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Stochastic Comparisons of Largest-Order Statistics and Ranges from Marshall–Olkin Bivariate Exponential and Independent Exponential Variables

Narayanaswamy Balakrishnan [1], Ghobad Saadat Kia (Barmalzan) [2,*] and Mohammad Mehrpooya [3]

[1] Department of Mathematics and Statistics, McMaster University, Hamilton, ON L8S 4L8, Canada; bala@mcmaster.ca
[2] Department of Mathematics, Kermanshah University of Technology, Kermanshah 6715685420, Iran
[3] Department of Mathematics, Faculty of Science, Kosar University of Bojnord, Bojnord 9415615458, Iran; sdmehrpooya@gmail.com
* Correspondence: gh.saadatkia@kut.ac.ir

Abstract: Sample range and the associated functions such as survival function and mean residual life function have found many important applications in the reliability field. In this work, we establish some results that are in two different directions. In the first part, we establish some conditions for comparing the largest-order statistics (in the sense of mean residual life order) arising from bivariate Marshall–Olkin exponential distribution. Then, in the second part, we present some sufficient conditions for comparing sample ranges (in the sense of usual stochastic order and reversed hazard rate order) arising from independent exponential random variables.

Keywords: usual stochastic order; mean residual life order; reversed hazard rate order; parallel systems; sample ranges; majorization orders

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1. Introduction

An r-out-of-n system, which functions if r out of n components in the system function, has found a lot of applications in practice. Well-known systems such as series-parallel and fail-safe are all its special cases. If $X_1, \cdots, X_n$ denote the lifetimes of the n components of the system and $X_{1:n} \leq \cdots \leq X_{n:n}$ denote corresponding ordered lifetimes, $X_{n-r+1:n}$ is evidently the lifetime of an r-out-of-n system. For this reason, the theory of order statistics plays a critical role in studying $(n-r+1)$-out-of-n systems and their distributional characteristics and properties.

In reliability theory, comparisons of lifetimes of technical systems are important as they may facilitate approximating a complex system using a simpler one. This in turn would enable one to obtain simpler bounds on some reliability characteristics of complex systems. Stochastic ordering theory is particularly useful in this context.

Comparisons of lifetimes of technical systems are a problem of interest in reliability theory as it would enable one to approximate complex systems with simpler ones and further to obtain bounds for some ageing characteristics of the complex system in terms of simpler ones. The theory of stochastic orderings is especially useful for this purpose.

As a parallel system is a commonly used system in practice, it is of interest to evaluate its performance based on the lifetimes of its components. Then, one is naturally interested in the survival function or the hazard rate of the system as a characteristic. Here, we focus especially on parallel systems consisting of two dependent components.

Much of the exiting research has focused on parallel systems with independent component; see [1–3] and the references therein. However, the independence assumption may not be realistic in common shock models, load-sharing models, stress-strength models, and many other practical problems, as the component lifetimes may be dependent. In the past two decades, some authors have discussed stochastic comparisons of parallel systems with dependent component lifetimes; see for example, [4–8]. In particular, [6] discussed the hazard rate order when the two components are jointly distributed as a Marshall–Olkin exponential distribution.

The Marshall–Olkin exponential (MOE) distribution has its joint survival distribution as (see [9,10])

$$\bar{F}(x_1, x_2) = e^{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 \max\{x_1, x_2\}},$$

where $x_1, x_2 > 0$ and $\lambda_1, \lambda_2, \lambda_3 > 0$, and it is usually denoted by $\mathcal{MOE}(\lambda_1, \lambda_2, \lambda_3)$.

Some authors have also discussed the stochastic comparison of the maximum of two random variables following jointly having an MOE distribution. In particular, the hazard rate order of a parallel system with two components having MOE has been studied in [6]. The MOE distribution has been generalized in different ways due to its practical use; one may refer to the monograph by Bernhart [11].

Two prominent generalized forms are as follows:

(i) Marshall–Olkin-type (MOT) distribution (see [12]):

$$\bar{F}(x_1, x_2) = e^{-\lambda_1 H(x_1) - \lambda_2 H(x_2) - \lambda_3 H(\max\{x_1, x_2\})}, \qquad x_1, x_2 \geq 0,$$

where $\lambda_1 > 0$, $i = 1, 2, 3$, and H increases with $H(0) = 0$ and $H(+\infty) = +\infty$;

(ii) Generalized Marshall–Olkin (GMO) distribution (see [13]):

$$\bar{F}(x_1, x_2) = e^{-H_1(x_1) - H_2(x_2) - H_3(\max\{x_1, x_2\})}, \qquad x_1, x_2 \geq 0,$$

where H_i increases with $H_i(0) = 0$ and $H_i(+\infty) = +\infty$, $i = 1, 2, 3$.

Hu and Li [14] discussed two-component parallel systems with the two component lifetimes following a general MOT distribution, instead of a particular model with exponential marginal components. They presented sufficient conditions for the hazard rate order of such parallel systems, which are generalizations of the results of Joo and Mi [6] and Cai and Xu [15] for the particular case of a model with an exponential marginal component.

Suppose a random variable X, with finite first moment and survival function $\bar{F}$, is the lifetime of an item. If the item functions at time t, its residual lifetime is given by the random variable $X_t = [X - t | X > t]$ with its survival function as $\bar{F}_t(u) = \bar{F}(u + t)/\bar{F}(t)$, $u \geq 0$. The mean residual life function of the item at time t is then given by

$$\mu(t) = \mathbb{E}[X - t | X > t] = \int_0^\infty \frac{\bar{F}(u + t)}{\bar{F}(t)} du.$$

This mean residual function has found many key applications in reliability analysis, survival analyses, and many other areas. For example, in a stop-loss agreement in the actuarial setting, the mean residual life function represents the expected amount paid by a reinsurer, provided that the retention t is reached. For this reason, the mean residual life function is referred to as the mean excess function in an actuarial field. In addition, the mean residual life function is a suitable measure for evaluating the thickness of tails of distributions; see [16]. In renewal theory, the hazard rate function of the stationary renewal distribution is the reciprocal of the mean residual life function. Some more interesting applications of mean residual life function in other areas such as extreme value theory, economics, and demography can be seen in [17–19].

The mean residual life order of largest-order statistics, unlike other orderings such as hazard rate order, usual stochastic order, reversed hazard rate order, and likelihood ratio order, has not been studied in detail. To the best of our knowledge, few results in this direction have focused on the exponential case. We now briefly describe these existing results. Let $(X_1, \ldots, X_m)$ and $(Y_1, \ldots, Y_m)$ be two random vectors of independent exponential random variables with scale vectors $(\delta_1, \ldots, \delta_m)$ and $(\lambda_1, \ldots, \lambda_m)$, respectively. For the case of $m = 2$, Zhao and Balakrishnan [20] showed under the assumption $\delta_1 \le \lambda_1 \le \lambda_2 \le \delta_2$ that

$$(\delta_1, \delta_2) \overset{rm}{\succeq} (\lambda_1, \lambda_2) \Longrightarrow X_{2:2} \ge_{mrl} Y_{2:2}. \tag{1}$$

Generalization of (1) to an arbitrary m could be achieved in two ways. In the first one discussed by Cheng and Wang [21], the complete heterogeneity of scale parameters is maintained with some restrictions on them. Specifically, when $\lambda_1 \ge \delta_1$ and $\delta_k \ge \lambda_k$ for $k = 2, \dots, m$, Cheng and Wang [21] proved that

$$(\delta_1, \dots, \delta_m) \overset{rm}{\succeq} (\lambda_1, \dots, \lambda_m) \Longrightarrow X_{m:m} \ge_{mrl} Y_{m:m}. \tag{2}$$

Another method is by reducing the heterogeneity of scale parameters to the case of multiple-outlier exponential models, wherein $\delta_1 = \cdots = \delta_n$, $\delta_{n+1} = \cdots = \delta_m$, $\lambda_1 = \cdots = \lambda_n$ and $\lambda_{n+1} = \cdots = \lambda_m$ with $n \in \{1, \dots, m-1\}$ and $\delta_1 \le \lambda_1 \le \lambda_m \le \delta_m$. In this setting, Cheng and Wang [21] have shown that

$$(\underbrace{\delta_1, \dots, \delta_1}_{n}, \underbrace{\delta_m, \dots, \delta_m}_{m-n}) \overset{rm}{\succeq} (\underbrace{\lambda_1, \dots, \lambda_1}_{n}, \underbrace{\lambda_m, \dots, \lambda_m}_{m-n}) \Longrightarrow X_{m:m} \ge_{mrl} Y_{m:m}. \tag{3}$$

A refinement of (1) has been obtained by [21], in which they have shown that the restriction $\delta_1 \le \lambda_1 \le \lambda_m \le \delta_m$ is not necessary for establishing $X_{2:2} \ge_{mrl} Y_{2:2}$. More recently, Haidari et al. [22] investigated mean residual life ordering between the largest-order statistics in multiple-outlier-scale models in the following form. Let $(X_1, \dots, X_m)$ and $(Y_1, \dots, Y_m)$ be two vectors of independent non-negative random variables such that $X_i \sim S(G; \delta_1)$, $X_j \sim S(G; \delta_2)$, $Y_i \sim S(G; \lambda_1)$, $Y_j \sim S(G; \lambda_2)$, for $i = 1, \dots, n$ and $j = n+1, \dots, m$, with $\delta_1 \le \lambda_1 \le \lambda_2 \le \delta_2$. The survival, density, hazard rate, and reversed hazard rate functions corresponding to G are denoted by $\bar{G}$, g, $h = g/\bar{G}$ and $r = g/G$, respectively. Suppose that the following conditions hold:

(i) $th(t)$ is increasing in $t \in \mathbb{R}^+$;
(ii) $h(t)$ is increasing in $t \in \mathbb{R}^+$;
(iii) $(tr(t))/\bar{G}(t)$ is increasing in $t \in \mathbb{R}^+$;
(iv) $(tr'(t))/r(t)$ is decreasing in $t \in \mathbb{R}^+$.

Then, under Conditions (ii)–(iv), Haidari et al. [22] established that

$$(\underbrace{\delta_1, \dots, \delta_1}_{n}, \underbrace{\delta_2, \dots, \delta_2}_{m-n}) \overset{rm}{\succeq} (\underbrace{\lambda_1, \dots, \lambda_1}_{n}, \underbrace{\lambda_2, \dots, \lambda_2}_{m-n}) \Longrightarrow X_{m:m} \ge_{mrl} Y_{m:m}. \tag{4}$$

These authors also studied mean residual life order between the largest-order statistics from independent heterogeneous scale variables. Suppose that $n_1, \dots, n_r$ are integer values such that $n_1 + \cdots + n_r = m$. Let $(X_1, \dots, X_m)$ and $(Y_1, \dots, Y_m)$ be two vectors of independent non-negative random variables following the scale models with common baseline distribution G and scale parameter vectors

$$(\underbrace{\delta_1, \dots, \delta_1}_{n_1}, \dots, \underbrace{\delta_r, \dots, \delta_r}_{n_r}) \quad \text{and} \quad (\underbrace{\lambda_1, \dots, \lambda_1}_{n_1}, \dots, \underbrace{\lambda_r, \dots, \lambda_r}_{n_r}),$$

respectively. If $\delta_1 \le \dots \le \delta_r$, $\lambda_1 \le \dots \le \lambda_r$ and $\delta_k \ge \lambda_k$ for $k = 2, \dots, m$, then under Condition (i) and (iv), they have established that

$$(\underbrace{\delta_1, \dots, \delta_1}_{n_1}, \dots, \underbrace{\delta_r, \dots, \delta_r}_{n_r}) \overset{rm}{\succeq} (\underbrace{\lambda_1, \dots, \lambda_1}_{n_1}, \dots, \underbrace{\lambda_r, \dots, \lambda_r}_{n_r}) \Longrightarrow X_{m:m} \ge_{mrl} Y_{m:m}. \tag{5}$$

Further, for the case when $n_1 = \cdots = n_r = 1$ and $\lambda_1 = \ldots, \lambda_r = \lambda$, they have shown that, under Conditions (i) and (iv),

$$\lambda \geq \left(\frac{\sum_{i=1}^{m} \delta_i^{-1}}{m}\right)^{-1} \Longrightarrow X_{m:m} \geq_{mrl} Y_{m:m}. \tag{6}$$

In this work, we identify some conditions to compare the largest-order statistics from Marshall–Olkin bivariate exponential distribution in the sense of mean residual life order. More precisely, for two parallel systems with component lifetimes being distributed as $(X_1, X_2) \sim \mathcal{MOE}(\lambda_1, \lambda_2, \lambda_3)$ and $(Y_1, Y_2) \sim \mathcal{MOE}(\mu_1, \mu_2, \lambda_3)$, we establish that the following implication holds:

$$((\lambda_1 + \lambda_3)^{-1}, (\lambda_2 + \lambda_3)^{-1}) \overset{m}{\succ} ((\mu_1 + \lambda_3)^{-1}, (\mu_2 + \lambda_3)^{-1}) \Rightarrow X_{2:2} \geq_{mrl} Y_{2:2}.$$

In addition to order statistics, the sample range defined as $R(\boldsymbol{X}, n) = X_{n:n} - X_{1:n}$, where $X_{1:n}$ and $X_{n:n}$ are, respectively, the smallest- and largest-order statistics arising from the set of random variables $X_1, \cdots, X_n$, has also been studied in detail. The sample range can be interpreted in the reliability as follows. Let n independent and identically distributed random variables $X_1, \cdots, X_n$ represent the lifetimes of components of a series system. When the system fails, that is, after the first failure, there exists $n - 1$ live components that can be used in some other systems. If the live components are placed in a parallel structure, then the lifetime of the new system can be described by $R(\boldsymbol{X}, n)$. In this regard, stochastic comparisons of sample ranges have been discussed a lot in the case of independent exponential random variables; for example, one may refer to [23–29].

Let $X_1, \cdots, X_n$ be independent exponential random variables with respective hazard rates $\theta_1, \cdots, \theta_n$, and the sample range of $\boldsymbol{X} = (X_1, \cdots, X_n)$ be $R(\boldsymbol{X}, n) = X_{n:n} - X_{1:n}$, where $X_{1:n}$ and $X_{n:n}$ are the smallest- and largest-order statistics. The distribution function of $R(\boldsymbol{X}, n)$ is then given by

$$F_{R(\boldsymbol{X},n)}(x) = \frac{\sum_{i=1}^{n} \theta_i \prod_{j \neq i}(1 - e^{-\theta_j x})}{\sum_{i=1}^{n} \theta_i}, \qquad x \geq 0.$$

If θ_is are replaced by η_is, then the distribution function of $R(\boldsymbol{Y}, n) = Y_{n:n} - Y_{1:n}$ is obtained, where $Y_1, \cdots, Y_n$ are independent exponential random variables with respective hazard rates $\eta_1, \cdots, \eta_n$. Ding et al. [28] proved the following results concerning the comparison of $R(\boldsymbol{X}, n)$ and $R(\boldsymbol{Y}, n)$:

$$(\log \theta_1, \cdots, \log \theta_n) \overset{m}{\succeq} (\log \eta_1, \cdots, \log \eta_n) \Longrightarrow R(\boldsymbol{X}, n) \geq_{st} R(\boldsymbol{Y}, n) \tag{7}$$

and

$$(\theta_1, \cdots, \theta_n) \overset{m}{\succeq} (\eta_1, \cdots, \eta_n) \Longrightarrow R(\boldsymbol{X}, n) \geq_{rh} R(\boldsymbol{Y}, n). \tag{8}$$

These authors also showed, by means of a counterexample, that the result in (7) cannot generalize when the majorization order is replaced with the weak majorization order. In other words, the result in (7) does not hold under the p-larger order between $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_n)$ and $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_n)$. Two questions arise naturally here. Does the result in (7) hold under p-larger order between $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ under more restrictions on their involved parameters? How can we reinforce the result in (8) by considering a version of weak majorization order between $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$? In what follows, we try to answer these two questions. To be specific, we find some sufficient conditions for comparing sample ranges arising from exponential random variables in terms of usual stochastic order and reversed hazard rate order. More precisely, under

$$\mathcal{E}_n^+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 < x_1 \leq \ldots \leq x_n\},$$

and

$$\Theta_n(\boldsymbol{\theta}) = \left\{ \boldsymbol{\eta} = (\eta_1, \cdots, \eta_n) \in \mathcal{E}_n^+ : \boldsymbol{\theta} \overset{p}{\succeq} \boldsymbol{\eta} \quad and \quad \theta_k \geq \eta_k \;\; for\ k = 2, \cdots, n \right\}$$

and

$$\Omega_n(\boldsymbol{\theta}) = \left\{ \boldsymbol{\eta} = (\eta_1, \cdots, \eta_n) \in \mathcal{E}_n^+ : \boldsymbol{\theta} \overset{w}{\succeq} \boldsymbol{\eta} \quad and \quad \theta_k \geq \eta_k \;\; for\ k = 2, \cdots, n \right\},$$

we establish that the following implication holds:

(i) For $\boldsymbol{\eta} \in \Theta_n(\boldsymbol{\theta})$, we have $R(\boldsymbol{X}, n) \geq_{st} R(\boldsymbol{Y}, n)$;
(ii) For $\boldsymbol{\eta} \in \Omega_n(\boldsymbol{\theta})$, we have $R(\boldsymbol{X}, n) \geq_{rh} R(\boldsymbol{Y}, n)$.

The remainder of this paper proceeds as follows. In Section 2, some essential concepts and definitions are introduced. Next, in Section 3, for the case of Marshall–Olkin bivariate exponential distribution, MRL orderings of largest-order statistics are established. In Section 4, the usual stochastic ordering and reversed hazard rate ordering of ranges from independent exponential random variables are established. Finally, in Section 5, some concluding remarks are presented.

2. Definitions and Notation

Some key stochastic ordering notions that are most relevant to this work are introduced here. For pertinent details, one may refer to [30–32].

Definition 1. *For $i = 1, 2$, let X_i be a non-negative random variables with distribution function F_i, survival function $\bar{F}_i = 1 - F_i$, reversed hazard function $r_i = f_i/F_i$, and mean residual life function μ_i. Then, we say X_1 is larger than X_2*

(i) In mean residual life order, written as $X_1 \geq_{mrl} X_2$, if $\mu_1(t) \geq \mu_2(t)$ for all $t \geq 0$;
(ii) In reversed hazard rate order, written as $X_1 \geq_{rh} X_2$, if $r_1(t) \geq r_2(t)$ for all $t \geq 0$;
(iii) In usual stochastic order, written as $X_1 \geq_{st} X_2$, if $\bar{F}_1(t) \geq \bar{F}_2(t)$ for all $t \geq 0$.

The following majorization notions are useful for comparing dispersion between two positive vectors.

Definition 2. *Let $\boldsymbol{a} = (a_1, \cdots, a_n)$ and $\boldsymbol{b} = (b_1, \cdots, b_n)$ be two vectors with corresponding increasing arrangements $a_{(1)} \leq \cdots \leq a_{(n)}$ and $b_{(1)} \leq \cdots \leq b_{(n)}$. Then,*

(i) $\boldsymbol{a}$ is said to majorize $\boldsymbol{b}$, denoted by $\boldsymbol{a} \overset{m}{\succeq} \boldsymbol{b}$, if $\sum_{j=1}^{i} a_{(j)} \leq \sum_{j=1}^{i} b_{(j)}$, for $i = 1, \cdots, n-1$, and $\sum_{j=1}^{n} a_{(j)} = \sum_{j=1}^{n} b_{(j)}$;
(ii) $\boldsymbol{a}$ is said to weakly majorize $\boldsymbol{b}$, denoted by $\boldsymbol{a} \overset{w}{\succeq} \boldsymbol{b}$, if $\sum_{j=1}^{i} a_{(j)} \leq \sum_{j=1}^{i} b_{(j)}$, for $i = 1, \cdots, n$;
(iii) $\boldsymbol{a} \in \mathbb{R}^{+n}$ is said to p-majorize $\boldsymbol{b} \in \mathbb{R}^{+n}$, denoted by $\boldsymbol{a} \overset{p}{\succeq} \boldsymbol{b}$, if $\prod_{j=1}^{i} a_{(j)} \leq \prod_{j=1}^{i} b_{(j)}$, for $i = 1, \cdots, n$.

It is easy to observe that $\boldsymbol{a} \overset{m}{\succeq} \boldsymbol{b}$ implies $\boldsymbol{a} \overset{w}{\succeq} \boldsymbol{b}$. Further, when $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^{+n}$, $\boldsymbol{a} \overset{w}{\succeq} \boldsymbol{b}$ implies $\boldsymbol{a} \overset{p}{\succeq} \boldsymbol{b}$. The converse is, however, not true. For example, $(3, 4) \overset{p}{\succeq} (2, 2.5)$, but clearly, the weak majorization order does not hold between these two vectors. The book by Marshall et al. [33] provides a detailed discussion on the theory of majorization.

Definition 3 ([33], Marshall et al.)**.** *Suppose that $\mathbb{A} \subseteq \mathbb{R}^n$. Then, a function $\phi : \mathbb{A} \longrightarrow \mathbb{R}$ is said to be Schur-convex on $\mathbb{A}$ if*

$$\boldsymbol{u} \overset{m}{\succeq} \boldsymbol{v} \Rightarrow \phi(\boldsymbol{u}) \geq \phi(\boldsymbol{v}) \qquad for\ any\ \boldsymbol{u}, \boldsymbol{v} \in \mathbb{A}.$$

ϕ is said to be Schur-concave function on $\mathbb{A}$ if $-\phi$ is Schur-convex on $\mathbb{A}$.

Some conditions for the characterization of Schur-convex and Schur-concave functions are presented in the following lemma:

Lemma 1 ([33], Marshall et al., p. 84)**.** *If $J \subset \mathbb{R}$ is an open interval and $\phi : J^n \to \mathbb{R}$ is a continuously differentiable function, then ϕ is Schur-convex (Schur-concave) on J^n if and only if*

(i) *ϕ is symmetric on J^n;*
(ii) *for all $i \neq j$ and all $\mathbf{z} \in J^n$,*

$$(z_i - z_j)\left(\frac{\partial\phi(\mathbf{z})}{\partial z_i} - \frac{\partial\phi(\mathbf{z})}{\partial z_j}\right) \geq 0\,(\leq 0),$$

where $\partial\phi(\mathbf{z})/\partial z_i$ denotes the partial derivative of ϕ with respect to its i-th argument.

3. Mean Residual Life Order of Largest-Order Statistics

In this section, we first establish some basic lemmas that are used subsequently in proving and establishing the main results on mean residual life order of the largest-order statistics from bivariate exponential random variables. Please see Appendix A for proof.

Lemma 2. *For $c > 0$, let $\varphi : (0,c) \times (0,c) \to \mathbb{R}^+$ be defined as*

$$\varphi(x_1, x_2) = \frac{x_1 e^{-x_1^{-1}} + x_2 e^{-x_2^{-1}} - \dfrac{e^{-(x_1^{-1}+x_2^{-1}-c)}}{x_1^{-1} + x_2^{-1} - c}}{e^{-x_1^{-1}} + e^{-x_2^{-1}} - e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}}.$$

Then, φ is Schur-convex on $(0,c) \times (0,c)$.

Lemma 3. *For $c > 0$, let $\phi : (0,c] \times (0,c] \to \mathbb{R}^+$ be defined as*

$$\varphi(x_1, x_2) = x_1 + x_2 - \frac{1}{x_1^{-1} + x_2^{-1} - c^{-1}}.$$

If $(x_1, x_2) \overset{m}{\succ} (y_1, y_2)$ on $(0,c] \times (0,c]$, then $\phi(x_1, x_2) \geq \phi(y_1, y_2)$.

The random vector (X_1, X_2) has bivariate Marshall–Olkin exponential distribution with parameters $(\lambda_1, \lambda_2, \lambda_3)$, written as $(X_1, X_2) \sim \mathcal{MOE}(\lambda_1, \lambda_2, \lambda_3)$, if the joint survival function is given by

$$\bar{F}_{X_1,X_2}(x_1, x_2) = exp\{-(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3(x_1 \vee x_2))\}, \qquad x_1 \geq 0, x_2 \geq 0,$$

where $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 \geq 0$, and $x_1 \vee x_2 = \max\{x_1, x_2\}$. The marginal survival functions of X_1 and X_2 are, respectively, given by

$$\bar{F}_{X_1}(x) = exp\{-(\lambda_1 + \lambda_3)x\} \quad \text{and} \quad \bar{F}_{X_2}(x) = exp\{-(\lambda_2 + \lambda_3)x\}, \qquad x \geq 0.$$

Moreover, the survival function of $X_{2:2} = \max\{X_1, X_2\}$, for $x \geq 0$, is given by

$$\begin{aligned}\bar{F}_{X_{2:2}}(x) &= \mathbb{P}(X_1 > x) + \mathbb{P}(X_2 > x) - \mathbb{P}(X_1 > x, X_2 > x)\\ &= exp\{-(\lambda_1 + \lambda_3)x\} + exp\{-(\lambda_2 + \lambda_3)x\} - exp\{-(\lambda_1 + \lambda_2 + \lambda_3)x\}.\end{aligned}$$

Theorem 1. *For two parallel systems with component lifetimes being distributed as $(X_1, X_2) \sim \mathcal{MOE}(\lambda_1, \lambda_2, \lambda_3)$ and $(Y_1, Y_2) \sim \mathcal{MOE}(\mu_1, \mu_2, \lambda_3)$, the following implication holds:*

$$((\lambda_1 + \lambda_3)^{-1}, (\lambda_2 + \lambda_3)^{-1}) \overset{m}{\succ} ((\mu_1 + \lambda_3)^{-1}, (\mu_2 + \lambda_3)^{-1}) \Rightarrow X_{2:2} \geq_{mrl} Y_{2:2}.$$

4. Usual Stochastic and Reversed Hazard Rate Orders of Sample Ranges

In this section, we establish the usual stochastic order and reversed hazard rate order of sample ranges for the case of independent exponential random variables.

Let $X_1, \cdots, X_n$ be independent exponential random variables with respective hazard rates $\theta_1, \cdots, \theta_n$. The sample range of $\boldsymbol{X} = (X_1, \cdots, X_n)$ is $R(\boldsymbol{X}, n) = X_{n:n} - X_{1:n}$, where $X_{1:n}$ and $X_{n:n}$ are the smallest- and largest-order statistics from the n underlying variables. Then, the distribution function of $R(\boldsymbol{X}, n)$ is given by

$$F_{R(\boldsymbol{X},n)}(x) = \frac{\sum_{i=1}^{n} \theta_i \prod_{j \neq i}(1 - e^{-\theta_j x})}{\sum_{i=1}^{n} \theta_i}, \qquad x \geq 0.$$

Let us now define

$$\mathcal{E}_n^+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 < x_1 \leq \ldots \leq x_n\}.$$

For $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_n) \in \mathcal{E}_n^+$, let us set

$$\Theta_n(\boldsymbol{\theta}) = \left\{ \boldsymbol{\eta} = (\eta_1, \cdots, \eta_n) \in \mathcal{E}_n^+ : \boldsymbol{\theta} \overset{p}{\succeq} \boldsymbol{\eta} \quad and \quad \theta_k \geq \eta_k \;\; for\ k = 2, \cdots, n \right\}$$

and

$$\Omega_n(\boldsymbol{\theta}) = \left\{ \boldsymbol{\eta} = (\eta_1, \cdots, \eta_n) \in \mathcal{E}_n^+ : \boldsymbol{\theta} \overset{w}{\succeq} \boldsymbol{\eta} \quad and \quad \theta_k \geq \eta_k \;\; for\ k = 2, \cdots, n \right\}.$$

Then, following two lemmas help us to examine the extrema of real-valued functions over the spaces $\Theta_n(\boldsymbol{\theta})$ and $\Omega_n(\boldsymbol{\theta})$.

Lemma 4 ([21], Cheng and Wang, Theorem 3.2)**.** *The function $\phi : \mathcal{E}_n^+ \longrightarrow \mathbb{R}$ satisfies*

$$\boldsymbol{\eta} \in \Theta_n(\boldsymbol{\theta}) \Longrightarrow \phi(\boldsymbol{\theta}) \geq (resp. \leq) \phi(\boldsymbol{\eta})$$

if it is decreasing (resp. increasing) along with the vectors $\boldsymbol{\alpha}_1 = (1, \underbrace{0, \ldots, 0}_{n-1})$ and $\boldsymbol{\alpha}_2 = (\theta_1, -\beta_2\theta_2, \cdots, -\beta_n\theta_n)$ wherein $\beta_2, \cdots, \beta_n$ are non-negative values such that $\sum_{i=2}^{n} \beta_i = 1$.

Lemma 5 ([34], Wang)**.** *The inequality $\phi(\boldsymbol{\theta}) \geq \phi(\boldsymbol{\eta})$ holds for any function $\phi : \mathcal{E}_n^+ \longrightarrow \mathbb{R}$, when $\boldsymbol{\eta} \in \Theta_n(\boldsymbol{\theta})$, if $\phi(\boldsymbol{\theta})$ is decreasing along the vectors $\boldsymbol{\alpha}_1 = (1, \underbrace{0, \ldots, 0}_{n-1})$ and $\boldsymbol{\alpha}_3 = (1, -\beta_2, \cdots, -\beta_n)$, where $\beta_2, \cdots, \beta_n$ are non-negative values such that $\sum_{i=2}^{n} \beta_i = 1$.*

Then, with the use of the above two lemmas, we establish the following theorem.

Theorem 2. *For $\boldsymbol{\eta} \in \Theta_n(\boldsymbol{\theta})$, we have $R(\boldsymbol{X}, n) \geq_{st} R(\boldsymbol{Y}, n)$.*

Theorem 3. *For $\boldsymbol{\eta} \in \Omega_n(\boldsymbol{\theta})$, we have $R(\boldsymbol{X}, n) \geq_{rh} R(\boldsymbol{Y}, n)$.*

5. Concluding Remarks

The comparison of important characteristics associated with lifetimes of technical systems is an important problem in reliability theory since it would enable one to approximate complex systems with simpler ones and subsequently enable one to obtain bounds for important ageing characteristics of the complex system in terms of simpler ones. A technique that is useful for this purpose is the theory of stochastic orderings.

A parallel system can be found most commonly in industrial engineering, and so, it is of interest to evaluate the performance of a parallel system based on component lifetimes. Naturally, one is often interested in the system survival function or the system hazard rate function.

The mean residual life function of largest-order statistics and survival function of the sample range have found important uses in reliability, life testing, and survival analysis. The mean residual life order of largest-order statistics, unlike other orderings such as hazard rate order, usual stochastic order, reversed hazard rate order, and likelihood ratio order, has not been studied in detail. To the best of our knowledge, few results in this direction have focused on the exponential case.

The sample range defined as $R(\boldsymbol{X}, n) = X_{n:n} - X_{1:n}$, where $X_{1:n}$ and $X_{n:n}$ are, respectively, the smallest and largest-order statistics arising from the set of random variables $X_1, \cdots, X_n$, has also been studied in detail. The sample range can be interpreted in terms of reliability as follows. Let n independent and identically distributed random variables $X_1, \cdots, X_n$ represent the lifetimes of components of a series system. When the system fails, that is, after the first failure, there exists $n-1$ live components that can be used in some other systems. If the live components are placed in a parallel structure, then the lifetime of the new system can be described by $R(\boldsymbol{X}, n)$.

In this work, results were established in two different directions. In the first part, MRL orderings of the largest-order statistics from Marshall–Olkin bivariate exponential distribution were established. In the second part, the usual stochastic ordering and reversed hazard rate ordering of ranges from independent exponential random variables were discussed.

Proceeding along these lines, it will be of interest to see whether the results for ranges established here can be extended to the case of Marshall–Olkin bivariate exponential distribution. It will also be of interest to check whether the results established for largest-order statistics from bivariate Marshall–Olkin distribution can be extended to the multivariate distribution. As present, we are working in the these directions and hope to present the results in a future paper.

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Appendix A

Proof of Lemma 2. First, observe that φ is a symmetric function. Then, upon taking the derivative of $\varphi(x_1, x_2)$ with respect to x_1, we find

$$\begin{aligned}
\frac{\partial \varphi(x_1,x_2)}{\partial x_1} \overset{sgn}{=} & \left[e^{-x_1^{-1}} + e^{-x_2^{-1}} - e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}\right] \\
& \times \left[e^{-x_1^{-1}} + x_1^{-1}e^{-x_1^{-1}} - \frac{x_1^{-2}e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}}{x_1^{-1}+x_2^{-1}-c^{-1}} - \frac{x_1^{-2}e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}}{(x_1^{-1}+x_2^{-1}-c^{-1})^2}\right] \\
& - \left[x_1^{-2}e^{-x_1^{-1}} - x_1^{-2}e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}\right]\left[x_1 e^{-x_1^{-1}} + x_2 e^{-x_2^{-1}} - \frac{e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}}{x_1^{-1}+x_2^{-1}-c^{-1}}\right] \\
= & \left[e^{-x_1^{-1}} + e^{-x_2^{-1}} - e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}\right]\left[e^{-x_1^{-1}} - \frac{x_1^{-2}e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}}{(x_1^{-1}+x_2^{-1}-c^{-1})^2}\right] \\
& + \left[e^{-x_1^{-1}} + e^{-x_2^{-1}} - e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}\right]\left[x_1^{-1}e^{-x_1^{-1}} - \frac{x_1^{-2}e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}}{x_1^{-1}+x_2^{-1}-c^{-1}}\right] \\
& - \left[x_1^{-2}e^{-x_1^{-1}} - x_1^{-2}e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}\right]\left[x_1 e^{-x_1^{-1}} + x_2 e^{-x_2^{-1}} - \frac{e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}}{x_1^{-1}+x_2^{-1}-c^{-1}}\right] \\
= & \left[e^{-x_1^{-1}} + e^{-x^{-1}} - e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}\right]\left[e^{-x_1^{-1}} - \frac{x_1^{-2}e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}}{(x_1^{-1}+x_2^{-1}-c^{-1})^2}\right] \\
& + x_1^{-1}e^{-(x_1^{-1}+x_2^{-1})} - \frac{x_1^{-2}e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}}{x_1^{-1}+x_2^{-1}-c^{-1}} - \frac{x_2}{x_1^2}e^{-(x_1^{-1}+x_2^{-1})} \\
& + \frac{x_2}{x_1^2}e^{-(x_1^{-1}+2x_2^{-1}-c^{-1})} \\
= & \left[e^{-x_1^{-1}} + e^{-x_2^{-1}} - e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}\right]\left[e^{-x_1^{-1}} - \frac{x_1^{-2}e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}}{(x_1^{-1}+x_2^{-1}-c^{-1})^2}\right] \\
& + \frac{e^{-(x_1^{-1}+x_2^{-1})}}{x_1+x_2-c^{-1}x_1x_2}\left[1 - c^{-1}x_2 - \frac{x_2^2}{x_1^2}(1-c^{-1}x_1)(1-e^{-(x_2^{-1}-c^{-1})})\right],
\end{aligned}$$

where $a \overset{sgn}{=} b$ means that both sides of an equality have the same sign. Similarly, the partial derivative of $\varphi(x_1, x_2)$ with respect to x_2 can be found to be

$$\begin{aligned}
\frac{\partial \varphi(x_1,x_2)}{\partial x_2} \overset{sgn}{=} & \left[e^{-x_2^{-1}} + e^{-x_2^{-1}} - e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}\right]\left[e^{-x_2^{-1}} - \frac{x_2^{-2}e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}}{(x_1^{-1}+x_2^{-1}-c^{-1})^2}\right] \\
& + \frac{e^{-(x_1^{-1}+x_2^{-1})}}{x_1+x_2-c^{-1}x_1x_2}\left[1 - c^{-1}x_1 - \frac{x_1^2}{x_2^2}(1-c^{-1}x_2)(1-e^{-(x_1^{-1}-c^{-1})})\right].
\end{aligned}$$

Based on the above two expressions, we have

$$\frac{\partial \varphi(x_1,x_2)}{\partial x_1} - \frac{\partial \varphi(x_1,x_2)}{\partial x_2} \overset{sgn}{=} \Delta_1 + \Delta_2,$$

where

$$\begin{aligned}
\Delta_1 = & \left[e^{-x_2^{-1}} + e^{-x_2^{-1}} - e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}\right]\left[(e^{-x^{-1}} - e^{-x_2^{-1}}) + (x_2^{-2} - x_2^{-2})\frac{e^{-(x_1^{-1}+x_2^{-1}-c^{-1})}}{(x_1^{-1}+x_2^{-1}-c^{-1})^2}\right], \\
\Delta_2 = & \frac{e^{-(x_1^{-1}+x_2^{-1})}}{x_1+x_2-c^{-1}x_1x_2}\left[c^{-1}(x_1-x_2) + \frac{x_1^2}{x_2^2}(1-c^{-1}x_2)(1-e^{-(x_1^{-1}-c^{-1})})\right. \\
& \left. - \frac{x_2^2}{x_1^2}(1-c^{-1}x_1)(1-e^{-(x_2^{-1}-c^{-1})})\right].
\end{aligned}$$

It is not difficult to show that $\Delta_1 \overset{sgn}{=} x_1 - x_2$. Moreover, since both $x/(1-c^{-1}x)$ and $x(1-e^{-(1/x-c^{-1})})$ are non-negative and increasing functions in $x \in \mathbb{R}^+$, we have that $x^2(1-e^{-(1/x-c^{-1})})/(1-c^{-1}x)$ increases in $x \in \mathbb{R}^+$. Using this fact, we can see that $\Delta_2 \overset{sgn}{=} x_1 - x_2$. Hence, we have

$$\frac{\partial\varphi(x_1,x_2)}{\partial x_1} - \frac{\partial\varphi(x_1,x_2)}{\partial x_2} \overset{sgn}{=} x_1 - x_2,$$

and now the required result follows readily from Lemma 1. □

Proof of Lemma 3. First, observe that ϕ is a symmetric function. Then, upon taking partial derivatives of $\phi(x_1,x_2)$ with respect to x_1 and x_2, respectively, we find

$$\frac{\partial\phi(x_1,x_2)}{\partial x_1} = 1 - \frac{x_1^{-2}}{(x_1^{-1}+x_2^{-1}-c^{-1})^2} \quad \text{and} \quad \frac{\partial\phi(x_1,x_2)}{\partial x_2} = 1 - \frac{x_2^{-2}}{(x_1^{-1}+x_2^{-1}-c^{-1})^2}.$$

We then find

$$\begin{aligned}(x_1-x_2)\left(\frac{\partial\phi(x_1,x_2)}{x_1} - \frac{\partial\phi(x_1,x_2)}{x_2}\right) &\overset{sgn}{=} (x_1-x_2)(x_2^{-2}-x_1^{-2}) \\ &\geq 0,\end{aligned}$$

which, based on Lemma 2, yields ϕ to be Schur-convex on $(0,c)\times(0,c)$, as required. □

Proof of Theorem 1. The mean residual function of $X_{2:2}$ is given by

$$\begin{aligned} m_{X_{2:2}}(x) &= \frac{\int_x^\infty \bar{F}_{X_{2:2}}(t)dt}{\bar{F}_{X_{2:2}}(x)} \\ &= \frac{\dfrac{e^{-(\lambda_1+\lambda_3)x}}{\lambda_1+\lambda_3} + \dfrac{e^{-(\lambda_2+\lambda_3)x}}{\lambda_2+\lambda_3} - \dfrac{e^{-(\lambda_1+\lambda_2+\lambda_3)x}}{\lambda_1+\lambda_2+\lambda_3}}{e^{-(\lambda_1+\lambda_3)x} + e^{-(\lambda_2+\lambda_3)x} - e^{-(\lambda_1+\lambda_2+\lambda_3)x}}, \quad x \geq 0. \end{aligned}$$

If we replace λ_i with μ_i, for $i=1,2$, in the above function, the mean residual function of $Y_{2:2}$ is similarly obtained. Now, we shall show that $m_{X_{2:2}}(x) \geq m_{Y_{2:2}}(x)$ for all $x \geq 0$ when $((\lambda_1+\lambda_3)^{-1},(\lambda_2+\lambda_3)^{-1}) \overset{m}{\succeq} ((\mu_1+\lambda_3)^{-1},(\mu_2+\lambda_3)^{-1})$. For this response, in Lemma 3, let us set

$$c = \lambda_3^{-1}, \quad x_i = (\lambda_i+\lambda_3)^{-1}, \quad y_i = (\mu_i+\lambda_3)^{-1}, \quad \text{for} \quad i=1,2.$$

Then, it readily follows that

$$((\lambda_1+\lambda_3)^{-1},(\lambda_2+\lambda_3)^{-1}) \overset{m}{\succeq} ((\mu_1+\lambda_3)^{-1},(\mu_2+\lambda_3)^{-1}) \Rightarrow m_{X_{2:2}}(0) \geq m_{Y_{2:2}}(0).$$

Now, consider the function φ in Lemma 2 with $c = (\lambda_3 x)^{-1}$, $x > 0$. Then, the mean residual functions of $X_{2:2}$ and $Y_{2:2}$ can be rewritten as, for $x > 0$,

$$\begin{aligned} m_{X_{2:2}}(x) &= x\varphi((\lambda_1 x+\lambda_3 x)^{-1},(\lambda_2 x+\lambda_3 x)^{-1}), \\ m_{Y_{2:2}}(x) &= x\varphi((\mu_1 x+\lambda_3 x)^{-1},(\mu_2 x+\lambda_3 x)^{-1}). \end{aligned}$$

Based on Lemma 2, it readily follows that

$$((\lambda_1+\lambda_3)^{-1},(\lambda_2+\lambda_3)^{-1}) \overset{m}{\succeq} ((\mu_1+\lambda_3)^{-1},(\mu_2+\lambda_3)^{-1}) \Rightarrow m_{X_{2:2}}(x) \geq m_{Y_{2:2}}(x),$$

which completes the proof of the theorem. □

Proof of Theorem 2. The distribution functions of $R(\boldsymbol{X}, n)$ and $R(\boldsymbol{Y}, n)$ can be expressed as

$$F_{R(\boldsymbol{X},n)}(x) = \phi(\boldsymbol{\theta} x) \qquad \textit{and} \qquad F_{R(\boldsymbol{Y},n)}(x) = \phi(\boldsymbol{\eta} x), \qquad x > 0,$$

where $\phi : \mathcal{E}_n^+ \longrightarrow \mathbb{R}$ is given by

$$\phi(\boldsymbol{\theta}) = \frac{\sum_{i=1}^n \theta_i \prod_{j\neq i}(1-e^{-\theta_j x})}{\sum_{i=1}^n \theta_i}, \qquad x \geq 0.$$

We then need to show that

$$\boldsymbol{\eta} \in \Theta_n(\boldsymbol{\theta}) \Longrightarrow \phi(\boldsymbol{\theta}) \leq \phi(\boldsymbol{\eta}).$$

To this end, in view of Lemma 4, it is enough to prove that $\phi(\boldsymbol{\theta})$ is increasing along the vectors $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$. The gradient of ϕ along $\boldsymbol{\alpha}_1$ is

$$\begin{aligned}
\nabla_{\boldsymbol{\alpha}_1}\phi &= \frac{\partial \phi(\boldsymbol{\theta})}{\partial \theta_1} \\
&\overset{sgn}{=} \left[\prod_{j\neq 1}(1-e^{-\theta_j}) + e^{-\theta_1}\sum_{i\neq 1}\theta_i \prod_{j\neq 1,i}(1-e^{-\theta_j})\right]\sum_{i=1}^n \theta_i - \sum_{i=1}^n \theta_i \prod_{j\neq i}(1-e^{-\theta_j}) \\
&= \prod_{j\neq 1}(1-e^{-\theta_j})\sum_{i=1}^n \theta_i + e^{-\theta_1}\sum_{i\neq 1}\theta_i \prod_{j\neq 1,i}(1-e^{-\theta_j})\sum_{i=1}^n \theta_i - \sum_{i=1}^n \theta_i \prod_{j\neq i}(1-e^{-\theta_j}) \\
&\geq \prod_{j\neq 1}(1-e^{-\theta_j})\sum_{i=1}^n \theta_i - \sum_{i=1}^n \theta_i \prod_{j\neq i}(1-e^{-\theta_j}) \\
&\overset{sgn}{=} \sum_{i=1}^n \theta_i \left[\frac{1}{1-e^{-\theta_1}} - \frac{1}{1-e^{-\theta_i}}\right] \\
&= \sum_{i=2}^n \theta_i \frac{(e^{-\theta_1} - e^{-\theta_i})}{(1-e^{-\theta_1})(1-e^{-\theta_i})} \\
&\geq 0 \quad ,
\end{aligned}$$

where the final inequality follows from the fact that $\theta_1 \leq \cdots \leq \theta_n$. So, $\phi(\boldsymbol{\theta})$ is increasing along the vector $\boldsymbol{\alpha}_1$. On the other hand, since $\sum_{i=2}^n \beta_i = 1$, we have

$$\begin{aligned}
\nabla_{\boldsymbol{\alpha}_2}\phi &= \theta_1 \frac{\partial \phi(\boldsymbol{\theta})}{\partial \theta_1} - \sum_{k=2}^n \beta_k \theta_k \frac{\partial \phi(\boldsymbol{\theta})}{\partial \theta_k} \\
&= \sum_{k=2}^n \beta_k \left[\theta_1 \frac{\partial \phi(\boldsymbol{\theta})}{\partial \theta_1} - \theta_k \frac{\partial \phi(\boldsymbol{\theta})}{\partial \theta_k}\right] \\
&= \sum_{k=2}^n \beta_k A_k, \quad \textit{say}.
\end{aligned}$$

But, as in the proof of [28], we find, for $k = 2, \cdots, n$,

$$\begin{aligned}
\Delta_k \overset{sgn}{=} \; & [A(\theta_1 + \theta_k + B) + B]\left[\frac{\theta_1 e^{-\theta_1}}{(1-e^{-\theta_1})} - \frac{\theta_1 e^{-\theta_k}}{(1-e^{-\theta_k})}\right] \\
+ \; & (\theta_k - \theta_1)(A-B)(1-e^{-\theta_1})(1-e^{-\theta_k}) + (\theta_1 + \theta_2 + B)\theta_1\theta_2(e^{-\theta_1} - e^{-\theta_2}),
\end{aligned}$$

where

$$A = \sum_{i\neq 1,k} \frac{\theta_i}{1-e^{-\theta_i}} \qquad \textit{and} \qquad B = \sum_{i\neq 1,k} \theta_i.$$

It is clear that $A \geq B$, which along with the assumption that $\theta_1 \leq \cdots \leq \theta_n$ and the decreasing property of $xe^{-\theta x}/(1-e^{-x})$ in $x \in (0,\infty)$, yields $\nabla_{\boldsymbol{\alpha}_2}\phi \geq 0$. Hence, $\phi(\boldsymbol{\theta})$ is also increasing along $\boldsymbol{\alpha}_2$, completing the proof of the lemma. $\square$

Proof of Theorem 3. The reversed hazard rate function of $R(\boldsymbol{X},n)$ is given by

$$r_{R(\boldsymbol{X},n)}(x) = \frac{\sum_{i\neq j} \frac{\theta_i}{1-e^{-\theta_i x}} \times \frac{\theta_j e^{-\theta_j x}}{1-e^{-\theta_j x}}}{\sum_{i=1}^n \frac{\theta_i}{1-e^{-\theta_i}}}, \qquad x>0.$$

We can rewrite the above reversed hazard rate function as $r_{R(\boldsymbol{X},n)}(x) = \phi(\boldsymbol{\theta}x)/x$, for $x>0$, where $\phi : \mathcal{E}_n^+ \longrightarrow \mathbb{R}$ is given by

$$\phi(\boldsymbol{\theta}) = \frac{\sum_{i\neq j} \frac{\theta_i}{1-e^{-\theta_i}} \times \frac{\theta_j e^{-\theta_j}}{1-e^{-\theta_j}}}{\sum_{i=1}^n \frac{\theta_i}{1-e^{-\theta_i}}}.$$

Similarly, the reversed hazard rate function of $R(\boldsymbol{Y},n)$ can be expressed as $r_{R(\boldsymbol{Y},n)}(x) = \phi(\boldsymbol{\eta}x)/x$, for $x>0$. Then, we have to prove that ϕ satisfies the conditions of Lemma 5. To compute the gradient of $\phi(\boldsymbol{\theta})$ along the vector $\boldsymbol{\alpha}_1$, we first rewrite $\phi(\boldsymbol{\theta})$ as

$$\begin{aligned}
\phi(\boldsymbol{\theta}) &= \frac{\frac{\theta_1}{1-e^{-\theta_1}}\sum_{j=2}^n \frac{\theta_j e^{-\theta_j}}{1-e^{-\theta_j}} + \frac{\theta_1 e^{-\theta_1}}{1-e^{-\theta_1}}\sum_{i=2}^n \frac{\theta_i}{1-e^{-\theta_i}} + \sum_{i=2}^n\sum_{j\neq 1,i} \frac{\theta_i}{1-e^{-\theta_i}} \times \frac{\theta_j e^{-\theta_j}}{1-e^{-\theta_j}}}{\frac{\theta_1}{1-e^{-\theta_1}} + \sum_{i=2}^n \frac{\theta_i}{1-e^{-\theta_i}}} \\
&= \frac{a(\theta_1)\sum_{j=2}^n b(\theta_j) + b(\theta_1)\sum_{i=2}^n a(\theta_i) + \sum_{i=2}^n\sum_{j\neq 1,i} a(\theta_i)b(\theta_j)}{a(\theta_1) + \sum_{i=2}^n a(\theta_i)} \\
&= \frac{a(\theta_1)B + b(\theta_1)A + C}{a(\theta_1)+A},
\end{aligned}$$

where

$$\begin{aligned}
a(\theta) &= \frac{\theta}{1-e^{-\theta}} \quad , \quad b(\theta) = \frac{\theta e^{-\theta}}{1-e^{-\theta}} \quad , \quad A = \sum_{i=2}^n a(\theta_i) \quad , \quad B = \sum_{j=2}^n b(\theta_j) \\
C &= \sum_{i=2}^n \sum_{j\neq 1,i} a(\theta_i)b(\theta_j).
\end{aligned}$$

Then, we find

$$\begin{aligned}
\nabla_{\boldsymbol{\alpha}_1}\phi &= \frac{\partial\phi(\boldsymbol{\theta})}{\partial\theta_1} \\
&\overset{sgn}{=} \left[a'(\theta_1)B + b'(\theta_1)A\right](a(\theta_1)+A) - a'(\theta_1)[a(\theta_1)B + b(\theta_1)A + C] \\
&= b'(\theta_1)A[a(\theta_1)+A] + a'(\theta_1)[AB - b(\theta_1)A - C] \\
&= I + II, \quad say.
\end{aligned}$$

Note that $b(x)$ is decreasing in $x \in (0,\infty)$, which yields $I \leq 0$. Furthermore, we can see that $AB - C = \sum_{i=2}^n a(\theta_i)b(\theta_i)$ and so

$$\begin{aligned}
AB - b(\theta_1)A - C &= \sum_{i=2}^n a(\theta_i)b(\theta_i) - b(\theta_1)\sum_{i=2}^n a(\theta_i) \\
&= \sum_{i=2}^n a(\theta_i)[b(\theta_i) - b(\theta_1)] \\
&\leq 0,
\end{aligned}$$

where the last inequality follows from the decreasing property of b and the assumption that $\theta_1 \leq \cdots \leq \theta_n$. Because $a(x)$ is increasing in $x \in (0, \infty)$, we can then conclude from the above discussion that $II \leq 0$. Hence, $\phi(\boldsymbol{\theta})$ is decreasing along $\boldsymbol{\alpha}_1$. On the other hand, since $\sum_{i=2}^{n} \beta_i = 1$, it follows that

$$\begin{aligned} \nabla_{\boldsymbol{\alpha}_3} \phi &= \frac{\partial \phi(\boldsymbol{\theta})}{\partial \theta_1} - \sum_{k=2}^{n} \beta_k \frac{\partial \phi(\boldsymbol{\theta})}{\partial \theta_k} \\ &= \sum_{k=2}^{n} \beta_k \left[\frac{\partial \phi(\boldsymbol{\theta})}{\partial \theta_1} - \frac{\partial \phi(\boldsymbol{\theta})}{\partial \theta_k} \right] \\ &= \sum_{k=2}^{n} \beta_k \psi_k, \quad say. \end{aligned}$$

Using an argument similar to the one used in the proof of Theorem 2 of [28], we find $\psi_k \leq 0$ for all $k = 2, \cdots, n$, which means that $\phi(\boldsymbol{\theta})$ is also decreasing along $\boldsymbol{\alpha}_3$, completing the proof of the theorem. □

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