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A More Flexible Extension of the Fréchet Distribution Based on the Incomplete Gamma Function and Applications

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Abstract: In this paper, a more flexible extension of the Fréchet distribution is introduced. The new distribution is defined by means of the stochastic representation as the quotient of two independent random variables, a Fréchet distribution and the power of a random variable, with uniform distribution in the interval (0, 1). We will call this new extension the slash Fréchet distribution and one of its main characteristics is that its tails are heavier than the Fréchet distribution. The general density of this distribution and some basic properties are determined. Its moments, skewness coefficients, and kurtosis are calculated. In addition, the estimation of the model parameters is obtained by the method of moments and maximum likelihood. Finally, three applications with real data are performed by fitting the new model and comparing it with the Fréchet distribution.

Keywords: Fréchet distribution; slash distribution; kurtosis coefficient; moment estimators; maximum likelihood estimator

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1. Introduction

The Fréchet distribution is named after Maurice Fréchet, the French mathematician who developed it in 1927 [1]. This model is also known as the inverse Weibull distribution and is a special case of the generalized distribution of extreme values. The Fréchet model is used to model maximum values in a dataset, such as flood analysis, maximum rainfall, survival analysis, and river discharge in hydrology. More details on the Fréchet distribution can be found in the work by Kotz and Nadarajah [2]. The probability density function of the Fréchet model (Fr) is defined as follows:

$$f_X(x;\alpha) = \alpha x^{-\alpha-1}\exp\{-x^{-\alpha}\},$$

where $x > 0$, and $\alpha > 0$ is the shape parameter, which we denote as $X \sim Fr(\alpha)$. Properties of this distribution are presented as follows:

1. $F_X(x;\alpha) = \exp\{-x^{-\alpha}\}$, where $F_X(\cdot)$ is the cumulative distribution function of X.
2. $Q(p) = (-\log(p))^{-1/\alpha}, \quad 0 < p < 1$. where $Q(\cdot)$ is the quantile function of X.
3. $E(X^r) = \Gamma\left(1 - \frac{r}{\alpha}\right), r = 1, 2, 3, ...$, is the r-th moment of X

Some extensions of the Fréchet distribution that are available in the literature are the exponentiated Fréchet distribution (Nadarajah and Kotz [3]), where the main objective is on providing a complete development of the mathematical properties of this distribution. Here, the theoretical analysis of the inverse Weibull distribution (Khan, M.S. et al. [4]) is performed; it is a flexible model that approximates different distributions when its shape parameter changes. The generalized inverse Weibull distribution (de Gusmão, F.R.S. et al. [5]), specifically the three-parameter version with both decreasing and unimodal failure rates, was studied. On the other hand, I. Elbatal and Hiba Z. Muhammed [6] presented the four-parameter exponential generalized inverse Weibull distribution (EGIW). Badr, M.M. [7]

presented a new class of distributions, called the Beta generalized exponentiated Fréchet distribution, based on the Beta-G family.

On the other hand, another important distribution for the development of this work is the slash distribution, which is represented as the quotient between two independent random variables, a normal distribution and a uniform power (see Johnson et al. [8]). Therefore, we say that X has a slash (S) distribution if its stochastic representation is given by

$$X = \frac{Z}{U^{\frac{1}{q}}},$$

where $Z \sim N(0,1)$ and $U \sim U(0,1)$ are independent random variables and $q > 0$ is the kurtosis parameter. It will be denoted as $X \sim S(q)$ and its density function has the following expression:

$$f_X(x;q) = \frac{2^{\frac{q-2}{2}} q}{\sqrt{\pi}|x|^{q+1}} \, \gamma\left(\frac{q+1}{2}; \frac{x^2}{2}\right), \quad x > 0.$$

where $\gamma(a,z) = \int_0^z w^{a-1} e^{-w} dw$, is the lower incomplete gamma function.

Rogers and Tukey [9] introduce the slash distribution as an alternative distribution to the standard normal distribution, but with heavier tails. Kadafar [10] proposes maximum likelihood estimators for location and scale parameters. Ref. [11] generalizes the slash distribution by introducing the family of slash-elliptic distributions. Genc [12] discusses the symmetric case of a generalization of the slash distribution. Reyes, Gómez, and Bolfarine [13] propose a modification to the classical slash distribution by changing the uniform distribution to an exponential distribution in its stochastic representation, and Rojas, Bolfarine, and Gómez [14] extend the slash distribution by considering a random variable with the Beta distribution in the denominator.

In this work, a new extension of the Fréchet distribution is introduced, with the objective that this new family presents greater flexibility in terms of the kurtosis of the Fr distribution, enabling it to model positive data that display atypical observations. It arises as the quotient of two independent random variables, one being the Fréchet distribution in the numerator and power of the uniform distribution in the denominator. The uniform distribution at (0, 1) produces a slash Fréchet distribution with heavier tails than the Fréchet distribution.

The paper is presented as follows. Section 2 presents the stochastic representation of the model, the density function, some basic properties, and moments and the coefficient of skewness and kurtosis. In Section 3, we obtain the parameter estimators by the method of moments (MM) and maximum likelihood (MV), ending with a simulation study to observe the asymptotic behavior of the MV estimators. In Section 4, we show three illustrations of real datasets. In Section 5, we provide some conclusions.

2. The Slash Fréchet Distribution

2.1. Density Function

Definition 1. *We will say that a random variable Y is slash Fréchet-distributed with shape parameter α and kurtosis parameter q, denoted by $Y \sim SFr(\alpha, q)$, if its stochastic representation is as follows:*

$$Y = \frac{X}{U^{\frac{1}{q}}}, \tag{1}$$

where $X \sim Fr(\alpha)$ and $U \sim U(0,1)$ are independent random variables with $\alpha > 0$ and $q > 0$.

The following proposition presents the density function of the SFr distribution.

Proposition 1. *Let $Y \sim SFr(\alpha, q)$, then the density function of Y is given by:*

$$f_Y(y; \alpha, q) = \frac{q}{y^{q+1}} \Gamma\left(1 - \frac{q}{\alpha}, y^{-\alpha}\right), \tag{2}$$

where $y > 0$, $\alpha > 0$, $q > 0$, $\alpha > q$ and $\Gamma(a, t) = \int_t^{\infty} w^{a-1} e^{-w} dw$ is the upper incomplete gamma function.

Proof. Using the stochastic representation given in (1) and using the random vector transformation method, it follows that

$$\left.\begin{array}{l} Y = \dfrac{X}{U^{\frac{1}{q}}} \\ W = U^{\frac{1}{q}} \end{array}\right\} \Rightarrow \left.\begin{array}{l} X = YW \\ U = W^q \end{array}\right\} \Rightarrow J = \begin{vmatrix} \dfrac{\partial x}{\partial y} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial w} \end{vmatrix} = \begin{vmatrix} w & y \\ 0 & q w^{q-1} \end{vmatrix} = q w^q.$$

Then, $f_{Y,W}(y, w) = |J| f_{X,U}(yw, w^q) = q w^q \alpha (yw)^{-(\alpha+1)} \exp\{-(yw)^{-\alpha}\}$, $0 < w < 1$, $y > 0$, by marginalizing with respect to the random variable W, we have that

$$f_Y(y) = q\alpha \int_0^1 w^q (yw)^{-(\alpha+1)} \exp\{-(yw)^{-\alpha}\} dw,$$

and making the change of variable $t = (yw)^{-\alpha}$, the result is obtained. □

Corollary 1. *If $q = 1$, we will say that Y is canonical slash Fréchet-distributed and its density function is as follows:*

$$f_Y(y; \alpha, 1) = \frac{1}{y^2} \Gamma\left(1 - \frac{1}{\alpha}, y^{-\alpha}\right),$$

where $\Gamma(a, t) = \int_t^{\infty} w^{a-1} e^{-w} dw$ is the upper incomplete gamma function and is denoted as $Y \sim SFr(\alpha, 1)$.

Proof. Making $q = 1$ in Proposition 1, the result is obtained. □

On the left side of Figure 1, the SFr and Fr distributions are shown for $\alpha = 1$ and different values of parameter q; on the right side is a zoomed-in view of the graphical representation of the tails; as the value of parameter q decreases, the density function of the SFr distribution exhibits greater kurtosis.

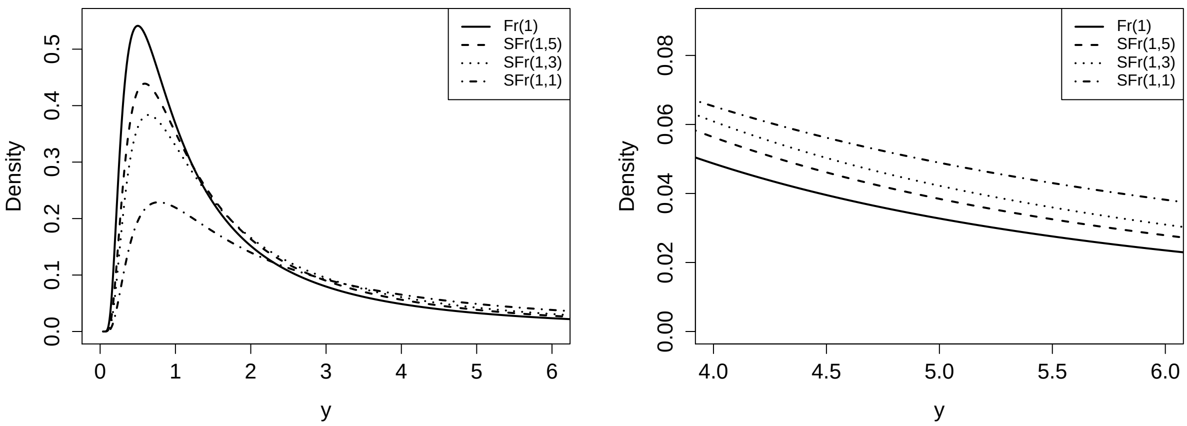

Figure 1. Graphical comparison of the density function of the Fréchet (Fr) and slash Fréchet (SFr) distributions for fixed alpha ($\alpha = 1$) and different values of q.

2.2. Properties

In this subsection, we show some properties of the SFr distribution.

Proposition 2. *Let $Y \sim SFr(\alpha, q)$, then the cumulative distribution function (cdf) of Y is given by:*

$$F_Y(t;\alpha,q) = \frac{q}{\alpha t^q}\Gamma\left(-\frac{q}{\alpha}, t^{-\alpha}\right),$$

where $t > 0$, $\alpha > 0$, $q > 0$, and $\Gamma(a,t) = \int_t^{\infty} w^{a-1}e^{-w}dw$ is the upper incomplete gamma function.

Proof. Using the definition of CDF, we obtain

$$\begin{aligned}
F_Y(t;\alpha,q) &= \int_0^t f_Y(y)dy \\
&= \int_0^t \frac{q}{y^{q+1}}\Gamma\left(1-\frac{q}{\alpha}, y^{-\alpha}\right)dy \\
&= \int_0^t \frac{q}{y^{q+1}}\left(\int_{y^{-\alpha}}^{\infty} t^{-\frac{q}{\alpha}}e^{-t}dt\right)dy.
\end{aligned}$$

Considering the following variable change, $z = y^{\alpha}t$, and developing the integral, the result is obtained. □

Figure 2 shows the graphical comparison of the cdf of the SFr model for ($\alpha = 1$) and different values of q, with the Fr distribution. It can be seen that for smaller values of the q parameter, the growth of the cdf in the SFr distribution is slower, which implies greater flexibility when working with data with high kurtosis.

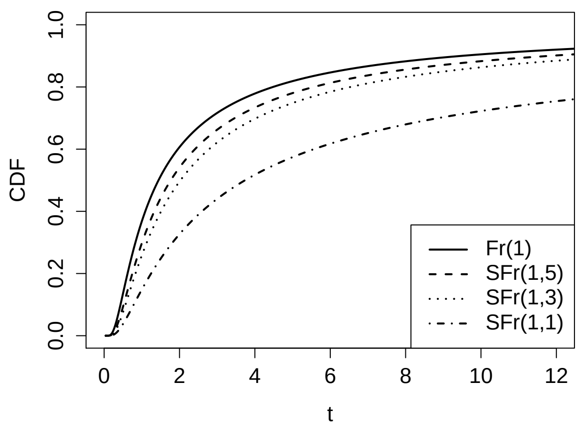

Figure 2. Graphical comparison of the CDF between the Fréchet (Fr) and slash Fréchet (SFr) distribution for the fixed alpha ($\alpha = 1$) and different values of q.

Proposition 3. *Let $Y \sim SFr(\alpha, q)$, then the survival function and the hazard function of Y, respectively, are given by*

$$S_Y(t;\alpha,q) = \frac{\alpha t^q - q\Gamma\left(-\frac{q}{\alpha}, t^{-\alpha}\right)}{\alpha t^q},$$

$$h_Y(t;\alpha,q) = \frac{\alpha q t^q\,\Gamma\left(1-\frac{q}{\alpha}, t^{-\alpha}\right)}{t^{q+1}\left(\alpha t^q - q\,\Gamma\left(-\frac{q}{\alpha}, t^{-\alpha}\right)\right)},$$

where $t > 0$, $\alpha > 0$, $q > 0$.

Proof. Using the definitions of the survival function and hazard function,

$$S_Y(t;\alpha,q) = 1 - F_Y(t;\alpha,q); \quad h_Y(t;\alpha,q) = \frac{f_Y(t;\alpha,q)}{1-F_Y(t;\alpha,q)}.$$

Substituting $f_Y(t;\alpha,q)$ and $F_Y(t;\alpha,q)$, we obtain the result. $\square$

Table 1 shows $P(Y > y)$ for different values of y for the mentioned models, where it is observed that the SFr distribution presents heavier tails than the Fr distribution.

Table 1. Comparison of values of the survival function between the SFr and Fr distributions for $\alpha = 1$ and q = 1, 3, 5, 10.

$P(Y > 10)$	$P(Y > 11)$	$P(Y > 12)$	$P(Y > 13)$	$P(Y > 14)$	$P(Y > 15)$
Fr (1)	0.0952	0.0869	0.0800	0.0740	0.0689
SFr (1, 10)	0.1051	0.0960	0.0884	0.0819	0.0763
SFr (1, 5)	0.1171	0.1070	0.0986	0.0914	0.0852
SFr (1, 3)	0.1368	0.1253	0.1157	0.1074	0.1002
SFr (1, 1)	0.2775	0.2605	0.2457	0.2327	0.2212

Figure 3 shows the survival function (left side) and the hazard function (right side) for $\alpha = 2$ and different values of q, compared to the Fr distribution. It can be seen that as parameter q increases, the SFr distribution has heavier tails than the Fr distribution.

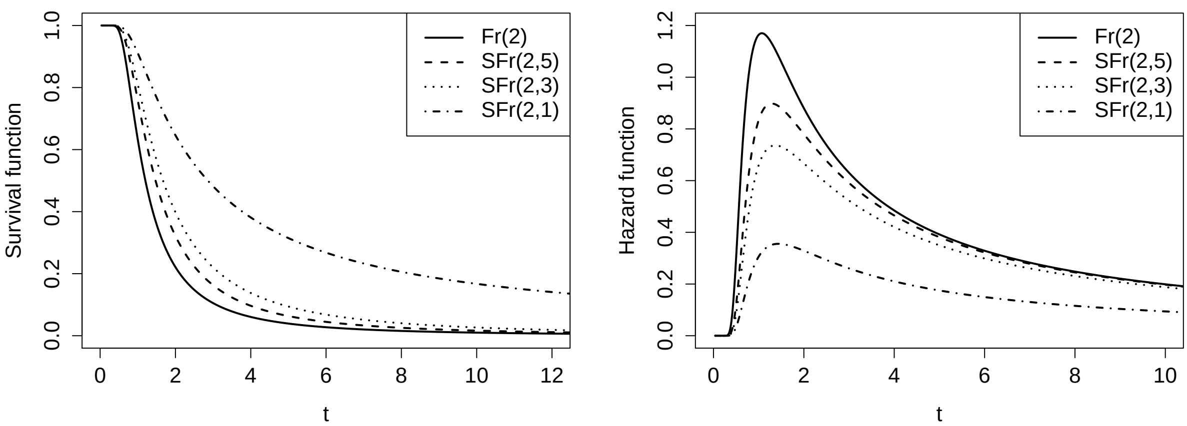

Figure 3. Graphs of the survival function and hazard function for the SFr distribution with $\alpha = 2$ and different values of q.

Proposition 4. *Let* $Y|W = w \sim Fr\left(\alpha, w^{-\alpha/q}\right)$ *and* $W \sim U(0,1)$*, then* $Y \sim SFr(\alpha, q)$*.*

Proof. The marginal density function of Y is given by:

$$\begin{aligned} f_Y(y;\alpha,q) &= \int_0^1 f_{Y|W}(y|w) \cdot f_W(w)\, dw \\ &= \int_0^1 w^{-\alpha/q} \alpha y^{-\alpha-1} e^{-w^{-\alpha/q} y^{-\alpha}} dw. \end{aligned}$$

Considering the change of variable $u = w^{-\alpha/q}$, the result is obtained. $\square$

Proposition 5. *Let* $Y \sim SFr(\alpha, q)$*. If* $q \to \infty$*, then* Y *converges in distribution to the random variable* $X \sim Fr(\alpha)$*.*

Proof. Let $Y \sim SFr(\alpha, q)$ and $Y = \frac{X}{U^{1/q}}$ given in (1). First, the probability convergence of $U^{1/q}$ is studied. We have that $U \sim U(0,1)$, and if $W = U^{1/q}$, then $W \sim Beta(q,1)$; therefore, the following is obtained:

$$E[(W-1)^2] = \frac{2}{(q+1)^2(q+2)},$$

where if $q \to \infty \Rightarrow E[(W-1)^2] \to 0$; therefore, $W \xrightarrow{P} 1$ (see Lehmann [15]), where $\xrightarrow{P}$ denotes convergence in probability. Finally, applying Slutsky's theorem [15] for $Y = \frac{X}{W}$, we have that $Y \xrightarrow{D} X \sim Fr(\alpha)$, where $\xrightarrow{D}$ denotes the convergence in distribution. □

2.3. Moments

Proposition 6. *Let $Y \sim SFr(\alpha, q)$, then the moment of order r of Y is given by*

$$\mu_r = E[Y^r] = \frac{q}{q-r}\Gamma\left(1 - \frac{r}{\alpha}\right), \text{ with } r = 1, 2, \ldots \text{ and } q, \alpha > r.$$

Proof. Using the stochastic representation given in (1) and considering that X and U are independent random variables, we have:

$$\begin{aligned}
\mu_r &= E[Y^r] \\
&= E\left[\left(\frac{X}{U^{\frac{1}{q}}}\right)^r\right] \\
&= E\left[X^r \cdot U^{-\frac{r}{q}}\right] \\
&= E[X^r] \cdot E[U^{-\frac{r}{q}}],
\end{aligned}$$

where $E[U^{-\frac{r}{q}}] = \frac{q}{q-r}$, $q > r$ and $E[X^r] = \Gamma(1 - \frac{r}{\alpha})$, $\alpha > r$, are the moments of order r of $U^{-\frac{1}{q}}$ and X, respectively, where $U \sim U(0,1)$ and $X \sim Fr(\alpha)$. □

Corollary 2. *If $Y \sim SFr(\alpha, q)$, then it follows that*

$$\mu_1 = E[Y] = \frac{q}{q-1}\Gamma\left(1 - \frac{1}{\alpha}\right), \qquad q, \alpha > 1. \tag{3}$$

$$\mu_2 = E[Y^2] = \frac{q}{q-2}\Gamma\left(1 - \frac{2}{\alpha}\right), \qquad q, \alpha > 2. \tag{4}$$

$$\mu_3 = E[Y^3] = \frac{q}{q-3}\Gamma\left(1 - \frac{3}{\alpha}\right), \qquad q, \alpha > 3.$$

$$\mu_4 = E[Y^4] = \frac{q}{q-4}\Gamma\left(1 - \frac{4}{\alpha}\right), \qquad q, \alpha > 4.$$

Proof. Replacing $r = 1, 2, 3, 4$ in Proposition 6, the result is obtained. □

Corollary 3. *If $Y \sim SFr(\alpha, q)$, then the expectation and variance of Y are given by*

$$E[Y] = \frac{q}{q-1}\Gamma\left(1 - \frac{1}{\alpha}\right), \qquad q, \alpha > 1$$

$$V(Y) = \frac{q}{q-2}\Gamma\left(1 - \frac{2}{\alpha}\right) - \left[\frac{q}{q-1}\Gamma\left(1 - \frac{1}{\alpha}\right)\right]^2, \qquad q, \alpha > 2.$$

Proof. Using μ_1 and μ_2 from Corollary 2, considering $E[Y] = \mu_1$ and $V(Y) = \mu_2 - \mu_1^2$, the result is obtained. □

Proposition 7. *Let $Y \sim SFr(\alpha, q)$, then the skewness coefficient of Y is given by*

$$\sqrt{\beta_1} = \frac{P_3 - 3qP_1P_2 + 2q^2P_1^3}{\sqrt{q}\left[P_2 - qP_1^2\right]^{3/2}}, \quad q > 3$$

where $P_r = \frac{\Gamma\left(1-\frac{r}{\alpha}\right)}{q-r}, \quad q, \alpha > r.$

Proof. Using the definition of the standardized skewness coefficient

$$\sqrt{\beta_1} = \frac{E[(Y - E(Y))^3]}{(V(Y))^{3/2}} = \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3}{(\mu_2 - \mu_1^2)^{3/2}},$$

and substituting μ_1, μ_2 and μ_3 in Corollary 2 and $P_r = \frac{\Gamma\left(1-\frac{r}{\alpha}\right)}{q-r}$, the result is obtained. □

The left side of Figure 4 shows the behavior of the skewness coefficient as a function of parameters α and q, where it is observed that as the value of q decreases, the value of the skewness coefficient increases. In addition, on the right side of Figure 4, it is shown that when parameter q tends to ∞, the value of the skewness coefficient of the SFr distribution tends to the value of the skewness coefficient of the Fr distribution.

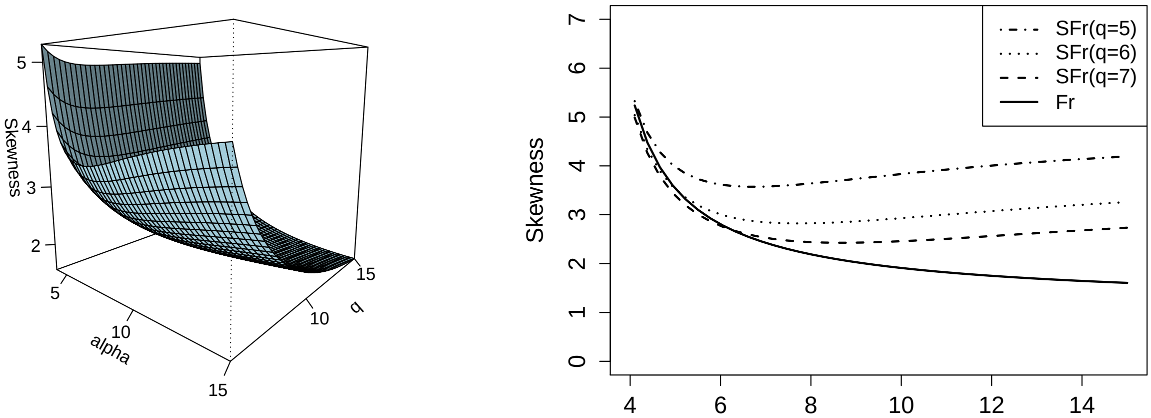

Figure 4. Skewness coefficient plot of the SFr model (**left side**). Comparison of the skewness coefficient between SFr and Fr for different values of q (**right side**).

Proposition 8. *Let $Y \sim SFr(\alpha, q)$, then the kurtosis coefficient of Y is given by*

$$\beta_2 = \frac{P_4 - 4qP_1P_3 + 6q^2P_1^2P_2 - 3q^3P_1^4}{q\left[P_2 - qP_1^2\right]^2}, \quad q > 4$$

where $P_r = \frac{\Gamma\left(1-\frac{r}{\alpha}\right)}{q-r}, \quad q, \alpha > r.$

Proof. Using the definition of the standardized kurtosis coefficient

$$\beta_2 = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_2\mu_1^2 - 3\mu_1^4}{(\mu_2 - \mu_1^2)^2},$$

and substituting the expressions obtained in Corollary 2 and $P_r = \frac{\Gamma\left(1-\frac{r}{\alpha}\right)}{q-r}$, the result is obtained. □

The left side of Figure 5 shows the behavior of the kurtosis coefficient as a function of parameters α and q, where it is observed that as the value of q decreases, the kurtosis coefficient increases. Furthermore, on the right side of Figure 5, it is observed that when parameter q tends to ∞, the value of the kurtosis coefficient of the SFr distribution tends to the value of the kurtosis coefficient of the Fr distribution.

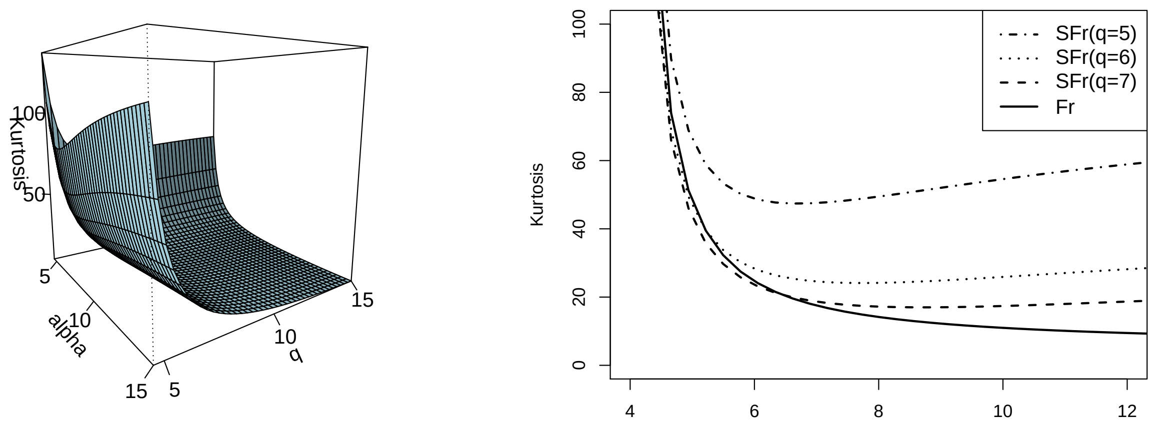

Figure 5. Plot of the kurtosis coefficient for the SFr model (**left side**). Comparison of the kurtosis coefficient between the SFr and Fr models for different values of q (**right side**).

2.4. Some Mathematical Properties

In this subsection, we show some mathematical properties of the SFr model, such as the order statistics, the first incomplete moment, and the Lorenz curve.

Proposition 9. *Let $Y_{(1)}, \ldots, Y_{(n)}$ denote the order statistics of a random variable of $Y_1, \ldots, Y_n$ with $Y \sim SFr(\alpha, q)$. Then, the pdf of $Y_{(j)}$ is as follows:*

$$f_{Y_{(j)}}(y) = \frac{n!}{(j-1)!(n-j)!} \frac{q}{y^{q+1}} G_0 \left[\frac{q}{\alpha y^q} G_1 \right]^{j-1} \left[1 - \frac{q}{\alpha y^q} G_1 \right]^{n-j},$$

In particular, the pdf of the minimum, $Y_{(1)}$, is

$$f_{Y_{(1)}}(y) = \frac{nq}{y^{q+1}} G_0 \left[1 - \frac{q}{\alpha y^q} G_1 \right]^{n-1},$$

and the pdf of the maximum, $Y_{(n)}$, is

$$f_{Y_{(n)}}(y) = \frac{nq}{y^{q+1}} G_0 \left[\frac{q}{\alpha y^q} G_1 \right]^{n-1},$$

where $G_0 = \Gamma\left(1 - \frac{q}{\alpha}, y^{-\alpha}\right)$ and $G_1 = \Gamma\left(-\frac{q}{\alpha}, y^{-\alpha}\right)$.

Proof. Since we are dealing with an absolutely continuous model, the pdf of the j-th order statistics is obtained by applying

$$f_{Y_{(j)}}(y) = \frac{n!}{(j-1)!(n-j)!} f(y) [F(y)]^{j-1} [1 - F(y)]^{n-j},$$

where F and f are the cdf and pdf of the SFr distribution. □

Proposition 10. *Let $Y \sim SFr(\alpha, q)$. Then, the first incomplete moment of Y is given by:*

$$m_1(y; \alpha, q) = \frac{q}{1-q} \left[y^{1-q} \Gamma\left(1 - \frac{q}{\alpha}, y^{-\alpha}\right) - \Gamma\left(1 - \frac{1}{\alpha}, y^{-\alpha}\right) \right], \qquad y > 0. \tag{5}$$

Proof. Using the definition of the first incomplete moment and substituting the density given in (2), we have

$$\begin{aligned} m_1(y;\alpha,q) &= \int_0^\infty t f_Y(t;\alpha,q)dt \\ &= q\int_0^\infty t^{-q}\Gamma\left(1-\frac{q}{\alpha}, t^{-\alpha}\right)dt. \end{aligned}$$

Then, integrating by parts using $u = \Gamma(1-\frac{q}{\alpha}, t^{-\alpha})$ and $v = \dfrac{t^{1-q}}{1-q}$, the result is obtained. □

Proposition 11. *Let $Y \sim SFr(\alpha,q)$. Then, the Lorenz curve, $L(y;\alpha,\beta)$, can be obtained*

$$L(y;\alpha,\beta) = \frac{q}{3(1-q)}\left[y^{1-q}\Gamma\left(1-\frac{q}{\alpha}, y^{-\alpha}\right) - \Gamma\left(1-\frac{1}{\alpha}, y^{-\alpha}\right)\right].$$

Proof. Using the definition of the Lorenz curve in terms of the first incomplete moment, we have

$$L(y;\alpha,\beta) = \frac{m_1(y;\alpha,q)}{\rho},$$

Replacing $m_1(y;\alpha,q)$ obtained in (5) and considering $\rho = 3$, the result is obtained. □

3. Estimation

In this section, we study two methods of estimating the parameters of the slash Fréchet distribution. First, the method of moments and the maximum likelihood method are used and then a simulation study is performed using the maximum likelihood method.

3.1. Moment Estimators

Proposition 12. *Let $Y_1, ..., Y_n$ be a random sample of the random variable Y with distribution $SFr(\alpha,q)$, the moment estimators for $\theta = (\alpha,q)$ can then be obtained by numerically solving the following nonlinear system of equations:*

$$\widehat{q}_M = \frac{\overline{Y}}{\overline{Y} - \Gamma\left(1-\frac{1}{\widehat{\alpha}_M}\right)}, \tag{6}$$

$$2\overline{Y^2}\left[\overline{Y} - \Gamma\left(1-\frac{1}{\widehat{\alpha}_M}\right)\right] - \overline{Y}\left[\overline{Y^2} - \Gamma\left(1-\frac{2}{\widehat{\alpha}_M}\right)\right] = 0, \tag{7}$$

where $\overline{Y}$ and $\overline{Y^2}$ are the first two sample moments of Y.

Proof. Using Equations (3) and (4) in Corollary 2 and equating the sample moments to the population moments, we have

$$\overline{X} = \frac{q}{q-1}\Gamma\left(1-\frac{1}{\alpha}\right), \tag{8}$$

$$\overline{X^2} = \frac{q}{q-2}\Gamma\left(1-\frac{2}{\alpha}\right), \tag{9}$$

by solving Equation (8) for q, we obtain $\widehat{q}_M$ given in (6). Then, substituting $\widehat{q}_M$ into Equation (9), we obtain the equation given in (7). By utilizing numerical methods and the `"uniroot"` function of the R software, we obtain $\widehat{\alpha}_M$; replacing $\widehat{\alpha}_M$ in Equation (6), we obtain $\widehat{q}_M$. □

3.2. Maximum Likelihood Estimators

Let $Y_1, \ldots, Y_n$, be a random sample of size n of a random variable Y with distribution $SFr(\alpha,q)$, then the log-likelihood function for $\theta = (\alpha,q)^T$ can be expressed as follows:

$$\ell(\theta, y_i) = n\log(q) - (q+1)\sum_{i=1}^{n}\log(y_i) + \sum_{i=1}^{n}\log G(y_i), \tag{10}$$

where $G(y_i) = \Gamma\left(1 - \frac{q}{\alpha}, y_i^{-\alpha}\right)$.

Partially deriving the log-likelihood function with respect to α and q, and setting them equal to zero, we obtain the normal equations:

$$\frac{\partial\ell(\theta, y_i)}{\partial\alpha} = \sum_{i=1}^{n}\frac{G_1(y_i)}{G(y_i)} = 0 \tag{11}$$

$$\frac{\partial\ell(\theta, y_i)}{\partial q} = \frac{n}{q} - \sum_{i=1}^{n}\log(y_i) + \sum_{i=1}^{n}\frac{G_2(y_i)}{G(y_i)} = 0 \tag{12}$$

where

$$G_1(y_i) = \frac{\partial G(y_i)}{\partial\alpha} = \frac{\partial\Gamma\left(1 - \frac{q}{\alpha}, y_i^{-\alpha}\right)}{\partial\alpha}$$

$$G_2(y_i) = \frac{\partial G(y_i)}{\partial q} = \frac{\partial\Gamma\left(1 - \frac{q}{\alpha}, y_i^{-\alpha}\right)}{\partial q}$$

The solutions for Equations (11) and (12) can be obtained using numerical methods, such as the Newton–Raphson algorithm. An alternative to obtaining the maximum likelihood estimator is to maximize Equation (10) using the `optim` function of the R software [16].

3.3. Simulation Study

In this section, we study the behaviors of the maximum likelihood estimators for parameters α and q. Moreover, 2000 samples of sizes 50, 100, 150, 200, 250, and 300 were used for the slash Fréchet distribution, and in each one, parameters α and q were estimated. In addition, the mean of the estimators ($\hat{\alpha}$ and $\hat{q}$), the mean of the standard errors (sd), and the coverage percentage (C) were calculated. The results are shown in Table 2. Next, the algorithm used to generate random samples of $Y \sim SFr(\alpha, q)$ is developed.

1. Generate $W \sim U(0,1)$.
2. Compute $X = (-\log(W))^{-1/\alpha}$.
3. Generate $U \sim U(0,1)$.
4. Compute $Y = \dfrac{X}{U^{1/q}}$.

Table 2 shows that as the sample size increases, the mean of the standard errors decreases and the values of the estimators approach the values of parameters α and q, indicating that the estimators are consistent. On the other hand, the coverage percentages approach the nominal values with which they were constructed (95%).

Table 2. Simulation of 2000 samples for the $SFr(\alpha, q)$ model.

n	α	q	$\hat{\alpha}$	$sd(\hat{\alpha})$	$C(\hat{\alpha})$	$\hat{q}$	$sd(\hat{q})$	$C(\hat{q})$
50	0.5	0.5	0.5544	0.1156	97.35	0.5329	0.1574	90.45
100			0.5225	0.0676	96.05	0.5141	0.0972	92.65
150			0.5148	0.0536	95.65	0.5074	0.0775	93.20
200			0.5109	0.0455	95.65	0.5061	0.0664	93.75
250			0.5068	0.0401	95.15	0.5052	0.0593	94.30
300			0.5068	0.0367	95.20	0.5035	0.0539	95.05
50	0.7	0.4	0.8964	0.4138	97.60	0.4106	0.0833	93.20
100			0.7438	0.1289	96.95	0.4059	0.0570	94.80
150			0.7307	0.1006	95.75	0.4032	0.0457	94.65
200			0.7205	0.0845	95.25	0.4025	0.0394	94.80
250			0.7160	0.0747	96.25	0.4022	0.0351	95.10
300			0.7151	0.0681	95.45	0.4007	0.0318	93.95

Table 2. *Cont.*

n	α	q	$\hat{\alpha}$	$sd(\hat{\alpha})$	$C(\hat{\alpha})$	$\hat{q}$	$sd(\hat{q})$	$C(\hat{q})$
50	1	1	1.1888	0.3298	96.90	1.0482	0.2915	90.1
100			1.0416	0.1342	95.75	1.0178	0.1910	94.1
150			1.0276	0.1066	95.40	1.0159	0.1550	93.3
200			1.0200	0.0909	95.00	1.0088	0.1322	92.8
250			1.0147	0.0805	95.35	1.0096	0.1187	94.4
300			1.0122	0.0731	95.30	1.0073	0.1078	94.4
50	3	2	3.5152	2.6864	97.25	2.0399	0.4426	93.25
100			3.1753	0.5104	96.30	2.0275	0.3054	94.80
150			3.1360	0.4020	96.20	2.0132	0.2444	93.75
200			3.0860	0.3375	95.45	2.0124	0.2113	94.85
250			3.0645	0.2973	96.05	2.0127	0.1885	94.60
300			3.0498	0.2694	96.25	2.0065	0.1712	95.50
50	5	3	6.0466	2.2275	96.55	3.0638	0.6347	93.05
100			5.3670	0.9570	96.45	3.0306	0.4333	94.10
150			5.1994	0.6965	95.70	3.0305	0.3513	94.80
200			5.1491	0.5910	96.35	3.0132	0.3003	95.25
250			5.1065	0.5197	95.25	3.0230	0.2699	94.65
300			5.0991	0.4726	95.45	3.0158	0.2451	95.30
50	2.3	2	2.5672	0.5551	96.65	2.0735	0.5295	91.60
100			2.3919	0.3308	96.00	2.0451	0.3593	93.55
150			2.3612	0.2624	95.85	2.0196	0.2835	94.75
200			2.3540	0.2253	95.65	2.0190	0.2449	95.20
250			2.3387	0.1994	95.05	2.0171	0.2186	95.35
300			2.3378	0.1816	95.25	2.0128	0.1983	95.15
50	4.5	5	4.8658	0.8912	96.75	5.2956	1.6178	91.10
100			4.6657	0.5677	96.05	5.1339	1.0337	92.45
150			4.6148	0.4520	95.50	5.1018	0.8302	92.35
200			4.5990	0.3896	94.40	5.0426	0.7024	93.25
250			4.5751	0.3441	95.25	5.0352	0.6242	93.85
300			4.5670	0.3126	94.20	5.0394	0.5719	93.45

Figure 6 shows the log-likelihood profile of the SFr distribution for a random sample of $n = 200$ for values of the parameter $q = 5$ (on the left side) and $q = 1$ (on the right side), where the parameter value shows the maximum likelihood estimator of q that maximizes the log-likelihood function, indicating the good performance of the MLE obtained in Table 2.

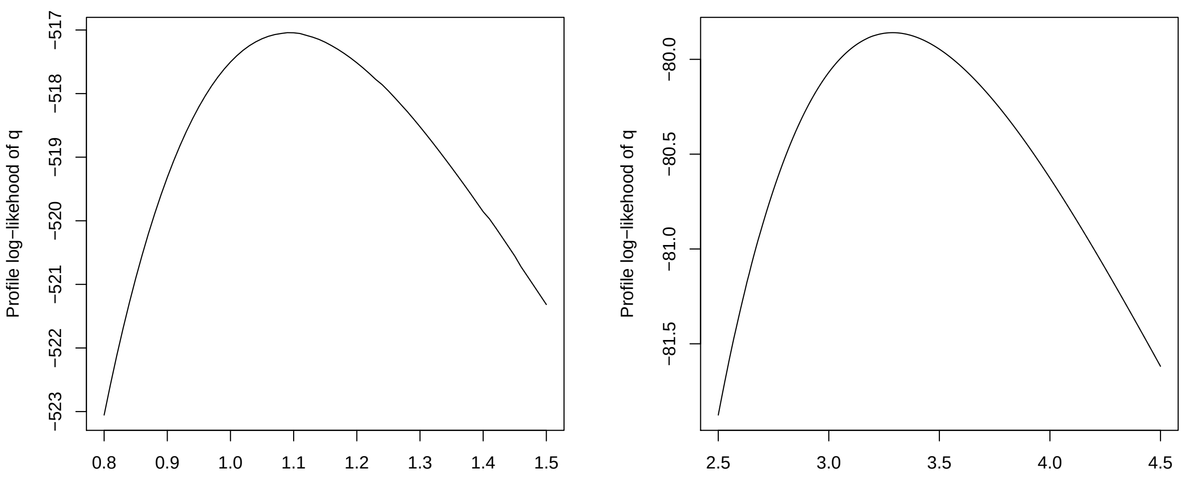

Figure 6. Profile of the log-likelihood of the SFr distribution.

4. Applications

In this section, three applications with real data are presented to compare the fit of the SFr distribution with the Fr model and with other slash distributions. The maximum likelihood method was used to obtain the estimators of the α and q parameters and their estimation errors were calculated through the Hessian matrix. To compare the distributions, the Akaike information criterion [17] (AIC), Bayesian information criterion [18] (BIC), Akaike information criterion consistent [19] (CAIC), and Hannan–Quinn information criterion [20] (HQIC) were considered.

4.1. Application 1 (Patients with Lung Cancer)

The first dataset corresponds to a study conducted by the US Veterans Administration, where the time elapsed between diagnosis and the start of the study (in months) of 137 patients with advanced lung cancer was recorded. This dataset was presented by Kalbfleisch [21] and is available in the `survival` R package [16], labeled as `veteran`.

Table 3 presents the descriptive statistics for this dataset: sample mean, sample standard deviation, sample skewness ($\sqrt{\beta_1}$), and sample kurtosis coefficient (β_2), where we highlight the high level of kurtosis of the data. On the other hand, Figure 7 shows the box plot of the data, showcasing the possible existence of outliers.

Table 3. Descriptive statistics for the dataset of patients undergoing lung cancer.

n	$\bar{x}$	S	$\sqrt{b_1}$	b_2
137	8.7737	10.6121	4.1055	26.3882

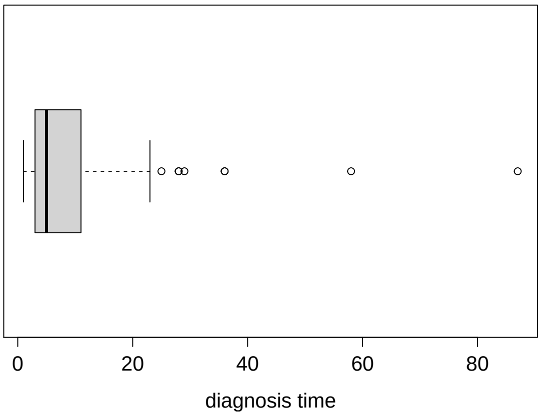

Figure 7. Box plot for the dataset of patients undergoing lung cancer.

Table 4 shows the results of the fit performed, comparing the SFr distribution with the Fréchet (Fr) distribution. It is concluded that the SFr distribution has the best fit for this dataset compared to the Fr distribution because it has lower values in the AIC, BIC, CAIC, and HQIC criteria.

Table 4. Estimates, SE in parenthesis, log-likelihood, AIC, BIC, CAIC, and HQIC values for the dataset of patients undergoing lung cancer.

Parameters	Fr	SFr
α	0.7452 (0.0540)	2.0245 (0.3805)
q	-	0.7382 (0.0812)
log-likelihood	-504.6068	-444.1976
AIC	1011.214	892.3952
BIC	1014.134	898.2351
CAIC	1015.134	900.2351
HQIC	1012.400	894.7684

In Figure 8, the histogram of the dataset of lung cancer patients fitted to the densities of the Fr and SFr distributions is presented. Note that the SFr model fit has heavier tails. Figure 9 illustrates the QQ plots, where it can be seen that the theoretical quantiles of the SFr model are close to the line, $y = x$, when compared to the Fr distribution.

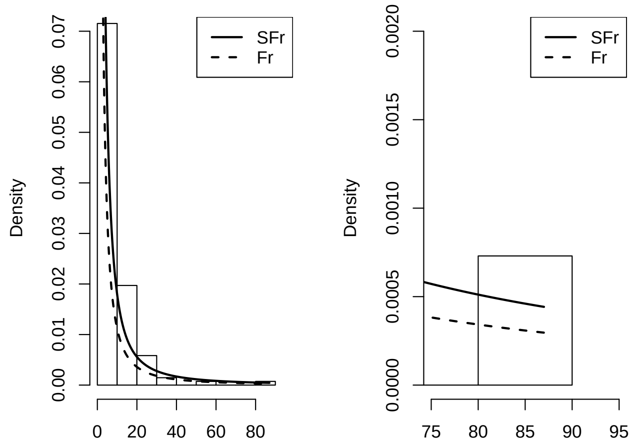

Figure 8. Density adjusted for the dataset of patients undergoing lung cancer in the Fr and SFr distributions.

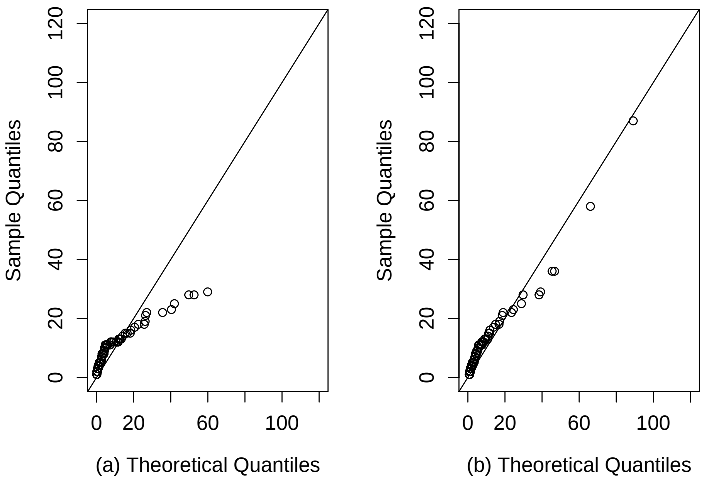

Figure 9. QQ plots for the dataset of patients undergoing lung cancer: (**a**) Fr Model; (**b**) SFr model.

4.2. Application 2 (Patients with Peritoneal Dialysis)

In this application, we consider the slash power-normal (SPN) distribution (see Chen, M. et al. [22]), whose density function is given by

$$f(x, \alpha, q) = q\alpha \int_0^1 [\Phi(xv)]^{\alpha-1} \phi(xv) v^q dv \qquad x, \alpha, q > 0. \tag{13}$$

The dataset presents the survival times (in months) of 64 patients on peritoneal dialysis who attended the University Clinical Hospital of Caracas between 1980 and 1997. This dataset can be obtained in Borges R. [23]. Table 5 presents the descriptive summary of the data and Figure 10 shows a box plot for the dataset of patients undergoing peritoneal dialysis, where atypical observations and high kurtosis can be seen.

Table 5. Descriptive statistics for the dataset of patients undergoing peritoneal dialysis.

n	$\bar{x}$	S	$\sqrt{b_1}$	b_2
64	27.9547	24.9442	1.5772	5.4244

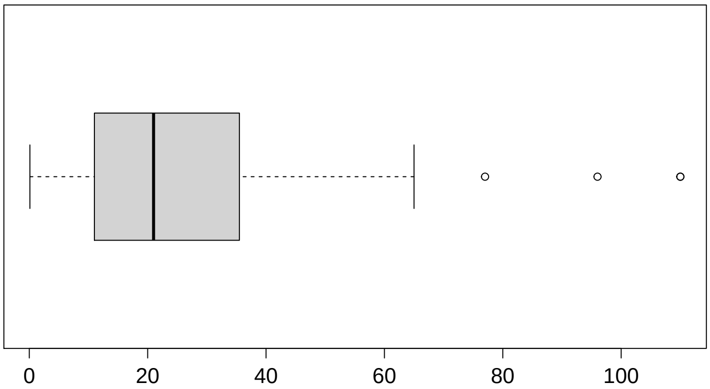

Figure 10. Box plot of the dataset of patients undergoing peritoneal dialysis.

Table 6 shows the results of the fit performed, showing that the SFr distribution fits best with this dataset compared to the Fr and SPN models, because it has lower values in the AIC, BIC, CAIC, and HQIC criteria.

Table 6. Estimates, SE in parenthesis, log-likelihood, AIC, BIC, CAIC, and HQIC values for the dataset of patients undergoing peritoneal dialysis.

Parameters	Fr	SPN	SFr
α	0.4377 (0.0446)	-	0.6679 (0.0767)
σ	-	8.6409 (1.9018)	-
q	-	0.3900 (0.0509)	0.5794 (0.1025)
log-likelihood	−336.0071	−319.3558	−315.4611
AIC	674.0141	642.7116	634.9221
BIC	676.1730	647.0294	639.2399
CAIC	677.1730	649.0294	641.2399
HQIC	674.8646	644.4126	636.6231

Figure 11 shows the histogram of the survival time for the dataset of patients undergoing dialysis, adjusted to the densities of the Fr, SPN, and SFr distributions, where it is evident that the SFr distribution performs a better fit than the other models, specifically on the right tail. On the other hand, Figure 12 shows QQ plots, where the good fit of the SFr distribution is visualized. Figure 13 presents profile log-likelihood functions for parameters α and q for the SFr distribution for application 2, indicating that the profiles behave well in the sense that there is a single maximum with a very pronounced value.

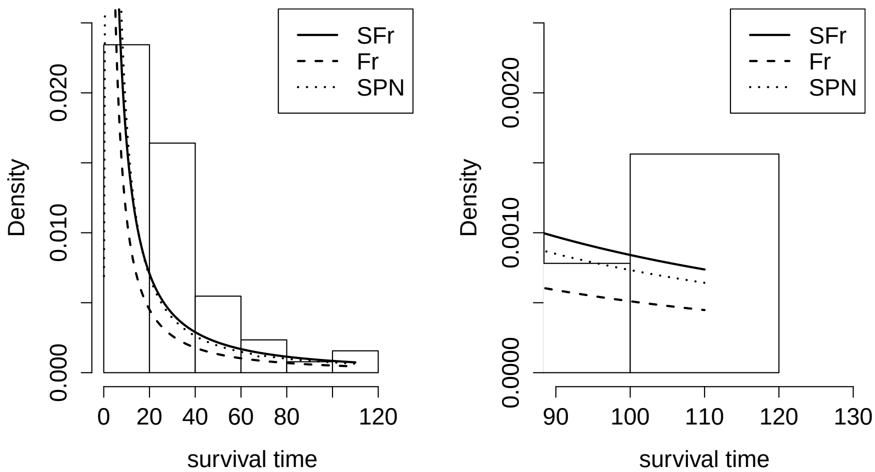

Figure 11. Density adjusted to the dataset of patients undergoing peritoneal dialysis in the Fr, SPN, and SFr distributions.

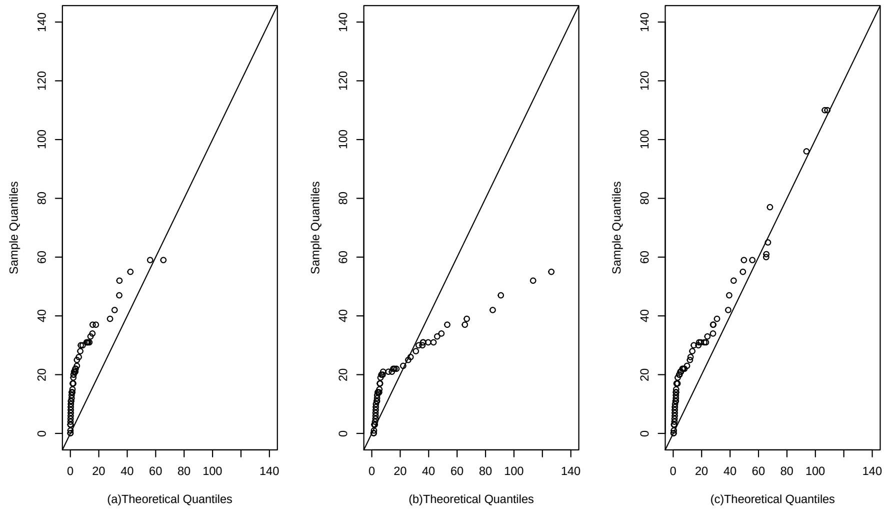

Figure 12. QQ plots for the dataset of patients undergoing peritoneal dialysis: (**a**) Fr model; (**b**) SPN model; (**c**) SFr model.

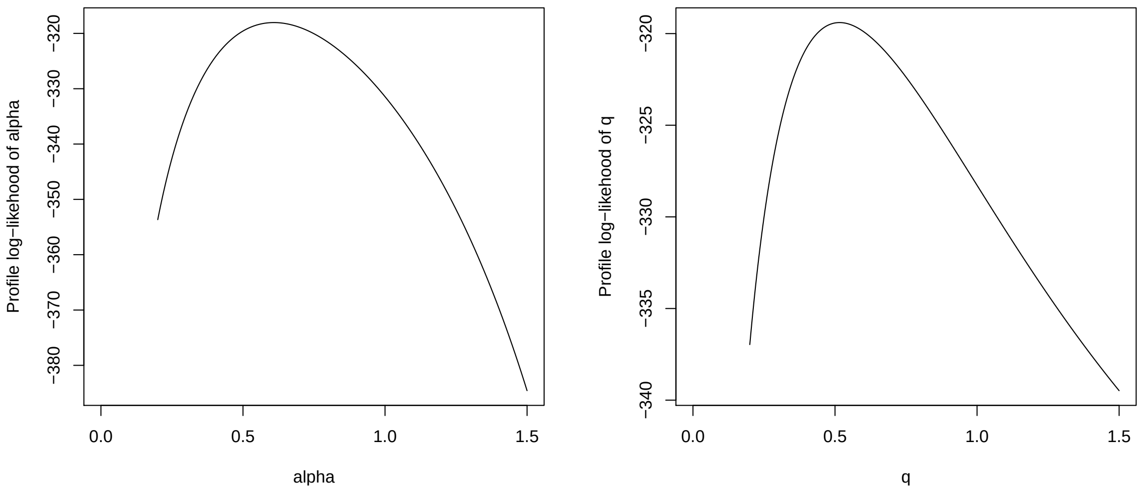

Figure 13. Profile log-likelihoods of α and q for the dataset of patients undergoing peritoneal dialysis.

4.3. Application 3 (Patients with Breast Cancer)

In the third application, we consider the slash half-normal (SHN) distribution (see Olmos, N.M. et al. [24]), whose density function is given by:

$$f(z,\sigma,q) = q\sqrt{\frac{2^q}{\pi}}\sigma^q\Gamma\left(\frac{q+1}{2}\right)z^{-(q+1)}G\left(z^2,\frac{q+1}{2},\frac{1}{2\sigma^2}\right) \qquad z,\sigma,q>0. \tag{14}$$

The dataset comes from the trial carried out between the years 1984 and 1989 by the "German Breast Cancer Study Group (GBSG)" on 686 patients with node-positive breast cancer. For the study, the descriptor of interest is the number of positive lymph nodes in each patient. The description of the study can be found in the work by Schumacher et al. [25] and the dataset is available in the R software package [16] "`survival`" with the database "`gbsg`".

Table 7 presents the descriptive statistics of the data and Figure 14 shows a box plot for the dataset of positive lymph nodes, where atypical observations and high kurtosis can be seen.

Table 7. Descriptive statistics for the dataset of patients undergoing breast cancer.

n	$\bar{x}$	S	$\sqrt{b_1}$	b_2
686	5.0102	5.4755	2.8784	16.2079

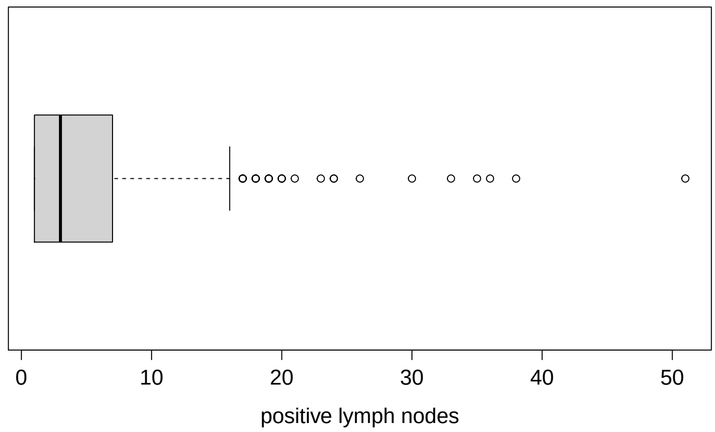

Figure 14. Box plot for the dataset of patients undergoing breast cancer.

Table 8 shows the results of the fit, comparing the SF distribution with the SHN and Fr distributions; in this case, it can be concluded that the SF distribution fits best with this dataset because it has lower values in the AIC, BIC, CAIC, and HQIC criteria.

Table 8. Estimates, SE in parenthesis, log-likelihood, AIC, BIC, CAIC, and HQIC values for the dataset of patients undergoing breast cancer.

Parameters	Fr	SHN	SFr
α	1.0452 (0.0348)	-	2.2209 (0.1934)
σ	-	3.2493 (0.2384)	-
q	-	1.9260 (0.2031)	1.1304 (0.0631)
log-likelihood	−1905.598	−1790.6070	−1712.0270
AIC	3813.196	3585.215	3428.054
BIC	3817.727	3594.277	3437.116
CAIC	3818.727	3596.277	3439.116
HQIC	3814.949	3588.721	3431.561

Figure 15 shows the histogram of the positive lymph node dataset fitted to the densities of the F, SHN, and SF distributions, where it can be seen that the SF distribution better captures the atypical data.

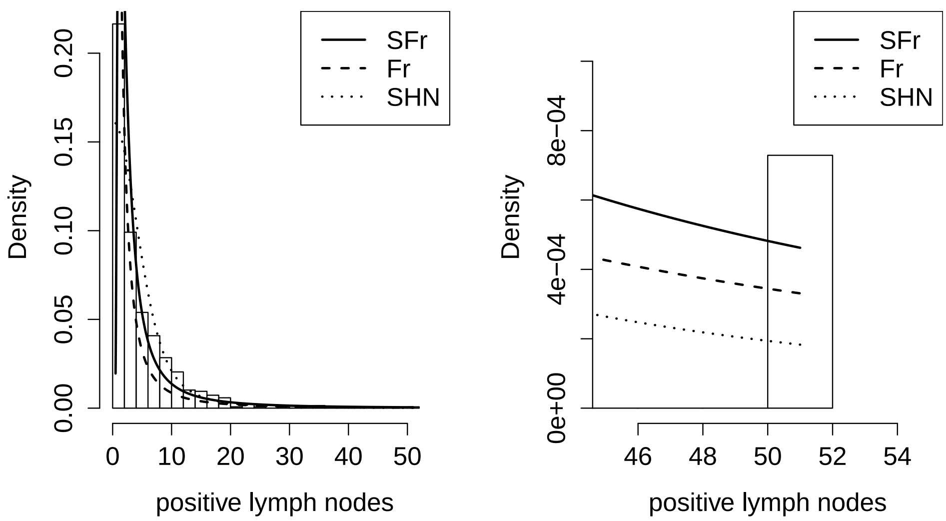

Figure 15. Density adjusted for the dataset of patients undergoing breast cancer in the Fr, SHN, and SFr distributions.

On the other hand, Figure 16 shows the QQ plots of the fitted models. From these results, it can be seen that the SFr distribution provides a better fit than the other distributions in comparison.

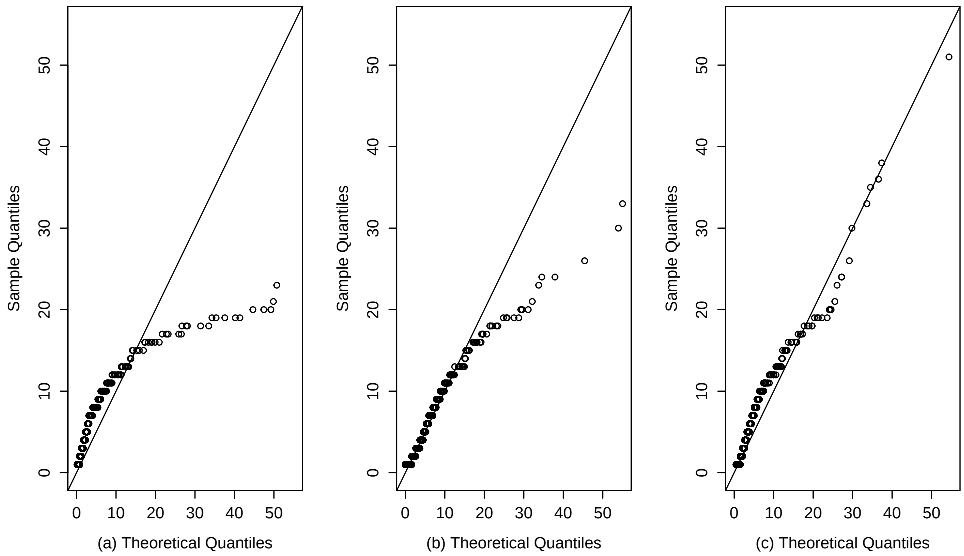

Figure 16. QQ plots for the dataset of patients undergoing breast cancer: (**a**) Fr model; (**b**) SHN model (**c**) SFr model.

5. Conclusions

In this work, a new distribution is studied that is an extension of the Fréchet distribution, which shows greater flexibility in the modeling of the kurtosis coefficient.

When carrying out the study of the slash Fréchet distribution, the following is concluded:

- A new extension of the Fréchet distribution with the density function, cumulative distribution function, survival function, and hazard function is obtained explicitly (closed) in terms of the incomplete gamma function.
- The moments, expectations, and variances of this new distribution were obtained, leading to closed expressions for all of them.
- By observing the skewness and kurtosis coefficients, it can be seen that the SFr model is more flexible than the Fr model. Furthermore, as shown in Table 1, the tails of the distribution become heavier when parameter q is smaller.
- Analyzing the stochastic representation of the SFr model, it is observed that the SFr distribution is a scale mixture of the Fr and $U(0,1)$ distribution.
- In the simulation study, it is observed that as the sample size increases, the maximum likelihood estimators are closer to the parameter values, suggesting consistent and stable estimators.
- In applications with real data, the SFr distribution demonstrates superior fits compared to the Fr model and other slash distributions, because it has lower values in the AIC, BIC, CAIC, and HQIC criteria.
- In future research, we plan to work on a new extension of the Fr distribution that is more flexible in the kurtosis coefficient than the SFr distribution. We will use this distribution in regression problems and survival analyses.

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