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New Perspectives of Symmetry Conferred by q-Hermite-Hadamard Type Integral Inequalities

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Abstract: The main goal of this work is to provide quantum parametrized Hermite-Hadamard like type integral inequalities for functions whose second quantum derivatives in absolute values follow different type of convexities. A new quantum integral identity is derived for twice quantum differentiable functions, which is used as a key element in our demonstrations along with several basic inequalities such as: power mean inequality, and Holder's inequality. The symmetry of the Hermite-Hadamard type inequalities is stressed by the different types of convexities. Several special cases of the parameter are chosen to illustrate the investigated results. Four examples are presented.

Keywords: q-calculus; convexity; midpoint inequalities; Hermite-Hadamard type inequalities

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1. Introduction

The concept of convexity plays a significant role in the theory of inequalities. Inequalities have increasing importance in modern mathematical analysis, and in many other mathematical disciplines. Moreover, it seems to have "a pivotal role in several pure and applied sciences" [1]. Integral inequalities, on the other hand, "plays a vital role in the theory of differential equations" [2]. The very famous Hermite-Hadamard inequality is a part of integral inequalities and have been intensively studied by numerous scholars in the last decades. Different approaches have been followed in order to obtain new improvements, generalizations and refinements of this inequality [3–9] and of classical inequalities like Ostrowski, Simpson, Gruss, Chebysev, Mercer, Jensen, Hardy, Opial, Bullen, Newton, Bernoulli, Popoviciu and so on. Hermite-Hadamard's inequality(H-H inequality) and its different variants have been established for newly studied concepts as Riemann-Liouville fractional integrals [10], (k,s)-fractional integrals, fuzzy environment, quantum calculus, (p,q)-calculus and for different types of convexities. For example, a new type of convexity is the n-polynomial convexity investigated by Toplu in [11] where new refinements of Hermite-Hadamard were given. A possible reason for the great interest given to the study of the famous Hermite-Hadamard inequality may be the symmetry from within. The utilisation of the properties of modulus in the proof of all Hermite-Hadamard type inequalities involves a symmetry between the two expression obtained, the left member and the right member.

From a historical point of view, one can say that quantum calculus, as a branch of mathematics, was founded by Euler which used "the parameter q in Newton's work on infinite series" [12]. However, Jackson [13], started to develop the theory of quantum calculus when defined the notions of "general q-integral and q-difference operator" [12]. The q-fractional derivative was introduced firstly by Agarwal [14] and Al-Salam [10] defined the quantum analogue of the q-fractional integral and "q-Riemann-Liouville fractional integral" [12]. "The q-calculus concepts on finite intervals" [15,16] were utilized to obtain q-analogues of classical mathematical objects. "New quantum analogues of the Ostrowski inequalities" [17] have been described by Noor et al. and some estimated "bounds for the left-hand-side(LHS) of quantum H-H inequalities" [18] were presented for convex and quasi-convex functions. For preinvex functions new q-analogues of the classical Simpson's inequality have been given [19]. The notion of right q-derivative, "^bD_q" and integral were introduced by Bermudo et al. [20]. The q-H-H inequality was also proven using the Green function [21,22], by Khan et al., and for recent research, see for example, [23–28]. Recently, new quantum Simpson's, quantum Newton's [29–32] and quantum Ostrowski's type inequalities [33] were developed for convex and coordinated convex functions. This theory knows a rapid development over the past few decades and have numerous applications in many areas of science such as quantum mechanics, approximation theory, statistics, and also in information theory, optimization, geometry function theory(GTF), as well as in cosmology and particle physics [34,35]. Quantum calculus was extended to (p,q)-calculus and recently to generalized quantum calculus [36].

Motivated by [37], our goal in this study is to present new parametrized q-Hermite-Hadamard like type inequalities for twice q-differentiable mappings by utilising an auxiliary q-integral identity. This identity is similar with the corresponding lemma from [37], concerning the q-left and right derivatives of order two. These inequalities are similar to those obtained in another study [37].

We need to recall the Lemma 2 from [37] which is the main tool in demonstrations from [37] and important in our study.

Lemma 1. *Alp et all [37] consider $\Psi : [a,b] \to \mathbb{R}$ be a q-differentiable function. If ${}_aD_q\Psi$ and ${}^bD_q\Psi$ are continuous and q-integrable over $[a,b]$, then the following new equality holds:*

$$\begin{aligned}
&\frac{\lambda(b-a)}{2}\int_0^1 qt[{}^bD_q\Psi((1-\lambda t)b+\lambda ta) - {}_aD_q\Psi((1-\lambda t)a+\lambda tb)]d_qt \\
&= \frac{1}{2\lambda(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a}\Psi(t)_a d_qt + \int_{\lambda a+(1-\lambda)b}^{b}\Psi(t)\ {}^bd_qt\right) \\
&\quad - \frac{\Psi(\lambda b+(1-\lambda)a)+\Psi(\lambda a+(1-\lambda)b)}{2}.
\end{aligned}$$

The case when the q-left and right derivatives of order three of the functions satisfies similar conditions was studied in [38] for convex functions. In all these inequalities we can see that in the expression from left member, the two integrals which appear are defined on different intervals, different from the corresponding intervals from inequalities [39] and [40] because of the parameter λ.

This approach could give interesting indications about how the variation of a quantity of the analyzed size varies(such as: utility, welfare economics, taxes, health or income inequalities) [41–43]. For the parameter, we choose $\lambda = 1$ and $\lambda = \frac{1}{2}$ in our examples, which validating the theoretical results.

The paper has been structured in four sections. In Section 2, it will be briefly resume the basic notions and definitions of q-calculus. The classical H-H integral inequality is presented. Section 3 is dedicated to formulations and demonstrations of the main results: Lemma 2, Theorem 4, Theorems 5–9. These theorems present new q-midpoints, trapezoidal and q-H-H-like type integral inequalities for mappings whose the second q-left and q-right derivatives in absolute value satisfies different type of convexities (i.e., convex, strongly convex, n-polynomial convex and strongly quasi-convex functions respectively). Many consequences have been established for some special choices of the parameter and the corresponding examples were discussed in detail. For the parameter we choose the values $\lambda = 1$ and $\lambda = \frac{1}{2}$ in inequality from Theorem 5. We used for figures and several calculus the Matlab R2023a software. Section 4 is dedicated to discussion and conclusions.

2. Outcomes

Here, we recall some different types of convexities which will be used below in this paper. The classical convexity, say that a function $\Psi : I \subset \mathbb{R} \to \mathbb{R}$ is convex if

$$\Psi(tx_1 + (1-t)x_2) \le t\Psi(x_1) + (1-t)\Psi(x_2),$$

for all $(x_1,\ x_2) \in I^2$ and $t \in [0,1]$.

Definition 1. *Khan et all [44] said that a function $\Psi : I \subset \mathbb{R} \to \mathbb{R}$ is called strongly convex with modulus $c > 0$ if*

$$\Psi(\tau m + (1-\tau)M) \le \tau\Psi(m) + (1-\tau)\Psi(M) - c\tau(1-\tau)(m-M)^2,$$

holds for all $(m,\ M) \in I^2$ and $\tau \in [0,1]$.

Definition 2. *Chu et all [45] consider $n \in \mathbb{N}$ and a nonnegative function $\Psi : I \subset \mathbb{R} \to \mathbb{R}$. This function is said then to be an n-polynomial convex function if for every $(x,y) \in I^2$ and $t \in [0,1]$, we have*

$$\Psi(tx + (1-t)y) \le \frac{1}{n}\sum_{s=1}^{n}[1-(1-t)^s]\Psi(x) + \frac{1}{n}\sum_{s=1}^{n}[1-t^s]\Psi(y).$$

Definition 3. *Kalsoom et all [46] and Ion [47] said that a function $\Psi : I \to \mathbb{R}$ with the modulus c is strongly quasi-convex function, if*

$$\Psi(\tau x_1 + (1-\tau)x_2) \le \max\{\Psi(x_1), \Psi(x_2)\} - \tau(1-\tau)c(x_1-x_2)^2$$

for all $(x_1, x_2) \in I^2,\ x_1 < x_2$ and $\tau \in [0,1]$.

The well-known Hermite-Hadamard's inequality can be stated as "if $\Psi : [a,b] \to \mathbb{R}$ is a convex function, then the following inequality holds:

$$\Psi\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \Psi(x)dx \le \frac{\Psi(a)+\Psi(b)}{2}, \tag{1}$$

and when Ψ is a concave function, then previous inequality holds but in the opposite direction" [48]. This inequality is known also as trapezium inequality.

Let suppose that $[a,b]$ is a real interval with $a < b$. In this paper, it will be assumed that $0 < q < 1$. It is well-known that the q-number is defined for any number n, $[n]_q = \frac{1-q^n}{1-q} = 1 + q + ... + q^{n-2} + q^{n-1}, \quad n \in \mathbb{N}$.

Further, several basic definitions, remarks and lemmas of the q-calculus will be presented because they will be used throughout this paper.

Definition 4. *Bermudo et all [20], and Alp et all [37] presents the right or q^b-derivative of $\Psi : [a,b] \to \mathbb{R}$ at $x \in [a,b]$ which is expressed as:*

$${}^bD_q\Psi(x) = \frac{\Psi(qx + (1-q)b) - \Psi(x)}{(1-q)(b-x)}, \quad x \ne b.$$

Definition 5. *Tariboon et all [16], and Alp et all [37] present the left or q_a-derivative of $\Psi : [a,b] \to \mathbb{R}$ at $x \in [a,b]$ which is expressed as:*

$${}_aD_q\Psi(x) = \frac{\Psi(x) - \Psi(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \ne a.$$

It would be appropriate to remind the classical definition of q-integral given in the treatise of Gasper and Rahman, [49], relation (1.11.3), page 23

$$\int_0^a \Psi(t)d_q(t) = (1-q)a\sum_{n=0}^{\infty} q^n\Psi(q^n a), (0<q<1).$$

Definition 6. *Bermudo et all [20], and Alp et all [37] present the right or q^b-integral of $\Psi : [a,b] \to \mathbb{R}$ at $x \in [a,b]$ which is defined as:*

$$\int_x^b \Psi(t)^b d_q t = (1-q)(b-x)\sum_{n=0}^{\infty} q^n\Psi(q^n x + (1-q^n)b) = (b-a)\int_0^1 \Psi(tb+(1-t)x)d_q t,$$

see also [49].

Definition 7. *Alp et all [50], and Rajkovic et all [37] state the left or q_a-integral of $\Psi : [a,b] \to \mathbb{R}$ at $x \in [a,b]$ which is defined as:*

$$\int_a^x \Psi(t)_a d_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n\Psi(q^n x + (1-q^n)a) = (b-a)\int_0^1 \Psi(tx+(1-t)a)d_q t.$$

Definition 8. *Alp et all [37] present the following equality for q_a- integrals*

$$\int_a^b (t-a)^\alpha{}_a d_q t = \frac{(b-a)^{\alpha+1}}{[\alpha+1]_q},$$

for $\alpha \in \mathbb{R} - \{-1\}$.

For the fundamental properties of these q-derivatives and q-integrals, see for example, [16,51,52]. Recently, new refinements and generalizations of q-Hermite-Hadamard integral inequalities for q-differentiable functions were given in [37].

From now, we suppose that $0 < \lambda \leq 1$.

By using Lemma 2 from [37] we will state again the following three theorems.

Theorem 1. *Alp et all [37] show that if the conditions of Lemma 2 [37] hold and the $|{}_aD_q\Psi|$ and $|{}^bD_q\Psi|$ are convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&|\frac{1}{2\lambda(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a} \Psi(t)_a d_q t + \int_{\lambda a+(1-\lambda)b}^b \Psi(t)^b d_q t\right) \\
&- \quad \frac{\Psi(\lambda b+(1-\lambda)a) + \Psi(\lambda a+(1-\lambda)b)}{2}| \\
&\leq \quad \frac{\lambda q(b-a)}{2[2]_q[3]_q}[([3]_q - \lambda[2]_q)[|{}^bD_q\Psi(b)| + |{}_aD_q\Psi(a)|] \\
&+ \quad \lambda[2]_q[|{}^bD_q\Psi(a)| + |{}_aD_q\Psi(b)|]].
\end{aligned}$$

Theorem 2. *Alp et all [37] prove that if the conditions of Lemma 2 [37] hold and if the $|{}_aD_q\Psi|^s$ and $|{}^bD_q\Psi|^s$, $s>1$ are convex, then the following inequality holds:*

$$|\frac{1}{2\lambda(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a} \Psi(t)_a d_q t + \int_{\lambda a+(1-\lambda)b}^b \Psi(t)^b d_q t\right)$$

$$
\begin{aligned}
&- \quad \frac{\Psi(\lambda b+(1-\lambda)a)+\Psi(\lambda a+(1-\lambda)b)}{2}| \\
&\leq \quad \frac{\lambda q(b-a)}{2}\left(\frac{1}{[r+1]_q}\right)^{\frac{1}{r}}[\left(\frac{[2]_q-\lambda}{[2]_q}|{}^bD_q\Psi(b)|^s+\frac{\lambda}{[2]_q}|{}^bD_q\Psi(a)|^s\right)^{\frac{1}{s}} \\
&+ \quad \left(\frac{[2]_q-\lambda}{[2]_q}|{}_aD_q\Psi(a)|^s+\frac{\lambda}{[2]_q}|{}_aD_q\Psi(b)|^s\right)^{\frac{1}{s}}],
\end{aligned}
$$

where $s^{-1}+r^{-1}=1$.

Theorem 3. *Alp et all [37] demonstrate that if the conditions of Lemma 2 [37] hold and if the* $|{}_aD_q\theta|^s$ *and* $|{}^bD_q\theta|^s$, $s\geq 1$ *are convex, then the following inequality holds:*

$$
\begin{aligned}
&|\frac{1}{2\lambda(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a}\Psi(t)_a d_q t+\int_{\lambda a+(1-\lambda)b}^b\Psi(t)^b d_q t\right) \\
&- \quad \frac{\Psi(\lambda b+(1-\lambda)a)+\Psi(\lambda a+(1-\lambda)b)}{2}| \\
&\leq \quad \frac{\lambda q(b-a)}{2[2]_q}[\left(\frac{([3]_q-\lambda[2]_q)|{}^bD_q\Psi(b)|^s+\lambda[2]_q|{}^bD_q\Psi(a)|^s}{[3]_q}\right)^{\frac{1}{s}} \\
&+ \quad \left(\frac{([3]_q-\lambda[2]_q)|{}_aD_q\Psi(a)|^s+\lambda[2]_q|{}_aD_q\Psi(b)|^s}{[3]_q}\right)^{\frac{1}{s}}].
\end{aligned}
$$

3. Results

A new quantum identity with parameter is given below as an important instrument in the demonstrations of the results of this section. Some new estimates of parametrized q-Hermite-Hadamard-type integral inequalities for twice q-differentiable functions are given below having as a starting point the results formulated in [37]. Moreover, several new consequences, applications and examples are presented in order to check the established results.

If $\Psi:[a,b]\to\mathbb{R}$ is a continuous function then the second q^b-derivative of Ψ at $t\in[a,b]$ is given as:

$$
\begin{aligned}
{}^bD_q^2\Psi(t) &= {}^bD_q\left({}^bD_q\Psi(t)\right) \\
&= \frac{\Psi(q^2t+(1-q^2)b)-[2]_q\Psi(qt+(1-q)b)+q\Psi(t)}{(1-q)^2q(b-t)^2}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
{}_aD_q^2\Psi(t) &= {}_aD_q\left({}_aD_q\Psi(t)\right) \\
&= \frac{\Psi(q^2t+(1-q^2)a)-[2]_q\Psi(qt+(1-q)a)+q\Psi(t)}{(1-q)^2q(t-a)^2}.
\end{aligned}
$$

We define

$$
{}_a^bS_q(\lambda)=\frac{(b-a)^2}{1+q}\int_0^1 q^3t^2[{}^bD_q^2\Psi((1-\lambda t)b+\lambda ta)+{}_aD_q^2\Psi((1-\lambda t)a+\lambda tb)]d_qt, \tag{2}
$$

The main purpose of this paper is to give inequalities for ${}_a^bS_q(\lambda)$.

Lemma 2. *Let $0 < \lambda \leq 1$, and $\Psi : [a, b] \to \mathbb{R}$ be a twice q-differentiable function. If, in addition, ${}_aD_q^2\Psi$ and ${}^bD_q^2\Psi$ are continuous and q-integrable functions over $[a, b]$ then the following equality takes place:*

$$\begin{aligned} {}_a^bS_q(\lambda) &= \frac{1}{\lambda^3(b-a)}[\int_{\lambda a+(1-\lambda)b}^{b} \Psi(t)^b d_q t + \int_a^{\lambda b+(1-\lambda)a} \Psi(t)_a d_q t] \\ &- \frac{1-q-q^2}{(1-q)(1+q)\lambda^2}[\Psi(\lambda a+(1-\lambda)b) + \Psi(\lambda b+(1-\lambda)a)] \\ &- \frac{q}{(1-q)(1+q)\lambda^2}[\Psi(\lambda qa+(1-q\lambda)b) + \Psi(\lambda qb+(1-q\lambda)a)]. \end{aligned}$$

In addition, we can obtain also,

$$\begin{aligned} {}_a^bS_q(\lambda) &= \frac{1}{\lambda^3(b-a)}[\int_{\lambda qa+(1-\lambda q)b}^{b} \Psi(t)^b d_q t + \int_a^{\lambda qb+(1-\lambda q)a} \Psi(t)_a d_q t] \\ &- \frac{q}{(1-q)(1+q)\lambda^2}[\Psi(\lambda qa+(1-\lambda q)b) + \Psi(\lambda qb+(1-\lambda q)a)] \\ &+ \frac{q^3}{(1-q)(1+q)\lambda^2}[\Psi(\lambda a+(1-\lambda)b) + \Psi(\lambda b+(1-\lambda)a)]. \end{aligned}$$

Proof. Denoting by I_1 the expression $\int_0^1 t^{2b}D_q^2\Psi((1-\lambda t)b+\lambda ta)d_qt$ and by I_2 the expression $\int_0^1 t^2{}_aD_q^2\Psi(a(1-\lambda t)+\lambda tb)d_qt$, we get ${}_a^bS_q(\lambda) = \frac{(b-a)^2}{1+q}q^3(I_1+I_2)$.

From Definition 4, of the right q-derivative of Ψ, we successively have

$$\begin{aligned} I_1 &= \int_0^1 t^2\, {}^bD_q^2\Psi(b(1-\lambda t)+\lambda ta)d_qt \\ &= \int_0^1 \frac{1}{(1-q)^2\lambda^2(b-a)^2q}[\Psi(\lambda tq^2a+b(1-\lambda tq^2)) \\ &- [2]_q\Psi(\lambda tqa+b(1-\lambda tq)) + q\Psi(\lambda ta+(1-\lambda t)b)]d_qt \\ &= \frac{1}{(1-q)(b-a)^2\lambda^2q}[\sum_{n=0}^{\infty} q^n\Psi(\lambda q^{n+2}a+b(1-\lambda q^{n+2})) \\ &- [2]_q\sum_{n=0}^{\infty} q^n\Psi(\lambda q^{n+1}a+b(1-\lambda q^{n+1})) + q\sum_{n=0}^{\infty} q^n\Psi(\lambda q^n a+b(1-\lambda q^n))] \\ &= \frac{1}{(1-q)(b-a)^2\lambda^2q}\{\frac{1}{q^2}\sum_{m=0}^{\infty} q^m\Psi(\lambda q^m a+b(1-\lambda q^m)) \\ &- \frac{\Psi(\lambda a+(1-\lambda)b)+q\Psi((1-\lambda q)b+\lambda qa)}{q^2} \\ &- [2]_q\frac{\sum_{m=0}^{\infty} q^m\Psi(\lambda q^m a+(1-\lambda q^m)b) - \Psi(b(1-\lambda)+\lambda a)}{q} \\ &+ q\sum_{m=0}^{\infty} q^m\Phi(\lambda q^m a+b(1-\lambda q^m))\}. \end{aligned}$$

By Definition 6, of the right q-integral of Ψ and calculus, it will be obtained,

$$\begin{aligned} I_1 &= \frac{1+q}{(b-a)^3\lambda^3q^3}\int_{\lambda a+(1-\lambda)b}^{b} \Psi(t)\,{}^bd_qt - \frac{1-q-q^2}{(1-q)(b-a)^2q^3\lambda^2}\Psi(b(1-\lambda)+\lambda a) \\ &- \frac{1}{(1-q)(b-a)^2q^2\lambda^2}\Psi(b(1-\lambda q)+\lambda qa). \end{aligned}$$

Similarly, by using Definition 5, of the left q-derivative of Ψ, we successively get

$$\begin{aligned} I_2 &= \int_0^1 t^2{}_aD_q^2\Psi(a(1-\lambda t)+\lambda tb)d_qt \\ &= \int_0^1 \frac{1}{q(1-q)^2\lambda^2(b-a)^2}[\Psi(q^2\lambda tb+a(1-q^2\lambda t)) \\ &- [2]_q\Psi(q\lambda tb+a(1-q\lambda t)+q\Psi(\lambda tb+a(1-\lambda t))]d_qt \\ &= \frac{1}{(1-q)(b-a)^2\lambda^2 q}[\sum_{n=0}^{\infty} q^n\Psi(\lambda q^{n+2}b+(1-\lambda q^{n+2})a) \\ &- [2]_q\sum_{n=0}^{\infty} q^n\Psi(\lambda q^{n+1}b+(1-\lambda q^{n+1})a) \\ &+ q\sum_{n=0}^{\infty} q^n\Psi(q^n\lambda b+a(1-\lambda q^n))] \\ &= \frac{1}{(1-q)(b-a)^2\lambda^2 q}[\frac{\sum_{m=0}^{\infty} q^m\Psi(q^m\lambda b+a(1-q^m\lambda))}{q^2} \\ &- \frac{\Psi(\lambda b+(1-\lambda)a)+q\Psi(\lambda qb+a(1-\lambda q))}{q^2} \\ &- [2]_q\frac{\sum_{m=0}^{\infty} q^m\Psi(\lambda q^m b+(1-\lambda q^m)a)-\Psi(\lambda b+(1-\lambda)a)}{q} \\ &+ q\sum_{m=0}^{\infty} q^m\Psi(\lambda q^m b+(1-\lambda q^m)a)]. \end{aligned}$$

By Definition 7, of the left q-integral of Ψ and calculus, we have

$$\begin{aligned} I_2 &= \frac{1+q}{(b-a)^3\lambda^3q^3}\int_a^{\lambda b+(1-\lambda)a}\Psi(t)_a d_qt - \frac{1-q-q^2}{(1-q)(b-a)^2\lambda^2q^3}\Psi(\lambda b+(1-\lambda)a) \\ &- \frac{1}{(b-a)^2(1-q)\lambda^2q^2}\Psi(q\lambda b+a(1-q\lambda)). \end{aligned}$$

Then multiplying the result of the sum I_1+I_2 by $\frac{(b-a)^2q^3}{1+q}$, it follows:

$$\begin{aligned} \frac{(b-a)^2}{1+q}q^3(I_1+I_2) &= \frac{1}{\lambda^3(b-a)}[\int_{\lambda a+(1-\lambda)b}^{b}\Psi(t)^b d_qt+\int_a^{\lambda b+(1-\lambda)a}\Psi(t)_a d_qt] \\ &- \frac{1-q-q^2}{(1-q^2)\lambda^2}[\Psi(\lambda a+b(1-\lambda))+\Psi(\lambda b+a(1-\lambda))] \\ &- \frac{q}{(1-q^2)\lambda^2}[\Psi(\lambda qa+b(1-\lambda q))+\Psi(\lambda qb+a(1-\lambda q))], \end{aligned}$$

and we obtain the first desired equality.

For the second expression of ${}_a^bS_q(\lambda)$, we have,

$$\begin{aligned} I_1 &= \frac{1}{(1-q)(b-a)^2\lambda^2 q}[\sum_{n=0}^{\infty} q^n\Psi(\lambda q^{n+2}a+b(1-\lambda q^{n+2})) \\ &- [2]_q\sum_{n=0}^{\infty} q^n\Psi(\lambda q^{n+1}a+b(1-\lambda q^{n+1}))+q\sum_{n=0}^{\infty} q^n\Psi(\lambda q^n a+b(1-\lambda q^n))] \\ &= \frac{1}{(1-q)(b-a)^2\lambda^2 q}[\frac{1}{q}\sum_{n=0}^{\infty} q^{n+1}\Psi(\lambda q^{n+2}a+b(1-\lambda q^{n+2})) \end{aligned}$$

$$-[2]_q\sum_{n=0}^{\infty} q^n\Psi(\lambda q^{n+1}a+b(1-\lambda q^{n+1}))$$

$$+q\Psi((1-\lambda)b+\lambda a)+q^2\sum_{n=1}^{\infty}q^{n-1}\Psi(\lambda q^n a+b(1-\lambda q^n))]$$

$$=\frac{1+q}{(b-a)^3\lambda^3q^3}\int_{\lambda qa+(1-\lambda q)b}^{b}\Psi(t)^b d_qt-\frac{\Psi(b(1-\lambda q)+\lambda qa)}{(1-q)(b-a)^2\lambda^2q^2}+\frac{\Psi((1-\lambda)b+\lambda a)}{(1-q)(b-a)^2\lambda^2}.$$

Similar we will obtain below the expression of I_2

$$I_2=\frac{1+q}{(b-a)^3\lambda^3q^3}\int_{a}^{\lambda qb+(1-\lambda q)a}\Psi(t)_a d_qt-\frac{\Psi(a(1-\lambda q)+\lambda qb)}{(1-q)(b-a)^2\lambda^2q^2}+\frac{\Psi((1-\lambda)a+\lambda b)}{(1-q)(b-a)^2\lambda^2}.$$

Then multiplying the result of the sum I_1+I_2 by $\frac{(b-a)^2q^3}{1+q}$, we get the second expression of ${}_a^bS_q(\lambda)$, which completes the proof. □

Remark 1. *First expression of ${}_a^bS_q(\lambda)$ has the coefficient $1-q-q^2$ which can take positive and also negative values, but the two integrals are defined on intervals which don't contain q. Also the second expression of ${}_a^bS_q(\lambda)$ is easier, but the integrals are defined on intervals more complicated which contain q.*

In addition, expression of ${}_a^bS_q(\lambda)$ from Lemma 2, but especially in the second expression is preserving a symmetry of coefficients and terms.

Theorem 4. *It will be assumed that the hypothesis of Lemma 2 are true. If $|{}_aD_q^2\Psi|$ and $|{}^bD_q^2\Psi|$ are convex functions on $[a,b]$ then we have:*

$$|{}_a^bS_q(\lambda)|\le\frac{(b-a)^2q^3}{[2]_q[3]_q[4]_q}[([4]_q-\lambda[3]_q)(|{}^bD_q^2\Psi(b)|$$
$$+|{}_aD_q^2\Psi(a)|)+\lambda[3]_q(|{}_aD_q^2\Psi(b)|+|{}^bD_q^2\Psi(a)|)]. \tag{3}$$

Proof. It will be used Lemma 2, obtaining:

$$\begin{aligned}|{}_a^bS_q(\lambda)| &\le \frac{(b-a)^2q^3}{[2]_q}\{\int_0^1 t^2|{}^bD_q^2\Psi((1-\lambda t)b+\lambda ta)|d_qt\\ &+\int_0^1 t^2|{}_aD_q^2\Psi((1-\lambda t)a+\lambda tb)|d_qt\}.\end{aligned}$$

Then taking into account the convexity of $|{}_aD_q^2\Psi|$ and $|{}^bD_q^2\Psi|$, we find that

$$\begin{aligned}|{}_a^bS_q(\lambda)| &\le \frac{q^3(b-a)^3}{[2]_q}\{\int_0^1 t^2[(1-\lambda t)|{}^bD_q^2\Psi(b)|+\lambda t|{}^bD_q^2\Psi(a)|]d_qt\\ &+\int_0^1 t^2[(1-\lambda t)|{}_aD_q^2\Psi(a)|+\lambda t|{}_aD_q^2\Psi(b)|]d_qt\}\end{aligned}$$

$$
\begin{aligned}
&= \frac{q^3(b-a)^2}{[2]_q}\{[|^bD_q^2\Psi(b)| + |_aD_q^2\Psi(a)|]\int_0^1 t^2(1-\lambda t)d_qt \\
&+ [|^bD_q^2\Psi(a)| + |_aD_q^2\Psi(b)|]\int_0^1 \lambda t^3 d_qt\} \\
&= \frac{q^3(b-a)^2}{[2]_q}\{(\frac{1}{[3]_q} - \frac{\lambda}{[4]_q})[|^bD_q^2\Psi(b)| + |_aD_q^2\Psi(a)|] \\
&+ \frac{\lambda}{[4]_q}[|^bD_q^2\Psi(a)| + |_aD_q^2\Psi(b)|]\}.
\end{aligned}
$$

So by calculus it will be obtained the inequality from previous theorem. □

Remark 2. *Now considering $\lambda = 1$ in Theorem 4 the following trapezoid type inequality takes place:*

$$
\begin{aligned}
|{}_a^bS_q(1)| &= |\frac{1}{b-a}[\int_a^b \Psi(t)\ {}^bd_qt + \int_a^b \Psi(t)\ {}_ad_qt] - \frac{1-q-q^2}{1-q^2}(\Psi(a)+\Psi(b)) \\
&- \frac{q}{1-q^2}[\Psi(b(1-q)+qa) + \Psi(qb+(1-q)a)]| \\
&\leq \frac{q^3(b-a)^2}{[2]_q[3]_q[4]_q}\{q^3[|^bD_q^2\Psi(b)| + |_aD_q^2\Psi(a)|] + [3]_q[|^bD_q^2\Psi(a)| + |_aD_q^2\Psi(b)|]\}.
\end{aligned}
$$

Remark 3. *Now taking $\lambda = \frac{1}{2}$ in Theorem 4 the following trapezoid type inequality holds:*

$$
\begin{aligned}
|\frac{{}_a^bS_q(\frac{1}{2})}{8}| &= |\frac{1}{b-a}[\int_{\frac{a+b}{2}}^b \Psi(t)\ {}^bd_qt + \int_a^{\frac{a+b}{2}} \Psi(t)\ {}_ad_qt] - \frac{1-q-q^2}{1-q^2}\Psi(\frac{a+b}{2}) \\
&- \frac{q}{2(1-q^2)}[\Psi(b(1-\frac{q}{2}) + \frac{q}{2}a) + \Psi(\frac{q}{2}b + (1-\frac{q}{2})a)]|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{q^3(b-a)^2}{16[2]_q[3]_q[4]_q}\{(2[4]_q - [3]_q)[|^bD_q^2\Psi(b)| + |_aD_q^2\Psi(a)|] \\
&+ [3]_q[|^bD_q^2\Psi(a)| + |_aD_q^2\Psi(b)|]\}.
\end{aligned}
$$

Remark 4. *Now we put $\lambda = \frac{1}{[2]_q}$ in Theorem 4 and we have:*

$$
|\frac{1}{b-a}[\int_{\frac{a+qb}{[2]_q}}^b \Psi(t)\ {}^bd_qt + \int_a^{\frac{b+qa}{[2]_q}} \Psi(t)\ {}_ad_qt] - \frac{1}{[2]_q}(\Psi(\frac{qb+a}{[2]_q}) + \Psi(\frac{qa+b}{[2]_q}))|
$$

$$
\begin{aligned}
&\leq \frac{q^3(b-a)^2}{[2]_q^5[3]_q[4]_q}\{([2]_q[4]_q - [3]_q)[|^bD_q^2\Psi(b)| + |_aD_q^2\Psi(a)|] \\
&+ [3]_q[|^bD_q^2\Psi(a)| + |_aD_q^2\Psi(b)|]\}.
\end{aligned}
$$

Theorem 5. *We suppose that the hypothesiss of Lemma 2 takes place. If $|_aD_q^2\Psi|^\sigma$ and $|^bD_q^2\Psi|^\sigma$ are strongly convex functions on $[a,b]$ for modulus c with $\sigma \geq 1$, then next inequality is true:*

$$
|{}_a^bS_q(\lambda)| \leq
$$

$$\leq \frac{(b-a)^2 q^3}{[2]_q [3]_q^{1-\frac{1}{\sigma}}} \{ [\frac{[4]_q - \lambda[3]_q}{[3]_q[4]_q} |^b D_q^2 \Psi(b)|^\sigma + \frac{\lambda}{[4]_q} |^b D_q^2 \Psi(a)|^\sigma - c(b-a)^2 \lambda \frac{[5]_q - \lambda[4]_q}{[4]_q[5]_q}]^{\frac{1}{\sigma}}$$
$$+ [\frac{[4]_q - \lambda[3]_q}{[3]_q[4]_q} |_a D_q^2 \Psi(a)|^\sigma + \frac{\lambda}{[4]_q} |_a D_q^2 \Psi(b)|^\sigma - c(b-a)^2 \lambda \frac{[5]_q - \lambda[4]_q}{[4]_q[5]_q}]^{\frac{1}{\sigma}} \},$$

Proof. It will be used the modulus properties, the strong convexity of $|_a D_q^2 \Psi|^\sigma$ and $|^b D_q^2 \Psi|^\sigma$ and the power mean inequality, obtaining:

$$|_a^b S_q(\lambda)| \leq$$

$$\leq \frac{(b-a)^2}{[2]_q} \{ \int_0^1 q^3 t^2 |^b D_q^2 \Psi((1-\lambda t)b + \lambda t a)| d_q t + \int_0^1 q^3 t^2 |_a D_q^2 \Psi((1-\lambda t)a + \lambda t b)| d_q t \}$$
$$\leq \frac{q^3 (b-a)^2}{[2]_q} \left(\int_0^1 t^2 d_q t \right)^{1-\frac{1}{\sigma}} [\left(\int_0^1 t^2 |^b D_q^2 \Psi((1-\lambda t)b + \lambda t a)|^\sigma d_q t \right)^{\frac{1}{\sigma}}$$
$$+ \left(\int_0^1 t^2 |_a D_q^2 \Psi((1-\lambda t)a + \lambda t b)|^\sigma d_q t \right)^{\frac{1}{\sigma}}],$$

and

$$|_a^b S_q(\lambda)| \leq \frac{q^3 (b-a)^2}{[2]_q} \left(\frac{1}{[3]_q} \right)^{1-\frac{1}{\sigma}} \times$$

$$\times [\left(\int_0^1 t^2 (1-\lambda t) |^b D_q^2 \Psi(b)|^\sigma d_q t + \int_0^1 \lambda t^3 |^b D_q^2 \Psi(a)|^\sigma d_q t - c\lambda(b-a)^2 \int_0^1 t^3 (1-\lambda t) d_q t \right)^{\frac{1}{\sigma}}$$
$$+ \left(\int_0^1 t^2 (1-\lambda t) |_a D_q^2 \Psi(a)|^\sigma d_q t + \int_0^1 \lambda t^3 |_a D_q^2 \Psi(b)|^\sigma d_q t - c\lambda(b-a)^2 \int_0^1 t^3 (1-\lambda t) d_q t \right)^{\frac{1}{\sigma}}]$$
$$= \frac{q^3 (b-a)^2}{[2]_q} \frac{1}{[3]_q^{1-\frac{1}{\sigma}}} \{ [\frac{[4]_q - \lambda[3]_q}{[3]_q[4]_q} |^b D_q^2 \Psi(b)|^\sigma + \frac{\lambda}{[4]_q} |^b D_q^2 \Psi(a)|^\sigma - c(b-a)^2 \lambda \frac{[5]_q - \lambda[4]_q}{[4]_q[5]_q}]^{\frac{1}{\sigma}}$$
$$+ [\frac{[4]_q - \lambda[3]_q}{[3]_q[4]_q} |_a D_q^2 \Psi(a)|^\sigma + \frac{\lambda}{[4]_q} |_a D_q^2 \Psi(b)|^\sigma - c(b-a)^2 \lambda \frac{[5]_q - \lambda[4]_q}{[4]_q[5]_q}]^{\frac{1}{\sigma}} \},$$

and the proof is completed. □

Theorem 6. *Under conditions of Lemma 2, if $|_a D_q^2 \Psi|^r$ and $|^b D_q^2 \Psi|^r$ are strongly convex functions on $[a, b]$ with modulus c, when $\frac{1}{s} + \frac{1}{r} = 1$ and $r > 1$ then the following inequality takes place:*

$$|_a^b S_q(\lambda)| \leq \frac{q^3 (b-a)^2}{[2]_q [2s+1]_q^{\frac{1}{s}}} \times$$

$$\times \{ [\frac{[2]_q - \lambda}{[2]_q} |^b D_q^2 \Psi(b)|^r + \frac{\lambda}{[2]_q} |^b D_q^2 \Psi(a)|^r - \lambda c(b-a)^2 \frac{[3]_q - \lambda[2]_q}{[2]_q[3]_q}]^{\frac{1}{r}}$$
$$+ [\frac{[2]_q - \lambda}{[2]_q} |_a D_q^2 \Psi(a)|^r + \frac{\lambda}{[2]_q} |_a D_q^2 \Psi(b)|^r - \lambda c(b-a)^2 \frac{[3]_q - \lambda[2]_q}{[2]_q[3]_q}]^{\frac{1}{r}} \}.$$

Proof. By applying now, the properties of modulus, and the Holder's inequality, it will be obtained,

$$|_a^b S_q(\lambda)| \leq$$

$$
\begin{aligned}
&\leq \frac{(b-a)^2q^3}{[2]_q}\{\int_0^1 t^2|^bD_q^2\Psi((1-\lambda t)b+\lambda ta)|d_qt+\int_0^1 t^2|_aD_q^2\Psi((1-\lambda t)a+\lambda tb)|d_qt\}\\
&\leq \frac{q^3(b-a)^2}{[2]_q}\left(\int_0^1 t^{2s}d_qt\right)^{\frac{1}{s}}\{\left(\int_0^1 |^bD_q^2\Psi((1-\lambda t)b+\lambda ta)|^r d_qt\right)^{\frac{1}{r}}\\
&+ \left(\int_0^1 |_aD_q^2\Psi((1-\lambda t)a+\lambda tb)|^r d_qt\right)^{\frac{1}{r}}\},
\end{aligned}
$$

and by using the strongly convexity of the functions $|_aD_q^2\Psi|^r$ and $|^bD_q^2\Psi|^r$ on $[a,b]$ with modulus c, we get,

$$
|_a^bS_q(\lambda)| \leq \frac{q^3(b-a)^2}{[2]_q}\frac{1}{[2s+1]_q^{\frac{1}{s}}}\times
$$

$$
\begin{aligned}
&\times\{\left(\int_0^1 ((1-\lambda t)|^bD_q^2\Psi(b)|^r+\lambda t|^bD_q^2\Psi(a)|^r)d_qt-\lambda c(b-a)^2\int_0^1 t(1-\lambda t)d_qt\right)^{\frac{1}{r}}\\
&+\left(\int_0^1 ((1-\lambda t)|_aD_q^2\Psi(a)|^r+\lambda t|_aD_q^2\Psi(b)|^r)d_qt-\lambda c(b-a)^2\int_0^1 t(1-\lambda t)d_qt\right)^{\frac{1}{r}}\}\\
&= \frac{q^3(b-a)^2}{[2]_q[2s+1]_q^{\frac{1}{s}}}\{[\frac{[2]_q-\lambda}{[2]_q}|^bD_q^2\Psi(b)|^r+\frac{\lambda}{[2]_q}|^bD_q^2\Psi(a)|^r-\lambda c(b-a)^2\frac{[3]_q-\lambda[2]_q}{[2]_q[3]_q}]^{\frac{1}{r}}\\
&\quad +[\frac{[2]_q-\lambda}{[2]_q}|_aD_q^2\Psi(a)|^r+\frac{\lambda}{[2]_q}|_aD_q^2\Psi(b)|^r-\lambda c(b-a)^2\frac{[3]_q-\lambda[2]_q}{[2]_q[3]_q}]^{\frac{1}{r}}\}.
\end{aligned}
$$

Thus, the proof is finished. □

Remark 5. *If we put $\lambda = 1$ in Theorem 5, then the following trapezoid type inequality holds:*

$$
|_a^bS_q(1)| \leq \frac{q^3(b-a)^2}{[2]_q[3]_q^{1-\frac{1}{\sigma}}}\times
$$

$$
\begin{aligned}
&\times \quad \frac{q^3(b-a)^2}{[2]_q[3]_q^{1-\frac{1}{\sigma}}}\{[\frac{q^3}{[3]_q[4]_q}|^bD_q^2\Psi(b)|^\sigma+\frac{1}{[4]_q}|^bD_q^2\Psi(a)|^\sigma-c(b-a)^2\frac{q^4}{[4]_q[5]_q}]^{\frac{1}{\sigma}}\\
&+ \quad [\frac{q^3}{[3]_q[4]_q}|_aD_q^2\Psi(a)|^\sigma+\frac{1}{[4]_q}|_aD_q^2\Psi(b)|^\sigma-c(b-a)^2\frac{q^4}{[4]_q[5]_q}]^{\frac{1}{\sigma}}\}.
\end{aligned}
$$

Remark 6. *If we take $\lambda = \frac{1}{2}$ in Theorem 5, then we obtain the following midpoint type inequality:*

$$
|_a^bS_q(\frac{1}{2})| \leq \frac{q^3(b-a)^2}{2^{\frac{1}{\sigma}}[2]_q[3]_q^{1-\frac{1}{\sigma}}[4]_q^{\frac{1}{\sigma}}}\times
$$

$$
\begin{aligned}
&\times\{[\frac{2[4]_q-[3]_q}{[3]_q}|^bD_q^2\Psi(b)|^\sigma+|^bD_q^2\Psi(a)|^\sigma-c(b-a)^2\frac{2[5]_q-[4]_q}{2[5]_q}]^{\frac{1}{\sigma}}\\
&+[\frac{2[4]_q-[3]_q}{[3]_q}|_aD_q^2\Psi(a)|^\sigma+|_aD_q^2\Psi(b)|^\sigma-c(b-a)^2\frac{2[5]_q-[4]_q}{2[5]_q}]^{\frac{1}{\sigma}}\}.
\end{aligned}
$$

Remark 7. *If it is assigned in Theorem 6, $\lambda = 1$, then we have next trapezoid type inequality:*

$$
|_a^bS_q(1)| \leq \frac{q^3(b-a)^2}{[2]_q[2s+1]_q^{\frac{1}{s}}}\times
$$

$$\times\{[q|^{b}D_q^2\Psi(b)|^r+|^{b}D_q^2\Psi(a)|^r-c(b-a)^2\frac{q^2}{[3]_q}]^{\frac{1}{r}}$$

$$+[q|_{a}D_q^2\Psi(a)|^r+|_{a}D_q^2\Psi(b)|^r-c(b-a)^2\frac{q^2}{[3]_q}]^{\frac{1}{r}}\}.$$

Remark 8. *If we take $\lambda=\frac{1}{2}$ in Theorem 6, then the following midpoint type inequality it is obtained:*

$$|{}_{a}^{b}S_q(\frac{1}{2})|\leq\frac{q^3(b-a)^2}{2^{\frac{1}{r}}[2]_q^{1+\frac{1}{r}}[2s+1]_q^{\frac{1}{s}}}\{[(2[2]_q-1)|^{b}D_q^2\Psi(b)|^r+|^{b}D_q^2\Psi(a)|^r-c(b-a)^2\frac{2[3]_q-[2]_q}{2[3]_q}]^{\frac{1}{r}}$$

$$+[(2[2]_q-1)|_{a}D_q^2\Psi(a)|^r+|_{a}D_q^2\Psi(b)|^r-c(b-a)^2\frac{2[3]_q-[2]_q}{2[3]_q}]^{\frac{1}{r}}\}.$$

Theorem 7. *We assume that the conditions of Lemma 2 are true. If $|_{a}D_q^2\Psi|^\sigma$ and $|^{b}D_q^2\Psi|^\sigma$ are n-polynomial convex functions on $[a,b]$, and $\sigma\geq 1$, then we have:*

$$|{}_{a}^{b}S_q(\lambda)|\leq\frac{q^3(b-a)^2}{[2]_q[3]_q^{1-\frac{1}{\sigma}}n^{\frac{1}{\sigma}}}\times$$

$$\times\{[|^{b}D_q^2\Psi(b)|^\sigma\sum_{s=1}^{n}(\frac{1}{[3]_q}-\frac{\lambda^s}{[s+3]_q})+|^{b}D_q^2\Psi(a)|^\sigma\sum_{s=1}^{n}(\frac{1}{[3]_q}-\sum_{k=0}^{n}\binom{k}{s}\frac{\lambda^{s-k}}{[s-k+3]_q})]^{\frac{1}{\sigma}}$$

$$+[|_{a}D_q^2\Psi(a)|^\sigma\sum_{s=1}^{n}(\frac{1}{[3]_q}-\frac{\lambda^s}{[s+3]_q})+|_{a}D_q^2\Psi(b)|^\sigma\sum_{s=1}^{n}(\frac{1}{[3]_q}-\sum_{k=0}^{n}\binom{k}{s}\frac{\lambda^{s-k}}{[s-k+3]_q})]^{\frac{1}{\sigma}}\},$$

where, $\binom{k}{s}=\frac{k!}{s!(k-s)!}$.

Proof. We can write as in the proof of Theorem 5 by using the power mean inequality and then the n-polynomial convexity of $|_{a}D_q^2\Psi|^\sigma$ and $|^{b}D_q^2\Psi|^\sigma$, that

$$|{}_{a}^{b}S_q(\lambda)|\leq\frac{q^3(b-a)^2}{[2]_q}\left(\frac{1}{[3]_q}\right)^{1-\frac{1}{\sigma}}[\left(\int_0^1 t^2|^{b}D_q^2\Psi((1-\lambda t)b+\lambda ta)|^\sigma d_qt\right)^{\frac{1}{\sigma}}$$

$$+\left(\int_0^1 t^2|_{a}D_q^2\Psi((1-\lambda t)a+\lambda tb)|^\sigma d_qt\right)^{\frac{1}{\sigma}}]\leq\frac{q^3(b-a)^2}{[2]_q}\left(\frac{1}{[3]_q}\right)^{1-\frac{1}{\sigma}}\times$$

$$\times\{[\int_0^1 t^2\left(|^{b}D_q^2\Psi(b)|^\sigma\frac{1}{n}\sum_{s=1}^{n}(1-(\lambda t)^s)+|^{b}D_q^2\Psi(a)|^\sigma\frac{1}{n}\sum_{s=1}^{n}(1-(1-\lambda t)^s)\right)d_qt]^{\frac{1}{\sigma}}$$

$$+[\int_0^1 t^2\left(|_{a}D_q^2\Psi(a)|^\sigma\frac{1}{n}\sum_{s=1}^{n}(1-(\lambda t)^s)+|_{a}D_q^2\Psi(b)|^\sigma\frac{1}{n}\sum_{s=1}^{n}(1-(1-\lambda t)^s)\right)d_qt]^{\frac{1}{\sigma}}\}.$$

Then following the calculus, we find that

$$|{}_{a}^{b}S_q(\lambda)|\leq\frac{q^3(b-a)^2}{[2]_q[3]_q^{1-\frac{1}{\sigma}}}\{[\frac{|^{b}D_q^2\Psi(b)|^\sigma}{n}\sum_{s=1}^{n}\int_0^1(t^2-\lambda^s t^{s+2})d_qt$$

$$+\frac{|^{b}D_q^2\Psi(a)|^\sigma}{n}\sum_{s=1}^{n}\int_0^1(t^2-t^2(1-\lambda t)^s)d_qt]^{\frac{1}{\sigma}}$$

$$+[\frac{|_{a}D_q^2\Psi(a)|^\sigma}{n}\sum_{s=1}^{n}\int_0^1(t^2-\lambda^s t^{s+2})d_qt+\frac{|_{a}D_q^2\Psi(b)|^\sigma}{n}\sum_{s=1}^{n}\int_0^1(t^2-t^2(1-\lambda t)^s)d_qt]^{\frac{1}{\sigma}}\}.$$

If we denote $A = \sum_{s=1}^{n} \int_0^1 (t^2 - \lambda^s t^{s+2}) d_q t$ and $B = \sum_{s=1}^{n} \int_0^1 (t^2 - t^2(1-\lambda t)^s) d_q t$, we get by calculus

$$A = \sum_{s=1}^{n} \left(\frac{1}{[3]_q} - \frac{\lambda^s}{[s+3]_q}\right)$$

and

$$B = \sum_{s=1}^{n} \left(\frac{1}{[3]_q} - \sum_{k=0}^{n} \frac{s!}{k!(s-k)!} \frac{\lambda^{s-k}}{[s-k+3]_q}\right) = \sum_{s=1}^{n} \left(\frac{1}{[3]_q} - \sum_{k=0}^{n} \binom{k}{s} \frac{\lambda^{s-k}}{[s-k+3]_q}\right),$$

which leads to desired inequality. □

Theorem 8. *We assume that the conditions of Lemma 2 are true and let* $s, \ r \in \mathbb{R}, \ r > 1$ *with* $\frac{1}{s} + \frac{1}{r} = 1$. *If* $|{}_aD_q^2\Psi|^s$ *and* $|{}^bD_q^2\Psi|^s$ *are n-polynomial convex functions on* $[a, b]$*, then we have:*

$$|{}_a^bS_q(\lambda)| \leq \frac{q^3(b-a)^2}{[2]_q [2s+1]_q^{\frac{1}{s}} n^{\frac{1}{r}}} \times$$

$$\times \{ [|{}^bD_q^2\Psi(b)|^r \sum_{k=1}^{n} (1 - \frac{\lambda^k}{[k+1]_q}) + |{}^bD_q^2\Psi(a)|^r \sum_{k=1}^{n} (1 - \sum_{i=0}^{k} \binom{k}{i} \frac{\lambda^{k-i}}{[k-i+1]_q})]^{\frac{1}{r}}$$

$$+ [|{}_aD_q^2\Psi(a)|^r \sum_{k=1}^{n} (1 - \frac{\lambda^k}{[k+1]_q}) + |{}_aD_q^2\Psi(b)|^r \sum_{k=1}^{n} (1 - \sum_{i=0}^{k} \binom{k}{i} \frac{\lambda^{k-i}}{[k-i+1]_q})]^{\frac{1}{r}} \}.$$

Proof. As in the proof of Theorem 6, by using Holder's inequality and then the n-polynomial convexity of $|{}_aD_q^2\Psi|^s$ and $|{}^bD_q^2\Psi|^s$, we have,

$$|{}_a^bS_q(\lambda)| \leq$$

$$\leq \frac{(b-a)^2 q^3}{[2]_q} \{ \int_0^1 t^2 |{}^bD_q^2\Psi((1-\lambda t)b + \lambda t a)| d_q t + \int_0^1 t^2 |{}_aD_q^2\Psi((1-\lambda t)a + \lambda t b)| d_q t \}$$

$$\leq \frac{q^3(b-a)^2}{[2]_q} \left(\int_0^1 t^{2s} d_q t \right)^{\frac{1}{s}} \{ \left(\int_0^1 |{}^bD_q^2\Psi((1-\lambda t)b + \lambda t a)|^r d_q t \right)^{\frac{1}{r}}$$

$$+ \left(\int_0^1 |{}_aD_q^2\Psi((1-\lambda t)a + \lambda t b)|^r d_q t \right)^{\frac{1}{r}} \} \leq \frac{q^3(b-a)^2}{[2]_q} \frac{1}{[2s+1]_q^{\frac{1}{s}}} \times$$

$$\times \{ \left(\int_0^1 (\frac{1}{n} \sum_{k=1}^{n} (1 - (\lambda t)^k) |{}^bD_q^2\Psi(b)|^r + \frac{1}{n} \sum_{k=1}^{n} (1 - (1-\lambda t)^k) |{}^bD_q^2\Psi(a)|^r) d_q t \right)^{\frac{1}{r}}$$

$$+ \left(\int_0^1 (\frac{1}{n} \sum_{k=1}^{n} (1 - (\lambda t)^k) |{}_aD_q^2\Psi(a)|^r + \frac{1}{n} \sum_{k=1}^{n} (1 - (1-\lambda t)^k) |{}_aD_q^2\Psi(b)|^r) d_q t \right)^{\frac{1}{r}} \},$$

or

$$|{}_a^bS_q(\lambda)| \leq \frac{q^3(b-a)^2}{[2]_q} \frac{1}{[2s+1]_q^{\frac{1}{s}} n^{\frac{1}{r}}} \times$$

$$\times \{ \left(|{}^bD_q^2\Psi(b)|^r \sum_{k=1}^{n} \int_0^1 (1 - \lambda^k t^k) d_q t + |{}^bD_q^2\Psi(a)|^r \sum_{k=1}^{n} \int_0^1 (1 - (1-\lambda t)^k) d_q t \right)^{\frac{1}{r}}$$

$$+ \left(|{}_aD_q^2\Psi(a)|^r \sum_{k=1}^{n} \int_0^1 (1 - \lambda^k t^k) d_q t + |{}_aD_q^2\Psi(b)|^r \sum_{k=1}^{n} \int_0^1 (1 - (1-\lambda t)^k) d_q t \right)^{\frac{1}{r}} \}.$$

By calculus, evaluating the last four integrals involved in previous inequality, we find the desired inequality. □

Theorem 9. *We suppose that all of the conditions of Lemma 2 are satisfied. If $|_aD_q^2\Psi|^r$ and $|^bD_q^2\Psi|^r$ are strongly quasi-convex convex functions on $[a,b]$ for modulus c, where $r>1$, with $\frac{1}{s}+\frac{1}{r}=1$ then we have:*

$$|_a^bS_q(\lambda)| \leq \frac{q^3(b-a)^2}{[2]_q[2s+1]_q^{\frac{1}{s}}}\times$$

$$\times\{[\max\{|^bD_q^2\Psi(b)|^r, |^bD_q^2\Psi(a)|^r - c\lambda(b-a)^2\frac{[3]_q-\lambda[2]_q}{[2]_q[3]_q}]^{\frac{1}{r}}$$

$$+[\max\{|_aD_q^2\Psi(a)|^r, |_aD_q^2\Psi(b)|^r - c\lambda(b-a)^2\frac{[3]_q-\lambda[2]_q}{[2]_q[3]_q}]^{\frac{1}{r}}\}.$$

Proof. We use the modulus properties, Holder's inequality and the definition of quasi-convex functions with modulus c for $|_aD_q^2\Psi|^r$ and $|^bD_q^2\Psi|^r$, having

$$|_a^bS_q(\lambda)| \leq \frac{q^3(b-a)^2}{[2]_q[2s+1]_q^{\frac{1}{s}}}\{[\int_0^1(\max\{|^bD_q^2\Psi(b)|^r, |^bD_q^2\Psi(a)|^r\} - c\lambda t(1-\lambda t)(b-a)^2)d_qt]^{\frac{1}{r}}$$

$$+[\int_0^1(\max\{|_aD_q^2\Psi(a)|^r, |_aD_q^2\Psi(b)|^r\} - c\lambda t(1-\lambda t)(b-a)^2)d_qt]^{\frac{1}{r}}\},$$

or

$$|_a^bS_q(\lambda)| \leq \frac{q^3(b-a)^2}{[2]_q[2s+1]_q^{\frac{1}{s}}}\times$$

$$\times\{[\max\{|^bD_q^2\Psi(b)|^r, |^bD_q^2\Psi(a)|^r - c\lambda(b-a)^2\int_0^1(t-\lambda t^2)d_qt]^{\frac{1}{r}}$$

$$+[\max\{|_aD_q^2\Psi(a)|^r, |_aD_q^2\Psi(b)|^r - c\lambda(b-a)^2\int_0^1(t-\lambda t^2)d_qt]^{\frac{1}{r}}\}.$$

An easy calculus shows that previous integral, $\int_0^1(t-\lambda t^2)d_qt = \frac{[3]_q-\lambda[2]_q}{[2]_q[3]_q}$, and the proof is completed. □

Example 1. *Let consider the function $\Psi:[0,1]\to\mathbb{R}$ defined by $\Psi(x)=x^4$ with $\lambda=1$, which satisfies the conditions of Theorem 4. By calculus, under these hypothesiss, we get for the left hand side of the inequality (3), the expression,*

$$|_a^bS_q(1)| = |\frac{1}{b-a}[\int_a^b\Psi(t)^bd_qt + \int_a^b\Psi(t)_ad_qt] - \frac{1-q-q^2}{1-q^2}[\Psi(a)+\Psi(b)]$$

$$-\frac{q}{1-q^2}[\Psi(b(1-q)+qa)+\Psi(qb+a(1-q))]|,$$

where we put $a=0$ and $b=1$, obtaining,

$$|_0^1S_q(1)| = |\int_0^1\Psi(t)^1d_qt + \int_0^1\Psi(t)_0d_qt - \frac{1-q-q^2}{1-q^2}[\Psi(0)+\Psi(1)]-$$

$$-\frac{q}{1-q^2}[\Psi(1-q)+\Psi(q)]|$$

$$= |\frac{1}{[5]_q} + (1-q)\sum_0^\infty q^n(1-q^n)^4 - \frac{1-q-q^2+q^5+q(1-q)^4}{1-q^2}|.$$

By calculus we obtain,

$$|{}_0^1S_q(1)| = |1 + \frac{2}{[5]_q} - \frac{4}{[2]_q} + \frac{6}{[3]_q} - \frac{4}{[4]_q} - \frac{1-q-q^2+q(1-q)^4}{1-q^2}|.$$

For the right hand side of the inequality (3), we have

$$\frac{q^3(b-a)^2}{[2]_q[3]_q[4]_q}\{([4]_q - [3]_q)[|{}^bD_q^2\Psi(b)| + |{}_aD_q^2\Psi(a)|] + [3]_q[|{}^bD_q^2\Psi(a)| + |{}_aD_q^2\Psi(b)|]\}$$

$$= \frac{q^3}{[2]_q[3]_q[4]_q}\{([4]_q - [3]_q)[|{}^1D_q^2\Psi(1)| + |{}_0D_q^2\Psi(0)|] + [3]_q[|{}^1D_q^2\Psi(0)| + |{}_0D_q^2\Psi(1)|]\}$$

$$= \frac{q^3}{[2]_q[3]_q[4]_q}\{6[2]_q([4]_q - [3]_q) + [3]_q(q^5 + 2q^4 - q^3 - 5q^2 + 3 + [3]_q[4]_q)\},$$

where we put $a = 0$ and $b = 1$.

On the other hand, by calculus, we have ${}_0D_q^2\Psi(x) = [4]_q[3]_qx^2$, thus ${}_0D_q^2\Psi(0) = 0$ and ${}_0D_q^2\Psi(1) = [4]_q[3]_q$. Using that,

$${}^1D_q^2\Psi(x) = \frac{(q^2x+1-q^2)^4 - [2]_q(qx+1-q)^4 + qx^4}{q(1-q)^2(1-x)^2}$$

we find ${}^1D_q^2\Psi(0) = q^5 + 2q^4 - q^3 - 5q^2 + 3$ and ${}^1D_q^2\Psi(1) = 6(q+1)$. The graphic of the function ${}^1D_q^2\Psi(x)$ considered as function of two variables x and q, is given in Figure 1b. We see that the function is positive as ${}_0D_q^2\Psi(x)$.

One can see the validity of the inequality (3) in Figure 1a, where the green line in graph represents the expression of the left member of the inequality (3) and the expression of the right member represents the magenta line in graph.

We used here the Matlab R2023a software for obtaining Figure 1 and also partial in calculus of last two derivatives.

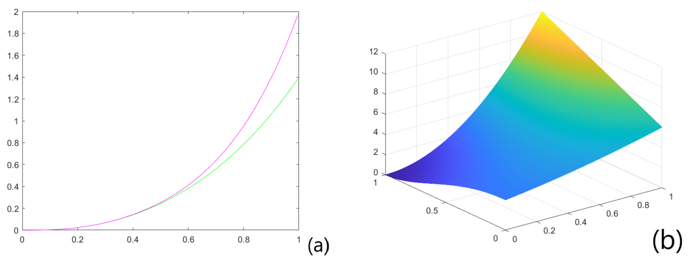

Figure 1. (**a**) An example for the inequality (3) from Theorem 4 for the function $\Psi(x) = x^4$ and $a = 0,\ b = 1$, when $\lambda = 1$; (**b**) Graphic for the functions ${}^1D_q^2\Psi(x)$, considered as function of two variables x and q, from Example 1 when $\Psi(x) = x^4$, $a = 0,\ b = 1$, and $\lambda = 1$.

Example 2. *For the same function $\Psi : [0,1] \to \mathbb{R}$ defined by $\Psi(x) = x^4$, but with $\lambda = \frac{1}{2}$, which satisfies the conditions of Theorem 4, we obtain for the left hand side of the inequality (3), by similar calculus, the expression,*

$$|{}_0^1S_q(\frac{1}{2})| = 8[\int_{\frac{1}{2}}^1 \Psi(t)^1d_qt + \int_0^{\frac{1}{2}} \Psi(t)_0d_qt] - 8\frac{1-q-q^2}{1-q^2}\frac{1}{2^4} - \frac{4q}{1-q^2}[(1-\frac{q}{2})^4 + \left(\frac{q}{2}\right)^4]$$

$$= 4 - \frac{8}{[2]_q} + \frac{6}{[3]_q} - \frac{2}{[4]_q} + \frac{1}{2[5]_q} - \frac{1-q-q^2}{2(1-q^2)} - \frac{4q}{1-q^2}[(1-\frac{q}{2})^4 + \left(\frac{q}{2}\right)^4].$$

The right hand side of the inequality (3) becomes for $\lambda = \frac{1}{2}$,

$$\frac{q^3}{[2]_q[3]_q[4]_q}\{([4]_q - \frac{[3]_q}{2})[|^1D_q^2\Psi(1)| + |_0D_q^2\Psi(0)|] + \frac{[3]_q}{2}[|^1D_q^2\Psi(0)| + |_0D_q^2\Psi(1)|]\}$$

$$= \frac{q^3}{[2]_q[3]_q[4]_q}\{6[2]_q([4]_q - \frac{[3]_q}{2}) + \frac{[3]_q}{2}(q^5 + 2q^4 - q^3 - 5q^2 + 3 + [3]_q[4]_q)\}.$$

Like in Example 1, the red line in Figure 2 represents the left hand side of the inequality (3) from Theorem 4, and the blue line represents the right hand side of (3) and the inequality is checked.

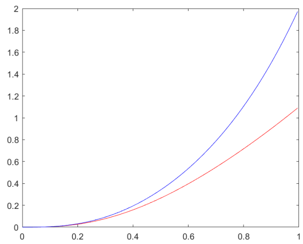

Figure 2. An example for the inequality (3) from Theorem 4 for the function $\Psi(x) = x^4$ and $a = 0,\ b = 1$, when $\lambda = \frac{1}{2}$.

Example 3. *Let consider the function $\Psi : [0,1] \to \mathbb{R}$ defined by $\theta(x) = x^6$ with $\lambda = 1$, which satisfies the conditions of Theorem 4. By calculus, under these hypothesis, we get for the left hand side of the inequality (3), the expression,*

$$|{}_a^bS_q(1)| = |\frac{1}{b-a}[\int_a^b \Psi(t)^b d_qt + \int_a^b \Psi(t)_a d_qt] - \frac{1-q-q^2}{1-q^2}[\Psi(a) + \Psi(b)]$$

$$-\frac{q}{1-q^2}[\Psi(b(1-q) + qa) + \Psi(qb + a(1-q))]|,$$

where we put $a = 0$ and $b = 1$, obtaining,

$$|{}_0^1S_q(1)| = |\int_0^1 \Psi(t)^1 d_qt + \int_0^1 \Psi(t)_0 d_qt - \frac{1-q-q^2}{1-q^2}[\Psi(0) + \Psi(1)] -$$

$$-\frac{q}{1-q^2}[\Psi(1-q) + \Psi(q)]|$$

$$= |\frac{1}{[7]_q} + (1-q)\sum_0^\infty q^n(1-q^n)^6 - \frac{1-q-q^2+q^7+q(1-q)^6}{1-q^2}|.$$

By calculus we obtain,

$$|{}_0^1S_q(1)| = |1 + \frac{2}{[7]_q} - \frac{6}{[2]_q} + \frac{15}{[3]_q} - \frac{20}{[4]_q} + \frac{15}{[5]_q} - \frac{6}{[6]_q} - \frac{1-q-q^2+q(1-q)^6+q^7}{1-q^2}|.$$

For the right hand side of the inequality (3), we have

$$\frac{q^3(b-a)^2}{[2]_q[3]_q[4]_q}\{(q^3[|^bD_q^2\Psi(b)| + |_aD_q^2\Psi(a)|] + [3]_q[|^bD_q^2\Psi(a)| + |_aD_q^2\Psi(b)|]\}$$

$$= \frac{q^3}{[2]_q[3]_q[4]_q}\{q^3[|^1D_q^2\Psi(1)| + |_0D_q^2\Psi(0)|] + [3]_q[|^1D_q^2\Psi(0)| + |_0D_q^2\Psi(1)|]\}$$

where we put $a = 0$ and $b = 1$.

On the other hand, by calculus, we have ${}_0D_q^2\Psi(x) = [6]_q[5]_q x^4$*, thus* ${}_0D_q^2\Psi(0) = 0$ *and* ${}_0D_q^2\Psi(1) = [6]_q[5]_q$*. Using that,*

$$ {}^1D_q^2\Psi(x) = \frac{(q^2x+1-q^2)^6 - [2]_q(qx+1-q)^6 + qx^6}{q(1-q)^2(1-x)^2} $$

we find ${}^1D_q^2\theta(0) = q^9 + 2q^8 - 3q^7 - 8q^6 + 2q^5 + 11q^4 + 5q^3 - 10q^2 - 5q + 5$ *and* ${}^1D_q^2\Psi(1) = 15q + 15$*. The graphic of the function* ${}^1D_q^2\Psi(x)$*, considered as function of two variables* x *and* q*, is given in Figure 3b.*

One can see the validity of the inequality (3) in Figure 3a, where the green line in graph represents the expression of the left member of the inequality (3) and the expression of the right member represents the magenta line in graph.

We used here the Matlab R2023a for obtaining Figure 3 and also partial in calculus of last two derivatives.

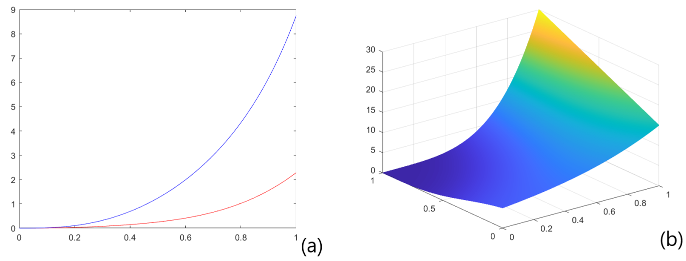

Figure 3. (**a**) An example for the inequality (3) from Theorem 4 for the function $\Psi(x) = x^6$, $a = 0$, $b = 1$ and $\lambda = 1$; (**b**) Graphic for the function ${}^1D_q^2\Psi(x)$ from Example 3, considered as function of two variable x and q, when $\Psi(x) = x^6$, $a = 0$, $b = 1$, and $\lambda = 1$.

Example 4. *For the function* $\Psi : [0,1] \to \mathbb{R}$ *defined by* $\theta(x) = (1-x)^2$*, but with* $\lambda = 1$*, which satisfies the conditions of Theorem 4, we obtain for the left hand side of the inequality (3), by similar calculus, the expression,*

$$ |{}_a^bS_q(1)| = |\frac{1}{b-a}[\int_a^b \Psi(t)^b d_qt + \int_a^b \Psi(t)_a d_qt] - \frac{1-q-q^2}{1-q^2}[\Psi(a)+\Psi(b)] $$

$$ -\frac{q}{1-q^2}[\Psi(b(1-q)+qa) + \Psi(qb+a(1-q))]|, $$

where we put $a = 0$ *and* $b = 1$*, obtaining,*

$$ |{}_0^1S_q(1)| = |\int_0^1 (1-t)^2\,{}^1d_qt + \int_0^1 (1-t)^2{}_0d_qt - \frac{1-q-q^2}{1-q^2} - $$

$$ -\frac{q}{1-q^2}[q^2 + (1-q)^2]| = \frac{2q^3}{q^2+q+1} $$

The right hand side of the inequality (3) becomes for $\lambda = 1$*,*

$$ \frac{q^3(b-a)^2}{[2]_q[3]_q[4]_q}\{(q^3[|{}^bD_q^2\Psi(b)| + |{}_aD_q^2\Psi(a)|] + [3]_q[|{}^bD_q^2\Psi(a)| + |{}_aD_q^2\Psi(b)|]\} $$

$$ = \frac{q^3}{[2]_q[3]_q[4]_q}\{q^3[|{}^1D_q^2\Psi(1)| + |{}_0D_q^2\Psi(0)|] + [3]_q[|{}^1D_q^2\Psi(0)| + |{}_0D_q^2\Psi(1)|]\} $$

where we put $a = 0$ *and* $b = 1$*.*

On the other hand, by calculus, we have

$$ {}_0D_q^2\Psi(x) = \frac{q(1-x)^2 - [2]_q(1-qx)^2 + (1-q^2x)^2}{qx^2(1-q)^2}, $$

thus by calculus, ${}_0D_q^2\Psi(x) = [2]_q$ *and from here,* ${}_0D_q^2\Psi(0) = [2]_q$ *and* ${}_0D_q^2\Psi(1) = [2]_q$*. We find,* ${}^1D_q^2\Psi(x) = [2]_q$, ${}^1D_q^2\theta(0) = [2]_q$ *and* ${}^1D_q^2\Psi(1) = [2]_q$*. Therefore the right member becomes*

$$ \frac{2q^3}{q^2+q+1}, $$

and we have equality.

4. Discussion and Conclusions

The main findings of this study prove some new parametrized q-Hermite-Hadamard like type integral inequalities for functions whose second left and right q-derivative satisfies several different types of convexities. Some basic inequalities as q-Holder's integral inequality and q-power mean inequality have been used in order to obtain the new estimated bounds. An auxiliary q-lemma was utilized as a main tool in our proofs. Symmetry can offer an advantage in study of many processes and phenomena from nature. Interesting consequences arise for special choices of the parameter and the corresponding cases were discussed in detail.

We used the Matlab R2023a software for figures and for some calculus in the examples. Several consequences, examples and applications were given to illustrate the outcome of the research.

Furthermore, it is interesting to extend such findings to other new kinds of convexities, (p,q)-calculus, and q-fractional inequalities, which could be some good generalizations.

Even between the concept of convexity and the concept of symmetry there is a strong correlation, these two having many common ground to develop. This shows that the conclusions reached are pretty consistent. The study could be useful for the analysis of utility, distribution of taxes and revenues.

Overall, we hope that our results will improve the existing literature in the field.

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