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On Primary Decomposition of Hermite Projectors

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Abstract: An ideal projector on the space of polynomials $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_d]$ is a projector whose kernel is an ideal in $\mathbb{C}[\mathbf{x}]$. Every ideal projector P can be written as a sum of ideal projectors $P^{(k)}$ such that the intersection of their kernels $\ker P^{(k)}$ is a primary decomposition of the ideal $\ker P$. In this paper, we show that P is a limit of Lagrange projectors if and only if each $P^{(k)}$ is. As an application, we construct an ideal projector P whose kernel is a symmetric ideal, yet P is not a limit of Lagrange projectors.

Keywords: ideal projector; hermite projector; smoothable ideals

1. Introduction

Let $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_d]$ denote the algebra of polynomials in d variables with complex coefficients. A projector P on $\mathbb{C}[\mathbf{x}]$ is a linear idempotent operator on $\mathbb{C}[\mathbf{x}]$. Such a projector is called an ideal projector if $\ker P$ is an ideal in $\mathbb{C}[\mathbf{x}]$. An ideal projector is called a Lagrange projector if $\ker P$ is a radical ideal in $\mathbb{C}[\mathbf{x}]$. If the range of P is N-dimensional, then P is a Lagrange projector if and only if there exist N distinct points $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{C}^d$ such that

$$\ker P = \{f \in \mathbb{C}[\mathbf{x}] : f(\mathbf{x}_1) = \ldots = f(\mathbf{x}_N) = 0\}$$

or equivalently

$$(Pf)(\mathbf{x}_j) = f(\mathbf{x}_j)$$

for all $j = 1, \ldots, N$ and all $f \in \mathbb{C}[\mathbf{x}]$. The last equivalence shows that Lagrange projectors interpolate at nodes $x_1, \ldots, x_N$ and therefore present a natural extension of the classical Lagrange interpolation theory to the multivariate setting.

The notion of an ideal projector was first introduced by Birkhoff in [1]. Since then, it was further studied, and connections to different branches of mathematics were explored (see [2–11]). In this paper, we consider exclusively finite dimensional ideal projectors.

In one variable, every Hermite interpolation projector is the limit of a sequence of classical Lagrange interpolation projectors. That allows us to extend the definition of the Hermite interpolation projectors to the multivariate setting as follows.

Definition 1. *An ideal projector P is called a Hermite projector if there exist a sequence of Lagrange projectors P_n on the range of P such that*

$$P_n f \to Pf$$

for every $f \in \mathbb{C}[\mathbf{x}]$. We do not specify type of convergence because $P_n f$ and Pf belong to the same finite-dimensional space; hence, all forms of convergence are equivalent.

In one variable setting, the ideal projectors are the same as classical Hermite projectors (see for example [10]). The natural question arises as to whether, in the multivariate setting, the same is true, i.e., is any ideal projector necessarily a limit of Lagrange projectors? Rather surprisingly, the resulting answer is positive in two variables (cf. [4]) but negative in three

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or more variables (cf. [12]). The question of a description of those ideal projectors that are Hermite was raised by Carl de Boor in [3]. Some partial results regarding this question were obtained in [8,9] and, in the very different language of algebraic geometry, in [13,14]. In this paper, we make a contribution to this problem by examining the primary decomposition of Hermite projectors.

Every finite-dimensional ideal projector P can be written as a finite (direct) sum of ideal projectors $P^{(k)}$

$$P = P^{(1)} \oplus P^{(2)} \oplus \ldots \oplus P^{(m)}, \tag{1}$$

where $P^{(k)}$ are ideal projectors such that the ideals $\ker P^{(k)}$ form the primary decomposition of the ideal $\ker P$. That is

$$\ker P = \bigcap_{k=1}^{m} \ker P^{(k)} \tag{2}$$

and, for each $P^{(k)}$, the variety

$$\mathcal{V}(\ker P^{(k)}) := \{\mathbf{x} \in \mathbb{C}^d : f(\mathbf{x}) = 0, \text{for all } f \in \ker P^{(k)}\} \tag{3}$$

consists of exactly one point.

The main result of this paper is

Theorem 1. *P is Hermite if and only if each $P^{(k)}$ is Hermite.*

Based on the above theorem, as an application, we will show the existence of a symmetric ideal projector (in three or more variables) that is not Hermite (see Theorem 7). Finally, we will showcase some problems in matrix theory (see Problem 2) that are related to the main result.

2. Preliminaries

Let $\mathbb{C}'[\mathbf{x}]$ denote the algebraic dual of $\mathbb{C}[\mathbf{x}]$, i.e., the space of all linear functionals on $\mathbb{C}[\mathbf{x}]$. For an ideal $J \subset \mathbb{C}[\mathbf{x}]$, let $\mathcal{V}(J)$ denote the affine variety associated with J:

$$\mathcal{V}(J) := \{\mathbf{x} \in \mathbb{C}^d : f(\mathbf{x}) = 0, \text{ for all } f \in J\}.$$

The ideal J has a finite codimension (0-dimensional) if and only if the set $\mathcal{V}(J)$ is finite (cf. [15]). Moreover, $|\mathcal{V}(J)| \leq \dim(\mathbb{C}[\mathbf{x}]/J)$ and $|\mathcal{V}(J)| = \dim(\mathbb{C}[\mathbf{x}]/J)$ if and only if the ideal J is radical i.e.,

$$J = \{f : f(\mathbf{x}) = 0, \text{ for all } \mathcal{V}(J)\}.$$

Let $J^{\perp}$ denote a subspace of $\mathbb{C}'[\mathbf{x}]$ of all functionals that vanish on J. Hence

$$\dim J^{\perp} = \dim(\mathbb{C}[\mathbf{x}]/J).$$

For $\mathbf{x} \in \mathbb{C}^d$, we use $\delta_{\mathbf{x}} \in \mathbb{C}'[\mathbf{x}]$ to denote the point evaluation functional:

$$\delta_{\mathbf{x}}(f) = f(\mathbf{x}).$$

It is easy to see that for any N-dimensional Lagrange interpolation projector P, the variety $\mathcal{V}(\ker P)$ is consisting of exactly N distinct points. Assuming $\mathcal{V}(\ker P) = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$, we have

$$\ker^{\perp} P = span\{\delta_{\mathbf{x}_1}, ..., \delta_{\mathbf{x}_N}\}.$$

Below, we will review relations between ideal projectors and the sequence of commuting matrices.

Let P be an N-dimensional ideal projector and let G be its range. Hence, $\mathbb{C}[\mathbf{x}] = G \oplus \ker P$ and $\ker P$ is an ideal of codimension N in $\mathbb{C}[\mathbf{x}]$. Thus, $\mathbb{C}[\mathbf{x}]/\ker P$ is an N-dimensional algebra. For each coordinate x_j of $\mathbf{x}$, we define a multiplication operator ($N \times N$ matrix) $M_j : \mathbb{C}[\mathbf{x}]/\ker P \to \mathbb{C}[\mathbf{x}]/\ker P$ associated with P by

$$M_j[f] = [x_j f] \tag{4}$$

where $[\cdot]$ represents a class equivalence in $\mathbb{C}[\mathbf{x}]/\ker P$. The set $(M_1, \ldots, M_d)$ forms a sequence of commuting matrices that are associated with the projector P. In fact, such a sequence uniquely defines P (see [4] for details). The matrices M_j represent the operators defined on G by

$$M_j(f) = P(x_j f).$$

The matrices $M_1, \ldots, M_d$ were introduced by Hans Stetter (cf. [11,16,17]) who discovered the main relation between common eigenvalues (eigentuples) of these matrices and the variety $\mathcal{V}(\ker P)$.

Definition 2. *A d-tuple of complex numbers $(\lambda_1, \ldots, \lambda_d)$ is called an eigentuple for $(M_1, \ldots, M_d)$ if there exists a non-zero vector $\mathbf{u} \in \mathbb{C}^N$ such that*

$$M_j \mathbf{u} = \lambda_j \mathbf{u}, \forall j = 1, \ldots, N.$$

Let $\sigma(M_1, \ldots, M_d)$ denote the set of all eigentuples of the commuting matrices $(M_1, \ldots, M_d)$.

Theorem 2 ([17])**.** $\sigma(M_1, \ldots, M_d) = \mathcal{V}(\ker P)$.

Theorem 3 (cf. [4])**.** *Suppose that we have a sequence of ideal projectors P_n onto the same space G and let $(M_1^{(n)}, \ldots, M_d^{(n)})$ be multiplication operators associated with P_n while $(M_1, \ldots, M_d)$ is the multiplication operators on G associated with P. Then, $P_n \to P$ if and only if $(M_1^{(n)}, \ldots, M_d^{(n)}) \to (M_1, \ldots, M_d)$.*

Next, we prove an extension of Theorem 5.2.1 in Artin [18] to the set of commuting matrices. Our proof is substantially different than the one presented there.

Theorem 4. *Let $(M_1^{(n)}, \ldots, M_d^{(n)})$ and $(M_1, \ldots, M_d)$ be a d-tuple of operators on an N-dimensional space G. Assume that $(M_1^{(n)}, \ldots, M_d^{(n)}) \to (M_1, \ldots, M_d)$. Then, the sets $\sigma(M_1^n, \ldots, M_d^n)$ are uniformly bounded and all accumulation points of $\sigma(M_1^n, \ldots, M_d^n)$ are in $\sigma(M_1, \ldots, M_d)$.*

Proof. Let $(\lambda_1^{(n)}, \ldots, \lambda_d^{(n)}) \in \sigma(M_1^n, \ldots, M_d^n)$, hence

$$M_j^{(n)} \mathbf{u}_n = \lambda_j^{(n)} \mathbf{u}_n, \forall j = 1, \ldots, d.$$

Assume without loss of generality that $\|u_n\| = 1$. Then

$$\left|\lambda_j^{(n)}\right| = \left\|\lambda_j^{(n)} \mathbf{u}_n\right\| = \left\|M_j^{(n)} \mathbf{u}_n\right\| \leq \left\|M_j^{(n)}\right\|$$

and since $M_j^{(n)}$ converges, the norms $\left\|M_j^{(n)}\right\|$ are uniformly bounded, which proves the first part of the theorem. Now, passing to a subsequence if necessary, we may assume that $\lambda_j^{(n)} \to \lambda_j$. Then, $M_j^{(n)} \mathbf{u}_n = \lambda_j^{(n)} \mathbf{u}_n$. The sequence $(\mathbf{u}_n)$ is uniformly bounded in a finite-dimensional space G; hence, it is compact. Passing to a subsequence if necessary, we may assume that $\mathbf{u}_n \to \mathbf{u} \in G$ and $\|\mathbf{u}\| = 1$. Finally, since $(M_1^{(n)}, \ldots, M_d^{(n)})$ are finite-dimensional operators, the convergence is uniform. Therefore

$$\lambda_j^{(n)} \mathbf{u}_n = M_j^{(n)} \mathbf{u}_n \to M_j \mathbf{u}.$$

In addition, $\lambda_j^{(n)}\mathbf{u}_n \to \lambda_j\mathbf{u}$, hence $(\lambda_j) \in \sigma(M_1,\ldots,M_d)$. $\square$

Combining the above with the Theorem 2, we obtain:

Corollary 1. *Let P be an N-dimensional Hermite projector and P_n be a sequence of Lagrange projectors such that $P_n \to P$. If $\mathcal{V}(\ker P) = \{\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N\}$ and $\mathcal{V}(\ker P_n) = \{\mathbf{x}_1^{(n)},\mathbf{x}_2^{(n)},\ldots,\mathbf{x}_N^{(n)}\}$. Then, there exists a constant C such that $|x_k^{(n)}| \le C$ for all k and n. Additionally, $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_K$ are the only possible limit points of the set $\bigcup_n \mathcal{V}(\ker P_n)$.*

Now, we will recollect a few facts regarding the convergence of ideal projectors. The main idea is that such a convergence depends only on their respective kernels. For more details and proofs, see [12].

Theorem 5 (cf. [12])**.** *Let P_n and P be ideal projectors onto a finite-dimensional space $G \subset \mathbb{C}[\mathbf{x}]$. Then, $P_n \to P$ if and only if for every functional $F \in \ker^\perp P$, there exists a sequence of functionals $F_n \in \ker^\perp P_n$ such that $F_n \to F$ in the weak-$\star$ topology. i.e.,*

$$F_n f \to F f,\ \textit{for all}\ f \in \mathbb{C}[\mathbf{x}]. \tag{5}$$

If each P_n is a Lagrange projector, then $\ker^\perp P_n$ is spanned by N point evaluation functionals $\delta_{\mathbf{x}_1^{(\mathbf{n})}}, \ldots, \delta_{\mathbf{x}_N^{(\mathbf{n})}}$, and each F_n can be written as their linear combination. Using the above theorem, we obtain the following.

Corollary 2. *An N-dimensional ideal projector P is Hermite if and only if every $F \in \ker^\perp P$ is the weak-$\star$ limit of linear combinations of N point evaluation functionals. That is, there exists sets $\mathcal{X}_n \subset \mathbb{C}^d$, each consisting of N distinct points such that for every $F \in \ker^\perp P$*

$$F(f) = \lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{X}_n} a_{\mathbf{x}}^{(n)} \delta_{\mathbf{x}}(f) = \lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{X}_n} a_{\mathbf{x}}^{(n)} f(\mathbf{x}) \tag{6}$$

for some coefficients $a_{\mathbf{x}}^{(n)}$ and for all $f \in \mathbb{C}[\mathbf{x}]$.

3. The Main Result

The main goal is to prove Theorem 1. One side is easy to establish and can be shown as follows. Let P be an N-dimensional ideal projector and assume that $\ker P$ has a primary decomposition

$$\ker P = \bigcap_{j=1}^{m} J_j$$

where the ideals J_j have codimensions equal to N_j, respectively. Since

$$\ker^\perp P = \oplus J_j^\perp$$

we have $\sum_{j=1}^{M} N_j = N$. Take any $F \in \ker^\perp P$. Then, $F = \sum_{j=1}^{m} F_j$ for some $F_j \in J_j^\perp$. If each J_j is the kernel of a Hermite projector then, by Corollary 2, there exists sets $\mathcal{X}_n^j \subset \mathbb{C}^d$ consisting of N_j distinct points such

$$F_j = \lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{X}_n^j} a_{\mathbf{x}}^{(n)} \delta_{\mathbf{x}}$$

It follows that

$$F = \sum_{j=1}^{m} F_j = \lim_{n\to\infty} \sum_{j=1}^{m} \sum_{\mathbf{x}\in\mathcal{X}_n^j} a_{\mathbf{x}}^{(n)} \delta_{\mathbf{x}} = \lim_{n\to\infty} \sum_{\mathbf{x}\in\cup\mathcal{X}_n^j} b_{\mathbf{x}}^{(n)} \delta_{\mathbf{x}}$$

and therefore, F is a weak-$\star$ limit of a linear combination of N point evaluations. By Corollary 2, P is Hermite.

The main result of this section is a proof of the converse statement. The main idea is as follows. Assume P is Hermite, $P = P^{(1)} \oplus P^{(2)} \oplus \ldots \oplus P^{(m)}$ and $\ker P$ has a primary decomposition $\ker P = \cap_{j=1}^{m} \ker P^{(j)}$. By Corollary 2, there exists sets $\mathcal{X}_n \subset \mathbb{C}^d$ such that (6) holds. Let $\mathcal{V}(\ker P^{(1)}) = \{\mathbf{y}\}$. We will decompose $\mathcal{X}_n = \mathcal{Y}_n \cup \mathcal{Z}_n$ so that all accumulation points of $\bigcup_{n\in\mathbb{N}} \mathcal{Z}_n$ are away from $\mathbf{y}$. For every functional $F \in \ker^{\perp} P$ (in particular, every functional $F \in \ker^{\perp} P_1$), we have

$$F(f) = \lim_{n\to\infty}\left(\sum_{\mathbf{x}\in\mathcal{Y}_n} a_{\mathbf{x}}^{(n)} f(\mathbf{x}) + \sum_{\mathbf{x}\in\mathcal{Z}_n} a_{\mathbf{x}}^{(n)} f(\mathbf{x})\right), \tag{7}$$

for some coefficients $a_{\mathbf{x}}^{(n)}$ and for all $f \in \mathbb{C}[\mathbf{x}]$. The main part of the proof is to show that the above implies that

$$F(f) = \lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{Y}_n} a_{\mathbf{x}}^{(n)} f(\mathbf{x}). \tag{8}$$

Thus, in (7), we can eliminate all point evaluations that do not accumulate at $\mathbf{y}$, and the number of points remaining is equal to $N_1 = \dim \ker^{\perp} P_1$. Hence, by Corollary 2, the projector P_1 is Hermite.

To carry the proof in detail, we need a few preliminary results. First, we will produce a multivariate analog of Lagrange fundamental polynomials that seems to be new.

Proposition 1. *Let $\mathcal{Y}$ be a finite set of m points in $\mathbb{C}^d$ and take $\mathbf{z} \in \mathbb{C}^d$ such that $\mathcal{Y}$ and $\mathbf{z}$ lie in the interior of a ball $B \subset \mathbb{C}^d$ of radius R. Let*

$$r = \min\{\|\mathbf{y} - \mathbf{z}\| : \mathbf{y} \in \mathcal{Y}\} > 0.$$

Then, there exists a constant $C(R,r)$ and polynomial $\omega(\mathbf{x}) = \omega_{\mathcal{Y},\mathbf{z}} \in \mathbb{C}[\mathbf{x}]$ of degree at most m such that

$$\omega(\mathbf{z}) = 1,\ \omega(\mathbf{y}) = 0, \text{ for all } \mathbf{y} \in \mathcal{Y}$$

and

$$\|\omega\|_B \le C(R,r) = \left(\frac{2R}{r}\right)^{2m}.$$

(here, $\|\omega\|_B$ denotes the supremum of the polynomial ω over the ball $B \subset \mathbb{C}^d$).

Proof. Let $< \mathbf{u}, \mathbf{v} >$ denote the Hermitian inner product in the space $\mathbb{C}^d$. Consider the following polynomial

$$\omega(\mathbf{x}) = \frac{\prod_{\mathbf{y}\in\mathcal{Y}} < \mathbf{x} - \mathbf{y}, \mathbf{z} - \mathbf{y} >}{\prod_{\mathbf{y}\in\mathcal{Y}} \|\mathbf{z} - \mathbf{y}\|^2}$$

Since $< \mathbf{x} - \mathbf{y}, \mathbf{z} - \mathbf{y} >$ is a linear polynomial in $\mathbb{C}[\mathbf{x}]$, $\omega(\mathbf{x})$ is a polynomial of degree at most m. Clearly, $\omega(\mathbf{y}) = 0$ for all $\mathbf{y} \in \mathcal{Y}$ and $\omega(\mathbf{z}) = 1$.

Since x, y, z lie in a ball of radius R

$$\left\|\prod_{\mathbf{y}\in\mathcal{Y}} < \mathbf{x} - \mathbf{y}, \mathbf{z} - \mathbf{y} >\right\| \le \prod_{\mathbf{y}\in\mathcal{Y}} ||x - y|| \cdot ||z - y|| \le (2R)^{2m}.$$

Additionally

$$\prod_{\mathbf{y}\in\mathcal{Y}} \|z - \mathbf{y}\|^2 \ge r^{2m}.$$

Combining these two inequalities together yields $\|\omega\|_B \leq \left(\frac{2R}{r}\right)^{2m}$. □

We will also need the following lemma.

Lemma 1. *Let* $(u_1^{(n)}, \ldots, u_m^{(n)})$ *and* $(\gamma_1^{(n)}, \ldots, \gamma_m^{(n)})$ *be two sequences in* $\mathbb{C}^m$ *such that*

$$\gamma_j^{(n)} \to 1 \quad \textit{and} \quad \sum_{j=1}^{m} u_j^{(n)} (\gamma_j^{(n)})^k \to 0, \textit{ as } n \to \infty \tag{9}$$

for all $k = 1, \ldots, m$. *Then,* $\sum_{j=1}^{m} u_j^{(n)} \to 0$ *as* $n \to \infty$.

Proof. By induction on m, if $m = 1$, then $u_1^{(n)}\gamma_1^{(n)} \to 0$ and $\gamma_1^{(n)} \to 1$ immediately implies that $u_1^{(n)} \to 0$.

Assume that the statement is true for a fixed m. Now, we need to show that the statement is true for $m+1$. Take any $(u_1^{(n)}, \ldots, u_{m+1}^{(n)})$ and $(\gamma_1^{(n)}, \ldots, \gamma_{m+1}^{(n)})$ such that $\gamma_j^{(n)} \to 1$ for $j = 1, \ldots, m+1$ and

$$\sum_{j=1}^{m+1} u_j^{(n)} (\gamma_j^{(n)})^k \to 0 \text{ as } n \to \infty, \quad \text{for all } k = 1, \ldots, m+1. \tag{10}$$

The goal is to show that $\sum_{j=1}^{m+1} u_j^{(n)} \to 0$. Since $\gamma_{m+1}^{(n)} \to 1$, from the above, we obtain

$$\gamma_{m+1}^{(n)} \sum_{j=1}^{m+1} u_j^{(n)} (\gamma_j^{(n)})^k = \sum_{j=1}^{m+1} u_j^{(n)} (\gamma_j^{(n)})^k \gamma_{m+1}^{(n)} \to 0,$$

for $k = 1, \ldots, m$. Hence

$$\sum_{j=1}^{m} u_j^{(n)} (\gamma_{m+1}^{(n)} - \gamma_j^{(n)})(\gamma_j^{(n)})^k = \sum_{j=1}^{m+1} u_j^{(n)} (\gamma_j^{(n)})^k \gamma_{m+1}^{(n)} - \sum_{j=1}^{m+1} u_j^{(n)} (\gamma_j^{(n)})^{k+1} \to 0,$$

for all $k = 1, \ldots, m$. Setting $\tilde{u}_j^{(n)} = u_j^{(n)}(\gamma_{m+1}^{(n)} - \gamma_j^{(n)})$ the above gives

$$\sum_{j=1}^{m} \tilde{u}_j^{(n)} (\gamma_j^{(n)})^k \to 0 \quad \text{for all } k = 1, \ldots, m.$$

By the inductive assumption applied to $(\tilde{u}_1^{(n)}, \ldots, \tilde{u}_m^{(n)})$ and $(\gamma_1^{(n)}, \ldots, \gamma_m^{(n)})$, we conclude that $\sum_{j=1}^{m} \tilde{u}_j^{(n)} \to 0$. Hence

$$\sum_{j=1}^{m} u_j^{(n)} \gamma_j^{(n)} - \gamma_{m+1}^{(n)} \sum_{j=1}^{m} u_j^{(n)} = \sum_{j=1}^{m} u_j^{(n)} (\gamma_j^{(n)} - \gamma_{m+1}^{(n)}) = \sum_{j=1}^{m} \tilde{u}_j^{(n)} \to 0.$$

Setting $k = 1$ in (10), we obtain

$$-\sum_{j=1}^{m} u_j^{(n)} \gamma_j^{(n)} - \gamma_{m+1}^{(n)} u_{m+1}^{(n)} = -\sum_{j=1}^{m+1} u_j^{(n)} \gamma_j^{(n)} \to 0.$$

Combining these two gives

$$-\gamma_{m+1}^{(n)} \sum_{j=1}^{m+1} u_j^{(n)} = \sum_{j=1}^{m} u_j^{(n)} \gamma_j^{(n)} - \gamma_{m+1}^{(n)} \sum_{j=1}^{m} u_j^{(n)} - \sum_{j=1}^{m} u_j^{(n)} \gamma_j^{(n)} - \gamma_{m+1}^{(n)} u_{m+1}^{(n)} \to 0.$$

Since $\gamma_{m+1}^{(n)} \to 1$, we conclude that $\sum_{j=1}^{m+1} u_j^{(n)} \to 0$ as required. □

We are now ready for the proof of the main theorem.

Theorem 6. *Let P be a Hermite projector onto an N-dimensional space $G \subset \mathbb{C}[\mathbf{x}]$. Suppose that*

$$P = P^{(1)} \oplus P^{(2)} \oplus \dots P^{(m)} \tag{11}$$

where $P^{(k)}$ are ideal projectors such that the ideals $\ker P^{(k)}$ form the primary decomposition of the ideal $\ker P$

$$\ker P = \bigcap_{k=1}^{m} \ker P^{(k)}. \tag{12}$$

Then, each $P^{(k)}$ is Hermite.

Proof. We will start with $P^{(1)}$. Assume that $\mathcal{V}(\ker P) = \{\mathbf{u}_1, \dots \mathbf{u}_m\}$ and $\ker P^{(m)} = \{\mathbf{u}_m\}$. Since P is Hermite, by Corollary 2, for every functional $F \in \ker^{\perp} P$

$$F(f) = \lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{X}_n} a_{\mathbf{x}}^{(n)} \delta_{\mathbf{x}}(f) = \lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{X}_n} a_{\mathbf{x}}^{(n)} f(\mathbf{x}) \tag{13}$$

for every $f \in \mathbb{C}[\mathbf{x}]$. In particular, if $F \in \cap_{j=1}^{m-1} \ker^{\perp} P^{(j)}$, then due to (12), $F \in \ker^{\perp} P$ and hence (13) holds. By Corollary 1, the sets $\mathcal{X}_n$ lie in some ball in $\mathbb{C}^d$ of radius R and $\{\mathbf{u}_1, \dots \mathbf{u}_m\}$ are the only accumulation points of $\cup_n \mathcal{X}_n$. Partition the points $\mathcal{X}_n = \mathcal{Y}_n \cup \mathcal{Z}_n$ so that for every sequence $\mathbf{x}_n \in \mathcal{Z}_n$, we have

$$\mathbf{x}_n \to \mathbf{u}_m \tag{14}$$

and, for sufficiently large n, the points $\mathbf{x}_n \in \mathcal{Y}_n$ are arbitrarily close to the set $\{\mathbf{u}_1, \dots \mathbf{u}_{m-1}\}$. In particular,

$$\|\mathbf{x}_n - \mathbf{u}_m\| \geq r > 0, \quad \text{for some } r \text{ and for all } \mathbf{x}_n \in \mathcal{Y}_n. \tag{15}$$

We rewrite (13) as

$$F(f) = \lim_{n\to\infty} \Bigg(\sum_{\mathbf{x}\in\mathcal{Y}_n} a_{\mathbf{x}}^{(n)} f(\mathbf{x}) + \sum_{\mathbf{x}\in\mathcal{Z}_n} a_{\mathbf{x}}^{(n)} f(\mathbf{x}) \Bigg), \tag{16}$$

where the points in $\mathcal{Y}_n$ and $\mathcal{Z}_n$ satisfy (15) and (14), respectively. Now, let p be a polynomial in $\cap_{j=1}^{m-1} \ker P^{(j)}$ such that $p(\mathbf{u}_m) = 1$. Such a polynomial exists since, otherwise, every polynomial in $\cap_{j=1}^{m-1} \ker P^{(j)}$ would vanish at $\mathbf{u}_m$ and hence $\mathbf{u}_m \in \mathcal{V}(\cap_{j=1}^{m-1} \ker P^{(j)}) = \{\mathbf{u}_1, \dots, \mathbf{u}_{m-1}\}$. Next, consider polynomials

$$h_{k,n} = (p \cdot \omega_{\mathcal{Y}_n, \mathbf{u}_m})^k f \tag{17}$$

for $k = 1, \dots, m$ where $\omega_{\mathcal{Y}_n, \mathbf{u}_m}$ is defined as in Proposition 1, and f is arbitrary. Since p is in the ideal $\cap_{j=1}^{m-1} \ker P^{(j)}$ so are $h_{k,n}$, hence $F(h_{k,n}) = 0$. By the same proposition and by (14), these polynomials are uniformly bounded and belong to a finite-dimensional space of polynomials of degree $\leq (mm + \deg p) + \deg f$. Thus, the convergence (16) on this space is uniform, and (16) gives

$$\lim_{n\to\infty} \Bigg(\sum_{\mathbf{x}\in\mathcal{Y}_n} a_{\mathbf{x}}^{(n)} h_{k,n}(\mathbf{x}) + \sum_{\mathbf{x}\in\mathcal{Z}_n} a_{\mathbf{x}}^{(n)} h_{k,n}(\mathbf{x}) \Bigg) = F(h_{k,n}) = 0. \tag{18}$$

Furthermore, since $\omega_{\mathcal{Y}_n, \mathbf{u}_m}$ vanishes on $\mathcal{Y}_n$, it follows that

$$\lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{Z}_n} a_{\mathbf{x}}^{(n)} (p(\mathbf{x}) \cdot \omega_{\mathcal{Y}_n, \mathbf{u}_m}(\mathbf{x}))^k f(\mathbf{x}) = \lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{Z}_n} a_{\mathbf{x}}^{(n)} h_{k,n}(\mathbf{x}) = 0, \tag{19}$$

for $k = 1, ...m$. Finally, since $\mathbf{u}_m$ is the limit point of $\mathcal{Z}_n$

$$\lim_{n\to\infty} \left(p(\mathbf{x}_n) \cdot \omega_{\mathcal{Y}_n,\mathbf{u}_m}(\mathbf{x}_n)\right) = 1 \text{ whenever } \mathbf{x}_n \in \mathcal{Z}_n.$$

Setting $\gamma_n = p(\mathbf{x}_n) \cdot \omega_{\mathcal{Y}_n,\mathbf{u}_m}(\mathbf{x}_n)$ for $x_n \in \mathcal{Z}_n$ and applying Lemma 1, we conclude that

$$\lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{Z}_n} a_{\mathbf{x}}^{(n)} f(\mathbf{x}) = 0$$

Thus, we eliminate the points that accumulate at $\mathbf{u}_m$ from the sum in (16). Since this implies (6) holds for $P^{(1)} \oplus \ldots \oplus P^{(m-1)}$, we can repeat this procedure, eliminating all points from $\mathcal{X}_n$ in the sum (13) that accumulates at $\mathbf{u}_{m-1}, \ldots, \mathbf{u}_2$. We arrive at the sequences of sets $\mathcal{X}_n^{(1)} = \{x_1^{(n)}, \ldots, x_{N_n^{(1)}}^{(n)}\} \subset \mathcal{X}_n$ that have an accumulation point at $\mathbf{u}_1$ such that every F in $\ker^\perp P^{(1)}$

$$F(f) = \lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{X}_n^{(1)}} a_{\mathbf{x}}^{(n)} \delta_{\mathbf{x}}(f) = \lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{X}_n^{(1)}} a_{\mathbf{x}}^{(n)} f(\mathbf{x}).$$

Thus, for sufficiently large n, the dimension of this space must be greater or equal to the dimension of the space $\ker^\perp P^{(1)}$. Hence $\left|\mathcal{X}_n^{(1)}\right| \geq \dim\ker^\perp P^{(1)}$. Repeating this procedure for the rest of the points $\mathbf{u}_j \in \mathcal{V}(\ker P)$, we will obtain a disjoint partition of $\mathcal{X}_n$:

$$\mathcal{X}_n = \cup_{j=1}^m \mathcal{X}_n^{(j)}$$

such that for every $F \in \ker^\perp P^{(j)}$

$$F(f) = \lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{X}_n^{(j)}} a_{\mathbf{x}}^{(n)} \delta_{\mathbf{x}}(f) = \lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{X}_n^{(j)}} a_{\mathbf{x}}^{(n)} f(\mathbf{x})$$

and

$$\left|\mathcal{X}_n^{(j)}\right| \geq \dim\ker^\perp P^{(j)}.$$

However, for every n

$$\sum_{j=1}^m \left|\mathcal{X}_n^{(j)}\right| = N = \sum_{j=1}^m \dim\ker^\perp P^{(j)}.$$

Hence, for sufficiently large n, we have $\left|\mathcal{X}_n^{(j)}\right| = \dim(\ker^\perp P^{(j)})$. By Corollary 2, we obtain that each $P^{(j)}$ is Hermite. □

4. Some Applications

The first application of the main theorem is to show the existence of a symmetric ideal projector (in three or more variables) that is not Hermite.

Definition 3. *An ideal J in $\mathbb{C}[\mathbf{x}]$ is called symmetric if for any polynomial $p(x_1, \ldots, x_d)$ in J and for any permutation σ on the set $\{1, \ldots, d\}$, the polynomial $p(x_{\sigma(1)}, \ldots, x_{\sigma(d)})$ is also in J. An ideal projector P is called symmetric if* $\ker P$ *is a symmetric ideal.*

Theorem 7. *In three or more variables, there exists a finite-dimensional symmetric ideal projector that is not Hermite.*

Proof. The result follows from the existence of a finite-dimensional non-Hermite ideal projector Q such that $\ker Q$ is primary, i.e., $\mathcal{V}(\ker Q)$ consists of one point. We chose such

Q so that $\mathcal{V}(\ker Q) = \{(1, 2, \ldots, d)\}$. Now, for every permutation σ, consider the ideal Q_σ with $\ker Q_\sigma = \{p(x_{\sigma(1)}, \ldots, x_{\sigma(d)}) : p(x_1, \ldots, x_d) \in \ker Q\}$. Then, clearly, the ideal

$$J := \bigcap_\sigma \ker Q_\sigma \tag{20}$$

is a symmetric ideal and (20) is a primary decomposition of this ideal. Let P be an ideal projector with $\ker P$. Then, P is a symmetric ideal projector. If P was a Hermite projector, then, by the Theorem 6, Q would also be Hermite, which gives us a contradiction. □

Problem 1. *Does there exist a non-Hermite ideal projector P such that P is symmetric and $\mathcal{V}(\ker P) = \{\mathbf{0}\}$, i.e.,* $\ker P$ *is primary?*

Our second application concerns linear algebra. A sequence of commuting $N \times N$ matrices $(M_1, \ldots, M_d)$ is called simultaneously diagonalizable if there exists an $N \times N$ matrix S such that the matrices $(S^{-1}M_1S, \ldots, S^{-1}M_dS)$ are diagonal matrices. We have the following.

Theorem 8 ([12])**.** *Let P be an ideal projector. Then, P is a Lagrange projector if and only if the sequence of matrices $\mathbf{M}_P = (M_1, \ldots, M_d)$ associated with P by (4) is simultaneously diagonalizable. The ideal projector P is Hermite if and only if the sequence of matrices $(M_1, \ldots, M_d)$ associated with P by (4) is a limit of a sequence $(M_1^{(n)}, \ldots, M_d^{(n)})$ of simultaneously diagonalizable matrices.*

Commuting matrices that are limits of simultaneously diagonalizable matrices have received a fair amount of attention (cf. [13,19,20]). The following result was proved in [5]:

Theorem 9. *Let P be an ideal projector onto the N-dimensional subspace V and let*

$$P = P^{(1)} \oplus P^{(2)} \oplus \ldots \oplus P^{(m)}$$

be the primary decomposition of P. Then,

(i) *$\mathbf{M}_P$ has a unique (up to order of blocks) block diagonalization $\mathbf{M}_P = diag(\mathbf{M}^{(j)})$ consisting of m blocks and m is the maximal number of blocks in any block-diagonalization of $\mathbf{M}_P$.*
(ii) *Each block $\mathbf{M}^{(j)}$ defines a distinct primary ideal.*

$$\ker P^{(j)} = \{p \in \mathbb{C}[\mathbf{x}] : p(\mathbf{M}^{(j)}) = 0\}$$

Under the assumptions of the above, we can set $\mathbf{M}^{(j)} = (M_1^{(j)}, \ldots, M_d^{(j)})$ (where all $M_1^{(j)}, \ldots, M_d^{(j)}$ have the same size) and $\mathbf{M}_P = (M_1, ..., M_d)$ where

$$M_j = \begin{pmatrix} M_j^{(1)} & 0 & \ldots & 0 \\ 0 & M_j^{(2)} & \ldots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & \ldots & 0 & M_j^{(m)} \end{pmatrix}, \text{for } j = 1, ..., d. \tag{21}$$

Observe that $\mathbf{M}_P = (M_1, ..., M_d)$ defines a sequence of commuting matrices.

It is clear that if each sequence $\mathbf{M}^{(j)} = (M_1^{(j)}, \ldots, M_d^{(j)})$ is a limit of simultaneously diagonalizable matrices $(M_{1,n}^{(j)}, \ldots, M_{d,n}^{(j)})$, then $\mathbf{M}_P$ is a limit of simultaneously diagonalizable matrices. Our main Theorem 6 asserts that the converse is also true. That is, if $\mathbf{M}_P$ is a limit of simultaneously diagonalizable matrices, then the sequences of maximal blocks $\mathbf{M}^{(j)}$ are also limits of simultaneously diagonalizable sequences. This leads to an interesting question about the extension of this result to an arbitrary commuting block-diagonal sequence of matrices.

Problem 2. *Let* $\mathbf{M} = (M_1, \ldots, M_d)$ *be a sequence of commuting matrices such that each* M_j *is block diagonal, i.e., of the form of (21). Let* $\mathbf{M}^{(j)} = (M_1^{(j)}, \ldots, M_d^{(j)})$ *be of the same size and commute. Suppose that* $\mathbf{M}$ *is a limit of simultaneously diagonalizable matrices. Does it imply that each sequence* $\mathbf{M}^{(j)}$ *is a limit of simultaneously diagonalizable matrices?*

Remark 1. *In the language of algebraic geometry, the ideals that serve as kernels of Hermite projectors are called "smoothable" (cf. [14]). Hence, the main result of this paper formulated in this language says that an ideal* J *is smoothable if and only if every ideal in the primary decomposition of* J *is smoothable.*

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