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New Results on a Fractional Integral of Extended Dziok–Srivastava Operator Regarding Strong Subordinations and Superordinations

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Abstract: In 2012, new classes of analytic functions on $U \times \overline{U}$ with coefficient holomorphic functions in $\overline{U}$ were defined to give a new approach to the concepts of strong differential subordination and strong differential superordination. Using those new classes, the extended Dziok–Srivastava operator is introduced in this paper and, applying fractional integral to the extended Dziok–Srivastava operator, we obtain a new operator $D_z^{-\gamma} H_m^l[\alpha_1, \beta_1]$ that was not previously studied using the new approach on strong differential subordinations and superordinations. In the present article, the fractional integral applied to the extended Dziok–Srivastava operator is investigated by applying means of strong differential subordination and superordination theory using the same new classes of analytic functions on $U \times \overline{U}$. Several strong differential subordinations and superordinations concerning the operator $D_z^{-\gamma} H_m^l[\alpha_1, \beta_1]$ are established, and the best dominant and best subordinant are given for each strong differential subordination and strong differential superordination, respectively. This operator may have symmetric or asymmetric properties.

Keywords: analytic function; strong differential subordination; strong differential superordination; fractional integral; Dziok–Srivastava operator

MSC: 30C45; 30A20; 34A40

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1. Introduction

Antonino and Romaguera [1] were the first who used the notion of strong differential subordination to study the Briot–Bouquet strong differential subordination. They introduced this concept as an extension of the classical differential subordination defined by Miller and Mocanu [2,3]. The well-known notions from the theory of differential subordination [4] were extended in 2009 [5], building the theory of strong differential subordination. The dual notion of differential subordination, differential superordination [6] was extended in 2009 [7], introducing the concept of strong differential superordination. The theories of strong differential subordination and superordination experienced a beautiful development. The best dominant for a strong differential subordination, and the dual notion, the best subordinant of a strong differential superordination, were established in [8], and first examples of strong differential subordinations and superordinations of analytic functions were given in [9]. Since then, there have been results regarding strong differential subordination and superordination with well-known operators: Liu–Srivastava operator [10], Sălăgean differential operator [11], Ruscheweyh derivative [12] and combinations between Sălăgean and Ruscheweyh operators [13], Komatu integral operator [14,15], multiplier transformation [16,17], general differential operators [18,19], and many others well-known operators [20–23].

Fractional calculus was associated with strong differential subordination theory in early studies [24], but this line of research was not followed. The development facilitated using quantum calculus and fractional calculus in geometric function theory is described in

Srivastava's paper [25]. A new integro-differential operator defined considering the meromorphic functions and the Mittag-Leffler function is studied in [26]. In Ref. [27], formulas for several fractional differintegral operators were developed and introduced by applying Riemann–Liouville fractional integrals. Ref. [28] described a qualitative analysis regarding a nonlinear Langevin integro-fractional differential equation. Fractional calculus is applied to Mittag-Leffler functions of one, two, three, and four parameters in [29]. Applying fractional calculus to the Mittag-Leffler function and confluent hypergeometric functions, new results were obtained in [30] regarding the Mittag-Leffler-confluent hypergeometric function. In [31], the authors applied fractional calculus to special functions to define and investigate new variants of the Gamma and Kummer functions for Mittag-Leffler functions. The Caputo–Katugampola fractional derivative was introduced and used to study a new class of fractional Volterra–Fredholm integro-differential equations in [32].

Starting with the results obtained in [33], which investigated the fractional integral of the Dziok–Srivastava operator using differential subordination theory, we intended to study this operator regarding the other types of differential subordinations and superordinations: strong and fuzzy. Therefore, in this paper we used the strong differential subordination and the strong differential superordination theories and we extended the Dziok–Srivastava operator to the class of analytic functions $\mathcal{A}_\tau^*$ and, applying fractional integral to this extended operator, we introduced a new operator $D_z^{-\gamma} H_m^l[\alpha_1, \beta_1]$ in $\mathcal{A}_\tau^*$, which has not been previously studied.

The novelty of this research is in the definition of the fractional integral of the extended Dziok–Srivastava operator introduced in Definition 6 and in the manner in which it is used to get new strong differential subordination results, together with the dual new strong differential superordinations. In each theorem, the best dominant or the best subordinant, respectively, is established. Replacing the functions considered as best subordinant or best dominant from the theorems with remarkable functions and using their geometric properties produce interesting corollaries.

Next, the main notions used in the research are recalled and basic lemmas applied to prove the main results.

The expression $\mathcal{H}(U \times \overline{U})$ represents the family of analytic functions from $U \times \overline{U}$, considering
$U = \{t \in \mathbb{C} : |t| < 1\}$ and its closure $\overline{U} = \{t \in \mathbb{C} : |t| \leq 1\}$. To avoid the similarity problem, the classical notations must be changed.

Both dual theories of strong differential subordination and superordination use specific subclasses of $\mathcal{H}(U \times \overline{U})$ extending the classical ones [34]:

$$\mathcal{A}_{n\tau}^* = \{f(t,\tau) = t + a_{n+1}(\tau)t^{n+1} + \dots\} \subset \mathcal{H}(U \times \overline{U}),$$

denoted by $\mathcal{A}_\tau^*$ for $n = 1$, and

$$\mathcal{H}^*[a,n,\tau] = \{f(t,\tau) = a + a_n(\tau)t^n + a_{n+1}(\tau)t^{n+1} + \dots\} \subset \mathcal{H}(U \times \overline{U}),$$

with $a_k(\tau)$ being holomorphic functions in $\overline{U}$, taking $n \in \mathbb{N}$, $k \geq n+1$ and $a \in \mathbb{C}$.

The notion of strong differential subordination is defined as follows:

Definition 1 ([5])**.** *Between the analytic functions $f(t,\tau)$ and $H(t,\tau)$ exists a strong differential subordination when there exists an analytic function u in U, with the properties $|u(t)| < 1$, $t \in U$ and $u(0) = 0$, such that $f(t,\tau) = H(u(t),\tau)$ for all $(t,\tau) \in U \times \overline{U}$, and this relation is denoted by $f(t,\tau) \prec\prec H(t,\tau)$.*

Remark 1 ([5])**.** *(i) For the univalent function $f(t,\tau)$ in U, with $\tau \in \overline{U}$, the relations from Definition 1 mean $f(U \times \overline{U}) \subset H(U \times \overline{U})$ and $f(0,\tau) = H(0,\tau)$, with $\tau \in \overline{U}$.*

(ii) The special case when $H(t,\tau) = H(t)$ and $f(t,\tau) = f(t)$ reduces the strong differential subordination to the classical differential subordination.

To investigate strong differential subordination, the following lemmas are needed.

Lemma 1 ([35]). *Considering the function $p \in \mathcal{H}^*[a,n,\tau]$ satisfying the strong differential subordination*

$$p(t,\tau)+\frac{1}{\eta}tp'_t(t,\tau) \prec\prec h(t,\tau), \quad t\in U,\ \tau\in\overline{U},$$

where $h(t,\tau)$ is a convex function such that $h(0,\tau)=a$, $\tau\in\overline{U}$ and $\eta\in\mathbb{C}^*$ with $Re\eta\geq 0$, we obtain the strong differential subordinations

$$p(t,\tau)\prec\prec q(t,\tau)\prec\prec h(t,\tau),$$

and the convex function $q(t,\tau)=\frac{\eta}{nt^{\frac{\eta}{n}}}\int_0^t h(x,\tau)x^{\frac{\eta}{n}-1}dx$ is the best dominant.

Lemma 2 ([35]). *Considering the holomorphic function*

$$p(t,\tau)=q(0,\tau)+p_n(\tau)t^n+p_{n+1}(\tau)t^{n+1}+\ldots,$$

in $U\times\overline{U}$ satisfying the strong differential subordination

$$p(t,\tau)+\eta tp'_t(t,\tau)\prec\prec h(t,\tau), \quad t\in U,\ \tau\in\overline{U},$$

where $q(t,\tau)$ is a convex function and

$$h(t,\tau)=q(t,\tau)+n\eta tq'_t(t,\tau),$$

for n a positive integer and $\eta>0$, we obtain the sharp strong differential subordination

$$p(t,\tau)\prec\prec q(t,\tau).$$

The notion of strong differential superordination is defined as follows:

Definition 2 ([7]). *Between the analytic functions $f(t,\tau)$ and $H(t,\tau)$ exists a strong differential superordination when there exists an analytic function u in U, with the properties $|u(t)|<1$, $t\in U$ and $u(0)=0$, such that $H(t,\tau)=f(u(t),\tau)$, for all $(t,\tau)\in U\times\overline{U}$, and this relation is denoted by $H(t,\tau)\prec\prec f(t,\tau)$.*

Remark 2 ([7]). *(i) For the univalent function $f(t,\tau)$ in U, with $\tau\in\overline{U}$, the relations from Definition 2 mean $H(U\times\overline{U})\subset f(U\times\overline{U})$ and $H(0,\tau)=f(0,\tau)$, with $\tau\in\overline{U}$.*

(ii) The special case when $H(t,\tau)=H(t)$ and $f(t,\tau)=f(t)$ reduces the strong differential superordination to the classical differential superordination.

Definition 3 ([36]). *The expression Q^* consists all the injective analytic functions on $\overline{U}\times\overline{U}\backslash E(f,\tau)$, satisfying $f'_t(x,\tau)\neq 0$ with $x\in\partial U\times\overline{U}\backslash E(f,\tau)$, and $E(f,\tau)=\{x\in\partial U:\lim_{t\to x}f(t,\tau)=\infty\}$. When $f(0,\tau)=a$, Q^* is denoted by $Q^*(a)$.*

To explore strong differential superordination, the following lemmas are needed.

Lemma 3 ([37]). *Considering the function $p\in\mathcal{H}^*[a,n,\tau]\cap Q^*$, satisfying the strong differential superordination*

$$h(t,\tau)\prec\prec p(t,\tau)+\frac{1}{\eta}tp'_t(t,\tau), \quad t\in U,\ \tau\in\overline{U},$$

and $p(t,\tau)+\frac{1}{\eta}tp'_t(t,\tau)$ is univalent in $U\times\overline{U}$, where $h(t,\tau)$ is a convex function such that $h(0,\tau)=a$, $\tau\in\overline{U}$ and $\eta\in\mathbb{C}^*$ with $Re\,\eta\geq 0$, we obtain the strong differential superordination

$$q(t,\tau)\prec\prec p(t,\tau),$$

and the convex function $q(t,\tau) = \frac{\eta}{nt^{\frac{\eta}{n}}}\int_0^t h(x,\tau)x^{\frac{\eta}{n}-1}dx$, $t \in U$, $\tau \in \overline{U}$, *is the best subordinant.*

Lemma 4 ([37]). *Considering the function* $p \in \mathcal{H}^*[a,n,\tau] \cap Q^*$ *satisfying the strong differential superordination*

$$q(t,\tau) + \frac{1}{\eta}tq_t'(t,\tau) \prec\prec p(t,\tau) + \frac{1}{\eta}tp_t'(t,\tau), \quad t \in U,\ \tau \in \overline{U},$$

and $p(t,\tau) + \frac{1}{\eta}tp_t'(t,\tau)$ *is univalent in* $U \times \overline{U}$, *where* $q(t,\tau)$ *is a convex function and*

$$h(t,\tau) = q(t,\tau) + \frac{1}{\eta}tq_t'(t,\tau),$$

for $\eta \in \mathbb{C}^*$ *with Re* $\eta \geq 0$, *we obtain the strong differential superordination*

$$q(t,\tau) \prec\prec p(t,\tau), \quad t \in U,\ \tau \in \overline{U},$$

and the convex function $q(t,\tau) = \frac{\eta}{nt^{\frac{\eta}{n}}}\int_0^t h(x,\tau)x^{\frac{\eta}{n}-1}dx$, $t \in U$, $\tau \in \overline{U}$, *is the best subordinant.*

We recall the definition of fractional integral used to obtain a new operator studied in this paper.

Definition 4 ([38,39]). *For an analytic function* f *in a simply connected region of the z-plane that contains the origin, the fractional integral of order* γ $(\gamma > 0)$ *is given by*

$$D_t^{-\gamma}f(t,\tau) = \frac{1}{\Gamma(\gamma)}\int_0^t \frac{f(x,\tau)}{(t-x)^{1-\gamma}}dx, \tag{1}$$

when $(t-x) > 0$, *removing the multiplicity of* $(t-x)^{\gamma-1}$ *by requiring* $\log(t-x)$ *to be real.*

2. Main Results

The Dziok–Srivastava operator was introduced in [40], and a lot of papers investigated the properties of this operator ([41–46]).

We extend the Dziok–Srivastava operator to the class of analytic functions $\mathcal{A}_\tau^*$.

Definition 5 ([47]). *For* $f \in \mathcal{A}_\tau^*$, *the extended Dziok–Srivastava operator is given by*

$$H_m^l(\alpha_1,\alpha_2,...,\alpha_m;\beta_1,\beta_2,...,\beta_l) : \mathcal{A}_\tau^* \to \mathcal{A}_\tau^*,$$

$$H_m^l(\alpha_1,\alpha_2,...,\alpha_m;\beta_1,\beta_2,...,\beta_l)f(t,\tau) = z + \sum_{j=2}^{\infty}\frac{(\alpha_1)_{j-1}(\alpha_2)_{j-1}...(\alpha_m)_{j-1}}{(\beta_1)_{j-1}(\beta_2)_{j-1}...(\beta_l)_{j-1}\Gamma(j)}a_j(\tau)t^j, \tag{2}$$

$\alpha_k \in \mathbb{C}$, $k = 1,2,...,m$, $\beta_i \in \mathbb{C}\backslash\{0,-1,-2,...\}$, $i = 1,2,...,l$, *and the Pochhammer symbol* $(x)_j$ *is defined by*

$$(x)_j = \frac{\Gamma(x+j)}{\Gamma(x)} = \begin{cases} 1, \text{ for } j = 0 \text{ and } x \in \mathbb{C}\backslash\{0\}, \\ x(x+1)...(x+j-1), \text{ for } j \in \mathbb{N} \text{ and } x \in \mathbb{C}. \end{cases}$$

For simplicity, we write

$$H_m^l[\alpha_1,\beta_1]f(t,\tau) = H_m^l(\alpha_1,\alpha_2,...,\alpha_m;\beta_1,\beta_2,...,\beta_l)f(t,\tau). \tag{3}$$

Applying fractional integral to the extended Dziok–Srivastava operator, we obtain a new operator studied in this paper.

Definition 6. *The fractional integral applied to the extended Dziok–Srivastava operator is defined by*

$$D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t, \tau) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{H_m^l[\alpha_1, \beta_1] f(x, \tau)}{(t-x)^{1-\gamma}} dx,$$

and, after a simple computation, can be written as follows:

$$D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t, \tau) = \frac{1}{\Gamma(2+\gamma)} t^{1+\gamma} +$$

$$\sum_{j=2}^{\infty} \frac{j}{(j+\gamma)!} \frac{(\alpha_1)_{j-1}(\alpha_2)_{j-1} \ldots (\alpha_m)_{j-1}}{(\beta_1)_{j-1}(\beta_2)_{j-1} \ldots (\beta_l)_{j-1}(j-1)!} a_j(\tau) t^{j+\gamma}, \tag{4}$$

considering the function $f(t, \tau) = t + \sum_{j=2}^{\infty} a_j(\tau) t^j \in \mathcal{A}_\tau^*$.

With a short computation, we get the relation

$$t\left(D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t, \tau)\right)_t' = \tag{5}$$

$$\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1 + 1, \beta_1] f(t, \tau) - [\alpha_1 - (1+\gamma)] D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t, \tau).$$

A similar result is found for the parameter β_1.

2.1. Strong Differential Subordination

In this subsection, we get strong differential subordinations involving the fractional integral of the extended Dziok–Srivastava operator.

Theorem 1. *Considering the convex function* $q(t, \tau)$ *with the property* $q(0, \tau) = 0$, *we take the function* $h(t, \tau) = q(t, \tau) + \gamma t q_t'(t, \tau)$, $t \in U$, $\tau \in \overline{U}$, *with* γ *a positive integer.*

If the strong differential subordination

$$\left(D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t, \tau)\right)_t' \prec\prec h(t, \tau), \tag{6}$$

is satisfied when $f \in \mathcal{A}_\tau^*$, *then we get the following sharp strong differential subordination:*

$$\frac{D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t, \tau)}{t} \prec\prec q(t, \tau).$$

Proof. Take $p(t, \tau) = \frac{D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t, \tau)}{t} \in \mathcal{H}^*[0, \gamma, \tau]$, $t \in U$, $\tau \in \overline{U}$, then $D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t, \tau) = t p(t, \tau)$ and, differentiating the relation with respect to t, we get $\left(D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t, \tau)\right)_t' = p(t, \tau) + t p_t'(t, \tau)$. Then, strong subordination (6) has the following form:

$$p(t, \tau) + t p_t'(t, \tau) \prec\prec h(t, \tau) = q(t, \tau) + \gamma t q_t'(t, \tau),$$

and, applying Lemma 2, we get

$$p(t, \tau) \prec\prec q(t, \tau), \quad \text{i.e.} \quad \frac{D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t, \tau)}{t} \prec\prec q(t, \tau).$$

□

Theorem 2. *If* $h(t, \tau)$ *is a convex function with* $h(0, \tau) = 0$, *satisfying the strong differential subordination:*

$$\left(D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t, \tau)\right)_t' \prec\prec h(t, \tau), \tag{7}$$

for $f \in \mathcal{A}^*_\tau$, then we get the following strong differential subordination

$$\frac{D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t,\tau)}{t} \prec\prec q(t,\tau),$$

and the convex function $q(t,\tau) = \frac{1}{t}\int_0^t h(x,\tau)dx$ is the best dominant.

Proof. Let $p(t,\tau) = \frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)}{t} \in \mathcal{H}^*[0,\lambda,\tau]$, $t \in U, \tau \in \overline{U}$.

Differentiating relation $D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t,\tau) = tp(t,\tau)$, with respect to t, it yields $\left(D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t,\tau)\right)'_t = p(t,\tau) + tp'_t(t,\tau)$, and the strong subordination (7) will be

$$tp'_t(t,\tau) + p(t,\tau) \prec\prec h(t,\tau),$$

and applying Lemma 1, we obtain

$$p(t,\tau) \prec\prec q(t,\tau), \quad \text{i.e.} \quad \frac{D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t,\tau)}{t} \prec\prec q(t,\tau) = \frac{1}{t}\int_0^t h(x,\tau)dx,$$

with q as the best dominant. □

Corollary 1. *Taking the convex function $h(t,\tau) = \frac{\tau+2(\lambda-\tau)t}{1+t}$, with $0 \le \lambda < 1$ satisfying the strong subordination*

$$\left(D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t,\tau)\right)'_t \prec\prec h(t,\tau), \tag{8}$$

for $f \in \mathcal{A}^*_\tau$, then we get the strong subordination

$$\frac{D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t,\tau)}{t} \prec\prec q(t,\tau),$$

and the convex function $q(t,\tau) = 2(\lambda-\tau) + 2(\tau-\lambda)\frac{\ln(1+t)}{t}$, $t \in U, \tau \in \overline{U}$ is the best dominant.

Proof. Repeating the steps made in the proof of Theorem 2, taking $p(t,\tau) = \frac{D_t^{-\lambda} H_m^l[\alpha_1,\beta_1] f(t,\tau)}{t}$, the strong subordination (8) takes the form

$$p(t,\tau) + tp'_t(t,\tau) \prec\prec h(t,\tau) = \frac{\tau + 2(\lambda-\tau)t}{1+t},$$

for which, applying Lemma 1, we get $p(t,\tau) \prec\prec q(t,\tau)$, i.e.,

$$\frac{D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t,\tau)}{t} \prec\prec q(t,\tau) = \frac{1}{t}\int_0^t h(x,\tau)dx =$$

$$\frac{1}{t}\int_0^t \frac{\tau+2(\lambda-\tau)x}{1+x}dx = 2(\lambda-\tau) + 2(\tau-\lambda)\frac{1}{t}\ln(t+1), \ \ t \in U, \ \tau \in \overline{U}.$$

□

Theorem 3. *Taking the convex function $q(t,\tau)$ with $q(0,\tau) = 0$, we consider the function $h(t,\tau) = q(t,\tau) + \frac{1}{\mu}tq'_t(t,\tau)$, with μ a positive integer, $t \in U, \tau \in \overline{U}$.*

If the strong subordination is accomplished

$$\left(\frac{D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t,\tau)}{t}\right)^{\mu-1} \left(D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t,\tau)\right)'_t \prec\prec h(t,\tau), \tag{9}$$

for $f \in \mathcal{A}^*_\tau$, then we get the following sharp strong subordination:

$$\left(\frac{D_t^{-\lambda} H_m^l[\alpha_1, \beta_1] f(t,\tau)}{t}\right)^\mu \prec\prec q(t,\tau).$$

Proof. Taking $p(t,\tau) = \left(\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)}{t}\right)^\mu \in \mathcal{H}^*[0,\gamma\mu,\tau]$, $t \in U$, $\tau \in \overline{U}$, and applying differentiation with respect to t, we have $tp_t'(t,\tau) = \mu\left(\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)}{t}\right)^{\mu-1}$ $\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)_t' - \mu\left(\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)}{t}\right)^\mu = \mu\left(\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)}{t}\right)^{\mu-1}$ $\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)_t' - \mu p(t,\tau)$,
written as $p(t,\tau) + \frac{1}{\mu} tp_t'(t,\tau) = \left(\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)}{t}\right)^{\mu-1} \left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)_t'$.

In these conditions, strong subordination (9) takes the form

$$p(t,\tau) + \frac{1}{\mu} tp_t'(t,\tau) \prec\prec h(t,\tau) = q(t,\tau) + \frac{1}{\mu} tq_t'(t,\tau),$$

and applying Lemma 2, we get

$$p(t,\tau) \prec\prec q(t,\tau), \quad \text{i.e.} \quad \left(\frac{D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t,\tau)}{t}\right)^\mu \prec\prec q(t,\tau).$$

□

Theorem 4. *Let the convex function $h(t,\tau)$ with $h(0,\tau) = 0$, verifying the strong subordination*

$$\left(\frac{D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t,\tau)}{t}\right)^{\mu-1} \left(D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t,\tau)\right)_t' \prec\prec h(t,\tau), \tag{10}$$

for $f \in \mathcal{A}^*_\tau$ and μ a positive integer, then we get the following strong subordination:

$$\left(\frac{D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t,\tau)}{t}\right)^\mu \prec\prec q(t,\tau),$$

and the convex function $q(t,\tau) = \frac{\mu}{t^\mu} \int_0^t h(x,\tau) x^{\mu-1} dx$ is the best dominant.

Proof. Let $p(t,\tau) = \left(\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)}{t}\right)^\mu \in \mathcal{H}^*[0,\gamma\mu,\tau]$, $t \in U$, $\tau \in \overline{U}$.

Applying the computation made in the proof of Theorem 3, we get

$$p(t,\tau) + \frac{1}{\mu} tp_t'(t,\tau) = \left(\frac{D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t,\tau)}{t}\right)^{\mu-1} \left(D_t^{-\gamma} H_m^l[\alpha_1, \beta_1] f(t,\tau)\right)_t',$$

and the strong subordination (10) becomes

$$p(t,\tau) + \frac{1}{\mu} tp_t'(t,\tau) \prec\prec h(t,\tau)$$

and verifies the conditions from Lemma 1, so we get the strong subordination

$$p(t,\tau) \prec\prec q(t,\tau), \quad \text{i.e.} \quad \left(\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\right)^{\mu} \prec\prec q(t,\tau) = \frac{\mu}{t^{\mu}} \int_0^t h(x,\tau)x^{\mu-1}dx,$$

with q as the best dominant. □

Theorem 5. *Let the convex function* $q(t,\tau)$ *with* $g(0,\tau) = \frac{1}{1+\gamma}$*, we take the function* $h(t,\tau) = q(t,\tau) + tq_t'(t,\tau)$*,* $t \in U$*,* $\tau \in \overline{U}$*.*

If the strong subordination

$$\frac{\alpha_1^2\left(D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(t,\tau)\right)^2 - \alpha_1(\alpha_1+1)D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\cdot D_t^{-\gamma} H_m^l[\alpha_1+2,\beta_1]f(t,\tau)}{\left(\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(t,\tau) - [\alpha_1-(1+\gamma)]D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)^2} +$$
$$\frac{2\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\cdot D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(t,\tau) - [\alpha_1-(1+\gamma)]\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)^2}{\left(\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(t,\tau) - [\alpha_1-(1+\gamma)]D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)^2}$$
$$\prec\prec h(t,\tau), \tag{11}$$

holds for $f \in \mathcal{A}_\tau^*$*, then we get the sharp strong subordination*

$$\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t'} \prec\prec q(t,\tau).$$

Proof. Considering $p(t,\tau) = \frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t'}$ and differentiating with respect to t yields $1 - \frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\cdot\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_{t^2}''}{\left[\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t'\right]^2} = p(t,\tau) + tp_t'(t,\tau)$.

After a short computation and applying relation (5), we get

$$1 - \frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\cdot\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_{t^2}''}{\left[\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t'\right]^2} =$$
$$\frac{\alpha_1^2\left(D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(t,\tau)\right)^2 - \alpha_1(\alpha_1+1)D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\cdot D_t^{-\gamma} H_m^l[\alpha_1+2,\beta_1]f(t,\tau)}{\left(\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(t,\tau) - [\alpha_1-(1+\gamma)]D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)^2} +$$
$$\frac{2\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\cdot D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(t,\tau) - [\alpha_1-(1+\gamma)]\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)^2}{\left(\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(t,\tau) - [\alpha_1-(1+\gamma)]D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)^2}$$

In these conditions, the strong subordination (11) becomes

$$p(t,\tau) + tp_t'(t,\tau) \prec\prec h(t,\tau) = q(t,\tau) + tq_t'(t,\tau),$$

and applying Lemma 2, we get the sharp strong subordination

$$p(t,\tau) \prec\prec q(t,\tau), \quad \text{i.e.} \quad \frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t'} \prec\prec q(t,\tau).$$

□

Theorem 6. *Let the convex function $h(t,\tau)$ with $h(0,\tau) = \frac{1}{1+\gamma}$, verifying the strong differential subordination*

$$\frac{\alpha_1^2\left(D_t^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(t,\tau)\right)^2 - \alpha_1(\alpha_1+1)D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\cdot D_t^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(t,\tau)}{\left(\alpha_1 D_t^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(t,\tau) - [\alpha_1-(1+\gamma)]D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)^2} +$$
$$\frac{2\alpha_1 D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\cdot D_t^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(t,\tau) - [\alpha_1-(1+\gamma)]\left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)^2}{\left(\alpha_1 D_t^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(t,\tau) - [\alpha_1-(1+\gamma)]D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)^2}$$
$$\prec\prec h(t,\tau), \tag{12}$$

for $f \in \mathcal{A}_\tau^*$, then we get the strong subordination

$$\frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t\left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)'_t} \prec\prec q(t,\tau),$$

and the convex function $q(t,\tau) = \frac{1}{t}\int_0^t h(x,\tau)dx$ is the best dominant.

Proof. Denote $p(t,\tau) = \frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t\left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)'_t}$.

Using the same steps and the computation used in the proof of Theorem 5, we get

$$p(t,\tau) + tp'_t(t,\tau) = 1 - \frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\cdot\left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)''_{t^2}}{\left[\left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)'_t\right]^2},$$

and the strong subordination (12) takes the form

$$p(t,\tau) + tp'_t(t,\tau) \prec\prec h(t,\tau),$$

and applying Lemma 1, we have the strong subordination

$$p(t,\tau) \prec\prec q(t,\tau), \quad \text{i.e.} \quad \frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t\left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)'_t} \prec\prec q(t,\tau) = \frac{1}{t}\int_0^t h(x,\tau)dx,$$

and the best dominant is the function q. □

Theorem 7. *Let the convex function $q(t,\tau)$ with $q(0,\tau) = 0$; we take the function $h(t,\tau) = q(t,\tau) + \gamma t q'_t(t,\tau\zeta)$, with γ a positive integer, $t \in U$, $\tau \in \overline{U}$.*

If the strong subordination

$$\alpha_1(\alpha_1+1)\frac{D_t^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(t,\tau)}{t} + \alpha_1(1-2\alpha_1+2\gamma)\frac{D_t^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(t,\tau)}{t}$$
$$+(\alpha_1-1-\gamma)^2\frac{D_t^{-\lambda}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t} \prec\prec h(t,\tau), \tag{13}$$

is accomplished for $f \in \mathcal{A}_\tau^*$, then we get the sharp strong subordination

$$\left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)'_t \prec\prec q(t,\tau).$$

Proof. Let
$$p(t,\tau)=\left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t'\in\mathcal{H}^*[0,\gamma,\tau]. \tag{14}$$
Using relation (5), we get
$$tp(t,\tau)=\alpha_1 D_t^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(t,\tau)-[\alpha_1-(1+\gamma)]D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau),$$
and differentiating it with respect to t, it yields
$$p(t,\tau)+tp_t'(t,\tau)=\alpha_1(\alpha_1+1)\frac{D_t^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(t,\tau)}{t}+$$
$$\alpha_1(1-2\alpha_1+2\gamma)\frac{D_t^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(t,\tau)}{t}+(\alpha_1-1-\gamma)^2\frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}.$$
In these conditions, the strong subordination (14) takes the form
$$p(t,\tau)+tp_t'(t,\tau)\prec\prec h(t,\tau)=q(t,\tau)+\gamma tq_t'(t,\tau),$$
and using Lemma 2, we get the sharp strong subordination
$$p(t,\tau)\prec\prec q(t,\tau),\quad i.e.,\quad \left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t'\prec\prec q(t,\tau).$$
□

Theorem 8. *Let the convex function $h(t,\tau)$ with $h(0,\tau)=0$, verifying the strong subordination*
$$\alpha_1(\alpha_1+1)\frac{D_t^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(t,\tau)}{t}+\alpha_1(1-2\alpha_1+2\gamma)\frac{D_t^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(t,\tau)}{t}$$
$$+(\alpha_1-1-\gamma)^2\frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\prec\prec h(t,\tau), \tag{15}$$
for $f\in\mathcal{A}_\tau^*$, then we get the strong subordination
$$\left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t'\prec\prec q(t,\tau),$$
and the convex function $q(t,\tau)=\frac{1}{t}\int_0^t h(x,\tau)dx$ is the best dominant.

Proof. Let $p(t,\tau)=\left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t'\in\mathcal{H}^*[0,\gamma,\tau]$, $t\in U$, $\tau\in\overline{U}$.
Using the computation used in the proof of Theorem 7, we get
$$tp_t'(t,\tau)+p(t,\tau)=\ \alpha_1(\alpha_1+1)\frac{D_t^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(t,\tau)}{t}+$$
$$\alpha_1(1-2\alpha_1+2\gamma)\frac{D_t^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(t,\tau)}{t}+(\alpha_1-1-\gamma)^2\frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t},$$
and the strong subordination (15) can be written as
$$p(t,\tau)+tp_t'(t,\tau)\prec\prec h(t,\tau),$$
which satisfies Lemma 1, getting
$$p(t,\tau)\prec\prec q(t,\tau),\quad \text{i.e.}\quad \left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t'\prec\prec q(t,\tau)=\frac{1}{t}\int_0^t h(x,\tau)dx,$$
and the best dominant is the function q. □

2.2. Strong Differential Superordination

In this subsection, we get strong differential superordinations involving the fractional integral of the extended Dziok–Srivastava operator.

Theorem 9. *Let the convex function $q(t,\tau)$ with $q(0,\tau)=0$, we take the function $h(t,\tau)=q(t,\tau)+\gamma t q'_t(t,\tau)$, for γ a positive integer, $t\in U$, $\tau\in\overline{U}$. Assume that $\frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\in Q^*\cap\mathcal{H}^*[0,\gamma,\tau]$ and $\left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)'_t$ is univalent and the strong differential superordination is accomplished*

$$h(t,\tau)\prec\prec\left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)'_t, \tag{16}$$

for $f\in\mathcal{A}^*_\tau$, then we get the strong superordination

$$q(t,\tau)\prec\prec\frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t},$$

and the convex function $q(t,\tau)=\frac{1}{t}\int_0^t h(x,\tau)dx$ is the best subordinant.

Proof. Consider $p(t,\tau)=\frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\in\mathcal{H}^*[0,\gamma,\tau]$, $t\in U$, $\tau\in\overline{U}$.

Differentiating the relation $D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)=tp(t,\tau)$ with respect to t, we get $\left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)'_t=p(t,\tau)+tp'_t(t,\tau)$.

The strong superordination (16) takes the following form:

$$h(t,\tau)=q(t,\tau)+\gamma t q'_t(t,\tau)\prec\prec p(t,\tau)+tp'_t(t,\tau),$$

and using Lemma 4, we get the strong superordination

$$q(t,\tau)\prec\prec p(t,\tau),\quad \text{i.e.}\quad q(t,\tau)=\frac{1}{t}\int_0^t h(x,\tau)dx\prec\prec\frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t},$$

and the best subordinant is the function q. □

Theorem 10. *If $h(t,\tau)$ is a convex function with $h(0,\tau)=0$, assume that $\frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\in Q^*\cap\mathcal{H}^*[0,\gamma,\tau]$ and $\left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)'_t$ is univalent and satisfies the strong differential superordination*

$$h(t,\tau)\prec\prec\left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)'_t, \tag{17}$$

for $f\in\mathcal{A}^*_\tau$, then we get the following strong superordination:

$$q(t,\tau)\prec\prec\frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t},$$

and the convex function $q(t,\tau)=\frac{1}{t}\int_0^t h(x,\tau)dx$ is the best subordinant.

Proof. Let $p(t,\tau)=\frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\in\mathcal{H}^*[0,\lambda,\tau]$, $t\in U$, $\tau\in\overline{U}$.

Applying differentiation with respect to t to the relation $D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)=tp(t,\tau)$, we get $\left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)'_t=p(t,\tau)+tp'_t(t,\tau)$, and the strong superordination (17) takes the form

$$h(t,\tau)\prec\prec tp'_t(t,\tau)+p(t,\tau),$$

for which, applying Lemma 3, we get

$$q(t,\tau) \prec\prec p(t,\tau), \quad \text{i.e.,} \quad q(t,\tau) = \frac{1}{t}\int_0^t h(x,\tau)dx \prec\prec \frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t},$$

and the best subordinant is the function q. □

Corollary 2. *Considering the convex function* $h(t,\tau) = \frac{\tau+2(\lambda-\tau)t}{1+t}$ *for* $0 \le \lambda < 1$, $f \in \mathcal{A}_\tau^*$, *we assume that* $\frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t} \in Q^* \cap \mathcal{H}^*[0,\gamma,\tau]$, $\left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t'$ *is univalent and the strong superordination*

$$h(t,\tau) \prec\prec \left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t', \tag{18}$$

is verified, then we get the strong superordination

$$q(t,\tau) \prec\prec \frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t},$$

and the convex function $q(t,\tau) = 2(\lambda-\tau) + 2(\tau-\lambda)\frac{\ln(1+t)}{t}$, $t \in U$, $\tau \in \overline{U}$ *is the best subordinant.*

Proof. Repeating the steps made in the proof of Theorem 10 considering $p(t,\tau) = \frac{D_t^{-\lambda}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}$, the strong superordination (18) takes the form

$$h(t,\tau) = \frac{\tau+2(\lambda-\tau)t}{1+t} \prec\prec p(t,\tau) + tp_t'(t,\tau).$$

Using Lemma 3, it yields $q(t,\tau) \prec\prec p(t,\tau)$, i.e.,

$$q(t,\tau) = \frac{1}{t}\int_0^t h(x,\tau)dx \prec\prec \frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}$$

and the best subordinant is the function

$$q(t,\tau) = \frac{1}{t}\int_0^t \frac{\tau+2(\lambda-\tau)x}{1+x}dx = 2(\lambda-\tau) + 2(\tau-\lambda)\frac{1}{t}\ln(t+1), \ \ t \in U, \ \tau \in \overline{U}.$$

□

Theorem 11. *Let the convex function* $q(t,\tau)$ *with* $q(0,\tau) = 0$, *we take the function* $h(t,\tau) = q(t,\tau) + \frac{1}{\mu}tq_t'(t,\tau)$, *with* μ *a positive integer,* $t \in U$, $\tau \in \overline{U}$. *Assume that for* $f \in \mathcal{A}_\tau^*$, $\left(\frac{D_t^{-\lambda}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\right)^\mu \in Q^* \cap \mathcal{H}^*[0,\gamma\mu,\tau]$, $\left(\frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\right)^{\mu-1}\left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t'$ *is univalent and the strong superordination*

$$h(t,\tau) \prec\prec \left(\frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\right)^{\mu-1}\left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t', \tag{19}$$

is verified, then we get the strong superordination

$$q(t,\tau) \prec\prec \left(\frac{D_t^{-\lambda}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\right)^\mu,$$

and the convex function $q(t,\tau) = \frac{\mu}{t^\mu}\int_0^t h(x,\tau)x^{\mu-1}dx$ *is the best subordinant.*

Proof. Consider $p(t,\tau) = \left(\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\right)^{\mu} \in \mathcal{H}^*[0,\gamma\mu,\tau]$, $t \in U$, $\tau \in \overline{U}$.

Differentiating with respect to t yields

$$tp_t'(t,\tau) =$$
$$\mu\left(\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\right)^{\mu-1}\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t' - \mu\left(\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\right)^{\mu} =$$
$$\mu\left(\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\right)^{\mu-1}\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t' - \mu p(t,\tau),$$

therefore $p(t,\tau) + \frac{1}{\mu}tp_t'(t,\tau) = \left(\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\right)^{\mu-1}\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t'$.

In these conditions, the strong superordination (19) can be written as

$$h(t,\tau) = q(t,\tau) + \frac{1}{\mu}tq_t'(t,\tau) \prec\prec p(t,\tau) + \frac{1}{\mu}tp_t'(t,\tau),$$

and by Lemma 4, we get the strong superordination

$$q(t,\tau) \prec\prec p(t,\tau), \quad \text{i.e.} \quad q(t,\tau) \prec\prec \left(\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\right)^{\mu},$$

and the best subordinant is the function $q(t,\tau) = \frac{\mu}{t^{\mu}}\int_0^t h(x,\tau)x^{\mu-1}dx$. □

Theorem 12. *Let the convex function $h(t,\tau)$ with $h(0,\tau) = 0$, and μ a positive integer, we assume for $f \in \mathcal{A}_\tau^*$ that $\left(\frac{D_t^{-\lambda} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\right)^{\mu} \in Q^* \cap \mathcal{H}^*[0,\gamma\mu,\tau]$, $\left(\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\right)^{\mu-1}\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t'$ is univalent and the strong superordination is satisfied*

$$h(t,\tau) \prec\prec \left(\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\right)^{\mu-1}\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t', \tag{20}$$

then the strong superordination

$$q(t,\tau) \prec\prec \left(\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\right)^{\mu},$$

holds and the convex function $q(t,\tau) = \frac{\mu}{t^{\mu}}\int_0^t h(x,\tau)x^{\mu-1}dx$ is the best subordinant.

Proof. Let $p(t,\tau) = \left(\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\right)^{\mu} \in \mathcal{H}^*[0,\gamma\mu,\tau]$, $t \in U$, $\tau \in \overline{U}$.

Using the computation used in the proof of Theorem 11 yields

$$p(t,\tau) + \frac{1}{\mu}tp_t'(t,\tau) = \left(\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}\right)^{\mu-1}\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t',$$

and the strong superordination (20) is written as

$$h(t,\tau) \prec\prec p(t,\tau) + \frac{1}{\mu}tp_t'(t,\tau).$$

By Lemma 3, we get the strong superordination

$$q(t,\tau) \prec\prec p(t,\tau), \quad \text{i.e.} \quad q(t,\tau) = \frac{\mu}{t^{\mu}} \int_0^t h(x,\tau) x^{\mu-1} dx \prec\prec \left(\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)}{t} \right)^{\mu},$$

and the best subordinant is the function q. □

Theorem 13. *Let the convex function $q(t,\tau)$ with $g(0,\tau) = \frac{1}{1+\gamma}$, we take the function $h(t,\tau) = q(t,\tau) + tq'_t(t,\tau)$, and assume that $f \in \mathcal{A}^*_\tau$, $\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)}{t\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)'_t} \in Q^* \cap \mathcal{H}^*\left[\frac{1}{1+\gamma}, 1, \tau\right]$, $\frac{\alpha_1^2\left(D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1] f(t,\tau)\right)^2 - \alpha_1(\alpha_1+1) D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau) \cdot D_t^{-\gamma} H_m^l[\alpha_1+2,\beta_1] f(t,\tau)}{\left(\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1] f(t,\tau) - [\alpha_1-(1+\gamma)] D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)^2} + \frac{2\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau) \cdot D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1] f(t,\tau) - [\alpha_1-(1+\gamma)]\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)^2}{\left(\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1] f(t,\tau) - [\alpha_1-(1+\gamma)] D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)^2}$ is univalent and verifies the strong differential superordination*

$$h(t,\tau) \prec\prec$$
$$\frac{\alpha_1^2\left(D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1] f(t,\tau)\right)^2 - \alpha_1(\alpha_1+1) D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau) \cdot D_t^{-\gamma} H_m^l[\alpha_1+2,\beta_1] f(t,\tau)}{\left(\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1] f(t,\tau) - [\alpha_1-(1+\gamma)] D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)^2}$$
$$+\frac{2\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau) \cdot D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1] f(t,\tau) - [\alpha_1-(1+\gamma)]\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)^2}{\left(\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1] f(t,\tau) - [\alpha_1-(1+\gamma)] D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)^2}, \tag{21}$$

then we get the strong differential superordination

$$q(t,\tau) \prec\prec \frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)}{t\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)'_t},$$

and the best subordinant is the convex function $q(t,\tau) = \frac{1}{t}\int_0^t h(x,\tau)dx$.

Proof. Differentiating the relation $p(t,\tau) = \frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)}{t\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)'_t}$ with respect to t yields $1 - \frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau) \cdot \left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)''_{t^2}}{\left[\left(D_{zt}^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)'_t\right]^2} = p(t,\tau) + tp'_t(t,\tau)$.

Making a short computation and applying relation (5), we get

$$1 - \frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau) \cdot \left(D_{zt}^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)''_{t^2}}{\left[\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)'_t\right]^2} =$$
$$\frac{\alpha_1^2\left(D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1] f(t,\tau)\right)^2 - \alpha_1(\alpha_1+1) D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau) \cdot D_t^{-\gamma} H_m^l[\alpha_1+2,\beta_1] f(t,\tau)}{\left(\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1] f(t,\tau) - [\alpha_1-(1+\gamma)] D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)^2} +$$
$$\frac{2\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau) \cdot D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1] f(t,\tau) - [\alpha_1-(1+\gamma)]\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)^2}{\left(\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1] f(t,\tau) - [\alpha_1-(1+\gamma)] D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)^2}$$

In these conditions, the strong superordination takes the form

$$h(t,\tau) = q(t,\tau) + tq'_t(t,\tau) \prec\prec p(t,\tau) + tp'_t(t,\tau),$$

and, applying Lemma 4, we get the strong superordination

$$q(t,\tau) \prec\prec p(t,\tau), \quad \text{i.e.} \quad q(t,\tau) \prec\prec \frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t\Big(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\Big)_t'},$$

and the best subordinant represents the function $q(t,\tau) = \frac{1}{t}\int_0^t h(x,\tau)dx$. □

Theorem 14. *Taking the convex function $h(t,\tau)$ such that $h(0,\tau) = \frac{1}{1+\gamma}$, we assume that $f \in \mathcal{A}_\tau^*$, $\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t'} \in Q^* \cap \mathcal{H}^*\left[\frac{1}{1+\gamma},1,\tau\right]$, the function $\frac{\alpha_1^2\left(D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(t,\tau)\right)^2 - \alpha_1(\alpha_1+1)D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\cdot D_t^{-\gamma} H_m^l[\alpha_1+2,\beta_1]f(t,\tau)}{\left(\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(t,\tau) - [\alpha_1-(1+\gamma)]D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)^2} + \frac{2\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\cdot D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(t,\tau) - [\alpha_1-(1+\gamma)]\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)^2}{\left(\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(t,\tau) - [\alpha_1-(1+\gamma)]D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)^2}$ is univalent and verifies the strong superordination*

$$\begin{aligned} & h(t,\tau) \prec\prec \\ & \frac{\alpha_1^2\Big(D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(t,\tau)\Big)^2 - \alpha_1(\alpha_1+1)D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\cdot D_t^{-\gamma} H_m^l[\alpha_1+2,\beta_1]f(t,\tau)}{\Big(\alpha_1 DtH_m^l[\alpha_1+1,\beta_1]f(t,\tau) - [\alpha_1-(1+\gamma)]D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\Big)^2} \\ & + \frac{2\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\cdot D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(t,\tau) - [\alpha_1-(1+\gamma)]\Big(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\Big)^2}{\Big(\alpha_1 D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(t,\tau) - [\alpha_1-(1+\gamma)]D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\Big)^2}, \end{aligned} \tag{22}$$

then we get the strong superordination

$$q(t,\tau) \prec\prec \frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t\Big(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\Big)_t'},$$

and the best subordinant is the convex function $q(t,\tau) = \frac{1}{t}\int_0^t h(x,\tau)dx$.

Proof. Let $p(t,\tau) = \frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)_t'}$.

Applying the computation made in the proof of Theorem 13, we get

$$p(t,\tau) + tp_t'(t,\tau) = 1 - \frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\cdot\Big(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\Big)_{t^2}''}{\left[\Big(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\Big)_t'\right]^2},$$

and the strong superordination (22) can be written as

$$h(t,\tau) \prec\prec p(t,\tau) + tp_t'(t,\tau),$$

and by Lemma 3, we get the strong superordination

$$q(t,\tau) \prec\prec p(t,\tau), \quad \text{i.e.,} \quad q(t,\tau) = \frac{1}{t}\int_0^t h(x,\tau)dx \prec\prec \frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t\Big(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\Big)_t'}$$

and the best subordinant is the function q. □

Theorem 15. *Let the convex function $q(t,\tau)$ with $q(0,\tau) = 0$, we take the function $h(t,\tau) = q(t,\tau) + \gamma t q'_t(t,\tau)$, when γ is a positive integer, $t \in U$, $\tau \in \overline{U}$.*

Assume that $\left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)'_t \in Q^* \cap \mathcal{H}^*[0,\gamma,\tau]$ and $\alpha_1(\alpha_1+1)\frac{D_t^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(t,\tau)}{t} + \alpha_1(1-2\alpha_1+2\gamma)\frac{D_t^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(t,\tau)}{t} + (\alpha_1-1-\gamma)^2\frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}$ is univalent for $f \in \mathcal{A}^*_\tau$ and verifies the strong superordination

$$h(t,\tau) \prec\prec \alpha_1(\alpha_1+1)\frac{D_t^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(t,\tau)}{t} +$$

$$\alpha_1(1-2\alpha_1+2\gamma)\frac{D_t^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(t,\tau)}{t} + (\alpha_1-1-\gamma)^2\frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}, \quad (23)$$

then we get the strong superordination

$$q(t,\tau) \prec\prec \left(DtH_m^l[\alpha_1,\beta_1]f(t,\tau)\right)'_t,$$

and the best subordinant is the convex function $q(t,\tau) = \frac{1}{t}\int_0^t h(x,\tau)dx$.

Proof. Let

$$p(t,\tau) = \left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)'_t \in \mathcal{H}^*[0,\gamma,\tau],\ t \in U,\ \tau \in \overline{U}. \quad (24)$$

Using relation (5) yields

$$tp(t,\tau) = \alpha_1 D_t^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(t,\tau) - [\alpha_1-(1+\gamma)]D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau),$$

and differentiating it with respect to t, we get

$$p(t,\tau) + tp'_t(t,\tau) = \alpha_1(\alpha_1+1)\frac{D_t^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(t,\tau)}{t} +$$

$$\alpha_1(1-2\alpha_1+2\gamma)\frac{D_t^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(t,\tau)}{t} + (\alpha_1-1-\gamma)^2\frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}.$$

In these conditions, the strong superordination (24) can be written as

$$h(t,\tau) = q(t,\tau) + \gamma t q'_t(t,\tau) \prec\prec p(t,\tau) + tp'_t(t,\tau),$$

and verifies Lemma 4. Therefore, we get the strong superordination

$$q(t,\tau) \prec\prec p(t,\tau), \quad i.e. \quad q(t,\tau) \prec\prec \left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)'_t,$$

and the best subordinant is represented by $q(t,\tau) = \frac{1}{t}\int_0^t h(x,\tau)dx$. □

Theorem 16. *Let the convex function $h(t,\tau)$ with $h(0,\tau) = 0$; we assume that $f \in \mathcal{A}^*_\tau$, $\left(D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)\right)'_t \in Q^* \cap \mathcal{H}^*[0,\gamma,\tau]$, the function $\alpha_1(\alpha_1+1)\frac{D_t^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(t,\tau)}{t} + \alpha_1(1-2\alpha_1+2\gamma)\frac{D_t^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(t,\tau)}{t} + (\alpha_1-1-\gamma)^2\frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}$ is univalent and verifies the strong differential superordination*

$$h(t,\tau) \prec\prec \alpha_1(\alpha_1+1)\frac{D_t^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(t,\tau)}{t} +$$

$$\alpha_1(1-2\alpha_1+2\gamma)\frac{D_t^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(t,\tau)}{t} + (\alpha_1-1-\gamma)^2\frac{D_t^{-\gamma}H_m^l[\alpha_1,\beta_1]f(t,\tau)}{t}, \quad (25)$$

then the strong superordination

$$q(t,\tau) \prec\prec \left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)_t',$$

holds and the best subordinant is the convex function $q(t,\tau) = \frac{1}{t}\int_0^t h(x,\tau)dx$.

Proof. Let $p(t,\tau) = \left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)_t' \in \mathcal{H}^*[0,\gamma,\tau]$, $t \in U$, $\tau \in \overline{U}$.

Using the computation from the proof of Theorem 15 yields

$$p(t,\tau) + tp_t'(t,\tau) = \alpha_1(\alpha_1+1)\frac{D_t^{-\gamma} H_m^l[\alpha_1+2,\beta_1] f(t,\tau)}{t} +$$

$$\alpha_1(1-2\alpha_1+2\gamma)\frac{D_t^{-\gamma} H_m^l[\alpha_1+1,\beta_1] f(t,\tau)}{t} + (\alpha_1-1-\gamma)^2\frac{D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)}{t},$$

and the strong superordination (25) can be written as

$$h(t,\tau) \prec\prec p(t,\tau) + tp_t'(t,\tau),$$

and using Lemma 3, we get the strong superordination

$$q(t,\tau) \prec\prec p(t,\tau), \quad \text{i.e.} \quad q(t,\tau) = \frac{1}{t}\int_0^t h(x,\tau)dx \prec\prec \left(D_t^{-\gamma} H_m^l[\alpha_1,\beta_1] f(t,\tau)\right)_t',$$

and the best subordinant is the function q. □

3. Conclusions

This paper is intended to propose a new line of investigation for strong differential subordination and its dual, strong differential superordination theories using fractional calculus.

As future research, the fractional integral of Dziok–Srivastava operator could be applied to quantum calculus to get differential subordinations and superordinations for it using q-fractional calculus. In addition, some classes of analytical functions can be introduced and investigated regarding the new defined operator. Conditions for univalence can be established for the defined classes, and coefficient studies could be done regarding those classes, like the Fekete–Szegő problem, Toeplitz determinants, or estimations for Hankel determinants of different orders.

In addition, fuzzy differential subordinations and fuzzy differential superordinations results were obtained involving the fractional integral of the Dziok–Srivastava operator in [48]. The fractional integral of the Dziok–Srivastava operator could be used for obtaining higher-order fuzzy differential subordinations, following study [49], regarding the classical theory of differential subordination. Hopefully, the fuzzy results obtained will have applications in future researches regarding real-life contexts.

The symmetry properties of the functions could be investigated to obtain solutions with particular properties for an equation or inequality. For the differential subordinations considered as inequalities, the investigation of special functions could get interesting results from applying their symmetry properties. Investigation regarding the symmetry properties of several functions involving quantum calculus could also be studied in a future paper.

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