) $g(h(Z,W), FTX) = g(h(Z,TX), \varphi W) - \cos^2\theta(X \ln f)g(Z,W)$;
(iii) $g(h(Z,W), FX) = g(h(Z,X), \varphi W) + \{(TX \ln f) + \eta_\alpha(X)\}g(Z,W)$;

for any $X, Y \in TM_\theta$ and $Z, W \in TM_\perp$.

Proof. (i) For any $X, Y \in TM_\theta$ and $Z \in TM_\perp$, by the Gauss formula, we get

$$g(h(X,Y), \varphi Z) = g\big(\overline{\nabla}_X Y, \varphi Z\big) = -g\big(\varphi \overline{\nabla}_X Y, Z\big)$$

Using the Equations (1), (5) and (7), we obtain

$$g(h(X,Y), \varphi Z) = -g\big(\overline{\nabla}_X \varphi Y, Z\big)$$

From (4) and (9)–(11), we find

$$g(h(X,Y), \varphi Z) = g\big(TY, \overline{\nabla}_X Z\big) + g(h(X,Z), FY)$$

Apply (8) and Lemma 7 (ii), we arrive at

$$g(h(X,Y), \varphi Z) = (X \ln f)g(TY, Z) + g(h(X,Z), FY)$$

From (4), and since $Z \in TM_\perp$, so $TZ = 0$ which proves our assertion.

(ii) For any $Z, W \in TM_\perp$ and $X \in TM_\theta$, we get

$$g(h(Z,W), FTX) = g(h(Z,W), \varphi TX) - g\Big(h(Z,W), T^2X\Big)$$

By the relation (15), we obtain

$$g(h(Z,W), FTX) = g(h(Z,W), \varphi TX) = g(\overline{\nabla}_Z W, \varphi TX) - g\Big(\nabla_Z W, T^2X\Big)$$

Then, using (4), (7) and (15), we find

$$g(h(Z,W), FTX) = -g(\varphi\overline{\nabla}_Z W, TX) - \cos^2\theta g(W, \nabla_Z X) + \cos^2\theta \eta_\alpha(X) g(W, \nabla_Z \xi_\alpha)$$

Applying the relations (1), (5), (7) and (13), we arrive at

$$g(h(Z,W), FTX) = -g(\overline{\nabla}_Z \varphi W, TX) - \cos^2\theta g(W, \nabla_Z X)$$

Hence, from (9), (10) and Lemma 7 (ii), we get

$$g(h(Z,W), FTX) = g(h(Z,TX), \varphi W) - \cos^2\theta (X \ln f) g(Z,W)$$

(iii) By interchanging X by TX in (ii), we get

$$g\Big(h(Z,W), FT^2X\Big) = g\Big(h\Big(Z, T^2X\Big), \varphi W\Big) - \cos^2\theta (TX \ln f) g(Z,W)$$

Note that, $FT^2X = -\cos^2\theta FX$. Then

$$-\cos^2\theta g(h(Z,W), FX) = g\Big(h\Big(Z, T^2X\Big), \varphi W\Big) - \cos^2\theta (TX \ln f) g(Z,W)$$

From the Equations (10) and (15), we find

$$g(h(Z,W), FX) = g(A_{\varphi W} Z, X) - \eta_\alpha(X) g(A_{\varphi W} Z, \xi_\alpha) + (TX \ln f) g(Z,W)$$

Using (10), (14) and the fact that $FZ = \varphi Z$ since $Z \in TN_\perp$, we arrive at

$$g(h(Z,W), FX) = g(h(Z,X), \varphi W) + (TX \ln f) g(Z,W) + \eta_\alpha(X) g(\varphi Z, \varphi W)$$

Apply the relations (2) and (3), we conclude that

$$g(h(Z,W), FX) = g(h(Z,X), \varphi W) + \{(TX \ln f) + \eta_\alpha(X)\} g(Z,W)$$

This proves the lemma completely. □

Now, we prove the following characterization theorem for a warped product hemi-slant submanifold by using a result of [39].

Theorem 6. *Let M be a proper hemi-slant submanifold of an S-manifold $\overline{M}$ such that ξ_α is a tangent to the slant distribution D_θ. Then, M is a locally warped product manifold of the form $M_\theta \times_\mu M_\perp$ such that M_θ is a proper slant submanifold and $M_\perp$ is an anti-invariant submanifold of $\overline{M}$ if and only if*

$$A_{\varphi Z} TX - A_{FTX} Z = \cos^2\theta X(\mu) Z,$$

for any $X \in D_\theta \oplus \langle \xi_\alpha \rangle$ and $Z \in D^\perp$, where μ is a function on M such that $W(\mu) = 0$, for any $W \in D^\perp$.

Proof. Let $M = M_\theta \times_f M_\perp$ be a warped product manifold submanifold of an S-manifold $\overline{M}$. Then, for any $X, Y \in TM_\theta$ and $Z \in TM_\perp$, we get

$$g\big(A_{\varphi Z}TX, Y\big) = g(h(TX, Y), \varphi Z)$$

Using Equations (1), (4), (5), (7) and (8), we find

$$g\big(A_{\varphi Z}TX, Y\big) = -g\big(\overline{\nabla}_Y \varphi TX, Z\big)$$

Applying (4), (6), (9), (11) and (15),

$$g\big(A_{\varphi Z}TX, Y\big) = \cos^2\theta g\big(\overline{\nabla}_Y X, Z\big) + g\big(A_{FTX}Z, Y\big)$$

Thus, from (4), (7) and (8), we get

$$g\big(A_{\varphi Z}TX - A_{FTX}Z, Y\big) = 0$$

Hence, $A_{\varphi Z}TX - A_{FTX}Z \in TM_\perp$ since $Y \in TM_\theta$. Also, from Lemma 8 (ii), and the fact that h is symmetry, we obtain

$$g(h(W, Z), FTX) = g(h(W, TX), \varphi Z) - \cos^2\theta(X \ln f)g(Z, W)$$

Therefore,

$$A_{\varphi Z}TX - A_{FTX}Z = \cos^2\theta X(\mu)Z,$$

for any $X \in D_\theta \oplus \langle\xi_\alpha\rangle$ and $Z \in D^\perp$, where $\mu = \ln f$ such that $W(\mu) = 0$, for any $W \in D^\perp$.

Conversely, let M be a proper hemi-slant submanifold with the slant distribution $D_\theta \oplus \langle\xi_\alpha\rangle$ and the anti-invariant distribution $D^\perp$ satisfying

$$A_{\varphi Z}TX - A_{FTX}Z = \cos^2\theta X(\mu)Z, \tag{23}$$

for any $X \in D_\theta \oplus \langle\xi_\alpha\rangle$ and $Z \in D^\perp$, where $\mu = \ln f$ such that $W(\mu) = 0$, for any $W \in D^\perp$. Then, by Lemma 6, we have

$$g(\nabla_X Y, Z) = \sec^2\theta\{g(h(X, TY), \varphi Z) - g(h(X, Z), FTY)\}, \tag{24}$$

for any $X, Y \in D_\theta \oplus \langle\xi_\alpha\rangle$ and $Z \in D^\perp$. By interchanging X by Y in (24), we find

$$g(\nabla_Y X, Z) = \sec^2\theta\{g(h(Y, TX), \varphi Z) - g(h(Y, Z), FTX)\}. \tag{25}$$

From (24) and (25), we get

$$\begin{aligned}\cos^2\theta g([X, Y], Z) = g(h(X, TY), \varphi Z) - g(h(Y, TX), \varphi Z) + g(h(Y, Z), FTX)\\ - g(h(X, Z), FTY)\end{aligned}$$

Using the fact that h is symmetry and (10), we have

$$\cos^2\theta g([X, Y], Z) = g\big(A_{\varphi Z}TY - A_{FTY}Z, X\big) - g\big(A_{\varphi Z}TX - A_{FTX}Z, Y\big)$$

Thus, by (23), and since M is a proper hemi-slant submanifold, we get $g([X, Y], Z) = 0$. Hence, $[X, Y] \in D_\theta \oplus \langle\xi_\alpha\rangle$ since $Z \in D^\perp$. This means, $D_\theta \oplus \langle\xi_\alpha\rangle$ is integrable. Also, from (24), the fact that h is symmetry and (10), we have

$$\cos^2\theta g(\nabla_X Y, Z) = g\big(A_{\varphi Z}TY - A_{FTY}Z, X\big)$$

Then, by (23), we get $g(\nabla_X Y, Z) = 0$. Thus, $\nabla_X Y \in D_\theta \oplus \langle\xi_\alpha\rangle$ since $Z \in D^\perp$. This mean that the leaves of the distribution $D_\theta \oplus \langle\xi_\alpha\rangle$ are totally geodesic in M. Therefore, M_θ is a totally geodesic submanifold of M. From Prposition 1, we have $D^\perp$ is integrable. If we consider $h^{\perp}$ to be the second fundamental form of a leaf $M_{\perp}$ of $D^{\perp}$ in M, then for any $X \in D_{\theta} \oplus \langle \xi_{\alpha} \rangle$ and $Z, W \in D^{\perp}$, we obtain

$$g\left(h^{\perp}(Z,W),X\right) = g(\overline{\nabla}_Z W, X) + g(\nabla_Z W, X) = g(\overline{\nabla}_Z W, X) = g(\varphi \overline{\nabla}_Z W, \varphi X)$$

Using (1), (4), (5) and (7), we find

$$g\left(h^{\perp}(Z,W),X\right) = g(\overline{\nabla}_Z \varphi W, \varphi X) = g(\overline{\nabla}_Z \varphi W, TX) + g(\overline{\nabla}_Z \varphi W, FX)$$

Then, using Formulas (4), (8) and the fact that h is the symmetry, (3), (5), (7) and (10), we derive

$$g\left(h^{\perp}(Z,W),X\right) = -g(A_{\varphi W} TX, Z) - g(\overline{\nabla}_Z W, \varphi FX)$$

Thus, by (4), (6), (8), (9), (12), (18) and (19) and the symmetry of the shape operator, we find that

$$\cos^2\theta g\left(h^{\perp}(Z,W),X\right) = -g(A_{\varphi W} TX - A_{FTX} W, Z)$$

Then, from (23), we derive

$$g\left(h^{\perp}(Z,W),X\right) = -X(\mu) g(W,Z) = -g(W,Z) g\left(\vec{\nabla}\mu, X\right)$$

which means that

$$h^{\perp}(Z,W) = -g(W,Z)\vec{\nabla}\mu$$

where $\vec{\nabla}\mu$ is the gradient of the fumction μ. Thus, $M_{\perp}$ is a totally umbilical submanifold of M with a mean curvature vector $H^{\perp} = -\vec{\nabla}\mu$. Now, we can show that $H^{\perp}$ is parallel with the normal connection ∇^F of $M_{\perp}$ in M. Consider for any $W \in D^{\perp}$ and $X \in D_{\theta} \oplus \langle \xi_{\alpha} \rangle$, we get

$$\begin{aligned} g\left(\nabla^F_W H^{\perp}, X\right) = -g\left(\nabla_W \vec{\nabla}\mu, X\right) &= -W g\left(\vec{\nabla}\mu, X\right) + g\left(\vec{\nabla}\mu, \nabla_W X\right) \\ &= -W(X\mu) + g\left(\vec{\nabla}\mu, [W,X]\right) + g\left(\vec{\nabla}\mu, \nabla_X W\right) \\ &= -X(W\mu) - g\left(\nabla_X \vec{\nabla}\mu, W\right) = 0, \end{aligned}$$

since $W(\mu) = 0, \forall\, W \in D^{\perp}$ and thus $\nabla_X \vec{\nabla}\mu \in D_{\theta} \oplus \langle \xi_{\alpha} \rangle$ and $D_{\theta} \oplus \langle \xi_{\alpha} \rangle$ begin totally geodesic. This means that the mean curvature $H^{\perp}$ of $M_{\perp}$ is parallel. Thus, the leaves of the distribution $D^{\perp}$ are totally umbilical with parallel mean curvature $H^{\perp}$ in M and hence $M_{\perp}$ is a totally umbilical submanifold with parallel mean curvature in M. That is, $M_{\perp}$ is an extrinsic sphere in M. Therefore, M is a locally warped product manifold of the form $M_{\theta} \times_{\mu} M_{\perp}$ by a result of Hiepko [39], which proves the theorem completely. □

As an application of the Theorem 6, if we put $s = 1$, then we have the following:

Theorem 7 ([36])**.** *Let M be a proper pseudo-slant submanifold of a Sasakian manifold $\overline{M}$ such that ξ is tangent to the slant distribution D_{θ}. Then, M is a locally warped product manifold of the form $M_{\theta} \times_{\mu} M_{\perp}$ such that M_{θ} is a proper slant submanifold and $M_{\perp}$ is an anti-invariant submanifold of $\overline{M}$ if and only if*

$$A_{\varphi Z} TX - A_{FTX} Z = \cos^2\theta X(\mu) Z,$$

for any $X \in D_{\theta} \oplus \langle \xi_{\alpha} \rangle$ and $Z \in D^{\perp}$, where μ is a function on M such that $W(\mu) = 0$, for any $W \in D^{\perp}$.

In the following, we construct an example of a warped product hemi-slant submanifold of an S-manifold.

Example 1. *Consider a submanifold M of R^{8+s} with the Cartesian coordinates $(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, t_1, \ldots, t_s)$ and the almost contact structure*

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \quad \varphi\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}, \quad \varphi\left(\frac{\partial}{\partial t_\alpha}\right) = 0,$$

for all $1 \leq i, j \leq 4$ and $\alpha = 1, \ldots, s$. For any vector field

$$X = \lambda_i \frac{\partial}{\partial x_i} + \mu_j \frac{\partial}{\partial y_j} + \sum_{\alpha=1}^{s} \nu_\alpha \frac{\partial}{\partial t_\alpha} \in TR^{8+s},$$

then we have

$$\varphi X = \lambda_i \frac{\partial}{\partial y_i} - \mu_j \frac{\partial}{\partial x_j} \quad \textit{and} \quad \varphi^2 X = -\lambda_i \frac{\partial}{\partial x_i} - \mu_j \frac{\partial}{\partial y_j} = -X + \sum_{\alpha=1}^{s} \eta_\alpha(X)\xi_\alpha$$

It is clear that

$$g(X, X) = \lambda_i^2 + \mu_j^2 + \sum_{\alpha=1}^{s} \nu_\alpha^2 \quad \textit{and} \quad g(\varphi X, \varphi X) = \lambda_i^2 + \mu_j^2$$

Therefore,

$$g(\varphi X, \varphi X) = g(X, X) - \sum_{\alpha=1}^{s} \eta_\alpha^2(X).$$

Hence, $(\varphi, \xi_\alpha, \eta_\alpha, g)$ is an S-structure on R^{8+s}. Now, let us consider the immersion ψ of M into R^{8+s} as

$$\psi(u, v, w, z, t_1, \ldots, t_s) = (u\cos w, v\cos w, z, z, u+v, u-v, u\sin w, v\sin w, t_1, \ldots, t_s).$$

Then, the tangent bundle TM of M is spanned by the following orthogonal vector fields:

$$e_1 = \cos w\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_3} + \sin w\frac{\partial}{\partial x_4}, \qquad e_2 = \cos w\frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_3} + \sin w\frac{\partial}{\partial y_4},$$
$$e_3 = -u\sin w\frac{\partial}{\partial x_1} - v\sin w\frac{\partial}{\partial y_1} + u\cos w\frac{\partial}{\partial x_4} + v\cos w\frac{\partial}{\partial y_4}, \qquad e_4 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2};$$
$$e_5 = \frac{\partial}{\partial t_1}, \ldots, e_{4+\alpha} = \frac{\partial}{\partial t_\alpha}.$$

Then, with respect to the given almost contact structure, we obtain

$$\varphi e_1 = \cos w\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_3} - \frac{\partial}{\partial x_3} + \sin w\frac{\partial}{\partial y_4}, \qquad \varphi e_2 = -\cos w\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_3} + \frac{\partial}{\partial x_3} - \sin w\frac{\partial}{\partial x_4},$$
$$\varphi e_3 = -u\sin w\frac{\partial}{\partial y_1} + v\sin w\frac{\partial}{\partial x_1} + u\cos w\frac{\partial}{\partial y_4} - v\cos w\frac{\partial}{\partial x_4}, \qquad \varphi e_4 = \frac{\partial}{\partial y_2} - \frac{\partial}{\partial x_2};$$
$$\varphi e_5 = 0, \ldots, \varphi e_{4+\alpha} = 0.$$

Since φe_3 and φe_4 are orthogonal to TM, then $D^\perp = Span\{e_3, e_4\}$ is an anti-invariant distribution, and $D_\theta = Span\{e_1, e_2\}$ is a proper slant distribution with slant angle $\theta = \arccos\left(\frac{1}{3}\right)$ such that $\xi_\alpha = e_{4+\alpha}$ is a tangent to D_θ. Hence, M is a proper hemi-slant submanifold of R^{8+s}. It is easy to observe that both the distributions are integrable. If we denote the integral manifolds of $D_\theta \oplus \langle\xi_\alpha\rangle$ and $D^\perp$ by M_θ and $M_\perp$, respectively, then the metric tensor g of the product manifold M is given by

$$g = 3du^2 + 3dv^2 + \sum_{\alpha=1}^{s} dt_\alpha^2 + \left(u^2 + v^2\right)dw^2 + 2dz^2 = g_1 + \left(\sqrt{\frac{u^2+v^2}{2}}\right)^2 g_2,$$

where $g_1 = 3(du^2 + dv^2) + \sum_{\alpha=1}^{s} dt_\alpha^2$ is the metric tensor of M_θ and $g_2 = 2dw^2 + \frac{4}{u^2+v^2}dz^2$ is the mertic tensor of $M_\perp$. Thus, M is a hemi-slant warped product of the form $M_\theta \times_f M_\perp$ with warping function $f = \sqrt{\frac{u^2+v^2}{2}}$, such that $u, v \neq 0$.

From the above example, if we put $s = 1$, then M is a warped product hemi-slant submanifold on a Sasakian manifold.

In [36], If we construct the example on an S-manifold, then we get the following:

Example 2. *Consider a submanifold M of R^{6+s} with the Cartesian coordinates $(x_1, y_1, x_2, y_2, x_3, y_3, t_1, \ldots, t_s)$ and the almost contact structure*

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \quad \varphi\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}, \quad \varphi\left(\frac{\partial}{\partial t_\alpha}\right) = 0,$$

for all $1 \leq i, j \leq 3$ and $\alpha = 1, \ldots, s$. Then, it is easy to show that $(\varphi, \xi_\alpha, \eta_\alpha, g)$ is an S-structure on R^{6+s}. Now, if we consider the immersion ψ of M into R^{6+s} as

$$\psi(u, v, w, t_1, \ldots, t_s) = \Big(u, v, u\cos w, \sqrt{3}v\cos w, u\sin w, \sqrt{3}v\sin w, t_1, \ldots, t_s\Big).$$

then the tangent bundle $TM = D^\perp \oplus D_\theta \oplus \langle\xi_\alpha\rangle$, where $D^\perp = Span\{e_3\}$ is an anti-invariant distribution and $D_\theta = Span\{e_1, e_2\}$ is a proper slant distribution with slant angle $\theta = \frac{5\pi}{12}$ such that $\xi_\alpha = e_{3+\alpha}$ tangent to D_θ. Hence, M is a proper hemi-slant submanifold of R^{6+s}. If we denote the integral manifolds of $D_\theta \oplus \langle\xi_\alpha\rangle$ and $D^\perp$ by M_θ and $M_\perp$, respectively, then the metric tensor g of the product manifold M is given by

$$g = 2du^2 + 4dv^2 + \sum_{\alpha=1}^{s} dt_\alpha^2 + \Big(u^2 + 3v^2\Big)dw^2 = g_1 + \Big(\sqrt{u^2 + 3v^2}\Big)^2 g_2,$$

where $g_1 = 2du^2 + 4dv^2 + \sum_{\alpha=1}^{s} dt_\alpha^2$ is the metric tensor of M_θ and $g_2 = dw^2$ is the metric tensor of $M_\perp$. Thus, M is a hemi-slant warped product of the form $M_\theta \times_f M_\perp$ with warping function $f = \sqrt{u^2 + 3v^2}$, such that $u, v \neq 0$.

5. Inequality for Warped Product Hemi-Slant Submanifold

In this section, we form a sharp inequality for the squared norm of the second fundamental form $\|h\|^2$ of a mixed totally geodesic warped product hemi-slant submanifold in terms of the gradient of the warping function and the slant angle. First, we construct the following frame fields for a warped product hemi-slant submanifold of an S-manifold to develop the main result of this section.

Let $M = M_\theta \times_f M_\perp$ be an m-dimensional warped product hemi-slant submanifold of an $(2n+s)$-dimensional S-manifold $\overline{M}$, where $M_\perp$ is an n_1-dimensional anti-invariant submanifold of $\overline{M}$, and M_θ is a proper slant submanifold of $\overline{M}$ with the dimension $n_2 = 2p + s$ such that ξ_α is tangent to M_θ. Let us consider the tangent spaces of $M_\perp$ and M_θ by $D^\perp$ and $D_\theta \oplus \langle\xi_\alpha\rangle$ instead of $TM_\perp$ and TM_θ, respectively. We set the orthonormal frame fields of $D^\perp$ and $D_\theta \oplus \langle\xi_\alpha\rangle$, respectively, as $\{e_1, e_2, \ldots, e_{n_1}\}$ and $\{e_{n_1+1} = e_1^*, \ldots, e_{n_1+p} = e_p^*, e_{n_1+p+1} = e_{p+1}^* = \sec\theta T e_1^*, \ldots, e_{n_1+2p} = e_{2p}^* = \sec\theta T e_p^*, e_{n_1+2p+1} = e_{2p+1}^* = \xi_1, \ldots, e_m = e_{2p+s}^* = \xi_s\}$, where θ is the slant angle of the immersion. Then, the orthonormal frame fields of the normal sub-bundles of $\varphi D^\perp$, FD_θ and ν, respectively, are $\{e_{m+1} = \bar{e}_1 = \varphi e_1, \ldots, e_{m+n_1} = \bar{e}_{n_1} = \varphi e_{n_1}\}$, $\{e_{m+n_1+1} = \bar{e}_{n_1+1} = \csc\theta F e_1^*, \ldots, e_{m+n_1+p} = \bar{e}_{n_1+p} = \csc\theta F e_p^*, e_{m+n_1+p+1} = \bar{e}_{n_1+p+1} = \csc\theta\sec\theta FT e_1^*, \ldots, e_{2m-s} = e_{m+n_1+2p} = \bar{e}_{m-s} = \csc\theta\sec\theta FT e_p^*\}$ and $\{e_{2m-s+1} = \bar{e}_{m-s+1}, \ldots, e_{2n+s} = \bar{e}_{2(n-m+s)}\}$. It is clear that the dimensions of the normal subspaces $\varphi D^\perp$, FD_θ and ν, respectively, are n_1, $2p$ and $2(n - m + s)$.

Theorem 8. *Let $M = M_\theta \times_f M_\perp$ be an m-dimensional mixed totally geodesic warped product hemi-slant submanifold of an S-manifold $\overline{M}$ such that $\xi_\alpha \in TM_\theta$, where M_θ is a proper slant submanifold of $\overline{M}$ with the dimension $n_2 = 2p + s$ and $M_\perp$ is an anti-invariant submanifold of dimension n_1 of $\overline{M}$. Then, we have the following:*

(i) *The squared norm of the second fundamental form of M satisfies*

$$\|h\|^2 \geq n_1 \cot^2\theta \|\vec{\nabla} \ln f\|^2 \tag{26}$$

where $\vec{\nabla} lnf$ is the gradient of $\ln f$ along M_θ.

(ii) *If the equality sign in (26) holds identically, then M_θ is totally geodesic and $M_\perp$ is totally umbilical submanifolds in $\overline{M}$.*

Proof. From the definition of h, we get

$$\|h\|^2 = \|h\left(D^\perp, D^\perp\right)\|^2 + \|h(D,D)\|^2 + 2\|h\left(D^\perp, D\right)\|^2$$

where $D = D_\theta \oplus \langle \xi_\alpha \rangle$. Since M is a mixed totally geodesic, hence the third term of right hand side should be identically zero, then we have

$$\|h\|^2 = \sum_{i,j=1}^{m} g\left(h\left(e_i, e_j\right), h\left(e_i, e_j\right)\right) = \sum_{r=m+1}^{2n+s} \sum_{i,j=1}^{m} g\left(h\left(e_i, e_j\right), e_r\right)^2$$

Using the orthonormal frame fields of $D^\perp$ and D, we have

$$\|h\|^2 = \sum_{r=m+1}^{2n+s} \sum_{l,k=1}^{n_1} g(h(e_l, e_k), e_r)^2 + \sum_{r=m+1}^{2n+s} \sum_{i,j=1}^{2p+s} g\left(h\left(e_i, e_j\right), e_r\right)^2$$

The above equation can be separated for the $\varphi D^\perp, FD_\theta$ and ν components as follows:

$$\begin{aligned} \|h\|^2 = & \sum_{r=1}^{n_1} \sum_{l,k=1}^{n_1} g(h(e_l, e_k), \bar{e}_r)^2 + \sum_{r=n_1+1}^{n_1+2p} \sum_{l,k=1}^{n_1} g(h(e_l, e_k), \bar{e}_r)^2 \\ & + \sum_{r=m+1-s}^{2(n-m+s)} \sum_{l,k=1}^{n_1} g(h(e_l, e_k), \bar{e}_r)^2 + \sum_{r=1}^{n_1} \sum_{i,j=1}^{2p+s} g\left(h\left(e_i, e_j\right), \bar{e}_r\right)^2 \\ & + \sum_{r=n_1+1}^{n_1+2p} \sum_{i,j=1}^{2p+s} g\left(h\left(e_i, e_j\right), \bar{e}_r\right)^2 + \sum_{r=m+1-s}^{2(n-m+s)} \sum_{i,j=1}^{2p+s} g\left(h\left(e_i, e_j\right), \bar{e}_r\right)^2 \end{aligned} \tag{27}$$

We shall leave all the terms except the second term in (27) to be evaluated, then we derive

$$\begin{aligned} \|h\|^2 \geq & \sum_{r=n_1+1}^{n_1+2p} \sum_{l,k=1}^{n_1} g(h(e_l, e_k), \bar{e}_r)^2 \\ = & \sum_{r=n_1+1}^{n_1+p} \sum_{l,k=1}^{n_1} g(h(e_l, e_k), \bar{e}_r)^2 + \sum_{r=n_1+p+1}^{n_1+2p} \sum_{l,k=1}^{n_1} g(h(e_l, e_k), \bar{e}_r)^2 \end{aligned}$$

From the orthonormal frame field of FD_θ, we arrive at

$$\|h\|^2 \geq \sum_{i=1}^{p} \sum_{l,k=1}^{n_1} g(h(e_l, e_k), \csc\theta F e_i^*)^2 + \sum_{i=1}^{p} \sum_{l,k=1}^{n_1} g(h(e_l, e_k), \csc\theta \sec\theta F T e_i^*)^2$$

Then, by Lemma 8 (ii)–(iii) , we find that

$$\begin{aligned}
\|h\|^2 &\geq \csc^2\theta \sum_{i=1}^{p}\sum_{l,k=1}^{n_1}[(Te_i^* \ln f)+\eta_\alpha(e_i^*)]^2 g(e_l,e_k)^2 + \cot^2\theta \sum_{i=1}^{p}\sum_{l,k=1}^{n_1}(e_i^* \ln f)^2 g(e_l,e_k)^2 \\
&= n_1\csc^2\theta \sum_{i=1}^{p}[(Te_i^* \ln f)+g(e_i^*,\xi_\alpha)]^2 + n_1\cot^2\theta\sum_{i=1}^{p}(e_i^* \ln f)^2 \\
&= n_1\csc^2\theta \sum_{i=1}^{p}(Te_i^* \ln f)^2 \pm n_1\csc^2\theta\sum_{i=p+1}^{2p+s}(Te_i^* \ln f)^2 + n_1\cot^2\theta\sum_{i=1}^{p}(e_i^* \ln f)^2 \\
&= n_1\csc^2\theta \sum_{i=1}^{2p+s}(Te_i^* \ln f)^2 - n_1\csc^2\theta\sum_{i=1}^{p} g\left(e_{p+i}^*, T\vec{\nabla}\ln f\right)^2 + n_1\cot^2\theta\sum_{i=1}^{p}(e_i^* \ln f)^2
\end{aligned}$$

Using the considered orthonormal frame fields, the above formula can be written as

$$\begin{aligned}
\|h\|^2 \geq{}& n_1\csc^2\theta \sum_{i=1}^{2p+s}(Te_i^* \ln f)^2 - n_1\csc^2\theta\sec^2\theta\sum_{i=1}^{p} g\left(Te_{p+i}^*, T\vec{\nabla}\ln f\right)^2 \\
&+ n_1\cot^2\theta\sum_{i=1}^{p}(e_i^* \ln f)^2
\end{aligned}$$

By (16), and the fact that for a warped product submanifold of an S-manifold, $\xi_\alpha \ln f = 0$, we arrive at

$$\|h\|^2 \geq n_1\csc^2\theta \sum_{i=1}^{2p+s}(Te_i^* \ln f)^2 - n_1\cot^2\theta\sum_{i=1}^{p}(e_i^* \ln f)^2 + n_1\cot^2\theta\sum_{i=1}^{p}(e_i^* \ln f)^2$$

Using the fact that $Te_i^* = \cos\theta e_i^*$, for $i = 1,\ldots,2p+s$, we find

$$\|h\|^2 \geq n_1\csc^2\theta\cos^2\theta\sum_{i=1}^{2p+s}(e_i^* \ln f)^2$$

To satisfy (22), the above expression can be simplified as

$$\|h\|^2 \geq n_1\cot^2\theta\|\vec{\nabla}\ln f\|^2$$

which is inequality (26). If the equality sign holds in (26), then from the leaving terms in (27), we get the following relations from the fifth and the sixth terms of (27) $g(h(D,D),FD_\theta) = 0$, $g(h(D,D),\nu) = 0$ which implies that

$$h(D,D) \perp FD_\theta, \quad h(D,D) \perp \nu \quad \Rightarrow \quad h(D,D) \in \varphi D^\perp \tag{28}$$

Also, from the fourth term of (27) and Lemma 8 (i) for a mixed totally geodesic warped product submanifold, we find $g\big(h(D,D),\varphi D^\perp\big) = 0$ which means that

$$h(D,D) \perp \varphi D^\perp \tag{29}$$

Thus, by using (28) and (29), we get $h(D,D) = 0$, using this relation with the fact that M_θ is totally geodesic in M [22], we conclude that M_θ is totally geodesic submanifold in $\overline{M}$. Furthermore, from the leaving first and third terms of (27), we get $g\big(h\big(D^\perp,D^\perp\big),\varphi D^\perp\big) = 0$, $g\big(h\big(D^\perp,D^\perp\big),\nu\big) = 0$, which implies that

$$h\left(D^\perp,D^\perp\right) \perp \varphi D^\perp, \quad h\left(D^\perp,D^\perp\right) \perp \nu \quad \Rightarrow \quad h\left(D^\perp,D^\perp\right) \in FD_\theta \tag{30}$$

Thus, since M is a mixed totally geodesic, from Lemma 8 (ii) and (30), we arrive at

$$g(h(Z,W),FTX) = -\cos^2\theta(X\ln f)g(Z,W) \tag{31}$$

for any $Z, W \in TM_\perp$ and $X \in TM_\theta$. Hence, by the relations (30), (31) and the fact that $M_\perp$ is totally umbilical in M [22], we find that $M_\perp$ is totally umbilical submanifold in $\overline{M}$. This completes the proof. □

As an application of the Theorem 8, if we put $s = 1$, then we have the following:

Theorem 9 ([36])**.** *Let $M = M_\theta \times_f M_\perp$ be an m-dimensional mixed totally geodesic warped product submanifold of a Sasakian manifold $\overline{M}$ such that $\xi \in TM_\theta$, where M_θ is a proper slant submanifold, and $M_\perp$ is an n_1-dimensional anti-invariant submanifold of $\overline{M}$. Then, we have the following:*

(i) *The squared norm of the second fundamental form of M satisfies*

$$\|h\|^2 \geq n_1 \cot^2\theta \|\vec{\nabla} \ln f\|^2 \tag{32}$$

where $\vec{\nabla} \ln f$ is the gradient of $\ln f$ along M_θ.

(ii) *If the equality sign in (32) holds identically, then M_θ is totally geodesic in $\overline{M}$, and $M_\perp$ is totally umbilical submanifold of $\overline{M}$.*

6. Conclusions

In this paper, we extend the study of the warped product submanifolds of an S-manifold. Firstly, we obtained the integrability conditions of distributions involved in the definition of a hemi-slant submanifold. After that, we proved interesting results for the existence of warped product hemi-slant submanifolds of the type $M_\theta \times_f M_\perp$ with $\xi_\alpha \in M_\theta$ of an S-manifold. Also, we proved the characterization theorem on the existence of such submanifolds and provided some examples. Finally, we formed an inequality for the squared norm of the second fundamental form in terms of the warping function and the slant angle. The case for equality is also considered.

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