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Three Alternative Model-Building Strategies Using Quasi-Hermitian Time-Dependent Observables

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Abstract: In the conventional (so-called Schrödinger-picture) formulation of quantum theory the operators of observables are chosen self-adjoint and time-independent. In the recent innovation of the theory, the operators can be not only non-Hermitian but also time-dependent. The formalism (called non-Hermitian interaction-picture, NIP) requires a separate description of the evolution of the time-dependent states $\psi(t)$ (using Schrödinger-type equations) as well as of the time-dependent observables $\Lambda_j(t)$, $j = 1, 2, \dots, K$ (using Heisenberg-type equations). In the unitary-evolution dynamical regime of our interest, both of the respective generators of the evolution (viz., in our notation, the Schrödingerian generator $G(t)$ and the Heisenbergian generator $\Sigma(t)$) have, in general, complex spectra. Only the spectrum of their superposition remains real. Thus, only the observable superposition $H(t) = G(t) + \Sigma(t)$ (representing the instantaneous energies) should be called Hamiltonian. In applications, nevertheless, the mathematically consistent models can be based not only on the initial knowledge of the energy operator $H(t)$ (forming a “dynamical” model-building strategy) but also, alternatively, on the knowledge of the Coriolis force $\Sigma(t)$ (forming a “kinematical” model-building strategy), or on the initial knowledge of the Schrödingerian generator $G(t)$ (forming, for some reason, one of the most popular strategies in the literature). In our present paper, every such choice (marked as “one”, “two” or “three”, respectively) is shown to lead to a construction recipe with a specific range of applicability.

Keywords: quantum theory of unitary systems; non-Hermitian interaction representation; non-stationary physical inner products; model-building classification



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1. Introduction

One of the sources of inspiration for our present study was a compact review [1] of the history of alternative formulations of quantum mechanics. In their paper dated 2002, the authors asked the question of how many formulations of quantum mechanics we have. For pedagogical reasons, nevertheless, their list remained incomplete. Surprisingly enough, it did not include Dirac’s “intermediate-” *alias* “interaction-picture” (IP) form of the Hermitian theory. The authors also avoided any reference to the innovative paper [2] in which the conventional lists of the available alternative formulations of quantum mechanics were complemented, as early as in 1992, by a manifestly non-Hermitian reformulation of Schrödinger picture (NSP, see also its more recent comprehensive review in [3]).

The latter omission was in fact not too surprising because the NSP (also known as “quasi-Hermitian” [2,4]) formalism has only been developed between the years 1998 and 2007 when Bender with multiple coauthors made the idea widely known and popular [5,6]. In spite of certain skepticism among specialists (as verbalized, e.g., by Streater [7] or, more recently, by several mathematicians [8–12]), Bender with his coauthors persuaded the quantum physics community that there exists a broad class of innovative stationary realizations of quantum theory (including, importantly, quantum field theory [13]) in which the manifestly non-Hermitian candidates H for the Hamiltonians with real spectra could

be phenomenologically appealing as well as mathematically sufficiently user-friendly (cf. also the newer reviews of the field in [14–20]).

Before the year 2008 the next-step transition to the non-stationary non-Hermitian theory has been considered impossible [21]. At the same time, the idea of the *stationary* unitary evolution “in non-Hermitian disguise” has been widely accepted. People realized that such a formulation of the theory remains equivalent to its standard textbook predecessors. For this reason, the presentation of the innovation could have started from the conventional stationary Schrödinger equation

$$i \frac{d}{dt} |\psi(t)\rangle = \mathfrak{h} |\psi(t)\rangle, \quad \mathfrak{h} = \mathfrak{h}^\dagger \neq \mathfrak{h}(t). \quad (1)$$

The generalization (attributed, often, to Dyson [22]) proved based just on a replacement of such an equation by its non-Hermitian (or, better, hidden Hermitian) upgrade mediated by an invertible time-independent mapping $\Omega \neq \Omega(t)$,

$$i \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle, \quad H = \Omega^{-1} \mathfrak{h} \Omega \neq H^\dagger, \quad H \neq H(t). \quad (2)$$

During the year 2008 the scientific community became prepared to accept the proposal of making the non-Hermitian theory *non-stationary* [23]. In the recent application of this approach to the so-called wrong-sign interaction potentials [24], we pointed out that within the corresponding form of quantum theory called non-Hermitian interaction picture (NIP, see also its compact review in [25]), the unitarity of the evolution of the so-called closed quantum systems can still be guaranteed in consistent manner.

In contrast to our preceding NIP-based paper [24], its present continuation will be example-independent. Only a few remarks on possible applications will be added, mainly in Appendix A. In the main body of our new, more methodically oriented paper, we discuss the three main model-building strategies. In a systematic manner, the presentation of our results starts in Section 2, in which we review the basic ideas behind the existing hidden Hermitian quantum theories. In Sections 3–5, we then outline the three respective construction options emphasizing, in each of them, the necessity of a clear separation of what is assumed and postulated from what is calculated, reconstructed, and deduced.

A compact summary of our considerations is finally added in Section 6.

2. The Abstract NIP Quantum Theory

2.1. The Concept of Non-Stationary Non-Hermitian Observables

In conventional textbooks, one often reads about the choice of a “picture” *alias* “representation” of quantum mechanics (cf., e.g., [26]). Let us temporarily return, therefore, to the Hermitian theory. One then usually mentions just the Schrödinger picture (SP) and the Heisenberg picture (HP). Sometimes, another option is presented under the name of “intermediate picture” (IP, cf. pp. 321–322 in [26]). In this case one is assumed to split a given self-adjoint Hamiltonian in its two separate self-adjoint components, $\mathfrak{h} = \mathfrak{h}_S + \mathfrak{h}_H$. Typically, $\mathfrak{h}_S$ is designed to control the evolution of states (i.e., it appears as a generator in a Schrödinger-type equation) while $\mathfrak{h}_H$ is interpreted as entering the Heisenberg-type equations for the relevant and, necessarily, time-dependent but still self-adjoint observables [26].

After one moves to the above-mentioned non-Hermitian (or, using a mathematically more precise terminology, quasi-Hermitian [2,4]) reformulations of quantum theory, a part of the terminology survives. In particular, in the non-Hermitian interaction picture (NIP [25]) we still encounter the Heisenberg’s generator (say, $\Sigma(t)$, i.e., the operator controlling the time-evolution of the non-stationary NIP observables [27]) as well as the Schrödinger’s generator (to be denoted as $G(t)$) entering the Schrödinger-type evolution equations.

The new feature of the more general non-Hermitian theory is that one can speak about a quantum Coriolis force $\Sigma(t)$ [28] filling the interval or space between its NSP extreme $\Sigma^{(NSP)}(t) = 0$ and its non-Hermitian HP extreme such that $\Sigma^{(NHP)}(t) = H^{(NHP)}(t)$ [29].

Moreover, one can formally define, not quite expectedly [30], the superposition of the generators

$$H(t) = G(t) + \Sigma(t). \quad (3)$$

This operator carries a clear physical meaning of an isospectral avatar of its self-adjoint SP partner Hamiltonian $\mathfrak{h}$ (cf. Equation (2)). The relevance of such a property has been emphasized in [24], where we paid attention to the very specific non-Hermitian anharmonic-oscillator models. We arrived there at the conclusion that one cannot easily transfer the “picture-selection” experience gained during the study of the special “wrong-sign” oscillators to the other non-stationary quasi-Hermitian quantum systems. Every NIP-described unitary quantum system has to be treated as specific.

In both of the above-mentioned NSP and NHP special cases, relation (3) degenerates to an identity. We either obtain the coincidence $G^{(NSP)}(t) \equiv H^{(NSP)}(t)$ with disappearing $\Sigma^{(NSP)}(t) = 0$, or $\Sigma^{(NHP)}(t) \equiv H^{(NHP)}(t)$ with disappearing $G^{(NHP)}(t) = 0$. Incidentally, it is worth adding that the widely used attribute “non-Hermitian” of the theory can be misleading because we mean non-Hermitian in our working space $\mathcal{H}_{(unphysical)}$ but not in the correct space of states $\mathcal{H}_{(physical)}$. A better name would certainly be “hiddenly Hermitian” theory, meaning that the operator $\Theta = \Omega^\dagger \Omega$ of the inner-product metric in $\mathcal{H}_{(physical)}$ is nontrivial, $\Theta \neq I$ [2,3].

In the most general NIP setting, the latter operator is also assumed manifestly time-dependent, $\Theta = \Theta(t)$. The flexibility is enhanced because $G^{(NIP)}(t) \neq 0 \neq \Sigma^{(NIP)}(t)$. At the same time, even in the NIP framework, one can distinguish between the different ways of encoding the input information about dynamics into operators. Thus, in our present paper, we speak about a “strategy number one” (cf. Section 3 below), a “strategy number two” (cf. Section 4), and “strategy number three” (discussed in Section 5). In other words, we are now going to propose that every choice of *one* of the operators in Equation (3) might be interpreted as leading to a specific eligible representation of the (by assumption, unitary) evolution.

Our forthcoming analysis of a triplet of refined NIP formulations of quantum theory was in fact motivated by the recent growth of the diversity of applications of the non-stationary versions of the non-Hermitian operators in the various branches of physics [16]. In these applications, the building of models appeared often restricted by a requirement of having an exact, non-numerical form of the model. Such a solvability requirement made the conclusions rather special and model-dependent. In what follows, we accept, therefore, a different attitude. We try to separate, clearly, the form and extent of the input information about the system from a systematic and consistent step-by-step reconstruction of the consequences of the assumptions.

As we already indicated, we arrive at three alternative model-building strategies. In the construction strategy number one, as described in Section 3, we accept the most traditional “dynamical” point of view of Scholtz et al. [2]. We emphasize that, in some sense, the observability property makes the energy-representing operator $H^{(NIP)}(t) = H_{(one)}^{(NIP)}(t)$ a unique candidate for being called Hamiltonian. Indeed, its time-dependent choice determines the quantum system’s dynamics in a phenomenologically satisfactory manner even in the non-stationary scenario.

In the alternative construction strategy number two as presented in Section 4, we start with the knowledge of the physical-Hilbert-space “kinematics”. Having knowledge of the Coriolis-force operator $\Sigma^{(NIP)}(t) = \Sigma_{(two)}^{(NIP)}(t)$ at all times at our disposal, we reconstruct the eligible forms of the dynamics in a way that is shown to be exceptionally straightforward.

In strategy number three, we assume that a key technical as well as the phenomenological role is played by the time-dependence of the states. Thus, what is assumed to be given in advance is the Schrödingerian generator $G^{(NIP)}(t) = G_{(three)}^{(NIP)}(t)$.

2.2. The Physical Inner-Product Metric

In the NIP picture, both the states ψ (i.e., the elements of a suitable Hilbert space) and the operators $\Lambda = \Lambda_j$ (representing observables) are allowed to vary with time. The respective generators of evolution, i.e., an operator $G(t)$ entering the Schrödinger-type equation for $\psi = \psi(t)$, and another operator $\Sigma(t)$ in the Heisenberg-type equation for $\Lambda_j = \Lambda_j(t)$ may be (and, in the literature, quite often are) both called “Hamiltonians”. For this reason, it may be useful to try to avoid misunderstandings by speaking, more explicitly, about a “non-Hermitian time-dependent “Schrödinger-equation Hamiltonian” $G(t) = G^{(NIP)}(t)$ in the former case. We may also need to amend the denotation of the “Heisenberg-equation-Hamiltonian”, $\Sigma(t) = \Sigma^{(NIP)}(t)$.

In the unitary evolution scenario, both of the auxiliary NIP generators $G(t)$ and $\Sigma(t)$ are just auxiliary and non-observable. Their spectra may be complex—for illustration, see, e.g., the schematic examples in [31,32]. Only their instantaneous-energy-representing sum (3) may be considered, in a way shown in [23,30], observable. For this reason, we call such an operator “observable Hamiltonian” or simply “Hamiltonian”.

We interpret the relationship (3) between the three eligible operators (i.e., between the “Hamiltonians” in a broader sense) as a starting point for theoretical constructive efforts. We propose that in such a setting, one picks up simply one of these operators as “known”, i.e., as an operator carrying a decisive portion of the “physical” input information about the unitary quantum system in question. In this manner, one arrives at the three alternative quantum-model-building strategies as described below.

As long as the form of the NIP Hamiltonian $H(t)$ of Equation (3) has to be flexible and, in particular, not necessarily Hermitian, the underlying “working” Hilbert space (say, $\mathcal{F}$) can be declared, in general, unphysical, playing just the role of a mathematical tool, $\mathcal{F} \equiv \mathcal{H}_{(unphysical)}$. In 1998, such an idea of working with observables in a mathematically friendlier non-Hermitian representation has been made particularly attractive and popular by Bender with Boettcher [5]. One of its main consequences is that our auxiliary, computation-friendly Hilbert space $\mathcal{F}$ (i.e., in many realistic models, just $L^2(\mathbb{R}^d)$ [14,15]) has to be complemented by another, correct physical alternative $\mathcal{H}_{(physical)} \equiv \mathcal{H} \neq \mathcal{F}$. In the words of the older comprehensive review [2], one only has to clarify the relationship between $\mathcal{F}$ and $\mathcal{H}$ by establishing “a criterion for a set of non-Hermitian operators” (i.e., for the above-mentioned set of operators Λ_j) “to constitute a consistent quantum mechanical system”, which “involves the construction of a [physical Hilbert-space] metric”, i.e., which involves a representation of $\mathcal{H}$ in $\mathcal{F}$. In our present notation this means that there must exist a suitable “inner-product-metric” operator Θ such that

$$\Lambda_j^\dagger \Theta = \Theta \Lambda_j, \quad j = 0, 1, \dots, J. \quad (4)$$

This relation would, indeed, render all of our observables Λ_j self-adjoint in $\mathcal{H}$ and, at the same time, non-Hermitian and, in a way specified by Definition (4), “quasi-Hermitian” in $\mathcal{F}$ [2,4].

This being said, the feasibility of the NIP-based model-building strategy is still marred by the fairly complicated nature of the description of the evolution using the two independent generators $G(t)$ and $\Sigma(t)$. This is one of the central questions and challenges in the theory. A simplification of the formalism is needed and sought in the various methodically (rather than phenomenologically) motivated restrictions of the admissible classes of eligible non-Hermitian Hamiltonians. In what follows, we analyze and describe, systematically, the possibilities of such a simplification.

3. The First, Dynamical-Input Strategy

In any model-building scenario reflecting relation (3), which connects the three different “Hamiltonians” one may start from an “input” knowledge of any one of them.

Still, the “input information” selection of the observable instantaneous energy, viz., of the operator

$$H_{(one)}^{(NIP)}(t) \neq 0 \quad (5)$$

looks most natural. Such an option could be called “dynamical”, being most closely connected with the philosophy of Scholtz et al. [2] who treated all of their generalized non-Hermitian quantum models as specified by their observables.

Once we restrict attention just to the observable Hamiltonian, we have to deliver, first of all, a rigorous proof of the reality of the energies. Secondly, a consistent probabilistic interpretation of the model requires a confirmation of the quasi-Hermiticity of the observable Hamiltonian. Thus, whenever the input information is encoded in the operator $H_{(one)}(t)$ (as well as in its conjugate form $H_{(one)}^\dagger(t)$), our first task is to solve the quasi-Hermiticity-constraint equation

$$H_{(one)}^\dagger(t) \Theta_{(one)}(t) = \Theta_{(one)}(t) H_{(one)}(t) \quad (6)$$

for an unknown metric $\Theta_{(one)}(t)$.

The solution of such a linear algebraic problem is non-unique but, conceptually, straightforward. At any time t , in a way indicated in [25], we may simply follow the notation convention of review [30] and initiate the search for all (or at least for some) of the admissible metric operators $\Theta_{(one)}(t)$ by solving the two instantaneous Schrödingerian eigenvalue problems

$$H_{(one)}(t) |\psi^{(one)}(t)\rangle = E_\psi(t) |\psi^{(one)}(t)\rangle, \quad H_{(one)}^\dagger(t) |\psi^{(one)}(t)\rangle = E_\psi(t) |\psi^{(one)}(t)\rangle. \quad (7)$$

In our notation, the symbol ψ can be read either as an index (numbering the elements of a complete set of states) or, more traditionally, as a letter that identifies a state in its two different and complementary (i.e., single-ket and doubled-ket) realizations.

In both of the equations in (7) the energy eigenvalues remain the same because they are, by assumption, observable (i.e., real), discrete (because they have to represent bound states [33]) and bounded from below (because the system in question is assumed stable [6]). Nevertheless, due to the non-Hermiticity $H_{(one)}(t) \neq H_{(one)}^\dagger(t)$ of the Hamiltonian, the respective two sets of the eigenvectors in (7) are different.

In the context of physics, the knowledge of both of them is necessary because both of them contribute to the probabilistic predictions, i.e., to the matrix elements

$$\langle\langle \psi(t) | \Lambda(t) | \psi(t) \rangle\rangle \quad (8)$$

in which $\Lambda(t)$ denotes any observable of interest and in which $t = t_f$ is the time of its measurement. In the language of mathematics, this means that what is needed for a definition of a state is in fact an elementary dyadic projector

$$\pi_\psi(t) = |\psi^{(one)}(t)\rangle \frac{1}{\langle\langle \psi^{(one)}(t) | \psi^{(one)}(t) \rangle\rangle} \langle\langle \psi^{(one)}(t) | \quad (9)$$

rather than just one of the two alternative versions of the state vector. This being clarified we may recall their biorthogonality property [34,35] and, via a suitable rescaling, we may upgrade it to a biorthonormality and bicompleteness,

$$\langle\langle \psi^{(one)}(t) | \phi^{(one)}(t) \rangle\rangle = \delta_{\psi\phi}, \quad \sum_\psi |\psi^{(one)}(t)\rangle \langle\langle \psi^{(one)}(t) | = I. \quad (10)$$

Formally, *all* of the metrics compatible with Equation (6) can be then expressed in terms of the wave-function solutions of the second, conjugate-operator equation in (7) [35],

$$\Theta_{(one)}(t) = \sum_{\psi} |\psi^{(one)}(t)\rangle \kappa_n^{(one)}(t) \langle \psi^{(one)}(t)|. \quad (11)$$

It is easy to verify that in such a formula, all of the parameters $\kappa_n^{(one)}$ are arbitrary. For the reasons as explained in [2], they only have to be real and positive. Also, for the sake of keeping the formalism reasonably tractable (see a more explicit formulation of this reason in [25]), their recommended choice will be time-independent, $\kappa_n^{(one)}(t) = \kappa_n^{(one)}(0)$.

The variability of the latter parameters can be interpreted either as a formal kinematical freedom of the theory (see, e.g., [36]) or, better, as a manifestation of the above-mentioned incompleteness of the dynamical-input information when restricted to the single observable $H_{(one)}(t)$. Indeed, the formalism admits (and also, for the sake of completeness, requires) additional information about dynamics simulated by the choice of parameters $\kappa_n^{(one)}$. More consequently and directly, such information could and should be, of course, provided by the introduction of additional observables—see a more detailed discussion of the suppression of the ambiguity in [2].

4. The Second, Coriolis-Choice Strategy

The oldest formulation of quantum mechanics, viz., the so called Heisenberg picture (HP [26]), which appeared in June 1925 [1] can be characterized as “kinematical” since a strict time-independence of the wave functions is required, $\psi(t) = \psi(0)$. In the Hermitian theory, one simply puts $G^{(HP)}(t) = 0$ so that relation (3) degenerates to the identity $H^{(HP)}(t) = \Sigma^{(HP)}(t)$. Just the above-discussed observable-Hamiltonian dynamical input is reobtained.

In a properly generalized non-Hermitian NIP setup, the situation is different [29]. A formulation of strategy number two becomes less straightforward. Although the “kinematical” design of models may still start from the initial specification of the NIP Coriolis force at all of the relevant times t ,

$$\Sigma_{(two)}^{(NIP)}(t) \neq 0 \quad (12)$$

one has to admit a non-triviality of $G_{(two)}^{(NIP)}(t) \neq 0$ and of $H_{(two)}^{(NIP)}(t) \neq \Sigma_{(two)}^{(NIP)}(t)$ in (3).

Once we pick up the kinematics (i.e., operator $\Sigma_{(two)}(t)$ and/or its conjugate partner $\Sigma_{(two)}^\dagger(t)$), we immediately imagine that this opens the way to the reconstruction of the unknown Dyson-mapping operator $\Omega_{(two)}(t)$. For this purpose, indeed, it is sufficient to recall its definition (see the detailed introduction of this concept in [30]) and to re-write it in the following equivalent form

$$i \frac{d}{dt} \Omega_{(two)}(t) = \Omega_{(two)}(t) \Sigma_{(two)}(t) \quad (13)$$

of the operator differential equation of the first order. Its solution yields the Dyson-map operator at all times from any preselected initial value at $t = 0$.

For the sake of symmetry, we may either conjugate the solution or solve the conjugate problem

$$i \frac{d}{dt} \Omega_{(two)}^\dagger(t) = -\Sigma_{(two)}^\dagger(t) \Omega_{(two)}^\dagger(t). \quad (14)$$

In the subsequent step, we become able to define the correct physical Hilbert-space metric as the product of the two Dyson maps,

$$\Theta_{(two)}(t) = \Omega_{(two)}^\dagger(t) \Omega_{(two)}(t). \quad (15)$$

The evaluation of this operator product enables us to specify all of the eligible Hamiltonians $H_{(two)}(t)$ as (naturally, non-unique [2,35]) solutions of the Dieudonné's [4] quasi-Hermiticity constraint

$$H_{(two)}^\dagger(t) \Theta_{(two)}(t) = \Theta_{(two)}(t) H_{(two)}(t) = A_{(two)}(t). \quad (16)$$

In a method inspired by the non-Hermitian random-matrix theories [37], we introduce here a new operator $A_{(two)}(t)$ encoding the input information about dynamics which is still missing.

It is worth emphasizing that the latter operator is almost arbitrary, restricted merely by the requirement (16) tractable as its Hermiticity, $A_{(two)}(t) = A_{(two)}^\dagger(t)$. This immediately yields the ultimate explicit definitions of both of the remaining unknown components of the model,

$$H_{(two)}(t) = \Theta_{(two)}^{-1}(t) A_{(two)}(t), \quad G_{(two)}(t) = H_{(two)}(t) - \Sigma_{(two)}(t). \quad (17)$$

The construction is completed.

5. The Third, State-Evolution Strategy

Although the Erwin Schrödinger's formulation of quantum mechanics is not the oldest one [1], its extreme conceptual as well as computational appeal and simplicity make it a dominant paradigm in textbooks [26]. For this reason, even Hynek Bíla, one of my previous PhD students refused the terminology and philosophy of my papers [23,30] and, even in the NIP regime, he insisted on using the dedicated name "Hamiltonian", strictly, just for the denotation of the Schrödinger's "input physical information" generator of the evolution of wave functions [38,39],

$$G_{(three)}^{(NIP)}(t) \neq 0. \quad (18)$$

The Bíla's convention was later accepted by a number of other researchers [3,40,41]. They were influenced by the widely accepted *stationary* non-Hermitian quantum theory of reviews [2,3,6] in which the time-independent Schrödinger's operator $G(t) = G(0)$ coincides with its stationary energy-representing partner $H(t) = H(0)$. Obviously, this operator still had a real spectrum and carried a fully consistent physical meaning of an observable.

Due to a rather naive straightforward transfer of terminology to non-stationary scenarios, the key role has been allocated to $G(t)$ even when $G(t) \neq H(t)$. Incidentally, the change in the convention appeared to also have several positive aspects and consequences. The main one was that after a shift of attention from the closed-system theory to the open-system theory [42] or even beyond the domain of quantum physics [43,44], the loss of the observability status of $G(t)$ (cf. its proof in Theorem 2 of review [3]) became irrelevant. Thus, in the study of non-unitary, open quantum systems, serendipitously, the Bíla's terminology proved inspiring. A number of interesting innovations of the traditional mathematical concepts have been revealed in this area: Cf., e.g., the new use of the Lewis-Riesenfeld invariants as described in [45], an innovative introduction of a generalized entropy in non-Hermitian systems in [46], a reinterpretation of the concept of $\mathcal{PT}$ -symmetry in [47] or, last but not least, a new wave of interest in non-linear theories, quantum (i.e., effective [14]) as well as non-quantum [15].

Even after one returns back to the description of the closed quantum systems, the initial selection of $G(t)$ need not destroy the internal consistency of the theory. A detailed description of the related technicalities may be found, e.g., in Section 5.3.2 of paper [48]. The point is that even the randomly emerging complexifications of the spectrum of $G(t)$ may be kept compatible with the unitarity of the evolution of the underlying closed quantum system. Via a few schematic non-stationary toy models, this was illustrated in [31,32].

A model-independent methodical support of the $G(t)$ -based approach can be based on our preceding considerations. First of all, we have to return to the concept of a biorthonor-

mal and bicomplete basis. Our assumption of the knowledge of operator (18) at all times implies that it is now sufficient to know the pure-state-representing projector (9) just at $t = 0$,

$$\pi_{\psi}(0) = |\psi^{(three)}(0)\rangle \frac{1}{\langle\langle\psi^{(three)}(0)|\psi^{(three)}(0)\rangle\rangle} \langle\langle\psi^{(three)}(0)|. \quad (19)$$

From the point of view of physics, this means that the theory admits the preparation of a more or less arbitrary initial state of the quantum system in question.

In the next step, we may now recall the availability of $G(t)$ and solve the two evolution-equation analogs

$$i \frac{d}{dt} |\psi^{(three)}(t)\rangle = G_{(three)}(t) |\psi^{(three)}(t)\rangle \quad (20)$$

and

$$i \frac{d}{dt} |\psi^{(three)}(t)\rangle\rangle = G_{(three)}^{\dagger}(t) |\psi^{(three)}(t)\rangle\rangle \quad (21)$$

of the two Schrödingerian eigenvalue problems (7) (see also the details in [23]). Next, via an appropriate rescaling of the initial-value vectors we may achieve their biorthonormality and bicompleteness,

$$\langle\langle\psi^{(three)}(0)|\phi^{(three)}(0)\rangle\rangle = \delta_{\psi\phi}, \quad \sum_{\psi} |\psi^{(three)}(0)\rangle \langle\langle\psi^{(three)}(0)| = I. \quad (22)$$

Finally, recalling the theory [25] we may extend the validity of these postulates to all times t ,

$$\langle\langle\psi^{(three)}(t)|\phi^{(three)}(t)\rangle\rangle = \delta_{\psi\phi}, \quad \sum_{\psi} |\psi^{(three)}(t)\rangle \langle\langle\psi^{(three)}(t)| = I. \quad (23)$$

Partial methodical parallels with the dynamical-input strategy emerge: Whenever our initial-time choice of the biorthonormalized and bicomplete basis of Equation (22) is made compatible with the input-information form of one of the observables, (i.e., say, of the energy operator) at $t = 0$,

$$H_{(three)}(0) = \sum_{\psi} |\psi^{(three)}(0)\rangle E_{\psi}^{(three)}(0) \langle\langle\psi^{(three)}(0)|, \quad (24)$$

We may immediately reconstruct the same operator at all times $t > 0$. Naturally, the construction of the metric acquires the explicit form

$$\Theta_{(three)}(t) = \sum_{\psi} |\psi^{(three)}(t)\rangle\rangle \kappa_n^{(three)}(t) \langle\langle\psi^{(three)}(t)|. \quad (25)$$

At this stage of development of the theory, it is useful to notice that many of its applications (cf., e.g., [27]) are just considered in a finite-dimensional Hilbert space. Then, many operators of interest (i.e., N by N matrices with $N < \infty$) may happen to form a representation of a suitable Lie algebra. This does not only render the constructions feasible but it also enables us to factorize the metric into a product of Dyson maps,

$$\Theta_{(three)}(t) = \Omega_{(three)}^{\dagger}(t) \Omega_{(three)}(t). \quad (26)$$

The latter formula may be compared with Equation (15) where the construction proceeded from Ω to Θ , i.e., in the opposite direction. In other words, the Dyson map may formally be written as the sum

$$\Omega_{(three)}(t) = \sum_{\psi} |\psi_{(three)}(t)\rangle \sqrt{\kappa_n^{(three)}(0)} \langle\langle\psi_{(three)}(t)| \quad (27)$$

where the new auxiliary basis $\{|\psi_{(three)}(t)\rangle\}$ may be chosen orthonormal.

On this level of reconstruction, one is already able to define the Coriolis force,

$$\Sigma_{(three)}(t) = \frac{i}{\Omega_{(three)}(t)} \dot{\Omega}_{(three)}(t) \quad (28)$$

where the dot represents the differentiation with respect to time. Now, the last step yielding, finally, the observable Hamiltonian and its decomposition

$$H_{(three)}(t) = G_{(three)}(t) + \Sigma_{(three)}(t) \quad (29)$$

is already trivial.

6. Summary

It is well known that the price to be paid for the generality and flexibility of the NIP formulation of non-stationary quantum theory in both its quantum-mechanical and field-theoretical versions is not too low. Only too many evolution equations have to be solved. In our present paper, we managed to show that a systematic subdivision of the related model-building strategies could simplify the picture thoroughly. The core of our message lies in the observation that the most natural interpretation of all of the eligible NIP-based mathematical constructions of quantum models should be based on a clear separation of the consistent implementation alternatives.

We have shown that an explicit guide to the choice from the menu has to be correlated with a context-dependent dominance of one of the operators $H(t)$, $\Sigma(t)$ or $G(t)$. We argued that such an identification of dominance leads directly to the three different NIP-implementation recipes. They may be characterized by their specific respective mathematical merits as well as by a natural subdivision and classification of the related phenomenological intentions. Thus, what we described are the three mutually complementary forms of the implementation of the abstract NIP quantum theory in situations where the set of the underlying unitary (i.e., closed) quantum systems can be subdivided according to the more detailed practical criteria.

The resulting construction process seems useful, enhancing the tractability of the systems living in a non-stationary dynamical regime in which the use of the hidden Hermitian representations of observables might throw new light also on the physical interpretation of the various important open questions, say, in cosmology [49]. In all of these contexts, a clear separation of the input information about the system from the resulting predictions seems to play, in non-stationary systems, a more important role than in their stationary predecessors because the increase in the complexity of mathematics is enormous. The traditional guidance by the stationary constructions using analogies with the techniques of linear algebra becomes, in the NIP framework, replaced by the necessity of solving complicated operator evolution equations. We believe that such a challenge has to be accepted. As a reward, indeed, the NIP formalism may be expected to open new methods of description of multiple deeply non-stationary phenomena.

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Appendix A. A Few Remarks on Applications

In full accord with the traditional textbooks on quantum mechanics [26] the observables have to be self-adjoint in a suitable Hilbert space. Nevertheless, in the non-stationary quantum NIP approach such a “correct” space (say, $\mathcal{L}_{(of\ textbooks)}$) is “hidden” (incidentally, one of the traces of its hidden existence may be seen in the emergence of the auxiliary basis $\{|\psi_{(three)}(t)\rangle\}$ in Formula (27)). Its role is symbolic, reduced to the mere hypothetical reference to the conventional probabilistic-interpretation contents of the model.

In [24], we paid attention to the applicability and application of the NIP evolution equations to several versions of the so-called “wrong-sign” quartic oscillator, with the choice inspired by a few older studies [33,50,51]. We emphasized that it is absolutely necessary to separate, very carefully, the study of the stationary systems from the study of their non-stationary analogs.

One of the reasons has already been known to Mostafazadeh who considered, on p. 1271 of his comprehensive 2010 review [3], a time-dependent and non-Hermitian “Schrödinger-equation Hamiltonian” $H^{(SEH)}(t)$ (abbreviated, in our present notation, as $G(t)$) and proved, in Theorem Nr. 2 of *loc. cit.*, that “if the time evolution of the system ... is unitary and [if] $H^{(SEH)}(t)$ is an observable for all $t \in (0, T)$, then the metric operator operator $[\Theta]$ defining $\mathcal{H}_{(physical)} \equiv \mathcal{H}$ does not depend on time”.

In our subsequent commentaries (cf., e.g., [25]), we fully agreed with the latter, strictly mathematical result. At the same time we emphasized that in the general quantum-mechanical NIP theoretical framework there exists absolutely no reason for the purely formalistic, fully redundant, and entirely unfounded requirement of the observability of the Schrödinger-equation-evolution generator $G(t) \equiv H^{(SEH)}(t)$. In [24], we complemented such a statement by its “wrong-sign” quartic-oscillator illustration.

In our present, methodically oriented continuation of the latter paper we just gave this statement a more explicit and more systematic form. We emphasized that both the Schrödinger-equation generators $G(t)$ and their Heisenberg-equation analogs $\Sigma(t)$ are just auxiliary concepts. They enter the respective NIP evolution equations but, in contrast to the widespread belief, their spectra are, in general, complex. For this reason, only their sum $H(t) = G(t) + \Sigma(t)$ is, from the point of view of physics, relevant, retaining the standard physical meaning of the instantaneous energy of the unitary quantum system in question. The postulate of its choice makes the construction strategy number one phenomenologically preferable.

From the point of view of mathematics, the situation is different because the experimental predictions are only given by the evaluation of the mean values of the form of overlap Equation (8). In the NIP framework, for this reason, one must evaluate, at the time of measurement $t = t_f$, both the (elementary) projector $\pi_\psi(t)$ of Equation (9) and the (complicated) operator $\Lambda(t_f)$. For this reason, the evaluation of the predictions (to be tested by a hypothetical experiment) requires not only the (usually, emphasized) solution of the two Schrödinger-type evolution equations (viz, Equations (20) and (21)) for vectors but also a parallel solution of the Heisenberg-type evolution equation for $\Lambda(t)$ as given, e.g., by equation Nr. 34 in [24].

In practice, the solvability of the two Schrödinger-type evolution equations (characterizing the state of the system via dyadic $\pi_\psi(t)$) is a comparatively easy part of the task because such a projector is defined just in terms of the two formally independent state vectors $|\psi(t)\rangle$ and $|\psi(t)\rangle\rangle$. In comparison, the necessary simultaneous construction of $\Lambda(t)$ is in fact by far the most difficult part of the task. Indeed, even the most simple, conservative-observable version of the related operator evolution equation having the familiar Heisenberg’s form

$$i \frac{\partial}{\partial t} Q(t) = Q(t) \Sigma(t) - \Sigma(t) Q(t) \quad (A1)$$

has to specify the solution which is an operator.

The latter observation could even be used as an argument in favor of the preference of the kinematic, Coriolis-based strategy number two. In it, indeed, one can fine-tune the operator $\Sigma(t)$ to make it as user-friendly as possible. Another analogous supportive argument could be also found in Fring’s and Tenney’s paper [51], in which the authors used a version of such a strategy for successful and explicit, non-numerical toy-model constructions. In fact, their procedure only deviated from the present, Coriolis-based one in a re-interpretation of Equation (13). In their case, a decisive role of success-yielding input

information has been played by a trial-and-error ansatz for $\Omega_{(two)}(t)$ (cf. Eq. Nr. (2.3) in *loc. cit.*) rather than by our above-recommended ansatz for $\Sigma_{(two)}(t)$.

In our comment [24] on the latter construction, we pointed out that in the case of the “wrong-sign” anharmonic toy models several equally compact algebraic results may be also obtained using directly the energy-based strategy number one. The key role in the description of dynamics has been played there by the “observable-Hamiltonian” operator sum $G(t) + \Sigma(t) = H(t)$. This enabled us to simplify the calculations, especially when the operators of observables remained time-independent, $\Lambda_j(t) = \Lambda_j(0)$.

During the development of the NIP theory after 2008 it became known that the condition of stationarity $H \neq H(t)$ appears to be not so easy to relax (see, e.g., [21,52,53] or Theorem Nr. 2 in review [3]). In fact, the emerging difficulties were of two types. The main ones were abstract and concerned with the consistency of the theory. The equally important obstacles emerged, in the context of the applicability of the formalism, as a consequence of the operator nature of the NIP evolution equations.

The answers to the abstract conceptual questions were not too difficult and their majority may be found in the early paper [23] and in review [30]. In contrast, the emergence of the NIP-related technical challenges resembled the opening of a Pandora’s box: In place of the mere construction of the metric in stationary models, it appeared necessary to solve several incomparably more complicated evolution equations [39].

The essence of the message was that one merely has to leave the over-restrictive Schrödinger-picture framework and that a consistent formulation of the non-stationary non-Hermitian theory requires just a transition to a non-Hermitian analog of intermediate picture (NIP). A consistent and unitary non-stationary *alias* NIP quantum theory has been formulated in which only the observability of the “observable non-stationary quasi-Hermitian Hamiltonian” $H(t) = G(t) + \Sigma(t)$ is in fact required and needed. Unfortunately, as long as all three operator components in the latter sum can be called “Hamiltonians”, a series of misunderstandings followed.

In 2010, in particular, one could still read, in the mathematically rigorous review [3], that in the time-dependent cases, “insisting on observability of the Hamiltonian operator” would be inconsistent (see Theorem 2 in *loc. cit.*). The source of such misunderstandings can be traced back to the terminology. Indeed, in a way paralleling the Bila’s 2009 proposal [39] and in a way used, later, also by Fring et al [40,51,54], the author of review [3] did not in fact have in mind the “observable non-stationary quasi-Hermitian Hamiltonian” $H(t) = G(t) + \Sigma(t)$ but rather just another, purely auxiliary operator representing, in our present notation, the time-dependent Schrödinger-equation generator *alias* “unobservable Hamiltonian” $G(t)$. The related dedicated discussions resolved the paradox and helped to clarify the puzzle.

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