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How to Pose Problems on Periodicity and Teaching with Problem Posing

Guoqiang Dang, Yufeng Guo * and Kai Li

School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China; dang@mail.bnu.edu.cn (G.D.); likai0930@126.com (K.L.)
* Correspondence: guoyufeng@bnu.edu.cn

Abstract: Research on how to pose good problems in mathematical science is rarely touched. Inspired by Kilpatrick's "Where do good problems come from?", the current research investigates the problem of the specific problem posed by mathematicians in mathematical sciences. We select a recent mathematical conjecture of Yang related to periodic functions in the field of functions of one complex variable. These problems are extended to complex differential equations, difference equations, differential-difference equations, etc. Through mathematical analysis, we try to reproduce the effective strategies or techniques used by mathematicians in posing these new problems. The results show that mathematicians often use generalization, constraint manipulation, and specialization when they pose new mathematical problems. Conversely, goal manipulation and targeting a particular solution are rarely used. The results of the study may have a potential impact and promotion on implementing problem-posing teaching in primary and secondary schools. Accordingly, teachers and students can be encouraged to think like mathematicians, posing better problems and learning mathematics better. Then, we give some examples of mathematical teaching at the high school level using problem-posing strategies, which are frequently employed by mathematicians or mathematical researchers, and demonstrate how these strategies work. Therefore, this is a pioneering research that integrates the mathematical problem posing by mathematicians and the mathematical problem posing by elementary and secondary school math teachers and students. In addition, applying constraint manipulation and analogical reasoning, we present four unsolved mathematical problems, including three problems of complex difference-related periodic functions and one problem with complex difference equations.

Keywords: Yang's conjecture; periodic function; entire function; problem posing; problem solving; problem-posing strategy

MSC: 97D50; 97I80; 30D20; 39A45

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1. Introduction

The development of mathematics is often driven by innovative problems. Many mathematicians have noted the importance of posing mathematical problems. The twenty-three most influential mathematical questions presented by David Hilbert stimulated a significant amount of progress [1]. Albert Einstein advocated that "posing a new question and a new possibility to regard an old problem from a new angle need creative imagination and led to a real advance in science" [2]. Paul Halmos advocated "I do believe that problems are the heart of mathematics" [3]. "Cantor and Klamkin be convinced that posing a problem is more important and valuable than solving" [4]. Recently, research on problem posing has been focused on students or teachers in school education; for instance, see [5]. However, problems related to how mathematicians find problems, pose problems, think about problems, and solve problems in mathematical science have rarely been touched upon until now. Therefore, "Where do good problems come from?" [6]. The motivation of the current paper is to start an attempt to uncover the magic box of how mathematicians find and pose problems, and what strategies are used in posing relevant mathematical problems, answering how to integrate problem posing into mathematics teaching, and how to encourage students to pose high-quality problems. Therefore, we aim to investigate how the problems or theorems of the periodicity of the entire functions of one complex variable and Yang's conjecture are formulated.

In Section 2, we conduct a literature review containing the concept of problem posing and the demonstration of the relevant studies. In Section 3, we introduce some mathematical definitions. In Section 4, we introduce some problems or theorems related to the periodicity of functions and Yang's conjecture selected from some recently published papers. In Section 5, we put the main results of how mathematicians pose better mathematical problems. In Section 6, we give some examples of mathematical teaching using the problem-posing strategies which are frequently employed by mathematicians or mathematical researchers, and describe how these strategies work. In Section 7, we give the discussions and limitations.

2. Literature Review

Let us start our statements from the famous Korteweg–de Vries (KdV) equation. In August 1834, a young Scottish engineer John Scott Russell (1808–1882) observed a mysterious phenomenon in a narrow channel (Union Canal at Hermiston, Edinburgh): the wave of translation (solitary wave). This led directly to the famous KdV equation. Throughout his life, Russell was convinced of the fundamental importance of his wave of translation.

We invite each reader of this article to imagine the mathematical problems we would ask if we happened to be standing by the same canal today and also saw this phenomenon.

The KdV equation reads $u_t + auu_x + u_{xxx} = 0$, and is a nonlinear partial differential equation arising in many areas of physics. The number a is arbitrary, and often equals to $a = \pm 6, \pm 1$. This equation depicts the long-time evolution of small amplitude and long wavelength in shallow water [7]. Many researchers have contributed to finding properties of the KdV equation, such as constructing exact solutions, soliton solutions, and the asymptotic behavior of solutions. At the same time, by modifying the form or numbers of the KdV equation, researchers converted this equation into a variety of KdV-like equations, for instance, see [8–10], such as the third-order KdV equations $u_t + g(u)u_x + u_{xxx} = 0$, where $g(u)$ can be $\alpha u, \alpha u^2, \alpha u^n, \alpha u_x, 2\alpha u - 3\beta u^2$; the $K(n,n)$ differential equation, such as $u_t + a(u^n)_x + b(u^n)_{xxx} = 0$; the modified KdV equation $u_t + 6\alpha u^2 u_x + u_{xxx} = 0$, $\alpha = \pm 1$; the generalized KdV equation $u_t + \sigma u^n u_x + u_{xxx} = 0, n > 2$; the potential KdV equation $u_t + au_x^2 + u_{xxx} = 0$; the Gardner equation $u_t + 2\alpha uu_x - 3\beta u^2 u_x + u_{xxx} = 0, \alpha > 0$, $\beta > 0$; the generalized KdV equation $u_t + (au^n - bu^{2n})u_x + u_{xxx} = 0$; the modified equal width equation $u_t + 3u^2 u_x - u_{xxt} = 0$; the Sawads–Kotera equation $u_t + \left(\frac{5}{3}u^3 + 5uu_{xx}\right)_x + u_{xxxxx} = 0$; the seventh-order KdV equation $u_t + 6uu_x + u_{3x} - u_{5x} + \alpha u_{7x} = 0$; the ninth-order KdV equation $u_t + 45u_x u_{6x} + 45uu_{7x} + 210u_{3x}u_{4x} + 210u_{2x}u_{5x} + 1575u_x(u_{2x})^2 + 3150uu_{2x}u_{3x} + 1260uu_x u_{4x} + 630u^2 u_{5x} + 9450u^2 u_x u_{2x} + 3150u^3 u_{3x} + 4725u^4 u_x + u_{9x} = 0$; and many other types. As a result, we can find that by modifying the forms of the KdV equation, changing even just one parameter will affect the properties of the solutions, and a very large number of new KdV-like differential equations can be obtained, greatly advancing the development of nonlinear differential equations. Consequently, many kinds of analytical or numerical methods for solving differential equations have been used, for instance, the Bäcklund transformation method [11], Darboux transformation method [12], physics-informed neural networks method [13], Hirota bilinear method [14], inverse scattering method [15], Homotopy analysis method [16], and complex method [17–20].

However, by changing the numbers and forms in a mathematical expression, a completely new mathematical expression can be obtained, for example, by modifying the numbers and forms of the KdV equation, which means that it will pose a new mathematical problem and quite possibly a new discovery in mathematics. Problem posing plays a key role in scientific research. In this section, we mainly introduce three aspects:

- What is the problem posing?
- Why problem posing?
- How does problem posing work?

2.1. What Is the Problem Posing

A few definitions specifically describe problem posing by mathematicians. Below are some selected definitions.

Silver (1994) [21] defined problem posing as “the generation of new problems based on a given situation or the reformulation of given problems”. Silver emphasized that problem posing can arrive before problem solving, during problem solving, or after solving a particular problem. This definition emphasizes the source of posing new problems.

Stoyanova (1997) [22] defined problem posing as “the process by which, on the basis of their mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful well structured mathematical problems”. The definition emphasizes the situation, which is classified as a free, semi-structured or structured situation.

Cai et al. (2013) [23] stated that “problem-posing tasks as those which require teachers or students to generate new problems based either on given situations or on mathematical expressions or diagrams”. They defined “problem posing” as consisting of three specific intellectual activities: “(1) teachers pose mathematical problems based on given situations or mathematical expressions or diagrams, (2) teachers predict the types of problems that students can pose based on given situations or mathematical expressions or diagrams, and (3) teachers design mathematical problem-posing tasks for students”.

Xu et al. (2013) [24] defined problem posing as the generation of new problems from a given problem situation in two aspects: students pose problems and teachers foresee the problems posed by students. The definitions from Xu et al. (2013) and Cai et al. (2013) are very similar; both consider the problem posed by teachers and students. Additionally, the teachers’ problem posing contains the following: (1) teachers pose mathematical problems according to given situations, and (2) teachers foresee the problems that students will likely present according to certain situations.

Considering problem posing in mathematics education, Cai and Hwang (2020) [25] pointed out that “problem posing are several related kinds of activity that help teachers and students formulating (or reformulating) and presenting a problem or task based on a particular situation (which we refer to as the problem context or problem situation)”. Their definition emphasizes problem posing as a teaching or learning activity. Their definition considers teachers’ and students’ problem posing by ways of formulations, expressions, and tasks. Cai and Hwang describe the definitional framework of mathematical problem posing as two separate aspects. For students, (1) they pose problems on the basis of given situations which may contain mathematical expressions or diagrams, and (2) they pose problems by changing (or reformulating) given problems. The following specific intellectual activities are defined as problem posing for teachers: (1) teachers pose mathematical problems based on given problem situations; (2) teachers predict the types of problems that students can pose based on given problem situations; (3) teachers pose problems by changing existing problems; (4) teachers create mathematical problem posing situations for students; and (5) teachers pose problems for students to solve [25].

Most recently, Cai (2022) [26] refined the definition of problem posing as follows: Problem posing is defined as four activities: (1) students pose problems on the basis of given problem situations, (2) students pose problems by changing (reformulating) existing problems, (3) teachers generate mathematical situations for students to pose problems, and (4) teachers predict problems that students can pose on the basis of given situations.

The current paper adopts the definition by Silver. This is because mathematicians often pose new problems on the basis of given conditions or by changing the given problems.

2.2. Why Problem Posing

Problem posing is located variously in different places in mathematical education. Mathematical problem posing is considered (1) an ingredient of problem solving, (2) a cognitive activity, (3) a target of learning unto itself, and (4) a way of teaching [5,27].

The first aspect sees mathematical problem posing as a means of solving mathematical problems or examining the relationship between them. For example, in Polya's problem-solving techniques, Polya suggests considering (posing) a related problem in the process of problem solving.

The second aspect focus deals with problem posing as a cognitive activity, such as the study of the capacity and process of posing problems with a focus on cognitive and affective aspects, and as a measurement tool to assess the thinking and creativity of humans by disclosing the process of posing problems when they pose problems.

The third aspect focuses on the products of the problem posers, and the posers are encouraged to learn how to pose problems and become good posers. The opening literature shows that both teachers and students can become better problem posers.

The fourth aspect focuses on school teaching mathematics by using problem posing. Considering how to include problem-posing tasks into mathematics instruction and study encourages teachers and students to pose problems in lessons, stimulates the problem-posing potential of teachers and students, and encourages research into the methods of using mathematical problem posing to improve mathematics learning and teaching.

In fact, there is no unique viewpoint about the classification of problem posing.

In the problem tasks, students will discover and pose problems on the basis of existing contexts. Students find something they want to know. In this framework, the problem-posing research focuses on what kinds of problems students can pose, what kinds of processes students use therein, and the assessments of thinking and creativity of the students. Therefore, researchers have not only examined the capacity of students and teachers to pose mathematical problems but also the cognitive and affective processes of problem posing [28]. However, the above definition of problem posing centers on the relationship between teachers and students. One shortcoming of the above definitions is that they do not take into account problem posing by mathematicians and mathematics researchers (not problem posing in mathematics education). Although little is known about how mathematicians identify problems and solve problems, it is necessary to extend the scope of problem posing to mathematicians and mathematical researchers. However, the mentioned definition of mathematical problem posing lacks mathematicians' problem posing.

2.3. How does Problem Posing Work

Many researchers are dedicated to developing heuristics and strategies of problem posing, aiming to motivate and empower people to pose better problems, occurring in school education (from primary school to high school) or educational investigations, such as service and pre-service teachers and students, or mathematics textbooks.

As we know, in traditional elementary and secondary math classrooms, in order for students to effectively acquire the teaching content, teachers will require students to pose problems, or students will pose problems to the teacher. These problems may include mathematical problems, and non-mathematical problems (e.g., problems due to difficulties in understanding mathematical contents, or problems without mathematical elements). In a first-year high school math problem posing task, the following problem situation was given:

Giving a function $f(x) = x + \frac{1}{x}$. From this, pose some mathematical problems.

From the problems posed by the students, we chose the following 13 responses:

1. *What is x?*
2. *If $x = 8$, find the maximum value of $f(x)$.*
3. *Find the values of $f(x)$ for $x = 5$ and $x = 7$.*
4. *Find out the domain of of $f(x)$.*

5. *Find out the range of $f(x)$.*
6. *What is the minimum value of $f(z)$?*
7. *Find out the monotonicity of $f(x)$.*
8. *Find out the monotonicity of f(x) on $[-1, \frac{\pi}{2}]$.*
9. *Find out the expression of $f(2x)$.*
10. *How's the image?*
11. *What quadrant is it in?*
12. *Find the image of $f(x)$ if it is shifted parallel to the left by y units.*
13. *If $x + \frac{1}{x} = 30$, find the value of $x^2 + \frac{1}{x^2}$.*

Responses 1–4 are not good mathematical problems, as they do not deepen thinking. Obviously, from the point of view of complexity and mathematical significance, compared to problems 1–4, problems 5–13 are better and are basic and important for high school students. Therefore, these are good problems. Responses 10–12 consider the geometric properties of $f(x)$, which need more knowledge of the rational function $f(x)$. Answering questions 5 and 6 requires the use of basic inequalities, and answering question 13 requires some transformation techniques. In order to promote the ability of students to pose good math problems, it is necessary to train students in problem-posing strategies or techniques. Therefore, we would like to demonstrate some common problem posing strategies next.

Heuristics are useful organizational units for planning, monitoring, and executive control [29]. The problem-posing strategies we use here refer to some specific techniques that problem posers can adopt when they pose better mathematical problems. They can be used independently or jointly. We should note that the strategies are not heuristics only, but also the approaches adopted by problem posers when they execute a task of problem posing. However, so far, problem posing has not resulted in a model comparable to the problem-solving model—Polya's problem-solving techniques. There are several problem-posing strategies scattered in the different opening works in the literature, for instance, see [30]. We collected some of the strategies as follows:

- Symmetry, for example, asking a new problem through the swap of conditions and goal between the given problem [31].
- Constraint manipulation, for example, posing a new problem using manipulations with the conditions or implicit assumptions involved in the problem situation [31], containing the following:
  - Numerical variation, for example, asking a new problem using changing the given numerical values into the new ones [32].
  - What-if-not, for example, posing a new problem through systematically asking the question "What if a particular condition or implicit assumption would be different" [33].
- Chaining, for example, we have the following:
  - Extending a given problem such that a solution to a new problem requires solving a given one first [31].
  - Extending a given problem such that a solution to the old problem requires solving a new one first. For example, we have an existing problem:
    Prove that (1) if $a > 0, b > 0$, then $\sqrt{ab} \leq \frac{a+b}{2}$.
    To prove this problem, we need the following:
    (2) If $a > 0, b > 0$, then $0 \leq (\sqrt{a} - \sqrt{b})^2$.
- Goal manipulation, for example, the formulation of a new problem by manipulating the goals of an existing or beforehand formulated problem, in which the original assumptions of the question are retained without adjustment [31].
- Targeting a specific solution, for example, constructing a new problem, whose solution requires using a specific theorem, solution, or mathematical approach [34].
- Generalization generates a problem for which the existing problem is a special case [35].
- Specialization considers some special, extreme, or exceptional situations.

- Association generates a new problem that closely connects the given problem. For example, if the existing problem is "A numerical series $a_n = 1, 2, 2^2, 2^3, \cdots, 2^{n-1}$, find the sum of the first n terms", then one may pose the problem "If $\{a_n\}$ is a geometric progression, then what series is $\{\log_2 a_n\}$?".
- Analogical reasoning uses analogical reasoning to create a new problem based on the existing problem. For example, according to "find the formula of the distance from a point to a straight line in the plane", create the new problem of "find the formula of the distance from a point to a straight line in the 3-dimensional space".

For more background on the mathematical problems posing, the readers can easily find relevant papers for further reading.

3. Preliminary

For the following discussion, we need to introduce some relevant mathematical concepts using the standard notations of Nevanlinna theory [36–38].

Define the growth order, low growth order and hyper order as

$$\rho(g) = \limsup_{r\to\infty} \frac{\log T(r,g)}{\log r}, \mu(g) = \liminf_{r\to\infty} \frac{\log T(r,g)}{\log r}, \rho_2(g) = \limsup_{r\to\infty} \frac{\log\log T(r,g)}{\log r}.$$

Define the shift by $g_c(z) = g(z+c)$, and define the first-order difference and n-th-order difference operators by

$$\triangle_c g(z) = g(z+c) - g(z), \triangle_c^n g(z) = \triangle_c^{n-1}(\triangle_c g(z)).$$

As is widely known, if $g(z+t) = g(z)$ for all $z \in \mathbb{C}$, where $t \in \mathbb{C}$, and $t \neq 0$, then we call $g(z)$ a periodic function with period t. Obviously, a non-constant polynomial is not a periodic function, for example, $w(z) = z^n$ is not periodic. We call a complex non-zero number t the basic period or fundamental period of the function $g(z)$ if all periods of $g(z)$ are integer multiples of t. Let kw and lw be two periods of $g(z)$, then their ratio l/k must be rational. No nonconstant entire function has two periods for which the ratio is not rational ([39], pp. 69–70).

4. Problems Related to the Periodicity of Functions

The periodicity of functions is an essential area in the study of functions theory. Many mathematicians have made important contributions to this field. Yang's conjecture gave an impetus for research in this area. In this section, we survey some of these results to facilitate our analysis of how mathematicians formulate their mathematical problems by applying the strategies. If more than one strategy is used in posing a problem, we prioritize the most central strategy.

In the mathematical research, researchers are very interested in exploring whether the sufficient necessary conditions for a mathematical statement and the inverse problem of a given problem are still correct. For example, for the entire function, $f(x) = \sin x$, then $(\sin x)' = \cos x$, $(\sin x)'' = (\cos x)' = -\sin x$; therefore, we have $(\sin x)'' \sin x = -\sin^2 x$. In this direction, we guess that Titchmarsh, by symmetry and generalization, posed the following result ([40], pp. 263).

Theorem 1 ([40], pp. 263)**.** *The differential equation $f(u)f''(u) = -\sin^2 u$ has no real entire functions of a finite order except $f(u) = \pm \sin u$.*

Applying generalization on Theorem 1, Titchmarsh posed the following problem.

Problem 1 ([40], pp. 267)**.** *Find out the real transcendental entire solutions of*

$$f(z)f^{(k)}(z) = m(z)\sin^2 z,$$

where $m(z)$ is a non-zero polynomial. The real entire function $f(z)$ means that $f : \mathbb{R} \to \mathbb{R}$.

By generalization on Theorem 1, Li, Lü, and Yang [41] removed the condition that $f(z)$ is real and of a finite order, and obtained the following result.

Theorem 2 ([41])**.** *Suppose that $f(z)$ is an entire function satisfying*

$$f(z)f''(z) = m(z)\sin^2 z,$$

where $m(z)$ is a non-zero polynomial with real coefficients and real zeros, then m(z) must be a non-zero constant p and $f(z) = a\sin z$, where a is a constant satisfying $a^2 = -p$.

Applying generalization on Theorem 2 with $ff^{(k)}$ and $f = \alpha\sin z$, it is easy to pose the following problem.

Problem 2. *If f is a transcendental entire function, k is a positive integer. If f is a periodic function, then $ff^{(k)}$ is also periodic.*

Applying symmetry on Problem 2, it is easy to obtain Problem 3.

Problem 3 (Yang's Conjecture, 2018 [42,43])**.** *Let f be a transcendental entire function and k a positive integer. If $ff^{(k)}$ is a periodic function, then f is also periodic.*

From the point of view of differential equations, applying the association on Problem 3, it is easy to find that the conjecture is connected to the equation $f(z+c)f^{(k)}(z+c) = f(z)f^{(k)}(z)$, which involves difference and differential operators. Therefore, applying association, it is easy to pose the following.

Problem 4 ([44])**.** *Solve the differential equation*

$$f(z)f^{(k)}(z) = f(z+c)f^{(k)}(k+c),$$

where $f(z)f^{(k)}(z)$ is a periodic function with period c, and c is a non-zero constant.

By using specialization, it is easy to reduce Problem 4 to the following special case:

Problem 5. *Solve $f(z+c) = f'(z)$, where c is a non-zero constant.*

By using constraint manipulation (numerical variation) on Problem 5, it is easy to pose the following special case, for example.

Problem 6. *Solve $f(z+c) = f''(z)$, where c is a non-zero constant.*

Applying constraint manipulation (what-if-not) on Problem 3, most likely, Wang and Hu [42] created and proved the following.

Theorem 3 ([42])**.** *Suppose that f is a transcendental entire function and assume that k is a positive integer. If $(f^2)^{(k)}$ is a periodic function, then f is also.*

Applying specialization on Problem 3, Theorem 3 is degraded into Theorem 4.

Theorem 4 ([42])**.** *Assuming that $k = 1$, then $(f^2)' = 2ff'$, and f^2 is a periodic function, and then f is also. This is a special case of Yang's conjecture (Problem 3) as $k = 1$.*

Applying specialization on Problem 3, if f is a transcendental entire function with a nonzero Picard exceptional value, Liu and Yu [43] prove the following result:

Theorem 5 ([43])**.** *Let f be a transcendental entire function with a nonzero Picard exceptional value, and let k be a positive integer. Suppose that $ff^{(k)}$ is a periodic function, then f is also.*

Applying generalization on Theorem 3, it is easy to obtain the following results.

Theorem 6 ([43])**.** *Suppose that f is a transcendental entire function, k and n are positive integers and $n \geq 2$. Supposing that $(f^n)^{(k)}$ is a periodic function, then f is also periodic.*

Applying constraint manipulation (what-if-not) on Problem 3, it is easy to obtain the following.

Problem 7 ([43])**.** *Suppose that $f(z)$ is a transcendental entire function, $n \geq 2$, k is a positive integer, a_i are constants, and $a_n \neq 0$. Is f also a periodic function if $(a_n f^n + \cdots + a_1 f)^{(k)} (n \geq 2)$ is a periodic function?*

Applying specialization on Problem 7, it is easy to obtain the following theorem.

Theorem 7 ([43])**.** *Let f be a transcendental entire function and k a positive integer. Let a_1 and a_2 be two constants and $a_2 \neq 0$. If $(a_2 f^2 + a_1 f)^{(k)}$ is a periodic function, then f is also periodic.*

Applying constraint manipulation (what-if-not) on Problem 7, it is easy to obtain the following theorem.

Theorem 8 ([43])**.** *Assume that k and n are positive integers with $n \geq 2$. Let f be a transcendental entire function with $\rho_2(f) < 1$ and $N(r, 1/f) = S(r, f)$. Suppose that $(a_n f^n + \cdots + a_1 f)^{(k)}$ is a periodic function and $a \neq 0$, then f is also periodic.*

Applying generalization on Problem 3, the generalized Yang's conjecture (Problem 8) was posed by Liu, Wei, and Yu [45].

Problem 8 (Generalized Yang's conjecture [45])**.** *Suppose that f is a transcendental entire function and n, k are two positive integers. Suppose that $f^n(z) f^{(k)}(z)$ is a periodic function, then f is also periodic.*

By specialization, Liu, Wei, and Yu posed and proved the following special case of Problem 8:

Theorem 9 ([45])**.** *Suppose that $f(z)$ is a transcendental entire function and n, k are positive integers. Suppose that one of the following conditions holds:*
(i) $k = 1$;
(ii) $f(z) = e^{h(z)}$, where $h(z)$ is a non-constant polynomial;
(iii) $f(z)$ has a non-zero Picard exceptional value, $f(z)$ is of a finite order, and $f(z)^n f^{(k)}(z)$ is also a periodic function with period c;
then, $f(z)$ is also a periodic function.

Applying associations on Problem 8, Liu et al. posed the following problem.

Problem 9 ([45])**.** *Supposing that $f^n f^{(k)}$ and $f^n f^{(l)}$ are periodic functions, do they have the same period?*

Applying the specialization to Problem 9, it is easy to obtain the following.

Theorem 10 ([45])**.** *Suppose that $f(z)$ is a transcendental entire function and n, k are positive integers. Suppose that $f(z)^n f^{(k)}(z)$ and $f(z)^n f^{(k+1)}(z)$ are periodic functions with the same principal period c, then $f(z)$ is also a periodic function with period $c, 2c$ or $(n+1)c$.*

Applying the association on Problem 8, it is easy to obtain Theorem 11.

Theorem 11 ([45]). *Supposing that $f(z)$ is a transcendental entire function and $n \geq 2$, k is a positive integer. Suppose that $f^n + f^{(k)}$ is a periodic function with period c and one of the following conditions holds:*
(i) $k = 1$;
(ii) $f(z+c) - f(z)$ without zeros;
(iii) The zero multiplicity of $f(z+c) - f(z)$ is greater than or equal to k; then $f(z)$ is also a periodic function with period c or $2c$.

Applying the specialization on Problem 8, it is easy to obtain the following.

Theorem 12 ([46]). *Suppose that f is a transcendental entire function of hyper-order strictly smaller than 1, and n, k are positive integers. If $f(z)$ has a finite Borel exceptional value b, and $f(z)^n f^{(k)}(z)$ is a periodic function, then $f(z)$ is also a periodic function.*

Applying generalization on Theorem 11, it is easy to obtain the following theorem.

Theorem 13 ([46]). *Let f be a transcendental entire function of hyper-order strictly less than 1, and $n(\geq 2), k(\geq 1)$ be integers. If $f(z)^n + a_1 f'(z) + \cdots + a_k f^{(k)}(z)$ is a periodic function, where $a_1, \cdots, a_k$ are constants, then $f(z)$ is also a periodic function.*

Applying generalization on Problem 8, it is easy to obtain the following problems.

Problem 10 ([47]). *Suppose that $f(z)$ is a transcendental entire function. Assuming that $f(z)^n f^{(k)}(z+\eta)$ is a periodic function, then is $f(z)$ also a periodic function, where n, k are integers?*

In Theorem 10, the case of $\eta = 0$ is the generalized Yang's conjecture.
Applying the specialization $k = 0$ on Problem 10, it is easy to obtain the following.

Theorem 14 ([47]). *Suppose that $f(z)$ is a transcendental entire function with $\rho_2(f) < 1$, and $n \geq 2$ is a positive integer. Assume that $f(z)^n f(z+\eta)$ is a periodic function with period c, then $f(z)$ is a periodic function with period $(n+1)c$.*

When $n = 1$, suppose that $f(z+\eta)f(z)$ is periodic and the period $c = \eta$, then $f(z)$ is also periodic with period 2η. Furthermore, applying constraint manipulation (what-if-not) mainly on Theorem 14, it is easy to obtain the following problem.

Problem 11 ([47]). *Supposing that $f(z)f(z+\eta)$ is a periodic function with period $c(\neq \eta)$, then $f(z)$ is also a periodic function.*

Applying generalization on Theorem 14, it is easy to obtain the following problem:

Problem 12 ([47]). *Suppose that $f(z)$ is a transcendental entire function and n, k are positive integers. Supposing that $[f(z)^n f(z+\eta)]^{(k)}$ is a periodic function, does it follow that $f(z)$ is also a periodic function?*

Applying the specialization on Problem 12, it is easy to obtain the following.

Theorem 15 ([47]). *Suppose that $f(z)$ is a transcendental entire function with $\rho_2(f) < 1$ and $n \geq 4$ is a positive integer. Assume that $[f(z)f(z+\eta)]^{(k)}$ is a periodic function with period c, and then $f(z)$ is a periodic function with period $(n+1)c$.*

Applying generalization on Theorem 15, it is easy to obtain the following.

Theorem 16 ([47]). *Assume that $[f(z)^n f(z+\eta)]^{(k)}$ is a periodic function with period η. If $f(z)$ is a transcendental entire function of finite order and $n \geq 1$, then $f(z)$ is a periodic function with period $(n+1)\eta$. If $f(z)$ is a transcendental entire function of infinite order and $n=1, k=1$, then $f(z)$ is a periodic function with period 2η.*

Applying analogical reasoning on Theorem 16, it is easy to obtain the following.

Problem 13 ([47]). *Suppose that $f(z)$ is a transcendental entire function and $\Delta_\eta := f(z+\eta) - f(z)$. If $[f(z)^n \Delta_\eta f]^{(k)}$ is a periodic function, does it follow that $f(z)$ is also a periodic function?*

Applying specialization on Problem 13, it is easy to obtain the following.

Theorem 17 ([47]). *Suppose that $f(z)$ is a transcendental entire function with $\rho_2(f) < 1$ and $n \geq 5$ is a positive integer. If $[f(z)^n \Delta_\eta f]^{(k)}$ is a periodic function with period η, then $f(z)$ is also a periodic function with period $(n+1)\eta$.*

Applying generalization on Problem 8, it is easy to obtain the following problem.

Problem 14 ([48]). *Suppose that $f(z)$ is a transcendental meromorphic function, $n \in Z$, and $k \in N$. If $f(z)^n f^{(k)}(z)$ is a periodic function, does it follow that $f(z)$ is also a periodic function?*

Applying specialization on Problem 14, it is easy to obtain the following theorem.

Theorem 18 ([48]). *Let $f(z)$ be a transcendental meromorphic function and $n \in \mathbb{Z}$. Suppose that $f(z)^n f'(z)$ is a periodic function with period c. Then $f(z)$ is also periodic, or a function of a periodic function.*

Applying generalization on Theorem 18, it is easy to obtain the following problem.

Problem 15 ([48]). *Suppose that $f(z)$ is a transcendental meromorphic function, and $n \in Z$, $k \in N$. If $f(z)^n f^{(k)}(z)$ is a periodic function, does it follow that $f(z)$ is also periodic?*

Applying constraint manipulation (what-if-not) on Problem 3, it is easy to obtain Theorem 19.

Theorem 19 ([48]). *Suppose that f is a transcendental entire function with a Picard exceptional value 0. If $\frac{f''}{f}$ is a periodic function with period c, then $f(z) = e^{h_1(z)+Az/(2c)}$, where $h_1(z)$ is a periodic function with period $2c$, and A is a constant.*

Applying generalization on Theorem 19, it is easy to obtain the following.

Theorem 20 ([48]). *Suppose that f is a transcendental entire function with a Picard exceptional value $\neq 0$. If $f^{(k)}/f$ is a periodic function with period c, then $f(z)$ is a periodic function with period c.*

Applying generalization on Theorem 20, it is easy to obtain the following.

Problem 16 ([48]). *Suppose that f is a transcendental entire function and n, k are positive integers. If $\frac{f^{(k)}}{f^n}$ is a periodic function, then f is also periodic.*

Applying constraint manipulation (what-if-not) on Problem 16, it is easy to obtain Theorem 21.

Theorem 21 ([48])**.** *Suppose that $f(z)$ is a transcendental meromorphic function, $n \in \mathbb{Z}$, and $k \in \mathbb{N}$. Suppose that $\frac{f^{(k)}(z)}{f(z)^n}$ and $\frac{f^{(k+1)}(z)}{f(z)^n}$ are periodic functions with period c. Then, $f(z)$ is also periodic or a function of a periodic function.*

Applying constraint manipulation (numerical variation and what-if-not) on Theorem 3, it is easy to obtain the following.

Theorem 22 ([49])**.** *Suppose that f is a transcendental entire function with finite order, zero is a Picard exceptional value of f, and k is a positive integer. If $(f^2)^{(k)}$ is a periodic function with a period of c, then f is also a periodic function with a period of $2c$; furthermore, $f(z) = e^{az+b}$, where $a \neq 0, b$ are constants and $e^{2ac} = 1$.*

Theorem 22 gives the expression of the function f.

Applying constraint manipulation (what-if-not) on Theorem 22, replace $(f^2)^{(k)}$ with $f(f^2)^{(k)}$, and Theorem 22 is changed into the following.

Theorem 23 ([49])**.** *Suppose that f is a transcendental entire function of a finite order, zero is a Picard exceptional value of f, and k is a positive integer. If $f(f^2)^{(k)}$ is a periodic function with a period of c, then f is also a periodic function with a period of $3c$; furthermore, $f(z) = e^{az+b}$, where $a \neq 0, b$ are constants and $e^{3ac} = 1$.*

Applying constraint manipulation (what-if-not) on Theorem 23, it is easy to obtain the following theorem.

Theorem 24 ([49])**.** *Suppose that f is a transcendental entire function of a finite order, zero is a Picard exceptional value of f, and k is a positive integer. If $f(f^{(k)})^2$ is a periodic function with a period of c, then f is also a periodic function with a period of $3c$.*

Applying constraint manipulation (what-if-not) on Theorem 24, it is easy to obtain the following.

Theorem 25 ([49])**.** *Suppose that f is a transcendental entire function with finite order, zero is a Picard exceptional value of f, and k is a positive integer. If $Q(f) = f^{n_0}(f')^{n_1}(f'')^{n_2} \cdots (f^{(k)})^{n_k}$ is a periodic function with a period of c, then f is also periodic with a period of $(n_0 + n_1 + \cdots + n_k)c$.*

Note that if use the strategy of specialization, set $n_0 = 1$, $n_k = 2$, and $n_1 = \cdots = n_{k-1}$ vanishes. Then, Theorem 25 degrades to Theorem 24.

Suppose that $f(z)$ is a periodic function with period c, then $f^{(k)}$ is also a periodic function with the same period c. But the symmetric problem is not true; see $f(z) = e^{e^z} + z$ and $f(z) = \sin z + z$. Therefore, applying generalization on Theorem 8, Wei, Liu, and Liu [50] obtained the following.

Theorem 26 ([50])**.** *If f is a transcendental entire function, k, n are positive integers and $a_1, a_2, \cdots, a_n (\neq 0)$ are constants. If $(a_n f^n + \cdots + a_1 f)^{(k)}$ is a periodic function and $n \geq 2$, then f is also a periodic function.*

Applying specialization on Theorem 26, the following corollary is obtained immediately by operation of the polynomial $\frac{1}{n+m+1} f^{n+m+1} - \frac{1}{n+1} f^{n+1}$ in the above Theorem 26.

Theorem 27 ([50])**.** *Let f be a transcendental entire function. If $f^n(f^m - 1)f'$ is a periodic function, then f is also a periodic function.*

Applying generalization on Theorem 27, it is easy to obtain Theorem 28.

Theorem 28 ([50]). *Let f be a transcendental entire function with n, m, and k being positive integers. If $f^n(f^m - 1)f^{(k)}$ is a periodic function with period c, and one of the following is satisfied, then $f(z)$ is also a periodic function:*
(i) $f(z) = e^{h(z)}$, where $h(z)$ is an entire function;
(ii) $f(z)$ has a non-zero Picard exceptional value and $f(z)$ is of a finite order;
(iii) $f^n(f^m - 1)f^{(k+1)}$ is a periodic function with period c;

Applying association on Theorem 28, it is easy to obtain the following problem.

Problem 17 ([50]). *If f is a transcendental entire function, k is a positive integer. If $\frac{f^{(k)}}{f^n(f^m-1)}$ $(m, n \geq 1)$ is a periodic function, is it true that f is also a periodic function?*

Applying Problem 17, it is easy to obtain the following theorem.

Theorem 29 ([50]). *Let f is a transcendental entire function. If $\frac{f'}{f(f-1)}$ is a periodic function, then f is also periodic.*

Applying generalization on Problem 29, it is easy to obtain the following theorem.

Theorem 30 ([50]). *Let f be a transcendental entire function and n, m, k be positive integers. If $\frac{f^{(k)}}{f^n(f^m-1)}$ and $\frac{f^{(k+1)}}{f^n(f^m-1)}$ are periodic functions, then f is also periodic.*

Applying analogical reasoning, the paper [51] considered the difference version of Problem 3, then obtained Theorem 31.

Theorem 31 ([51]). *Let f be a transcendental entire function with finite order, and d be a Picard exceptional value of f. If $f(z)\Delta_c f(z)$ is a periodic function, then f is also.*

Applying analogical reasoning on Theorem 22, substituting $(f^2)^{(k)}$ to $\Delta_c(f^2)$, Deng and Yang [51] obtained the following.

Theorem 32 ([51]). *Let f be a transcendental entire function with finite order, and d be a Picard exceptional value of f. If $\Delta_c(f^2)$ is a periodic function, then f is also periodic.*

Theorems 31 and 32 indicate the connections of periodicity between a transcendental entire function and its differences. Furthermore, applying generalization, Ren and Dang [49] put out the following.

Theorem 33 ([49]). *Let f be a transcendental entire function of a finite order and d be a Picard exceptional value of f. If $f(z)\Delta_c^n f(z)$ is a periodic function, then f is also a periodic function.*

In this direction, applying constraint manipulation on Theorem 33, we would like to pose the following new problems for the readers regarding future research.

Problem 18. *Suppose that f is a transcendental entire function of a finite order and d is a Picard exceptional value of f. If $f'(z) + \Delta_c^n f(z)$ is a periodic function, then f is also periodic.*

Problem 19. *Let f be a transcendental entire function of a finite order and d be a Picard exceptional value of f. If $f'(z)\Delta_c^n f(z)$ is a periodic function, then f is also a periodic function.*

Problem 20. *Let f be a transcendental entire function with finite order and d be a Picard exceptional value of f. If $\frac{\Delta_c^n f(z)}{f(z)}$ is a periodic function, then f is also periodic.*

Applying analogical reasoning on Problem 4, we obtain the following new problem for the readers regarding future research.

Problem 21. *Solve the following difference equation:*

$$f(z)\Delta_c^n f(z) = f(z+\eta)\Delta_c^n f(z+\eta),$$

where $f(z)\Delta_c^n f(z)$ is a periodic function with period η.

5. Results

There are many published papers on the study of the periodicity of entire functions and Yang's conjecture, and we analyzed only a few of them. In total, there are 54 problems or theorems collected in the current research. The mathematicians may use more than one strategy when posing their problems. However, we assume that they usually use one main strategy when creating new problems or theorems. Then, the following Table 1 is obtained.

Table 1. The statistics of used strategies.

Strategies [1]	Times
symmetry	2
constraint manipulation	15
goal manipulation	0
targeting a particular solution	0
generalization	17
specialization	12
association	4
analogical reasoning	4
Total	54

[1] We treat the authors posing multiple theorems and problems in one paper as though the authors are using the chaining strategy once. So, we omit chaining here.

The results show that mathematicians often use constraint manipulation, generalization, and specialization to make new problems of mathematical science, and association, analogical reasoning, and symmetry are less used. Generalization being used more frequently is due to mathematicians' proclivity to generalize existing results to more general situations, giving proof if they are correct and counterexamples if they are not. The constraint manipulation is used many times when mathematicians solve difficult problems in mathematics, such as Yang's conjecture in this paper. This is because, in solving difficult problems, mathematicians often have to restrict the situations to make the problem specific and easier to solve.

The number of times that goal manipulation and targeting a particular solution are used here is zero. Actually, it is not surprising because the two strategies are often used when students learn math and take exams. Teachers use them to teach students to pose more math problems, solve a series of exercises, or acquire mathematics knowledge.

6. Examples

Today, more and more mathematics teachers are recognizing that integrating problem posing into their teaching has many advantages, but they also encounter many difficulties, such as a lack of confidence in implementing problem posing, a lack of knowledge about how to create high-level problems, and a lack of knowledge about how to respond to the challenges that arise in problem-driven lessons [5]. For students, they also face challenges, such as whether they can adapt to being integrated into the classroom with problem posing, whether they can pose high-quality mathematical problems, and whether problem posing will have a proactive impact on their academic performance.

Most recently, to facilitate the easy integration of problem posing into mathematics instruction, Dang et al. developed a model of problem-driven teaching (PDT) [5]. The model includes three stages: "(1) The preparation of the problems (generating new problems based on textbooks; generating new problems based on mathematical, scientific, and life situations; and imagining the solutions of the prepared problems; (2) The implementation of teaching (teachers teaching based on prepared problems and posing new problems in real-time, and students solving them or posing new problems); (3) The evaluation and reflection stage (evaluating the quality of teaching and the quality of problems, improving the instruction)".

We divided the problems into two categories: mathematical and non-mathematical problems. In the first stage, mathematical teachers should prepare for the given situations of problems or the given problems according to the teaching goal of a lesson using the model of PDT. We suggest that mathematics teachers obtain problems or situations of problems from mathematics textbooks, instructional materials, exercises, examinations, or other related sources. Teachers need to transform the existing problems in these materials into problem-posing tasks designed to integrate math instruction based on instructional goals. Importantly, teachers should be able to foresee that in prepared problem-posing tasks, teachers and students should be able to pose problems using the strategies or techniques often used by mathematicians as mentioned above.

In the second stage, the math teacher begins to implement classroom instruction. The math teacher presents a problem-posing task (including a given problem or problem situation). The teacher then guides students to generate problems and pose problems. This is a good time to use the strategies mentioned above, which are often used by mathematicians. The teacher evaluates the problem posing, guides students to improve the problem, and then selects high-quality problems for problem solving based on the instructional goals. Problem solving can be performed by the teacher, by the students, or jointly. There may be more than one problem-posing task in a lesson. Students will acquire math knowledge and achieve math understanding through the interweaving of problem posing and problem solving.

Next, we give some cases of problem posing.

6.1. The Quadratic Equations and Roots

This case demonstrates how math exercises from instructional materials can be transformed into a class of problem posing. The initial problem is as follows:

Problem A. *Find the range of values of m when a quadratic equation $(m-1)x^2 + 2(m+1) - m = 0$ with respect to x is assumed to satisfy the following conditions, respectively:*

1. *Both roots are greater than* 0.
2. *One root is greater than* -1 *and the other is less than* -1.
3. *One root is in* $(1, 2)$, *the other in* $(-1, 0)$.
4. *One root is in* $(-1, 1)$ *and the other is not in* $(-1, 1)$.
5. *One root is less than* 1 *and the other is greater than* 2.
6. *Both roots are in* $[-1, 3)$.
7. *Both roots are less than* 1.
8. *There is a root in* $(1, 2)$.
9. *Both roots are in* $(1, 2)$.

The problem examines the distribution of roots of quadratic equations. Students have previously studied knowledge, such as Veda's theorem, quadratic functions, and quadratic inequalities. The learning goal of this lesson is to understand the distribution of roots of quadratic equations in order to achieve a deep understanding of quadratic equations. Knowledge about Veda's theorem, the formula for finding the roots of a quadratic equation, the image of a quadratic function, and fractional inequalities, is used in solving these problems.

However, in the actual process of teaching mathematics, teachers often do not integrate problem posing into teaching but directly show the behavior of the distribution of the roots of quadratic equations and the relevant theorems, and students have significant obstacles and confusion in understanding and memorizing this knowledge. This way of teaching and learning is very direct, memorized and force fed. It makes it difficult for students to understand and memorize the knowledge and techniques, theorems and conclusions. The learning effect of the students and the effectiveness of the teacher may not be high, and it is difficult to achieve the intended teaching goals.

Therefore, we suggest that teachers begin their instruction by showing a simple distribution of Problem B in front of Problem A as follows.

Problem B. *Suppose that the two real roots of a quadratic equation* $ax^2 + bx + c = 0$, $(a \neq 0)$ *are* x_1, x_2, *and* $x_1 < x_2$.

(The teacher anticipates the following sub-problems, which the students pose according to the given situation, aiming to explore the sufficient and necessary conditions. Let k, k_1, k_2 *be constants,* $k_1 < k_2$.*)*

1. *Both roots are positive,* $x_1 > 0, x_2 > 0$.
2. *Both roots are negative,* $x_1 < 0, x_2 < 0$.
3. *Both roots are greater than k,* $x_1 > k, x_2 > k$.
4. *Both roots are smaller than k,* $x_1 < k, x_2 < k$.
5. *One root is smaller than k and the other is greater than k,* $x_1 < k < x_2$.
6. *If and only if* $k_1 < x_1 < k_2$.
7. *If and only if* $k_1 < x_2 < k_2$.
8. *There is and only one root in* (k_1, k_2).
9. *Both roots are in* (k_1, k_2).

In Problem B, the nine problems do not need to be presented by the teacher all at once. Instead, students are required to identify the problems themselves on the basis of the problem situation and the sample problem given by the teacher (e.g., the teacher gives only the sub-problem 1 "Both roots are positive"). Then, the students pose problems that may be contained within the scope of these nine problems or maybe other problems that are out of scope. At this point, the teacher needs to consider how to evaluate the students' problems posing, explain how these strategies work well and, further, select some high-quality problems for problem solving.

In Problem B, we can see that applying constraint manipulation (what-if-not, if it is not 0) on 1 will result in 2, applying generalization (from 0 to k) on 1 will result in 3, applying generalization (from 0 to k) on 2 will result in 4, and applying constraint manipulation (what-if-not if it is not k, $x_1 > k, x_2 > k$, but others) on 3 and 4 will result in 5 to 9. Teachers need to consider how to help students practice using these strategies in their problem-posing tasks.

After solving the basic Problem B, the teacher is ready to lead their students to posing and solving Problem A. This kind of pedagogical process, which imitates mathematicians' process of identifying, problem posing, and problem solving, can facilitate good action for students' mathematical understanding.

6.2. Linear Programming

We are going to give another example of how a math exercise can be transformed into a problem-posing task and what problem-posing strategies might be used. The exercise is as follows.

Problem C. *Consider that* x, y *satisfy the following constraint:*

$$\begin{cases} x \geq 1 \\ x - 3y + 4 \leq 0 \\ 3x + 5y - 30 \leq 30. \end{cases}$$

1. *Calculate the minimum and maximum values of the function* $z = 2x - y$.
2. *Calculate the minimum and maximum values of the function* $z = 2x + y$.
3. *If there are infinitely many optimal solutions for the function* $z = ax + y$ *to maximize, find the value of the real number a.*
4. *Calculate the range of the function* $z = \frac{y+5}{x+5}$.
5. *Calculate the range of the function* $z = x^2 + y^2$.
6. *Calculate the range of the function* $z = |x + y + 1|$.

During instruction, it is recommended that math teachers show students only the problem situation and the first sub-problem, and not the last five problems that follow. When students pose problems with the situation and the sample sub-problems, teachers can encourage students to use the strategies on the basis of the given information and the given problems.

However, in order to accomplish the teaching and learning goals, teachers should remind students of the following six points. First, the problem posing should focus on the learning content of the lesson (e.g., viable domain for linear programming, optimal solutions, and ranges of relevant expressions). Second, students are encouraged to use the frequent problem-posing strategies mentioned above, such as constraint manipulation, generalization, specialization, and so on. Third, when students have difficulties in problem posing, teachers should give more hints and examples. Fourth, evaluate the quality by comparing students' products with the original problems in the exercise and give them positive feedback. Fifth, solve the problems in the exercise. Sixth, select some novel problems posed by students for problem solving. The following are some problems that students may pose:

1. *Calculate the minimum and maximum values of the function* $z = 2x - 2y$. *(Constraint manipulation.)*
2. *Calculate the minimum and maximum values of the function* $z = 2x + 3y$. *(Constraint manipulation.)*
3. *If there are infinitely many optimal solutions for the function* $z = x + by$ *to maximize, find the value of the real number b. (Constraint manipulation and goal manipulation.)*
4. *If there are infinitely many optimal solutions for the function* $z = x + ay$ *to minimize, find the value of the real number a. (Constraint manipulation.)*
5. *Calculate the range of the function* $z = \frac{y+1}{x+2}$. *(Constraint manipulation.)*
6. *Calculate the range of the function* $z = |x + 2y + 1|$. *(Constraint manipulation.)*
7. *Calculate the area of the variable domain. (Association.)*

After the given problem has been solved, mathematical problems can also be posed by students according to the solutions or their experience in the processes of problem solving. For example, the answer to sub-problem 1 is $z_{max} = 7, z_{min} = -\frac{17}{5}$. Teachers can suggest the following prompt:

> If we make use of the symmetry strategy and switch the knowns and unknowns of the sub-problem 1, will we get a new problem?

In other words, teachers and students need to reorganize the following statements to generate new mathematical problems (for example, see Problem D and Problem E) through strategies such as symmetry, constraint manipulation, goal manipulation.

Given that x, y *satisfy the constraint that*

$$\begin{cases} x \geq\ 1 \\ x - 3y + 4 \leq\ 0 \\ 3x + 5y - 30 \leq 0. \end{cases}$$

the maximum and minimum values of the objective function $z = 2x - y$ *are* $z_{max} = 7$, $z_{min} = -\frac{17}{5}$.

Problem D. *Given that x, y satisfy the constraint that*

$$\begin{cases} x \geq\ k \\ x - 3y + 4 \leq\ 0 \\ 3x + 5y - 30 \leq 0. \end{cases}$$

the maximum and minimum values of the objective function $z = 2x - y$ are $z_{max} = 7$, $z_{min} = -\frac{17}{5}$. Then, find the value (scope) of k.

Problem E. *Given that x, y satisfy the constraint that*

$$\begin{cases} x \geq\ 1 \\ x - 3y + 4 \leq\ 0 \\ 3x + 5y - 30 \leq 0. \end{cases}$$

find the integer solutions (x, y).

The teacher encourages students that they can go along with the idea of posing more problems. Further, after training in problem posing, the level and complexity of the students' mathematical products will increase.

6.3. The Tangent

We intend to distill a case of problem posing from a Chinese high school mathematics textbook in order to show teachers and students how a simple mathematical formula can be expanded into a more meaningful one through utilizing the strategies. By simply using the strategies, the original problem is expanded into a more complex one. Therefore, the process involved in mathematics problem posing is also a kind of experience of discovering mathematical principles. It is the process by which students improve their mathematical performance, and mathematical thinking develops and evolves.

As we all know, for each high school student, the following formula is very familiar:

$$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}.$$

Then, applying specialization, we can obtain

$$\tan(1^\circ + 44^\circ) = \frac{\tan 1^\circ + \tan 44^\circ}{1 - \tan 1^\circ \tan 44^\circ}.$$

So, what is the relationship between $\tan 1^\circ + \tan 44^\circ$ and $\tan 1^\circ \tan 44^\circ$?

Applying constraint manipulation (numerical variation), we can pose another problem: what is the relationship between $\tan(-1^\circ) + \tan 46^\circ$ and $\tan(-1^\circ) \tan 46^\circ$?

Applying generalization, we can pose the following problem: if $a + b = \frac{\pi}{2}$, then $\tan a + \tan b + \tan a \tan b =?$

Applying constraint manipulation (numerical variation), we can pose the following problem: if $a + b = \frac{\pi}{2}$, then $(1 + \tan a)(1 + \tan b) =?$

Applying generalization, we can pose the following problem:

$$(1 + \tan 1^\circ)(1 + \tan 2^\circ) \cdots (1 + \tan 43^\circ)(1 + \tan 44^\circ) =?$$

Teachers can continue with the following question: can students pose any other relevant problems?

Consider this one:

$$(1 + \tan -1^\circ)(1 + \tan -2^\circ) \cdots (1 + \tan 47^\circ)(1 + \tan 46^\circ) =?$$

To high school students, these questions seem very simple, but they are much more interesting. If students are trained in problem-posing strategies, they will be capable of posing these or similar questions when they are faced with this problem-posing task. Perhaps they will be challenged to pose high-quality problems. With positive encouragement and guidance from the teacher, every student will be able to come up with good math problems of their own. As a result, every student in the problem posing classroom will be able to develop accordingly.

7. Discussions and Limitations

7.1. Discussions

We investigate which strategies are frequently used by mathematicians in their works. According to the opening literature on problem posing that we analyzed, there are relatively very few studies on problem formulation for mathematicians, so our results fill the gap. At the micro level, the results of the study show how mathematicians apply these strategies to pose complex mathematical problems and solve them, even mathematical conjectures. In particular, at the end of Section 4, the authors of this paper also posed four interesting mathematical problems that remain to be solved.

These 54 theorems and problems reflect well the progress in the study of the periodicity and Yang's conjecture, as well as the path of research. The strategies used here are also positive for solving other mathematics problems.

At the same time, the results also provide some help and real examples of effective mathematical problem solving and even mathematical conjecture solving. This is because problem posing as a component of problem solving can effectively contribute to problem solving under the framework of Polya's problem-solving technique.

Cai and Hwang (2020) [25] put forward two key questions: Are teachers equipped to pose important and valuable math problems? How can teachers be supported in posing important and valuable math problems? In Section 4, we pose four new pure mathematics problems on the periodicity of entire functions. In Section 6, by reformulating and restructuring pre-existing problem situations or pre-existing problems that are slightly randomly selected in the math textbooks or exercises, new math problems are posed. Therefore, our results give partial answers to the questions asked by Cai and Hwang. Also, we give some classical examples of transforming "fixed" teaching resources into "active" problem-posing tasks in math teaching.

There is a great need for research on how to encourage secondary school students to imitate mathematicians in posing worthwhile problems, and how to integrate mathematical problem posing in instruction using these strategies. Subsequent research may therefore focus on how to foster primary and secondary school students to raise good mathematical problems in math classroom—just like mathematicians do.

7.2. Limitations

There are some limitations of the current research. First, since there is a large body of literature on the topic of the periodicity of complex functions, this paper analyzes only some of them. Second, the chaining strategy is more often used to prove theorems and solve problems. Therefore, we omit it here. Third, in many cases, we do not know what is really in the mind of the mathematician at the time of posing these mathematical problems or theorems. Therefore, when analyzing the strategies used to formulate these problems, we determine them based on the mathematical expressions, conditions, and conclusions in the problems or theorems. Four, although some researchers have shown cases of problem posing in statistics courses, for instance, see [5], few case studies have integrated problem posing into the teaching of geometry, probability, or other topics. Therefore, future research should cover a wider range of cases, or extend problem posing to university math teaching. This paper attempts to provide a case study research paradigm for studying how mathematicians formulate and pose new mathematical problems and gives some interesting examples of teaching with problem posing. Therefore, this is a pioneering study that integrates the mathematical problem posing of mathematicians with the mathematical problem posing by elementary and secondary school math teachers and students, which is of great importance in the field of mathematical problem posing.

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