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The Cross-Intersecting Family of Certain Permutation Groups

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Abstract: Two subsets X and Y of a permutation group G acting on Ω are cross-intersecting if for every $x \in X$ and every $y \in Y$ there exists some point $\alpha \in \Omega$ such that $\alpha^x = \alpha^y$. Based on several observations made on the cross-independent version of Hoffman's theorem, we characterize in this paper the cross-intersecting families of certain permutation groups. Our proof uses a Cayley graph on a permutation subgroup with respect to the derangement. By carefully analyzing the cross-independent version of Hoffman's theorem, we obtain a useful theorem to consider cross-intersecting subsets of certain kinds of permutation subgroups, such as $\mathrm{PGL}(2,q)$, $\mathrm{PSL}(2,q)$ and S_n.

Keywords: cross-intersecting family; permutation group; point-stabilizer

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1. Introduction

Let $[n] = \{1,2,3,\cdots,n\}$. For two permutations $\phi, \varphi \in \mathrm{S}_n$, we say ϕ and φ intersect if $\phi(i) = \varphi(i)$ for some $i \in [n]$. Similarly, they are said to k-intersect if $\phi(i_t) = \varphi(i_t)$ for $t = 1,2,\cdots,k$, where $i_1, i_2, \cdots, i_k \in [n]$. A subset $A \subset \mathrm{S}_n$ is said to be k-intersecting if any two permutations in A k-intersect. Similarly, two subsets $A, B \subset \mathrm{S}_n$ are said to be k-cross-intersecting if any permutation in A and any permutation in B k-intersect. Specifically, 1-cross-intersecting is called cross-intersecting for short.

The Erdős–Ko–Rado theorem [1] is a fundamental theorem in extremal set theory. Many similar theorems have been proved for other mathematical objects. We can briefly understand them by consulting with [2–5]. A version of the EKR theorem for permutations stems from [6], in which Frankl and Deza proved that the maximal size of intersecting families of permutations is $(n-1)!$. They conjectured that for any $n \in \mathbb{N}$, the 1-coset is the only intersecting subsets of S_n with the size $(n-1)!$, and if n is large enough depending on k, the k-coset is the only largest k-intersecting subsets of S_n. The first part was solved by Cameron and Ku [5], and there are some other proofs such as [7,8]. D. Ellis, Friedgut and H. Pilpel [9] proved the second conjecture, in which they also have a similar result concerning k-cross-intersecting subsets of S_n. In recent years, considering the intersecting families of the subgroups of S_n and cross-intersecting family in other mathematical objects have result in much attention. See [10–14] for versions of intersecting family for A_n, $\mathrm{GL}(n,q)$, $\mathrm{PGL}(2,q)$, $\mathrm{PSL}(2,q)$ and so on. See [15–17] for a version of a cross-intersecting family. In the past two years, some relevant comprehensive articles have also been published, see [18,19].

The object of this paper is to establish the relationship between intersecting families and cross-intersecting families of certain kinds of permutation subgroups, which is a great help when considering cross-intersecting families. Our proof uses a Cayley graph on a permutation subgroup with respect to the derangement, which appears in [5]. By carefully analyzing the cross-independent version of Hoffman's theorem [9], we obtain a useful theorem to consider cross-intersecting subsets of a certain kind of permutation subgroups, such as $\mathrm{PGL}(2,q)$, $\mathrm{PSL}(2,q)$ and S_n.

2. Preliminary Results

In this section, we present some preliminary results, which will be helpful for our main results. Let G be a group and S a subset of G which does not contain the identity element 1.

The Cayley digraph $\mathrm{Cay}(G,S)$ (on G with respect to S) is defined on G such that x is adjacent to y if and only if $yx^{-1} \in S$. It is easily shown that $\mathrm{Cay}(G,S)$ is connected if and only if $G = \langle S \rangle$, that is, S is a generating set of the underlying group G. If $S = S^{-1} := \{s^{-1} \mid s \in S\}$ the $\mathrm{Cay}(G,S)$ may be viewed as a graph by identifying every two opposite directed edges as an edge; the resulting graph is called a Cayley graph of G and is also denoted by $\mathrm{Cay}(G,S)$. For each $g \in G$, define

$$R(g) : G \to G,\ x \mapsto xg,\ \forall x \in G.$$

It follows that, $R(g) \in \mathrm{Aut}\Gamma$, and $R(G) = \{R(g) | g \in G\}$ is a regular subgroup of $\mathrm{Aut}\Gamma$ isomorphic to G, which yields that Γ is a vertex-transitive.

For a permutation subgroup G, we construct a Cayley graph $\mathrm{Cay}(G,S)$ where S is all the derangements of the group G, and this graph, denoted by Γ_G for short, is called the derangement graph of the group G. Then, the characterization of a cross-intersecting family of permutation groups equals the characterization of two subsets of the derangement graph for which there are no edges between them.

Theorem 1 ([10])**.** *Every independent set M of the derangement graph of* $\mathrm{PGL}(2,q)$ *acting on the projective line $\mathbb{P}_q$ has size at most $q(q-1)$. The bound is attained if and only if M is a coset of the stabilizer of a point.*

Theorem 2 ([11])**.** *Every independent set M of the derangement graph of* $\mathrm{PSL}(2,q)$ *where q is even, acting on the projective line $\mathbb{P}_q$ has size at most $q(q-1)$. The bound is attained if and only if M is the coset of the stabilizer of a point.*

Theorem 3 ([5])**.** *Let $n \geq 2$ and $M \subset \mathrm{S}_n$ be an intersecting set of permutations such that $|M| = (n-1)!$. Then M is a coset of a stabilizer of one point.*

The eigenvalues of the derangement graph of $\mathrm{PGL}(2,q)$, $\mathrm{PSL}(2,q)$ and S_n are also needed for our proof. Then we present the eigenvalues of these derangement graphs as follows.

Theorem 4 ([10])**.** *1. If q is even, then the spectrum of $\Gamma_{\mathrm{PGL}(2,q)}$ is*

$$\begin{pmatrix} \frac{q^2(q-1)}{2} & \frac{-q(q-1)}{2} & q & 0 \\ 1 & q^2 & \frac{q(q-1)^2}{2} & \frac{(q+1)^2(q-2)}{2} \end{pmatrix}$$

2. If q is odd, then the spectrum of $\Gamma_{\mathrm{PGL}(2,q)}$ is

$$\begin{pmatrix} \frac{q^2(q-1)}{2} & \frac{-q(q-1)}{2} & \frac{q-1}{2} & q & 0 \\ 1 & q^2+1 & q^2 & \frac{(q-1)^3}{2} & \frac{(q+1)^2(q-3)}{2} \end{pmatrix}$$

Theorem 5 ([11])**.** *1. If q is even, then the spectrum of $\Gamma_{\mathrm{PSL}(2,q)}$ is*

$$\begin{pmatrix} \frac{q^2(q-1)}{2} & \frac{-q(q-1)}{2} & q & 0 \\ 1 & q^2 & \frac{q(q-1)^2}{2} & \frac{(q+1)^2(q-2)}{2} \end{pmatrix}$$

2. If q is odd, then the spectrum of $\Gamma_{\mathrm{PSL}(2,q)}$ is

$$\begin{pmatrix} \frac{q(q-1)^2}{4} & \frac{-(q-1)^2}{4} & q & 0 \\ 1 & q^2 & \frac{(q-1)^3}{4} & \frac{(q+1)^2(q-3)}{4} \end{pmatrix}$$

Theorem 6 ([9,20])**.** *The second largest absolute eigenvalue is the smallest eigenvalue of the derangement graph of* Γ_{S_n}*, where* $n \geq 5$*. And the smallest eigenvalue is given by* $\eta = \frac{-d_n}{n-1}$*, where* d_n *is the number of the derangements of* S_n*.*

In the following section, we show the main result of this paper. Let G be one of the following permutation groups: S_n (with $n \geq 5$) acts on $\{1,2,3,\cdots,n\}$, and $\mathrm{PGL}(2,q)$ (with $q \geq 4$) acts on the projective line $\mathrm{PG}(1,q)$. Let H be a point-stabilizer in G. Then, it is proven that if X and Y are cross-intersecting in G then $|X||Y| \leq |H|^2$, and the equality holds if and only if $X = Y$ is a right coset of some conjugation of H in G. For S_n, our proof is different from [9] presented by Ellis, Friedgut and Pilpel.

3. Main Results and Discussions

In this section, we show the main result of this paper. As our results mainly concerned the relationship of intersecting families and cross-intersecting families of a certain kind of permutation subgroups, we will first give some information about The Erdős–Ko–Rado Theorem.

Let $\binom{[n]}{k}$ denote the collection of all k-subsets of $[n] = \{1,2,3\cdots,n\}$. A family $\mathcal{A} \subseteq \binom{[n]}{k}$ is called intersecting if $A \cap B \neq \emptyset$ for all A, $B \in \mathcal{A}$, and two families $\mathcal{A}$, $\mathcal{B} \subseteq \binom{[n]}{k}$ are called cross-intersecting if $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$. The Erdős–Ko–Rado Theorem 1 says that for $n \geq 2k$ if a family $\mathcal{A} \subseteq \binom{[n]}{k}$ is intersecting then $|\mathcal{A}| \leq \binom{n-1}{k-1}$, and the equality holds for $n > 2k$ if and only if $\mathcal{A}$ consists of all the k-subsets containing one fixed element. Pyber gave a generalization of this result for cross-intersecting families.

Theorem 7 ([21,22])**.** *Let* $n \geq 2k$*. If* $\mathcal{A}$*,* $\mathcal{B} \subseteq \binom{[n]}{k}$ *are cross-intersecting then* $|\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{k-1}^2$*, and the equality holds for* $n > 2k$ *if and only if* $\mathcal{A} = \mathcal{B}$ *consists of all the k-subsets containing one fixed element.*

Theorem 7 has a natural extension to permutations. In this paper, we are concerned with analogues of the above result for certain permutation groups. Let G be a permutation group on a finite set Ω, that is, G is a subgroup of the symmetric group $\mathrm{Sym}(\Omega)$. A point-stabilizer in G is the subgroup fixes some given point $\alpha \in \Omega$. Two subsets X and Y of G are called cross-intersecting if for every $x \in X$ and every $y \in Y$ there exists some point $\alpha \in \Omega$ such that $\alpha^x = \alpha^y$.

Leader [23] conjectured that for $|\Omega| \geq 4$ if X, $Y \subseteq \mathrm{Sym}(\Omega)$ are cross-intersecting then $|X||Y| \leq ((|\Omega| - 1)!)^2$. Ellis, Friedgut and Pilpel [9] proved this conjecture. Note that all point-stabilizers in $\mathrm{Sym}(\Omega)$ are conjugates; in particular, they have the same order $(|\Omega| - 1)!$. Then we formulate the following result from (Theorem 4 in [9]).

Theorem 8. *Let H be a point-stabilizer in* $G = \mathrm{Sym}(\Omega)$*. For* $|\Omega| \geq 5$*, if* X*,* $Y \subseteq G$ *are cross-intersecting then* $|X||Y| \leq |H|^2$*. Equality holds if and only if* $X = Y$ *is a right coset of some conjugation of H in G.*

In this paper, we are concerned with analogues of the above result for a certain kind of permutation groups. Our main results are stated as follows.

Theorem 9. *Let* $G = \mathrm{PGL}(2,q)$ *act on the projective line* $\mathrm{PG}(1,q)$*, and let H be a point-stabilizer in G. Assume that* X*,* $Y \subseteq \mathrm{PGL}(2,q)$ *are cross-intersecting.*

1. *If* $q > 2$ *then* $|X||Y| \leq |H|^2 = q^2(q-1)^2$*. Equality holds for* $q \geq 4$ *if and only if* $X = Y$ *is a coset of some conjugation of H in G.*
2. *If X and Y are cross-intersecting in* $\mathrm{PGL}(2,q)$ *where* $q \neq 2$*, then* $|X||Y| \leq q^2(q-1)^2$*. For* $q \geq 4$*, the bound is attained if and only if* $X = Y$ *is a coset of the stabilizer of a point of* $\mathbb{P}_q$*.*

Theorem 10. *Let $G = \mathrm{PSL}(2,q)$ act on the projective line $\mathrm{PG}(1,q)$, and let H be a point-stabilizer in G. Assume that X, $Y \subseteq \mathrm{PSL}(2,q)$ are cross-intersecting. If q is odd and $q \geq 7$ then $|X||Y| \leq |H|^2 = \frac{q^2(q-1)^2}{4}$, and the equality yields that $X = Y$.*

It was conjectured in [10] that, for $\mathrm{PSL}(2,q)$ acting on $\mathrm{PG}(1,q)$, only the cosets of point-stabilizers are the biggest intersecting families. Thus, we have the following conjecture.

Conjecture 1. *Let $G = \mathrm{PSL}(2,q)$ act on $\mathrm{PG}(1,q)$, and let H be a point-stabilizer in G, where q is odd and $q \geq 7$. Assume that $X, Y \subset \mathrm{PSL}(2,q)$ are cross-intersecting. If $|X||Y| = \frac{q^2(q-1)^2}{4}$ then $X = Y$ is a coset of some conjugation of H in G.*

4. Comparison and Proof

In this section, we give the proof of main theorems. For convenience, we restate the cross-independent version of Hoffman's theorem as follows, which bounds the maximum possible size of the cross-independent set in a d-regular graph.

4.1. The Cross-Independent Set of a d-Regular Graph

Let $\Gamma = (V,E)$ be a d-regular graph of order m, where $d \geq 1$. Consider the space $\mathbb{R}[V]$ of real-valued functions on V. Define an inner product

$$< \mathbf{v}, \mathbf{w} >= \frac{1}{|V|} \sum_{\alpha \in V} \mathbf{v}(\alpha)\mathbf{w}(\alpha).$$

This induces the following Euclidean norm:

$$||\mathbf{v}||_2 = \sqrt{< \mathbf{v}, \mathbf{v} >} = \sqrt{\frac{1}{|V|}} \sqrt{\sum_{\alpha \in V} \mathbf{v}(\alpha)^2}.$$

For a subset $X \subseteq V$, denote by $\mathbf{1}_X$ the characteristic function of X. Take an orthonormal system

$$\{\mathbf{1}_V = \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m\}$$

of eigenvectors of Γ, with corresponding eigenvalues

$$d = \lambda_1, \lambda_2, \cdots, \lambda_m$$

ordered by descending absolute value.

Let $\lambda_{\min}$ be the minimum eigenvalue of Γ. Then $|\lambda_2| \geq -\lambda_{\min}$. Let I be an independent set of Γ. Then, by (Theorem 11 in [9]),

$$|I| \leq \frac{-\lambda_{\min}}{d - \lambda_{\min}} |V| \leq \frac{|\lambda_2|}{d + |\lambda_2|} |V|, \tag{1}$$

and the first equality implies that $\mathbf{1}_I - \frac{|I|}{|V|}\mathbf{1}_V$ is an eigenvector of Γ with corresponding eigenvalue $\lambda_{\min}$.

Take subsets X, $Y \subseteq V$ such that there are no edges of Γ between X and Y. Write

$$\mathbf{1}_X = \sum_{i=1}^{m} a_i \mathbf{v}_i, \ \ \mathbf{1}_Y = \sum_{i=1}^{m} b_i \mathbf{v}_i.$$

It follows that,

$$a_1 =< \mathbf{1}_X, \mathbf{v}_1 >= \frac{|X|}{|V|}, \ \ b_1 =< \mathbf{1}_Y, \mathbf{v}_1 >= \frac{|Y|}{|V|}.$$

Let A be the adjacency matrix of Γ. Note that $\mathbf{1}_X^T A \mathbf{1}_Y$ is equal to the number of edges of Γ between M and N. Then,

$$0 = \mathbf{1}_X^T A \mathbf{1}_Y = |V| \sum_{i=1}^{m} \lambda_i a_i b_i = |V|(da_1 b_1 + \sum_{i=2}^{m} \lambda_i a_i b_i). \quad (2)$$

Note that

$$\sum_{i=2}^{m} \lambda_i a_i b_i = < \sum_{i=2}^{m} \lambda_i a_i \mathbf{v}_i, \sum_{i=2}^{m} b_i \mathbf{v}_i > .$$

By the Cauchy–Schwarz inequality,

$$|\sum_{i=2}^{m} \lambda_i a_i b_i| \le ||\sum_{i=2}^{m} \lambda_i a_i \mathbf{v}_i||_2 \cdot ||\sum_{i=2}^{m} b_i \mathbf{v}_i||_2 = \sqrt{\sum_{i=2}^{m} \lambda_i^2 a_i^2} \sqrt{\sum_{i=2}^{m} b_i^2}. \quad (3)$$

Equality holds if and only if $\sum_{i=2}^{m} \lambda_i a_i \mathbf{v}_i$ and $\sum_{i=2}^{m} b_i \mathbf{v}_i$ are parallel vectors, that is, there is a non-zero real number c such that $b_i = c\lambda_i a_i$ for $2 \le i \le m$.

By (2) and (3), we have

$$a_1 b_1 = -\frac{1}{d} \sum_{i=2}^{m} \lambda_i a_i b_i = \frac{1}{d} |\sum_{i=2}^{m} \lambda_i a_i b_i| \le \frac{1}{d} \sqrt{\sum_{i=2}^{m} \lambda_i^2 a_i^2} \sqrt{\sum_{i=2}^{m} b_i^2}. \quad (4)$$

Similarly,

$$a_1 b_1 \le \frac{1}{d} \sqrt{\sum_{i=2}^{m} a_i^2} \sqrt{\sum_{i=2}^{m} \lambda_i^2 b_i^2}. \quad (5)$$

Equality holds if and only if there is a non-zero real number c' such that $a_i = c' \lambda_i b_i$ for $2 \le i \le m$. By (4) or (5), we have

$$\begin{aligned} a_1 b_1 &\le \tfrac{|\lambda_2|}{d} \sqrt{\textstyle\sum_{i=2}^{m} a_i^2} \sqrt{\textstyle\sum_{i=2}^{m} b_i^2} \\ &= \tfrac{|\lambda_2|}{d} \sqrt{< \mathbf{1}_X, \mathbf{1}_X > - a_1^2} \sqrt{< \mathbf{1}_Y, \mathbf{1}_Y > - b_1^2} \\ &= \tfrac{|\lambda_2|}{d} \sqrt{a_1 - a_1^2} \sqrt{b_1 - b_1^2}. \end{aligned}$$

It follows that

$$\sqrt{\frac{a_1 b_1}{(1-a_1)(1-b_1)}} \le \frac{|\lambda_2|}{d}. \quad (6)$$

Note that

$$(1-a_1)(1-b_1) = 1 - (a_1 + b_1) + a_1 b_1 \le 1 - 2\sqrt{a_1 b_1} + a_1 b_1 = (1 - \sqrt{a_1 b_1})^2.$$

It follows from (6) that

$$\sqrt{a_1 b_1} \le \frac{|\lambda_2|}{d + |\lambda_2|}.$$

Thus, we obtain the next result, which is a slight improvement of the cross-independent version of Hoffman's Theorem given in [9].

Theorem 11. *Using the above notation, we have*

$$|X||Y| \le (\frac{|\lambda_2|}{d + |\lambda_2|} |V|)^2.$$

Equality implies that $|X| = |Y|$ *and, for* $i \ge 2$,

(i) $a_i = 0$ *if and only if* $b_i = 0$;
(ii) *if* $a_i \neq 0$ *then* $\lambda_i = \pm|\lambda_2|$;
(iii) *there are non-zero real numbers* c *and* c' *such that* $cc' = \frac{1}{\lambda_2^2}$, $a_i = c\lambda_i b_i$ *and* $b_i = c'\lambda_i a_i$;
(iv) *if further* $X = Y$ *then* $|\lambda_2| = -\lambda_{\min}$.

Corollary 1. *Continue the above notation. Suppose that* $\lambda_i = \lambda_2$ *provided* $i \geq 2$ *and* $|\lambda_i| = |\lambda_2|$. *If* $|X||Y| = (\frac{|\lambda_2|}{d+|\lambda_2|}|V|)^2$. *Then either*

(i) $X = Y$, *in this case,* $\lambda_2 = \lambda_{\min}$ *and* X *is a maximum independent set of* Γ; *or*
(ii) $V = X \cup Y$ *and* Γ *are not connected. If* $|X| = \frac{1}{2}|V|$ *and* $1_X \neq 1_Y$, *then* $1_X + 1_Y = 1$.

Proof. First of all, we prove the first part. Assume that $|X||Y| = (\frac{|\lambda_2|}{d+|\lambda_2|}|V|)^2$. Recall that

$$\mathbf{1}_X = \sum_{i=1}^{m} a_i \mathbf{v}_i = a_1 \mathbf{1}_V + \sum_{i \geq 2, |\lambda_i| = |\lambda_2|} a_i \mathbf{v}_i = a_1 \mathbf{1}_V + \sum_{i \geq 2, \lambda_i = \lambda_2} a_i \mathbf{v}_i.$$

Then, by Theorem 11 we have

$$\mathbf{1}_X = a_1 \mathbf{1}_V + \sum_{i \geq 2, \lambda_i = \lambda_2} c\lambda_i b_i \mathbf{v}_i = a_1 \mathbf{1}_V - c\lambda_2 b_1 \mathbf{1}_V + c\lambda_2 \mathbf{1}_Y.$$

Thus, $\mathbf{1}_X - c\lambda_2 \mathbf{1}_Y = (a_1 - c\lambda_2 a_1)\mathbf{1}_V$ as $a_1 = b_1$. It follows that either $X = Y$ or $V = X \cup Y$. The former case yields item (i) of this corollary.

Now assume that $V = X \cup Y$. Then $|X| = |Y| = \frac{|V|}{2}$. It follows that $\frac{|\lambda_2|}{d+|\lambda_2|} = \frac{1}{2}$, and so $d = |\lambda_2|$. Suppose that Γ is connected. Then, the only possibility is that $\lambda_2 = -d$. Thus, Γ is a connected bipartite graph, and $|\lambda_i| < d$ for $i > 2$. Let U and W be the bipartition subsets of Γ. We may choose $\mathbf{v}_2 = \mathbf{1}_U - \mathbf{1}_W$. Then $\mathbf{1}_X = a_1\mathbf{1}_V + a_2(\mathbf{1}_U - \mathbf{1}_W)$. It follows that $X = U$ or W (note that Γ is not empty, and so $X \neq V$ and $Y \neq V$). Similarly, $Y = U$ or W. Since $V = X \cup Y$, we have $\{X, Y\} = \{U, W\}$. Thus there are ages of Γ between X and Y, a contradiction. This completes the proof. □

This theorem characterizes a cross-independent set by establishing a connection between a certain kind of cross-independent set and an independent set of a regular graph.

4.2. Proofs of the Main Results

By Theorem 11, we easily obtain the bounds of the cross-intersecting family. But we cannot describe all the families that meet these bounds. In this section, we present a connection between cross-intersecting family and intersecting family by optimizing the previous cross-independent version of Hoffman's theorem.

To describe the construction of this connection, we present some claims. Notice that the bound in Theorem 11 is obtained if and only if all the equalities in Theorem 11 are met at the same time.

Claim 1: If $|X||Y| = (\frac{|\lambda_2|}{\lambda_1+|\lambda_2|}|V|)^2$, then $1 - a_1 - b_1 + a_1b_1 = 1 - 2\sqrt{a_1b_1} + a_1b_1$, and $|X| = |Y|$.

Proof. Obviously, $0 < a_1 < 1$, $0 < b_1 < 1$ and $a_1 + b_1 - 2\sqrt{a_1b_1} = 0$. Hence, $|X| = |Y|$. □

Claim 2: If $|X||Y| = (\frac{|\lambda_2|}{\lambda_1+|\lambda_2|}|V|)^2$, then $\sum_{i=2}^{n} |a_ib_i| = \sqrt{\sum_{i=2}^{n} a_i^2}\sqrt{\sum_{i=2}^{n} b_i^2}$. We have $a_i = 0$ if and only if $b_i = 0$, and $\frac{|a_i|}{|b_i|} = \frac{|a_j|}{|b_j|}$ for all $i \neq j$ where $a_i \neq 0$ and $a_j \neq 0$.

Proof. It is easy to see that

$$(\sum_{i=2}^{n} a_i^2)(\sum_{i=2}^{n} b_i^2) - (\sum_{i=2}^{n} |a_i||b_i|)^2 = \frac{1}{2}\sum_{i=2}^{n}\sum_{j=2}^{n}(|a_ib_j| - |a_jb_i|)^2.$$

If $\sum_{i=2}^{n}|a_ib_i| = \sqrt{\sum_{i=2}^{n}a_i^2}\sqrt{\sum_{i=2}^{n}b_i^2}$. Then $|a_ib_j| = |b_ia_j|$ for $2 \le i, j \le n$.

For $a_i = 0$, we have $|a_ib_j| = |a_jb_i| = 0$, for all $2 \le j \le n$. Then we will prove $b_i = 0$. Suppose for a contradiction that $b_i \ne 0$, then $a_2 = a_3 = \cdots = a_n = 0$. We have $1_X = a_1v_1$, and so $1_X = v_1$. Therefore $1_Y = v_1$; it is impossible. For $b_i = 0$, we have $a_i = 0$ by the same way. It is obvious that $\frac{|a_i|}{|b_i|} = \frac{|a_j|}{|b_j|}$ for all $i \ne j$ where $a_i \ne 0$ and $a_j \ne 0$. □

Claim 3: If $|X||Y| = (\frac{|\lambda_2|}{\lambda_1+|\lambda_2|}|V|)^2$, then $\sum_{i=2}^{n}|a_ib_i\lambda_i| = |\lambda_2|\sum_{i=2}^{n}|a_ib_i|$, and if $|\lambda_i| \ne |\lambda_2|$ then $a_i = b_i = 0$.

Proof. We have $\sum_{i=2}^{n}|a_ib_i\lambda_i| = |\lambda_2|\sum_{i=2}^{n}|a_ib_i|$, and hence $\sum_{i=2}^{n}(|\lambda_2| - |\lambda_i|)|a_ib_i| = 0$. Thus $(|\lambda_2| - |\lambda_i|)|a_ib_i| = 0$ for $2 \le i \le n$. Suppose $|\lambda_i| \ne |\lambda_2|$, we have $a_i = b_i = 0$. □

By Claim 3, if $|X||Y| = (\frac{|\lambda_2|}{\lambda_1+|\lambda_2|}|V|)^2$, then $1_X, 1_Y \in Span(\{v_1\} \cup \{v_i : |\lambda_i| = |\lambda_2|\})$.

Claim 4: If $|X||Y| = (\frac{|\lambda_2|}{\lambda_1+|\lambda_2|}|V|)^2$, then $|\sum_{i=2}^{n}a_ib_i\lambda_i| = \sum_{i=2}^{n}|a_ib_i\lambda_i|$. Hence $a_ib_i\lambda_i$ are all non-positive. Then for positive eigenvalue λ_i, $a_ib_i \le 0$, but for negative eigenvalue λ_j, $a_jb_j \ge 0$.

Finally, we comment on a feature of a certain kind of cross-independent set of a regular graph.

Proof of Theorem 9. Let $G = \mathrm{PGL}(2,q)$ act on the projective line $\Omega := \mathrm{PG}(1,q)$, and let H be a point-stabilizer in G. By [10,24], the eigenvalues of $\mathrm{Cay}(G, D(G))$ are listed in Tables 1 and 2.

Table 1. Eigenvalues of $\mathrm{Cay}(G, D(G))$ for even q.

Eigenvalue	$\frac{q^2(q-1)}{2}$	$\frac{-q(q-1)}{2}$	q	0
Dimension	1	q^2	$\frac{q(q-1)^2}{2}$	$\frac{(q+1)^2(q-2)}{2}$

Table 2. Eigenvalues of $\mathrm{Cay}(G, D(G))$ for odd q.

Eigenvalue	$\frac{q^2(q-1)}{2}$	$\frac{-q(q-1)}{2}$	$\frac{q-1}{2}$	q	0
Dimension	1	q^2+1	q^2	$\frac{(q-1)^3}{2}$	$\frac{(q+1)^2(q-3)}{2}$

Assume that X, $Y \subseteq \mathrm{PGL}(2,q)$ are cross-intersecting. If $q > 2$ then $|X||Y| \le |H|^2 = q^2(q-1)^2$. Equality holds for $q \ge 4$ if and only if $X = Y$ is a coset of some conjugation of H in G.

By the derangement graph of the permutation groups $\mathrm{PGL}(2,q)$ and $\mathrm{PSL}(2,q)$, we now establish some results about the cross-intersecting family of $\mathrm{PGL}(2,q)$ and $\mathrm{PSL}(2,q)$ based on Corollary 1.

Lemma 1. *If $X, Y \subset \mathrm{PGL}(2,q)$ are cross-intersecting, then $|X||Y| \le q^2(q-1)^2$ when $q \ne 2$.*

Proof. Assume that $q \neq 2$. For the eigenvalues of the derangement graph of $\Gamma_{\mathrm{PGL}(2,q)}$, we have

$$|X||Y| \leq (\frac{|\lambda_2|}{\lambda_1 + |\lambda_2|})^2 |V|^2 = (\frac{\frac{q(q-1)}{2}}{\frac{q^2(q-1)}{2} + \frac{q(q-1)}{2}})^2 q^2(q^2-1)^2 = q^2(q-1)^2.$$

□

Theorem 12. *If X and Y are cross-intersecting in* $\mathrm{PGL}(2,q)$ *where $q \neq 2$, then $|X||Y| \leq q^2(q-1)^2$. For $q \geq 4$, the bound is attained if and only if $X = Y$ is a coset of the stabilizer of a point of $\mathbb{P}_q$.*

Proof. Assume $q \neq 2$ and $q \neq 3$. Notice that $|X| = |Y| = q(q-1) \neq \frac{1}{2}|\mathrm{PGL}(2,q)|$, and all $\lambda_i's$ are negative. By Corollary 1, we have $1_X = 1_Y$.

Observe that X and Y are cross-intersecting, indeed $X = Y$ is the biggest intersecting family in $\mathrm{PGL}(2,q)$. We complete the proof. □

And this finished the proof of Theorem 9. □

Proof of Theorem 10. Also, we firstly give some results needed for the proof.

Lemma 2. *If $X, Y \subset$* $\mathrm{PSL}(2,q)$ *are cross-intersecting, then $|X||Y| \leq q^2(q-1)^2$ where q is even and $q \geq 4$, and $|X||Y| \leq \frac{q^2(q-1)^2}{4}$ where q is odd and $q \geq 7$.*

Proof. Assume that $q \neq 2$, $q \neq 3$ and $q \neq 5$. For the eigenvalues of the derangement graph of $\Gamma_{\mathrm{PSL}(2,q)}$, we have

$|X||Y| \leq (\frac{|\lambda_2|}{\lambda_1+|\lambda_2|})^2 |V|^2 = (\frac{\frac{q(q-1)}{2}}{\frac{q^2(q-1)}{2} + \frac{q(q-1)}{2}})^2 q^2(q^2-1)^2 = q^2(q-1)^2$, where q is even and $q \geq 4$; and $|X||Y| \leq (\frac{|\lambda_2|}{\lambda_1+|\lambda_2|})^2 |V|^2 = (\frac{\frac{(q-1)^2}{4}}{\frac{q(q-1)^2}{4} + \frac{(q-1)^2}{4}})^2 \cdot \frac{q^2(q^2-1)^2}{4} = \frac{q^2(q-1)^2}{4}$, where q is odd and $q \geq 7$. □

Theorem 13. *If $X, Y \subset$* $\mathrm{PSL}(2,q)$ *are cross-intersecting where q is even and $q \geq 4$, then $|X||Y| \leq q^2(q-1)^2$ with equality only if $X = Y$ is a coset of the stabilizer of one point.*

Proof. Assume that q is even, and $q \neq 2$. Notice that $|X| = |Y| = q(q-1) \neq \frac{1}{2}|\mathrm{PSL}(2,q)|$. If $q \neq 2$, then all $\lambda_i's$ are negative, and hence $1_X = 1_Y$.

Observe that X and Y are cross-intersecting, indeed $X = Y$ is the biggest intersecting family in $\mathrm{PSL}(2,q)$. We complete the proof. □

Theorem 14. *If $X, Y \subset$* $\mathrm{PSL}(2,q)$ *are cross-intersecting where q is odd and $q \geq 7$, then $|X||Y| \leq \frac{q^2(q-1)^2}{4}$. And if the equality is met, $X = Y$ is the biggest intersecting family in* $\mathrm{PSL}(2,q)$.

Proof. Assume that q is odd, and $q \geq 7$. Notice that $|X| = |Y| = \frac{q(q-1)}{2} \neq \frac{1}{2}|\mathrm{PSL}(2,q)|$. If $q \geq 7$, then all $\lambda_i's$ are negative, and therefore $1_X = 1_Y$.

Observe that X and Y are cross-intersecting, indeed $X = Y$ is the biggest intersecting family in $\mathrm{PSL}(2,q)$. We complete this proof. □

From the above analysis, we can conclude the result of Theorem 10. □

Finally, we deal with several some special cases of $\mathrm{PGL}(2,q)$ and $\mathrm{PSL}(2,q)$, which are not taken into account above. Recall that the Cayley graph is a vertex-transitive graph, so we only need to consider the cross-intersecting subsets which contain an identity element.

(1) Theorems 8 and 9 do not hold for S_3 (noting that $PGL(2,2) \cong S_3$). Let $G = S_3$. Then $\lambda_1 = \lambda_2 = 2 \neq \lambda_{\min} = -1$. Thus (2) of Corollary 1 holds, and $\{X,Y\} = \{\{(1),(123),(132)\},\{(23),(12),(13)\}\}$. In fact, in this case, $D(G) = \{(1\,2\,3),(1\,3\,2)\}$ and $\Gamma = \mathrm{Cay}(G, D(G))$ is the vertex-disjoint union of two 3-cycles.

(2) Let $G = S_4$. Then, $D(G)$ consists of the six odd permutations of order 4 and the three even permutations of order 2. In this case, $(|G|\frac{|\lambda_2|}{d(G)+|\lambda_2|})^2 = 36 = |H|^2$, and $\pm\lambda_2$ are eigenvalues. If $|X||Y| = 36$, then $X = Y$ is a coset of some conjugation of H in G. (Verified by GAP.)

(3) Theorem 10 does not hold for $PSL(2,3)$. Let $G = PSL(2,3)$. Then $\lambda_1 = \lambda_2 = 3 \neq \lambda_{\min} = -1$. In this case, $\Gamma = \mathrm{Cay}(G, D(G))$ is the vertex-disjoint union of three copies of the complete graph K_4, which yields that $|X||Y| \leq 18 < 36 = (|G|\frac{|\lambda_2|}{d(G)+|\lambda_2|})^2$.

(4) Let $G = PSL(2,5)$. Then $\lambda_2 = 5 \neq \lambda_{\min} = -4$, and λ_1 is simple, and hence $\Gamma = \mathrm{Cay}(G, D(G))$ is connected. Thus, by Theorem 11 and Corollary 1, $|X||Y| < 144 = (|G|\frac{|\lambda_2|}{d(G)+|\lambda_2|})^2$. This finishes the proof of the main theorems.

5. Conclusions

In this article, we have successfully established a connection between intersecting families and cross-intersecting families of certain kinds of permutation subgroups. This is a great help when considering cross-intersecting families. And this also gives us a new direction for considering the properties of some permutation groups. In the future, we intend to extend our exploration to more permutation groups, such as the unitary group, orthogonal and so on. In the process of solving the problems, we used properties of Cayley graphs on a permutation subgroup with respect to the derangement. The symmetry of these graphs will also aid in proving the upper bounds of the subgroups, and the use of these inequalities will be helpful for understanding the symmetry of corresponding graph classes. The research will promote the development of the related fields.

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