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Robustness Analysis of Exponential Synchronization in Complex Dynamic Networks with Time-Varying Delays and Random Disturbances

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Abstract: This paper aims to investigate the robustness of exponential synchronization in complex dynamic networks (CDNs) with time-varying delays and random disturbances. Via the Gronwall–Bellman lemma and partial inequality methods, by calculating the transcendental equations, the delays limits and maximum disturbance size of the CDNs are estimated. This means that the perturbed system achieves exponential synchronization if the disturbance strength is within our estimation range. The theoretical results are illustrated by several simulations.

Keywords: complex dynamic network; exponential synchronization; time-varying delays; random disturbances; robustness

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1. Introduction

Complex dynamic networks (CDNs) consist of a large number of nodes or edges; each node or edge represents a basic dynamic system with a specific content. CDNs have become an interesting research field in various fields, such as ecological networks [1], biological neural networks [2], and communication networks [3,4]. With the continuous exploration of the dynamic behavior of CDNs, the analysis of complex behavior of CDNs has been recognized as a research hotspot [5–8].

Synchronization, as one of the topics worth thinking about in CDNs, has been used in various industries, such as face recognition, information security, and so on [9,10]. The earliest research on synchronization dates back to the time of Huygens in 1655 [11]. In recent years, synchronization control of CDNs, such as global Mittag–Leffler synchronization [2], adaptive synchronization [12] and lag synchronization [13], etc., has attracted great attention as an interesting direction of control systems.

Generally speaking, because of the different characteristics of CDN nodes, it is difficult to realize synchronization. Then, the synchronous implementation of the CDNs still needs some control strategy. Therefore, many scholars have produced some effective controllers to synchronize CDNs. To the best of our knowledge, controllers are divided into linear controllers [14,15] and nonlinear controllers [16–19]. The synchronization problem of the CDN is studied using a hierarchical controller in [20]. The authors of [21,22] studied the synchronization of a CDN with delays under the adaptive control strategy.

Because of the limitation of information transmission rate between nodes, CDNs are inevitably affected by time-varying delays and random noises [19]. The existence of time-varying delays may cause delays in information transfer between CDN nodes, and may cause the CDN to lose synchronization [23]. In general, the main types of time delays are leakage delays [24], discrete delays [25], and state-dependent delays [26], etc. For the random disturbances, the disturbance process is complex, and noise interference may lead to the deviation of information transmission in CDNs. For CDNs interfered by time-varying delays and external noise, the synchronization phenomenon has attracted the attention of scholars at home and abroad in recent years [19,27–29].

It is noted that CDNs may lose synchronization if the time-varying delays and noise intensity exceed certain limits [30]. We know that CDNs may still have exponential synchronization (ESy) if the intensity of time-varying delays and random noise disturbance is small when ESy has been achieved in CDNs [31]. Many interesting results were obtained for CDN synchronization interfered by time-varying delays [32–34] and random noise in [16,19,21,23,28,35–37]. It should be noted that the above work pays little attention to the robustness of ESy in CDNs. Therefore, how much time-varying delay and noise intensity CDNs can withstand without losing synchronization is an interesting question.

Robustness means that the system maintains certain characteristics under the certain disturbance of external factors. It has important reference value for the analysis and design of complex systems. In [38], the robustness of exponential stability in recurrent neural networks is explored for the first time. In recent years, there have been many interesting results on the robustness of stability [39]. In addition, with the development of CDNs, the Lyapunov stability theory and the linear matrix inequality methods have been used to analyze the synchronization of CDNs extensively, but the robustness of ESy is rarely studied by estimating the upper bound of noise intensity and delays. The robustness of ESy in CDNs with piecewise constant parameters is studied for the first time in [40]. In addition, in [41], Zhang et al. investigated the robustness of ESy in CDNs with random disturbances.

Inspired by the above works, this paper discusses the robustness of ESy in CDNs with time-varying delays and noise interference. The main works are as follows.

- Using the Gronwall–Bellman lemma and inequality methods, the upper bound of a CDN subject to both delays and random noise intensity is obtained. By calculating the transcendental equations, The allowable ESy range of a CDN with both delay and external noise is estimated.
- In the systems discussed in this paper, the parameter configuration matrices without delay and with delay are not necessarily symmetric.
- Compared with the [23,41], this article comprehensively considers the influence of time-varying delays and random disturbance on ESy of CDNs. This paper provides a judgment basis for the analysis and designs of CDNs.
- In [39], the robustness of neural network stability was thoroughly studied. Unlike the aforementioned paper, we investigate the robustness of ESy in CDNs with time-varying delays and random disturbances.

The structure of this paper is as follows: Section 2 introduces the preparatory knowledge. Section 3 is the results. In Section 4, some simulations are given. In Section 5 is the summary of this paper.

2. Problem Statement and Model

2.1. Notation

In this paper, $\mathfrak{R}^m$ is sets of m-dimensional Euclidean spaces, and $\mathfrak{R}^{m\times m}$ represents $m\times m$-dimensional matrices composed of spaces or sets. I_m is an identity matrix of $m\times m$. $\theta\otimes\vartheta$ is the Kronecker product of matrices θ and ϑ. Let $(\Omega,\mathfrak{F},\{\mathfrak{F}_t\}_{t\geqslant 0})$ be a complete probability space with filtering $\{\mathfrak{F}_t\}_{t\geqslant 0}$, and the filter contains all P-null sets and is right-continuous. $\varrho(t)$ stands for time delays. $\mathsf{B}(t)$ is a Brownian motion. $||*||$ is the Euclidean norm. $E\{\star\}$ stands for a mathematical expectation operator in probability measure P.

A graph $G=(\mathfrak{V},\mathcal{E},\mathbf{A})$ has three elements. $\mathfrak{V}=\{1,\ldots,m\}$ is the node set, and the edge set $\mathcal{E}\subset\mathfrak{V}\times\mathfrak{V}$. Coupling matrix is $\mathbf{A}=(a_{ij})_{m\times m}$, where a_{ij} represents the coupling weight from the i-th node to the j-th node. If the information is from the j-th node to the i-th node ($i\neq j$), then the $a_{ij}\neq 0$; otherwise, $a_{ij}=0$.

2.2. Problem Formulation

Consider the following CDN model consisting of m nodes with time delays

$$\begin{cases} \dot{z}_i(t) = \Theta(z_i(t), t) + k\sum_{j=1}^{m} a_{ij}z_j(t) + k\sum_{j=1}^{m} b_{ij}z_j(t-\varrho(t)) + c_i(t), \\ z_i(t_0) = z_{i0} \in \Re^m, \end{cases} \tag{1}$$

where $z_i(t) = (z_{i1}, \ldots, z_{im})^T \in \Re^m$ is the state vector of the i-th node; Θ: $\Re^m \to \Re^m$ is vector-valued function; k is called the coupling strength; $c_i(t) \in \Re^m$ is the latter needed the design of the controllers. a_{ij}, b_{ij} represents the matrix $\mathbf{A}$, $\mathbf{B}$, and satisfying $a_{ii} = -\sum_{j=1, j\neq i}^{m} a_{ij}$ and $b_{ii} = -\sum_{j=1, j\neq i}^{m} b_{ij}$; $\varrho(t)$ are delays that satisfy $\varrho(t) : [t_0, +\infty), \varrho'(t) \leq \vartheta < 1, \psi = \{\psi(s) : -\tilde{\varrho} \leq s \leq 0\} \in C([-\tilde{\varrho}, 0], \Re^m)$.

The dynamic equation of isolated nodes of CDN (1) is

$$\dot{\chi}(t) = \Theta(\chi(t), t), \tag{2}$$

where $\chi(t)$ is any expected state.

In addition, when CDN (1) has no time delay interference, the following form is obtained:

$$\begin{cases} \dot{\hat{z}}_i(t) = \Theta(\hat{z}_i(t), t) + k\sum_{j=1}^{m} a_{ij}\hat{z}_j(t) + k\sum_{j=1}^{m} b_{ij}\hat{z}_j(t) + c_i(t), \\ \hat{z}_i(t_0) = \hat{z}_{i0} \in \Re^m. \end{cases} \tag{3}$$

Defining system (2) and (3), the error is $l_i(t) = \hat{z}_i(t) - \chi(t)$, and subtracting (2) from system (3) gives the following error system:

$$\begin{cases} \dot{l}_i(t) = \Upsilon(l_i(t), t) + k\sum_{j=1}^{m} a_{ij}l_j(t) + k\sum_{j=1}^{m} b_{ij}l_j(t) + c_i(t), \\ l_i(t_0) = l_{i0} \in \Re^m, \end{cases} \tag{4}$$

where $\Upsilon(l_i(t), t) = \Theta(\hat{z}_i(t), t) - \Theta(\chi(t), t)$, and the function $\Upsilon(l_i(t), t)$ satisfies the following assumptions.

(**H1**) The function $\Upsilon(\star, t)$ will satisfy the following condition:

$$||\Upsilon(l(t), t) - \Upsilon(q(t), t)|| \leq \Phi||l(t) - q(t)||, \tag{5}$$

where Φ is a known constant.

To obtain synchronization of (1) and (2), the linear controller is expressed as

$$c_i(t) = \check{S}l_i(t),\ i = 1, \ldots, m, \tag{6}$$

where $\check{S} \in \Re^{m\times m}$ is the feedback gain matrix.

Substitute (6) into (4):

$$\begin{cases} \dot{l}_i(t) = \Upsilon(l_i(t), t) + k\sum_{j=1}^{m} a_{ij}l_j(t) + k\sum_{j=1}^{m} b_{ij}l_j(t) + \check{S}l_i(t), \\ l_i(t_0) = l_{i0} \in \Re^m. \end{cases} \tag{7}$$

Denote $l(t) = (l_1(t), l_2(t), \ldots, l_m(t))^T$, $l(t-\varrho(t)) = (l_1(t-\varrho(t)), \ldots, l_m(t-\varrho(t)))^T$, $\Upsilon(l(t), t) = (\Upsilon(l_1(t), t)^T, \ldots, \Upsilon(l_m(t), t)^T)^T$, $\mathbf{A} = (a_{ij})_{m\times m}$, $\mathbf{B} = (b_{ij})_{m\times m}$, $A = I_m \otimes \mathbf{A}$, $B = I_m \otimes \mathbf{B}$, $S = I_m \otimes \check{S}$.

Then, CDN (7) can be rewritten into the following matrix form:

$$\begin{cases} \dot{l}(t) = \Upsilon(l(t), t) + kAl(t) + kBl(t) + Sl(t), \\ l(t_0) = l_0 \in \mathfrak{R}^{m \times m}. \end{cases} \tag{8}$$

Define system (1) and (2); the synchronization error is $q_i(t) = z_i(t) - \chi(t)$. Subtracting (2) from (1) gives the following error system:

$$\begin{cases} \dot{q}_i(t) = \Upsilon(q_i(t), t) + k\sum_{j=1}^{m} a_{ij}q_i(t) + k\sum_{j=1}^{m} b_{ij}q_i(t - \varrho(t)) + \check{S}q_i(t), \\ q_i(t_0) = q_{i0} \in \mathfrak{R}^m. \end{cases} \tag{9}$$

The matrix form of the error system (9) is

$$\begin{cases} \dot{q}(t) = \Upsilon(q(t), t) + kAq(t) + kBq(t - \varrho(t)) + Sq(t), \\ q(t_0) = q_0 \in \mathfrak{R}^{m \times m}, \end{cases} \tag{10}$$

For the convenience of the analysis, this paper first provides some basic definitions, and abbreviates some nouns (see Table 1).

Table 1. This article deals with abbreviated tables.

Full Name	Abbreviation
Exponential stability	ESt
Mean-square exponential stability (synchronization)	MSESt (MSESy)
Almost surely exponential stability (synchronization)	ASESt (ASESy)

Definition 1. *If the error system (8) achieved ESt (Table 1), then the CDN (2) and (3) said to achieve ESy. That is, for* $\forall l_0 \in \mathfrak{R}^m$, $\exists\ \mathfrak{L} > 0, \mathfrak{D} > 0$, *such that for any* $t \in \mathfrak{R}^+$, *satisfying*

$$||l(t)|| \leq \mathfrak{L}||l_0|| \exp(-\mathfrak{D}(t - t_0)).$$

3. Main Results

A controller is an algorithm or rule that makes the system reach the desired state under certain control rules. Next, we discuss the robustness of ESy of CDNs with time delays and noise interference under linear and nonlinear controllers, respectively.

We first discuss how much time delay CDN can withstand without losing synchronization under linear and nonlinear controllers.

3.1. Effect of Time Delays on CDN Synchronization

3.1.1. The Linear Controller

In this section, we will investigate the robustness of ESy for CDNs with time delays under linear controllers. For CDNs that have reached ESy, how much external time delay can the system withstand without losing synchronization? That is, the CDN is still synchronous if the limits of the system being disturbed by external time delays is within our estimate. From this, we can obtain the following theorem:

Theorem 1. *If Assumption* **H1** *is trues, and (8) is ESt, then (10) is ESt. That is, systems (1) and (2) can realize ESy under (6), if* $\tilde{\varrho} < \min(\Delta/2, \breve{\varrho})$, *where* $\breve{\varrho}$ *is the only nonnegative solution of the following transcendental equation.*

$$\mathfrak{L} \exp(-\mathfrak{D}(\Delta - \tilde{\varrho})) + \mathcal{L}_4 \exp(2\Delta\mathcal{L}_3) = 1, \tag{11}$$

where

$$
\begin{aligned}
&\Delta > \ln \mathfrak{L}/\mathfrak{D}, \\
&\mathcal{L}_3 = (\Phi + k||A|| + ||S|| + k||B||) + k||B||\left[\Phi + k||A|| + ||S|| + k||B||(1-\vartheta)^{-1}\right]\breve{\varrho}, \\
&\mathcal{L}_4 = k||B||[\breve{\varrho} + \breve{\varrho}(1-\vartheta)^{-1} + k||B||\breve{\varrho}^2(1-\vartheta)^{-1}] + k\breve{\varrho}||B||[\Phi + k||A|| + ||S|| \\
&\qquad + k||B||(1-\vartheta)^{-1}]\mathfrak{L}/\mathfrak{D}.
\end{aligned}
$$

Proof. Denote $l(t; t_0, \psi(0)), p(t; t_0, \psi(0))$ is $l(t), q(t)$ and $l_0 = q_0$. According to the systems (8) and (10), the differential equation is transformed into an integral equation, and we have

$$
\begin{aligned}
||l(t) - q(t)|| =& \left|\left| \int_{t_0}^{t} (\Upsilon(l(s), s) - \Upsilon(q(s), s)) + kA(l(s) - q(s)) + kB(l(s) \right.\right. \\
&\left.\left. - q(s - \varrho(s))) + S(l(s) - q(s))\, ds \right|\right| \\
\leq& \int_{t_0}^{t} ||(\Upsilon(l(s), s) - \Upsilon(q(s), s))|| + ||kA(l(s) - q(s))|| \\
&+ ||kB(l(s) - q(s - \varrho(s)))|| + ||S(l(s) - q(s))||\, ds.
\end{aligned}
\tag{12}
$$

From (5),

$$
\begin{aligned}
||l(t) - q(t)|| \leq& \int_{t_0}^{t} \Phi||l(s) - q(s)|| + k||A||\ ||l(s) - q(s)|| + k||B||\ ||l(s) \\
&- q(s - \varrho(s))|| + ||S||\ ||(l(s) - q(s))||\, ds \\
\leq& \int_{t_0}^{t} (\Phi + k||A|| + k||B|| + ||S||)||l(s) \\
&- q(s)|| + k||B||\ ||q(s) - q(s - \varrho(s))||\, ds \\
=& (\Phi + k||A|| + k||B|| + ||S||) \\
&\times \int_{t_0}^{t} ||l(s) - q(s)||ds + k||B|| \int_{t_0}^{t} ||q(s) - q(s - \varrho(s))||ds.
\end{aligned}
\tag{13}
$$

Furthermore, when $t \geq t_0 + \tilde{\varrho}$, from (5) and (9),

$$
\begin{aligned}
\int_{t_0+\tilde{\varrho}}^{t} ||q(s) - q(s - \varrho(s))||ds \leq& \int_{t_0+\tilde{\varrho}}^{t} ds \int_{s-\tilde{\varrho}}^{s} \left[(\Phi + k||A|| + ||S||)\ ||q(r)|| \right. \\
&\left. + k||B||\ ||q(r - \varrho(r))||\right] dr.
\end{aligned}
\tag{14}
$$

Exchange integration order, one has

$$
\begin{aligned}
&\int_{t_0+\tilde{\varrho}}^{t} ds \int_{s-\tilde{\varrho}}^{s} (\Phi + k||A|| + ||S||)\ ||q(r)||dr \\
&\qquad = \int_{t_0}^{t} dr \int_{\max(t_0+\tilde{\varrho}, r)}^{\min(r+\tilde{\varrho}, t)} (\Phi + k||A|| + ||S||)\ ||q(r)||ds \\
&\qquad \leq (\Phi + k||A|| + ||S||)\tilde{\varrho} \int_{t_0}^{t} ||q(r)||dr.
\end{aligned}
\tag{15}
$$

Similarly, one has

$$
\begin{aligned}
&\int_{t_0+\tilde{\varrho}}^{t} ds \int_{s-\tilde{\varrho}}^{s} k||B||\,||q(r-\varrho(r))||dr \\
&\qquad = \int_{t_0}^{t} dr \int_{\max(t_0+\tilde{\varrho},r)}^{\min(r+\tilde{\varrho},t)} k||B||\,||q(r-\varrho(r))||ds \\
&\qquad \leq k||B||\tilde{\varrho}(1-\vartheta)^{-1}\left(\tilde{\varrho}\sup_{t_0-\tilde{\varrho}\leq s\leq t_0}||q(s)||+\int_{t_0}^{t}||q(u)||\,du\right).
\end{aligned} \tag{16}
$$

When $t \geq t_0 + \tilde{\varrho}$, substitute (14) and (15) into (13), and one has

$$
\begin{aligned}
\int_{t_0+\tilde{\varrho}}^{t} ||q(s)-q(s-\varrho(s))||ds \leq & k||B||\tilde{\varrho}^2(1-\vartheta)^{-1}\left(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0}||q(s)||\right) \\
& + (\Phi + k||A|| + ||S|| + k||B||\tilde{\varrho}(1-\vartheta)^{-1}) \\
& \times \int_{t_0}^{t}||q(s)||\,ds.
\end{aligned} \tag{17}
$$

When $t \geq t_0 + \tilde{\varrho}$, substitute (16) into (12), and one has

$$
\begin{aligned}
||l(t)-q(t)|| \leq & (\Phi + k||A|| + ||S|| + k||B||)\int_{t_0}^{t}||l(s)-q(s)||\,ds \\
& + k||B||\left[\tilde{\varrho}+\tilde{\varrho}(1-\vartheta)^{-1}\right]\left(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0+\tilde{\varrho}}||q(s)||\right) \\
& + k^2||B||^2\tilde{\varrho}^2(1-\vartheta)^{-1}\left(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0}||q(s)||\right) \\
& + k||B||\left[\Phi + k||A|| + ||S|| + k||B||(1-\vartheta)^{-1}\right]\tilde{\varrho}\int_{t_0}^{t}||q(s)||\,ds.
\end{aligned} \tag{18}
$$

By Definition 1, further available is

$$
\begin{aligned}
||l(t)-q(t)|| \leq & \Bigg\{(\Phi + k||A|| + ||S|| + k||B||) + k||B|| \\
& \left[\Phi + k||A|| + ||S|| + k||B||(1-\vartheta)^{-1}\right]\tilde{\varrho}\Bigg\}\int_{t_0}^{t}||l(s)-q(s)||\,ds \\
& + k||B||\left\{\tilde{\varrho}+\tilde{\varrho}(1-\vartheta)^{-1}+k||B||\tilde{\varrho}^2(1-\vartheta)^{-1}\right\} \\
& \times\left(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0+\tilde{\varrho}}||q(s)||\right) \\
& + k\tilde{\varrho}||B||\left[\Phi + k||A|| + ||S|| + k||B||(1-\vartheta)^{-1}\right]\mathfrak{L}/\mathfrak{D}||l_0|| \\
\leq & \mathcal{L}_1\int_{t_0}^{t}||l(s)-q(s)||\,ds + \mathcal{L}_2\left(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0+\tilde{\varrho}}||q(s)||\right),
\end{aligned} \tag{19}
$$

where

$$
\begin{aligned}
\mathcal{L}_1 = & (\Phi + k||A|| + ||S|| + k||B||) + k||B||\left[\Phi + k||A|| + ||S|| + k||B||(1-\vartheta)^{-1}\right]\tilde{\varrho}, \\
\mathcal{L}_2 = & k||B||[\tilde{\varrho}+\tilde{\varrho}(1-\vartheta)^{-1}+k||B||\tilde{\varrho}^2(1-\vartheta)^{-1}] + k\tilde{\varrho}||B||[\Phi + k||A|| + ||S|| \\
& + k||B||(1-\vartheta)^{-1}]\mathfrak{L}/\mathfrak{D}.
\end{aligned}
$$

Using the Gronwall–Bellman lemma [42], when $t \leq t_0 + 2\Delta$,

$$||l(t) - q(t)|| \leq \mathcal{L}_2 \exp(2\Delta\mathcal{L}_1)\Big(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0+\tilde{\varrho}} ||q(s)||\Big).$$

Hence, when $t_0 + \tilde{\varrho} \leq t \leq t_0 + 2\Delta$,

$$\begin{aligned} ||q(t)|| \leq& ||l(t)|| + ||l(t) - q(t)|| \\ \leq& ||l(t)|| + \mathcal{L}_2 \exp(2\Delta\mathcal{L}_1)\Big(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0+\tilde{\varrho}} ||q(s)||\Big). \end{aligned} \tag{20}$$

Notice $\tilde{\varrho} < \min(\Delta/2, \check{\varrho})$, due to $t_0 - \tilde{\varrho} + \Delta \leq t \leq t_0 - \tilde{\varrho} + 2\Delta$,

$$\begin{aligned} ||q(t)|| \leq& \big[\mathfrak{L}\exp(-\mathfrak{D}(\Delta - \tilde{\varrho})) + \mathcal{L}_2 \exp(2\Delta\mathcal{L}_1)\big]\Big(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0+\tilde{\varrho}} ||q(s)||\Big) \\ =& \hat{\mathcal{C}}\Big(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0+\tilde{\varrho}} ||q(s)||\Big), \end{aligned} \tag{21}$$

where $\hat{\mathcal{C}} = \mathfrak{L}\exp(-\mathfrak{D}(\Delta - \tilde{\varrho})) + \mathcal{L}_2 \exp(2\Delta\mathcal{L}_1)$.

From (10), when $\tilde{\varrho} < \min(\Delta/2, \check{\varrho})$, $\hat{\mathcal{C}} < 1$.

Choose

$$\Xi = -\ln\hat{\mathcal{C}}/\Delta.$$

From (21), one has

$$\begin{aligned} &\sup\nolimits_{t_0-\tilde{\varrho}\leq t\leq t_0-\tilde{\varrho}+\Delta} ||q(t;t_0,\psi)|| \\ &\leq \exp(-\Xi\Delta)\Big(\sup_{t_0-\tilde{\varrho}\leq t\leq t_0-\tilde{\varrho}+\Delta} ||q(t;t_0,\psi)||\Big). \end{aligned} \tag{22}$$

Then, for any nonnegative integer $d = 1, 2, \ldots$, when $t \geq t_0 + (d-1)\Delta$, one has

$$q(t;t_0,\psi) = q(t;t_0 + (d-1)\Delta, q(t_0 + (d-1)\Delta; t_0,\psi)). \tag{23}$$

From (22) and (23),

$$\begin{aligned} &\sup\nolimits_{t_0+d\Delta\leq t\leq t_0+(d+1)\Delta} |q(t;t_0,\psi)| = \\ &\Big(\sup_{t_0+(d-1)\Delta+\Delta\leq t\leq t_0+(d-1)\Delta+2\Delta} |q(t;t_0 + (d-1)\Delta, q(t_0 + (d-1)\Delta; t_0,\psi))|\Big) \\ &\leq \exp(-\Xi\Delta)\Big(\sup_{t_0+(d-1)\Delta\leq t\leq t_0+2d\Delta} |q(t;t_0,\psi)|\Big) \\ &\quad\ldots \\ &\leq \exp(-\Xi d\Delta)\Big(\sup_{t_0\leq t\leq t_0+\Delta} |q(t;t_0,\psi)|\Big) \\ &\doteq c\exp(-\Xi d\Delta), \end{aligned}$$

where $c = \sup_{t_0\leq t\leq t_0+\Delta} |q(t;t_0,\psi)|$. So, for $\forall t > t_0 + \Delta$, there is a nonnegative integer d, such than $t_0 + d\Delta \leq t \leq t_0 + (d+1)\Delta$, and we have

$$\begin{aligned} |p(t;t_0,q_0)| \leq& c\exp(-\Xi t + \Xi t_0 + 2\Xi\Delta) \\ =& (c\exp(2\Xi\Delta))\exp(-\Xi(t - t_0)). \end{aligned} \tag{24}$$

The condition is also trues when $t_0 \leq t \leq t_0 + \Delta$. So, the error system (10) achieves ESt. Thus, systems (1) and (2) achieve ESy under linear controller (6). □

Remark 1. *Theorem 1 shows that the error system (10) still achieves ESt when the magnitude of delays is within the range of our derivation. Thus, systems (1) and (2) are ESy under linear controller (6).*

3.1.2. The Nonlinear Controller

In the previous section, we discussed for how much time delay the CDN is still ESy under linear controllers. Now, we discuss the robustness of ESy in CDNs with time delays under nonlinear controllers.

To obtain synchronization of (1) and (2), the nonlinear controllers are expressed as

$$c_i(t) = -\Theta(l_i(t), t) + \Theta(\chi(t), t) + \breve{O}l_i(t),\ i = 1, \ldots, m, \tag{25}$$

Combining (4) and (25), one has

$$\begin{cases} \dot{l}_i(t) = k\sum_{j=1}^{m} a_{ij}l_j(t) + k\sum_{j=1}^{m} b_{ij}l_j(t) + \breve{O}l_i(t), \\ l_i(t_0) = l_{i0} \in \mathfrak{R}^m, \end{cases} \tag{26}$$

The matrix form is

$$\begin{cases} \dot{l}(t) = kAl(t) + kBl(t) + Ol(t), \\ l(t_0) = l_0 \in \mathfrak{R}^{m\times m}, \end{cases} \tag{27}$$

where $O = I_m \otimes \breve{O}$.

The system (27) disturbed by the time delays will take the following form:

$$\begin{cases} \dot{q}(t) = kAq(t) + kBq(t - \varrho(t)) + Oq(t), \\ q(t_0) = q_0 \in \mathfrak{R}^{m\times m}, \end{cases} \tag{28}$$

where $q(t) = (q_1(t), \ldots, q_m(t))^T$, $q(t - \varrho(t)) = (q_1(t - \varrho(t)), \ldots, q_m(t - \varrho(t)))^T$, $O = I_m \otimes \breve{O}$.

The following theorem describes how much time delay CDN can withstand under a nonlinear controller that is ESy.

Theorem 2. *If Assumption* **H1** *is trues, and (27) is ESt, then (26) is ESt. That is, systems (1) and (2) can realize ESy under (25), if $\tilde{\varrho} < \min(\Delta/2, \breve{\varrho})$, where $\breve{\varrho}$ is the only nonnegative solution of the following transcendental equation.*

$$\mathfrak{L}\exp(-\mathfrak{D}(\Delta - \tilde{\varrho})) + \mathcal{T}_4\exp(2\Delta\mathcal{T}_3) = 1, \tag{29}$$

where

$$\begin{aligned} &\Delta > \ln\mathfrak{L}/\mathfrak{D} > 0, \\ \mathcal{T}_3 =& (k||A|| + ||O|| + k||B||) + k||B||\left[k||A|| + ||O|| + k||B||(1-\vartheta)^{-1}\right]\breve{\varrho}, \\ \mathcal{T}_4 =& k||B||[\breve{\varrho} + \breve{\varrho}(1-\vartheta)^{-1} + k||B||\breve{\varrho}^2(1-\vartheta)^{-1}] + k\breve{\varrho}||B||[k||A|| + ||O|| \\ &+ k||B||(1-\vartheta)^{-1}]\mathfrak{L}/\mathfrak{D}. \end{aligned}$$

Proof. Similar to Theorem 1, from (8) and (10), we have

$$\begin{aligned} ||l(t)-q(t)|| \leq & \int_{t_0}^{t} k||A||\,||l(s)-q(s)|| + k||B||\,||l(s)-q(s-\varrho(s))|| \\ & + ||O||\,||l(s)-q(s)||\,ds \\ = & \int_{t_0}^{t} (k||A||+||O||)\,||l(s)-q(s)|| + k||B||\,||l(s)-q(s-\varrho(s))||\,ds \\ \leq & (k||A||+k||B||+||O||)\int_{t_0}^{t} ||l(s)-q(s)||\,ds \\ & + k||B||\int_{t_0}^{t} ||q(s)-q(s-\varrho(s))||\,ds. \end{aligned} \tag{30}$$

In addition, when $t \geq t_0+\tilde{\varrho}$, from (28) and (5),

$$\begin{aligned} \int_{t_0+\tilde{\varrho}}^{t} ||q(s)-q(s-\varrho(s))||\,ds \leq \int_{t_0+\tilde{\varrho}}^{t} ds \int_{s-\tilde{\varrho}}^{s} & (k||A||+||O||)||q(r)|| \\ & + k||B||\,||q(r-\varrho(r))||\,dr. \end{aligned} \tag{31}$$

Similarly to (15),

$$\begin{aligned} \int_{t_0+\tilde{\varrho}}^{t} ds \int_{s-\tilde{\varrho}}^{s} (k||A||+||O||)\,||q(r)||dr = & \int_{t_0}^{t} dr \int_{\max(t_0+\tilde{\varrho},s)}^{\min(s+\tilde{\varrho},t)} (k||A||+||O||)\,||q(r)||ds \\ \leq & \ (k||A||+||O||)\tilde{\varrho}\int_{t_0}^{t} ||q(r)||dr. \end{aligned} \tag{32}$$

Similarly to (16),

$$\begin{aligned} & \int_{t_0+\tilde{\varrho}}^{t} ds \int_{s-\tilde{\varrho}}^{s} k||B||\,||p(r-\varrho(r))||dr \\ & \qquad = \int_{t_0}^{t} dr \int_{\max(t_0+\tilde{\varrho},s)}^{\min(s+\tilde{\varrho},t)} k||B||\,||q(r-\varrho(r))||ds \\ & \qquad \leq k||B||\tilde{\varrho}(1-\vartheta)^{-1}\bigg(\tilde{\varrho}\sup_{t_0-\tilde{\varrho}\leq s\leq t_0} ||q(s)|| + \int_{t_0}^{t} ||q(u)||\,du\bigg). \end{aligned} \tag{33}$$

When $t \geq t_0+\tilde{\varrho}$, substitute (32) and (33) into (31):

$$\begin{aligned} \int_{t_0+\tilde{\varrho}}^{t} ||q(s)-q(s-\varrho(s))||ds \leq & k||B||\tilde{\varrho}^2(1-\vartheta)^{-1}\bigg(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0} ||q(s)||\bigg) \\ & + (k||A||+||O||+k||B||\tilde{\varrho}(1-\vartheta)^{-1})\int_{t_0}^{t} ||q(s)||\,ds. \end{aligned} \tag{34}$$

For $t \geq t_0+\tilde{\varrho}$, substitute (34) into (30):

$$\begin{aligned} ||l(s)-q(s)|| \leq & (k||A||+||O||+k||B||)\int_{t_0}^{t} ||l(s)-q(s)||\,ds \\ & + k||B||\big[\tilde{\varrho}+\tilde{\varrho}(1-\vartheta)^{-1}\big]\bigg(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0+\tilde{\varrho}} ||q(s)||\bigg) \\ & + k^2||B||^2\tilde{\varrho}^2(1-\vartheta)^{-1}\bigg(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0} ||q(s)||\bigg) \\ & + k||B||\Big[k||A||+||O||+k||B||(1-\vartheta)^{-1}\Big]\tilde{\varrho}\int_{t_0}^{t} ||q(s)||\,ds. \end{aligned} \tag{35}$$

By Definition 1, one has

$$\begin{aligned}
||l(s)-q(s)|| \leq & k||A||+||O||+k||B||+k||B||\Big[k||A||+||O|| \\
& +k||B||(1-\vartheta)^{-1}\Big]\tilde{\varrho}\int_{t_0}^{t}||l(s)-q(s)||\,ds \\
& +k||B||\big[\tilde{\varrho}+\tilde{\varrho}(1-\vartheta)^{-1}\big]\Big(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0+\tilde{\varrho}}||q(s)||\Big) \\
& +k^2||B||^2\tilde{\varrho}^2(1-\vartheta)^{-1}\Big(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0}||q(s)||\Big) \\
& +k||B||\Big[k||A||+||O||+k||B||(1-\vartheta)^{-1}\Big]\tilde{\varrho}\int_{t_0}^{t}||l(s)||\,ds \\
\leq & \mathcal{T}_1\int_{t_0}^{t}||l(s)-q(s)||\,ds+\mathcal{T}_2\Big(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0+\tilde{\varrho}}||q(s)||\Big),
\end{aligned} \tag{36}$$

where

$$\begin{aligned}
\mathcal{T}_1 =&(k||A||+||O||+k||B||)+k||B||\big[k||A||+||O||+k||B||(1-\vartheta)^{-1}\big]\tilde{\varrho}, \\
\mathcal{T}_2 =&k||B||[\tilde{\varrho}+\tilde{\varrho}(1-\vartheta)^{-1}+k||B||\tilde{\varrho}^2(1-\vartheta)^{-1}]+k\tilde{\varrho}||B||[k||A||+||O|| \\
&+k||B||(1-\vartheta)^{-1}]\mathfrak{L}/\mathfrak{D}.
\end{aligned}$$

For $t\leq t_0+2\Delta$, using the Gronwall–Bellman lemma [42],

$$||l(t)-q(t)||\leq \mathcal{T}_2\exp(2\Delta\mathcal{T}_1)\Big(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0+\tilde{\varrho}}||q(s)||\Big).$$

Due to $\tilde{\varrho}<\min(\Delta/2,\breve{\varrho})$, and $t_0-\tilde{\varrho}+\Delta\leq t\leq t_0-\tilde{\varrho}+2\Delta$,

$$\begin{aligned}
||q(t)|| \leq&\big[\mathfrak{L}\exp(-\mathfrak{D}(\Delta-\tilde{\varrho}))+\mathcal{T}_2\exp(2\Delta\mathcal{T}_1)\big]\Big(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0+\tilde{\varrho}}||q(s)||\Big) \\
=&\hat{\mathcal{H}}\Big(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0+\tilde{\varrho}}||q(s)||\Big),
\end{aligned} \tag{37}$$

where $\hat{\mathcal{H}}=\mathfrak{L}\exp(-\mathfrak{D}(\Delta-\tilde{\varrho}))+\mathcal{T}_2\exp(2\Delta\mathcal{T}_1)$. From (29), when $\tilde{\varrho}<\min(\Delta/2,\breve{\varrho})$, $\hat{\mathcal{H}}<1$. Select

$$\mathfrak{C}=-\ln\hat{\mathcal{H}}/\Delta.$$

From (37), one has

$$\sup_{t_0-\tilde{\varrho}+\Delta\leq t\leq t_0-\tilde{\varrho}+2\Delta}||q(t;t_0,\psi)||\leq\exp(-\mathfrak{C}\Delta)\Big(\sup_{t_0-\tilde{\varrho}\leq t\leq t_0-\tilde{\varrho}+\Delta}||q(t;t_0,\psi)||\Big). \tag{38}$$

The following proof is similar to Theorem 1. □

Remark 2. *Theorem 2 shows that CDNs achieve ESy under nonlinear controllers. When the time delays does not exceed the deduced range, the CDNs (1) and (2) are still exponentially synchronized.*

Remark 3. *Theorems 1 and 2 show that CDNs achieve ESy under the controllers. When the time delays of external interference is within the range we deduced, that is, $\tilde{\varrho}<\min(\Delta/2,\breve{\varrho})$, CDNs still remain ESy.*

3.2. Effect of Time Delays and Random Disturbances on CDN Synchronization

Consider the following CDN model consisting of m nodes with time delays and random disturbances:

$$\begin{cases} dz_i(t) = \Big[\Theta(z_i(t),t) + k\sum_{j=1}^{m} a_{ij}z_j(t) + c_i(t) + k\sum_{j=1}^{m} b_{ij}z_j(t-\varrho(t))\Big]dt \\ \qquad\qquad + \delta\sum_{j=1}^{m} v_{ij}z_j(t)d\mathrm{B}(t), \\ z_i(t_0) = z_{i0} \in \mathfrak{R}^m, \end{cases} \tag{39}$$

where δ is the random interference intensity, and v_{ij} represents the coupling matrix $\mathcal{V}$, satisfying $v_{ii} = -\sum_{j=1, j\neq i}^{m} v_{ij}$.

When CDN (39) is not interfered by time delays and external noise, system (39) will become the following form:

$$\begin{cases} \dot{\hat{z}}_i(t) = \Theta(\hat{z}_i(t),t) + k\sum_{j=1}^{m} a_{ij}\hat{z}_j(t) + k\sum_{j=1}^{m} b_{ij}\hat{z}_j(t) + c_i(t), \\ \hat{z}_i(t_0) = \hat{z}_{i0} \in \mathfrak{R}^m. \end{cases} \tag{40}$$

Similarly, (2) is defined as the isolated node equation of CDN (39) and (40).

Similar to system (4), we have

$$\begin{cases} \dot{l}_i(t) = \Upsilon(l_i(t),t) + k\sum_{j=1}^{m} a_{ij}l_j(t) + k\sum_{j=1}^{m} b_{ij}l_j(t) + c_i(t), \\ l_i(t_0) = l_{i0} \in \mathfrak{R}^m, \end{cases} \tag{41}$$

where $\Upsilon(l_i(t),t)$ is defined in (4).

Substituting the linear controller (6) into (41), we have

$$\begin{cases} \dot{l}_i(t) = \Upsilon(l_i(t),t) + k\sum_{j=1}^{m} a_{ij}l_j(t) + k\sum_{j=1}^{m} b_{ij}l_j(t) + \check{S}l_i(t), \\ l_i(t_0) = l_{i0} \in \mathfrak{R}^m. \end{cases} \tag{42}$$

The matrix form of system (42) is

$$\begin{cases} \dot{l}(t) = \Upsilon(l(t),t) + kAl(t) + kBl(t) + Sl(t), \\ l(t_0) = l_0 \in \mathfrak{R}^{m\times m}. \end{cases} \tag{43}$$

Similar to system (9), we have

$$\begin{cases} dq(t) = \big[\Upsilon(q(t),t) + kAq(t) + kBq(t-\varrho(t)) + Sq(t)\big]dt + \delta Vq(t)d\mathrm{B}(t), \\ q(t_0) = q_0 \in \mathfrak{R}^{m\times m}, \end{cases} \tag{44}$$

where $V = I_m \otimes \mathcal{V}$, $\mathcal{V} = (v_{ij})_{m\times m}$.

For the error system (39), we have the following definitions.

Definition 2 ([42,43])**.** *CDN (44) is called to be ASESt, if* $\forall t_0 \in \mathfrak{R}^+$, $q_0 \in \mathfrak{R}^m$, *the Lyapunov exponent*

$$\limsup_{t\to\infty} \frac{1}{t}(\ln ||q_i(t;t_0,q_0)||) < 0, \ i = 1,2,\ldots,m,$$

then, the CDN (39) with (2) is called to be ASESy. That is, if $\exists M > 0$*, and the* $\wp > 0, \varsigma > 0$*, such that*

$$||q_i(t)|| \leq \daleth \exp(-\wp\varsigma(t-t_0)),\ i = 1, 2, \ldots, m,$$

where $\daleth > 0$ *is constant.*

Definition 3 ([29,42])**.** *CDN (44) is called to be MSESt, if* $\forall t_0 \in \Re^+$*,* $q_0 \in \Re^m$*, the Lyapunov exponent*

$$\limsup_{t\to\infty}(\ln \frac{1}{t}(E||q_i(t;t_0,q_0)||^2)) < 0,\ i = 1, 2, \ldots, m,$$

then, the CDN (39) with (2) is called to be MSESy, That is, if $\exists G > 0$*, and the* $\wp > 0, \varsigma > 0$*, such that*

$$E||l_i(t)||^2 \leq \daleth_1 \exp(-\wp\varsigma(t-t_0)),\ i = 1, 2, \ldots, m,$$

where $\daleth_1 > 0$ *is constant.*

From Definitions 2 and 3, MSESt can be derived from ASESt, but vice versa is not trues. However, if (5) is assumed to hold, the MSESt of (44) implies the ASESt of the CDN (44) [42].

3.2.1. The Linear Controller

In this section, we will analyze the effects of time delays and random disturbance on ESy of CDNs under linear controllers. For CDNs that have reached ESy, how much external delay and random noise interference can the system withstand without losing synchronization? In other words, if the disturbance of the system by external delays and random noise is within the range of our estimation, the CDNs are still ESy.

Theorem 3. *If assumption* **H1** *is trues, and (43) is ESt, then (44) is ESt. That is, systems (39) and (2) can realize ESy under (6), if* $|\delta| \leq \tilde{\delta}, \tilde{\varrho} < \min(\Delta/2, \breve{\varrho})$*, where* $\tilde{\delta}$ *is the only nonnegative solution of the following transcendental equation.*

$$\begin{aligned} &2\mathfrak{L}^2\exp(-2\mathfrak{D}\Delta) + 4\mathfrak{L}^2\tilde{\delta}^2||V||^2/\mathfrak{D}\exp\Big\{8\Delta\Big[4\Delta(||\Phi||^2 \\ &\quad + k^2||A||^2 + k^2||B||^2 + ||S||^2) + \tilde{\delta}||V||^2\Big]\Big\} = 1, \end{aligned} \tag{45}$$

and $\breve{\varrho}$ *is the only nonnegative solution of the following transcendental equation.*

$$2\mathcal{N}_2\exp(\Delta\mathcal{N}_1) + 2\mathfrak{L}^2\exp(-2\mathfrak{D}(\Delta-\tilde{\varrho})) = 1, \tag{46}$$

where

$$\begin{aligned} &\Delta > \ln(2\mathfrak{L}^2)/(2\mathfrak{D}), \\ &\mathcal{N}_1 = 16\Delta(||\Phi||^2 + k^2||A||^2 + 2k^2||B||^2 + ||S||^2) + 2\tilde{\delta}^2||V||^2 + 64\Delta k^2||B||^2\Big(8\tilde{\varrho}^2\Big[||\Phi||^2 \\ &\qquad + k^2||A||^2 + ||S||^2 + k^2||B||^2(1-\vartheta)^{-1}\Big] + 2\tilde{\varrho}\tilde{\delta}^2||V||^2\Big), \\ &\mathcal{N}_2 = 64\Delta k^2||B||^2\Big[\tilde{\varrho} + \tilde{\varrho}(1-\vartheta)^{-1}\Big] + 2\Delta k^2||B||^2\Big[8\tilde{\varrho}^3k^2||B||^2(1-\vartheta)^{-1} + \Big(8\tilde{\varrho}^2\Big[||\Phi||^2 \\ &\qquad + k^2||A||^2 + ||S||^2 + k^2||B||^2(1-\vartheta)^{-1}\Big] \\ &\qquad + 2\tilde{\varrho}\tilde{\delta}^2||V||^2\Big)||V||^2\mathfrak{L}^2/\mathfrak{D}\Big] + 2\tilde{\delta}^2\mathfrak{L}^2/\mathfrak{D}. \end{aligned}$$

Proof. Similar to Theorem 1, from (43) and (44), we have

$$l(t)-q(t)=\int_{t_0}^{t}\Big[(\Upsilon(l(s),s)-\Upsilon(q(s),s))+kA(l(s)-q(s))+kB(l(s)-q(s-\varrho(s)))+S(l(s)-q(s))\Big]\,ds-\int_{t_0}^{t}\delta Vq(s)\,d\mathsf{B}(s). \tag{47}$$

For $t\le t_0+\Delta$, based on the *Hölder* inequality, combine (5) and the ESt of (43):

$$\begin{aligned}E||l(t)-q(t)||^2\le&2E\bigg|\bigg|\int_{t_0}^{t}(\Upsilon(l(s),s)-\Upsilon(q(s),s))+kA(l(s)-q(s))\\&+kB(l(s)-q(s-\varrho(s)))+S(l(s)-q(s))\,ds\bigg|\bigg|^2\\&+2E\bigg|\bigg|\int_{t_0}^{t}\delta Vq(s)\,d\mathsf{B}(s)\bigg|\bigg|^2\\\le&[16\Delta(||\Phi||^2+k^2||A||^2+2k^2||B||^2+||S||^2)+4\delta^2||V||^2]\\&\times\int_{t_0}^{t}E||l(s)-q(s)||^2ds\\&+2\Delta k^2||B||^2\int_{t_0}^{t}E||q(s)-q(s-\varrho(s))||^2\,ds\\&+4\delta^2||V||^2\int_{t_0}^{t}E||l(s)||^2\,ds.\end{aligned} \tag{48}$$

In addition, when $t\ge t_0+\tilde{\varrho}$, from (44) and (5), one has

$$\begin{aligned}\int_{t_0+\tilde{\varrho}}^{t}E||q(s)-q(s-\varrho(s))||^2ds\le\int_{t_0+\tilde{\varrho}}^{t}ds\int_{s-\tilde{\varrho}}^{s}\bigg\{\Big[&8\tilde{\varrho}(||\Phi||^2+k^2||A||^2\\&+||S||^2)+2\delta^2||V||^2\Big]E||q(r)||^2\\&+8\tilde{\varrho}k^2||B||^2E||q(r-\varrho(r))||^2\bigg\}dr.\end{aligned} \tag{49}$$

Like (14) and (15), combined with (49),

$$\begin{aligned}\int_{t_0+\tilde{\varrho}}^{t}E||q(s)-q(s-\varrho(s))||^2ds\le&8\tilde{\varrho}^3k^2||B||^2(1-\vartheta)^{-1}\Big(\sup_{t_0-\tilde{\varrho}\le s\le t_0}E||q(s)||^2\Big)\\&+\bigg\{8\tilde{\varrho}^2\Big[||\Phi||^2+k^2||A||^2+||S||^2\\&+k^2||B||^2(1-\vartheta)^{-1}\Big]+2\tilde{\varrho}\delta^2||V||^2\bigg\}\\&\times\int_{t_0}^{t}E||q(s)||^2ds.\end{aligned} \tag{50}$$

Substituting (50) into (48),

$$\begin{aligned}E||l(t)-q(t)||^2 \leq & \big[16\Delta(||\Phi||^2+k^2||A||^2+2k^2||B||^2+||S||^2)+4\delta^2||V||^2\big] \\ & \times \int_{t_0}^{t} E||l(s)-q(s)||^2 ds \\ & + 64\Delta k^2||B||^2\left[\tilde{\varrho}+\tilde{\varrho}(1-\vartheta)^{-1}\right]\left(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0+\tilde{\varrho}} E||q(s)||^2\right) \\ & + 2\Delta k^2||B||^2\Bigg\{8\tilde{\varrho}^3k^2||B||^2(1-\vartheta)^{-1}\left(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0} E||q(s)||^2\right) \\ & + \Bigg(8\tilde{\varrho}^2\Big[||\Phi||^2+k^2||A||^2+||S||^2 \\ & + k^2||B||^2(1-\vartheta)^{-1}\Big]+2\tilde{\varrho}\delta^2||V||^2\Bigg) \\ & \times \int_{t_0}^{t} E||l(s)-q(s)+l(s)||^2 ds\Bigg\} \\ & + 2\delta^2||V||^2\mathfrak{L}^2/\mathfrak{D}\left(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0} E||q(s)||^2\right).\end{aligned} \tag{51}$$

Further obtained by (51),

$$E||l(t)-q(t)||^2 \leq \mathcal{N}_3\int_{t_0}^{t} E||l(s)-q(s)||^2 ds+\mathcal{N}_4\left(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0+\tilde{\varrho}} E||q(s)||^2\right), \tag{52}$$

where

$$\begin{aligned}\mathcal{N}_3 = & 16\Delta(||\Phi||^2+k^2||A||^2+2k^2||B||^2+||S||^2)+4\delta^2||V||^2 \\ & + 64\Delta k^2||B||^2\left(8\tilde{\varrho}^2\left[||\Phi||^2+k^2||A||^2+||S||^2+k^2||B||^2(1-\vartheta)^{-1}\right]+2\tilde{\varrho}\delta^2||V||^2\right), \\ \mathcal{N}_4 = & 64\Delta k^2||B||^2\left[\tilde{\varrho}+\tilde{\varrho}(1-\vartheta)^{-1}\right]+2\Delta k^2||B||^2\times\Bigg[8\tilde{\varrho}^3k^2||B||^2(1-\vartheta)^{-1} \\ & + \Bigg(8\tilde{\varrho}^2\left[||\Phi||^2+k^2||A||^2+||S||^2+k^2||B||^2(1-\vartheta)^{-1}\right] \\ & + 2\tilde{\varrho}\delta^2||V||^2\Bigg)\mathfrak{L}^2/\mathfrak{D}\Bigg]+2\delta^2||V||^2\mathfrak{L}^2/\mathfrak{D}.\end{aligned}$$

For $t+\tilde{\varrho}\leq t_0+\Delta$, using the Gronwall–Bellman lemma [42],

$$E||l(t)-q(t)||^2 \leq \mathcal{N}_4\exp(\Delta\mathcal{N}_3)\left(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0+\tilde{\varrho}} E||q(s)||^2\right). \tag{53}$$

Hence,

$$\begin{aligned}E||q(t)||^2 \leq & 2E||l(t)-q(t)||^2+2E||l(t)||^2 \\ \leq & \left[2\mathcal{N}_4\exp(\Delta\mathcal{N}_3)+2\mathfrak{L}^2\exp(-2\mathfrak{D}(t-t_0))\right]\left(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0+\tilde{\varrho}} E||q(s)||^2\right).\end{aligned} \tag{54}$$

When $t_0 - \tilde{\varrho} + \Delta \leq t \leq t_0 - \tilde{\varrho} + \Delta$,

$$\begin{aligned} E||q(t)||^2 \leq& \Big[2\mathcal{N}_4 \exp(2\Delta\mathcal{N}_3) + 2\mathfrak{L}^2 \exp(-2\mathfrak{D}(\Delta - \tilde{\varrho}))\Big] \Bigg(\sup_{t_0 - \tilde{\varrho} \leq s \leq t_0 - \tilde{\varrho} + \Delta} E||q(s)||^2 \Bigg) \\ =:& \hbar(\delta, \tilde{\varrho}) \Bigg(\sup_{t_0 - \tilde{\varrho} \leq s \leq t_0 - \tilde{\varrho} + \Delta} E||q(s)||^2 \Bigg), \end{aligned} \tag{55}$$

where $\hbar(\delta, \tilde{\varrho}) = 2\mathcal{N}_4 \exp(\Delta\mathcal{N}_3) + 2\mathfrak{L}^2 \exp(-2\mathfrak{D}(\Delta - \tilde{\varrho}))$.

Due to $\hbar(0,0) < 1$, $\hbar(\infty, 0) > 1$ and $\hbar(\delta, 0)$ is strictly monotonically increasing with respect to δ, there is a only solution $\tilde{\delta}$ such that $\hbar(\tilde{\delta}, 0) = 1$, that is (45) is trues. When $|\delta| < \tilde{\delta}$, $\hbar(\delta, \infty) > 1$, $\hbar(\delta, 0) < 1$, and $\hbar(\delta, \tilde{\varrho})$ is strictly monotonically increasing with respect to $\tilde{\varrho}$, there is a only $\breve{\varrho}$, such that $\hbar(\tilde{\delta}, \breve{\varrho}) = 1$, that is, (46) is trues. From (45) and (46), for $|\delta| < \tilde{\delta}$ and $\tilde{\varrho} < \min(\Delta/2, \breve{\varrho})$, then $\hbar(\delta, \tilde{\varrho}) < 1$.

Choose

$$\eth = -\ln \hbar(\delta, \tilde{\varrho})/\Delta,$$

From (55), we have

$$\sup_{t_0 - \tilde{\varrho} + \Delta \leq t \leq t_0 - \tilde{\varrho} + \Delta} E|q(t; t_0, \psi)|^2 \leq \exp(-\eth\Delta) \Bigg(\sup_{t_0 - \tilde{\varrho} \leq t \leq t_0 - \tilde{\varrho} + \Delta} E|q(t; t_0, \psi)|^2 \Bigg). \tag{56}$$

The rest of the proof is exactly like Theorem 1. □

Remark 4. *According to Theorem 3, the CDN can realize ESy under (6). When the delays and random disturbances intensity are satisfied, $|\delta| < \tilde{\delta}$ and $\tilde{\varrho} < \min(\Delta/2, \breve{\varrho})$, that is, the CDN is still ESy within the range of time delays and noise intensity.*

3.2.2. The Nonlinear Controller

In the previous section, we discussed the effect of time delays and random disturbances on the ESy of CDNs under linear controllers. In this section, we will explore the influence of nonlinear controllers on the ESy of CDNs.

To obtain synchronization of systems (39) and (2), (25) is still the nonlinear controller. Substitute (25) into (41):

$$\begin{cases} \dot{l}_i(t) =& k \sum_{j=1}^{m} a_{ij} l_j(t) + k \sum_{j=1}^{m} b_{ij} l_j(t) + \check{O} l_i(t), \\ l_i(t_0) =& l_{i0} \in \mathfrak{R}^m. \end{cases} \tag{57}$$

Written in matrix form is as follows:

$$\begin{cases} \dot{l}(t) = kAl(t) + kBl(t) + Ol(t), \\ l(t_0) = l_0 \in \mathfrak{R}^{m \times m}. \end{cases} \tag{58}$$

Similarly, subtract (2) from (39) and write in matrix form

$$\begin{cases} dq(t) =& [kAq(t) + kBq(t - \varrho(t)) + Oq(t)]dt + \delta Vq(t) d\mathrm{B}(t), \\ q(t_0) =& q_0 \in \mathfrak{R}^{m \times m}, \end{cases} \tag{59}$$

the definition of ESy of the (58) is similar to Definition 1.

The following theorem describes how much time delay and random disturbance a CDN can withstand under a nonlinear controller that is ESy.

Theorem 4. *If Assumption* **H1** *is trues, and (58) is ESt, then (59) is ESt. That is, systems (39) and (2) can realize ESy under (25), if* $|\delta| \leq \tilde{\delta}, \tilde{\varrho} < \min(\Delta/2, \breve{\varrho})$, *where* $\tilde{\delta}$ *is the only solution to the following transcendental equation.*

$$2\mathfrak{L}^2 \exp(-2\mathfrak{D}\Delta) + 4\mathfrak{L}^2\tilde{\delta}^2||B||^2/\mathfrak{D} \exp\Bigg\{2\Delta\Bigg[6\Delta (k^2||A||^2 + 2k^2||B||^2 + ||O||^2) + 2\tilde{\delta}||B||^2\Bigg]\Bigg\} = 1, \tag{60}$$

and $\breve{\varrho}$ *is the only solution to the following transcendental equation.*

$$2\mathcal{N}_6 \exp(2\Delta\mathcal{N}_5) + 2\mathfrak{L}^2 \exp(-2\mathfrak{D}(\Delta - \tilde{\varrho})) = 1, \tag{61}$$

where

$$\begin{aligned}
&\Delta > \ln(2\mathfrak{L}^2)/(2\mathfrak{D}),\\
\mathcal{N}_5 =& 12\Delta(k^2||A||^2 + 2k^2||B||^2 + ||O||^2) + 2\tilde{\delta}^2||V||^2\\
&+ 48\Delta k^2||B||^2\Bigg(6\tilde{\varrho}^2\Bigg[k^2||A||^2 + ||S||^2 + k^2||B||^2(1-\vartheta)^{-1}\Bigg] + 2\tilde{\varrho}\tilde{\delta}^2||V||^2\Bigg),\\
\mathcal{N}_6 =& 48\Delta k^2||B||^2\Bigg[\tilde{\varrho} + \tilde{\varrho}(1-\vartheta)^{-1}\Bigg] + 24\Delta k^2||B||^2\Bigg[4\tilde{\varrho}^3k^2||B||^2(1-\vartheta)^{-1}\\
&+ \Bigg(4\tilde{\varrho}^2\Bigg[k^2||A||^2 + ||O||^2 + k^2||B||^2(1-\vartheta)^{-1}\Bigg]\\
&+ 2\tilde{\varrho}\tilde{\delta}^2||V||^2\Bigg)||V||^2\mathfrak{L}^2/\mathfrak{D}\Bigg] + 2\tilde{\delta}^2||V||^2\mathfrak{L}^2/\mathfrak{D}.
\end{aligned}$$

Proof. Similar to Theorem 1, from (58) and (59), one has

$$\begin{aligned}
l(t) - q(t) = &\int_{t_0}^{t}\Bigg[kA(l(s) - q(s)) + kB(l(s) - q(s - \varrho(s))) + O(l(s) - q(s))\Bigg]ds\\
&- \int_{t_0}^{t}\delta Vq(s)\,d\mathrm{B}(s).
\end{aligned} \tag{62}$$

For $t \leq t_0 + 2\Delta$, using the *Hölder* inequality, define the ESt of system (58):

$$\begin{aligned}
E||l(t) - q(t)||^2 \leq& \big[12\Delta(k^2||A||^2 + 2k^2||B||^2 + ||O||^2) + 4\delta^2||V||^2\big]\int_{t_0}^{t}E||l(s) - q(s)||ds\\
&+ 24\Delta k^2||B||^2\int_{t_0}^{t}E||q(s) - q(s - \varrho(s))||^2\,ds\\
&+ 4\delta^2||V||^2\int_{t_0}^{t}E||l(s)||^2\,ds.
\end{aligned} \tag{63}$$

Furthermore, when $t \geq t_0 + \tilde{\varrho}$, from (59),

$$\begin{aligned}
\int_{t_0+\tilde{\varrho}}^{t}E||q(s) - q(s-\varrho(s))||^2ds \leq& \int_{t_0+\tilde{\varrho}}^{t}ds\int_{s-\tilde{\varrho}}^{s}\Bigg\{\Bigg[4\tilde{\varrho}(k^2||A||^2 + ||O||^2)\\
&+ 2\delta^2||V||^2\Bigg]E||q(r)||^2 + 4\tilde{\varrho}k^2||B||^2E||q(r-\varrho(r))||^2\Bigg\}dr.
\end{aligned} \tag{64}$$

Similarly to (14) and (15),

$$E||l(t) - q(t)||^2 \leq \mathcal{N}_7\int_{t_0}^{t}E||l(s) - q(s)||^2ds + \mathcal{N}_8\Bigg(\sup_{t_0-\tilde{\varrho}\leq s\leq t_0+\tilde{\varrho}}E||q(s)||^2\Bigg), \tag{65}$$

where

$$\mathcal{N}_7 = 12\Delta(k^2||A||^2 + 2k^2||B||^2 + ||O||^2) + 4\delta^2||V||^2 + 48\Delta k^2||B||^2\Bigg(4\tilde{\varrho}^2\Big[k^2||A||^2 + ||O||^2 + k^2||B||^2(1-\vartheta)^{-1}\Big] + 2\tilde{\varrho}\delta^2||V||^2\Bigg),$$

$$\mathcal{N}_8 = 48\Delta k^2||B||^2\Big[\tilde{\varrho} + \tilde{\varrho}(1-\vartheta)^{-1}\Big] + 24\Delta k^2||B||^2\Bigg[4\tilde{\varrho}^3k^2||B||^2(1-\vartheta)^{-1} + \Bigg(4\tilde{\varrho}^2\Big[k^2||A||^2 + ||O||^2 + k^2||B||^2(1-\vartheta)^{-1}\Big] + 2\tilde{\varrho}\delta^2||V||^2\Bigg)||V||^2\mathfrak{L}^2/\mathfrak{D}\Bigg] + 2\delta^2||V||^2\mathfrak{L}^2/\mathfrak{D}.$$

For $t + \tilde{\varrho} \le t_0 + 2\Delta$, using the Gronwall–Bellman lemma [42],

$$E||l(t) - q(t)||^2 \le \mathcal{N}_8 \exp(2\Delta\mathcal{N}_7)\Bigg(\sup_{t_0-\tilde{\varrho}\le s\le t_0+\tilde{\varrho}} E||q(s)||^2\Bigg). \quad (66)$$

Like (54), (55), and when $t_0 - \tilde{\varrho} + \Delta \le t \le t_0 - \tilde{\varrho} + 2\Delta$,

$$\begin{aligned} E||q(t)||^2 &\le \Big[2\mathcal{N}_8\exp(2\Delta\mathcal{N}_7) + 2\mathfrak{L}^2\exp(-2\mathfrak{D}(\Delta-\tilde{\varrho}))\Big]\Bigg(\sup_{t_0-\tilde{\varrho}\le s\le t_0-\tilde{\varrho}+\Delta} E||q(s)||^2\Bigg) \\ &=: \hbar_0(\delta,\tilde{\varrho})\Bigg(\sup_{t_0-\tilde{\varrho}\le s\le t_0-\tilde{\varrho}+\Delta} E||q(s)||^2\Bigg), \end{aligned} \quad (67)$$

where $\hbar_0(\delta,\tilde{\varrho}) = 2\mathcal{N}_8\exp(2\Delta\mathcal{N}_7) + 2\mathfrak{L}^2\exp(-2\mathfrak{D}(\Delta-\tilde{\varrho}))$.

Similar to Theorem 3, $\hbar_0(\delta,\tilde{\varrho})$ is strictly monotonically increasing in terms of δ and $\tilde{\varrho}$. Therefore, there exist only solutions $\tilde{\delta}$ and $\breve{\varrho}$, such that $\hbar_0(\tilde{\delta},0) = 1$, $\hbar_0(\tilde{\delta},\breve{\varrho}) = 1$, that is, (60) and (61) hold. For $|\delta| < \tilde{\delta}$ and $\tilde{\varrho} < \min(\Delta/2,\breve{\varrho})$, then $\hbar_0(\delta,\tilde{\varrho}) < 1$.

Select

$$\aleph = -\ln\hbar_0(\delta,\tilde{\varrho})/\Delta,$$

from (67),

$$\sup_{t_0-\tilde{\varrho}+\Delta\le t\le t_0-\tilde{\varrho}+2\Delta} E|q(t;t_0,\psi)|^2 \le \exp(\aleph\Delta)\Bigg(\sup_{t_0-\tilde{\varrho}\le t\le t_0-\tilde{\varrho}+\Delta} E|q(t;t_0,\psi)|^2\Bigg). \quad (68)$$

The rest of the proof is exactly like Theorem 1. □

Remark 5. *From Theorem 4, when the external delays and random interference intensity are satisfied,* $|\delta| < \tilde{\delta}$ *and* $\tilde{\varrho} < \min(\Delta/2,\breve{\varrho})$*, that is, the CDN is still ESy within the range of external delays and random interference intensity.*

Remark 6. *Theorems 3 and 4 show that CDNs achieve ESy under the controllers. As long as the external delays and the random interference are within the range of our derivation, the CDNs can still achieve ESy.*

4. Numerical

In this section, some examples are given to illustrate the correctness of the results in this paper.

Remark 7. *In Table 2, we represent random disturbances as RD, deviating argument as DAr, time-varying delays as TvD and asymptotic synchronization as AsSy, respectively.*

Table 2. Comparing other articles with this one.

	Controllers	RP	DAr	TvD	ESy	AsSy	Robustness
Shen [16]	✓	–	✓	–	–	✓	–
Zhang [23]	✓	✓	–	✓	✓	–	–
Wang [29]	✓	✓	–	–	✓	–	–
Lia [40]	✓	–	✓	–	✓	–	✓
Zhang [41]	✓	✓	–	–	✓	–	✓
This paper	✓	✓	–	✓	✓	–	✓

Example 1. *Consider CDN consisting of three nodes with time delays in the linear controller*

$$\begin{cases} \dot{\hat{z}}_i(t) = \Theta(\hat{z}_i(t), t) + k\sum_{j=1}^{3} a_{ij}\hat{z}_j(t) + k\sum_{j=1}^{3} b_{ij}\hat{z}_j(t-\varrho(t)) + c_i(t), \\ \hat{z}_i(t_0) = \hat{z}_{i0} \in \mathfrak{R}^{3\times 3}, \\ c_i(t) = \check{S}l_i(t), \\ \dot{\chi}(t) = \Theta(\chi(t), t), \end{cases} \tag{69}$$

without delay interference, system (69) becomes

$$\begin{cases} \dot{\hat{z}}_i(t) = \Theta(\hat{z}_i(t), t) + k\sum_{j=1}^{3} a_{ij}\hat{z}_j(t) + k\sum_{j=1}^{3} b_{ij}\hat{z}_j(t) + c_i(t), \\ \hat{z}_i(t_0) = \hat{z}_{i0} \in \mathfrak{R}^{3\times 3}, \\ c_i(t) = \check{S}l_i(t), \\ \dot{\chi}(t) = \Theta(\chi(t), t), \end{cases} \tag{70}$$

where $i \in 1,2,3$, define the error as $l_i(t) = \hat{z}_i(t) - \chi(t)$, and $l_i(t) = (l_{i1}(t), l_{i2}(t), l_{i3}(t))^T \in \mathfrak{R}^{3\times 3}$ is the i state vector of the CDN. $\varrho(t)$ is time delays and satisfies $\varrho(t) : [t_0, +\infty), \varrho'(t) \leq \vartheta < 1, \psi = \{\psi(s) : -\dot{\varrho} \leq s \leq 0\} \in C([-\tilde{\varrho}, 0], \mathfrak{R}^m)$. The matrix forms of system (69) and (70) are, respectively,

$$\begin{cases} \dot{l}(t) = \Upsilon(l(t), t) + kAl(t) + kBh(t-\varrho(t)) + Sl(t), \\ l(t_0) = l_0 \in \mathfrak{R}^{3\times 3}, \end{cases} \tag{71}$$

and

$$\begin{cases} \dot{l}(t) = \Upsilon(l(t), t) + kAl(t) + kBl(t) + Sl(t), \\ l(t_0) = l_0 \in \mathfrak{R}^{3\times 3}. \end{cases} \tag{72}$$

Suppose the coupling matrix is

$$A = \begin{bmatrix} -0.4 & 0.1 & 1 \\ 0.1 & -0.4 & 1 \\ 0.1 & -0.16 & -0.064 \end{bmatrix}, \quad B = \begin{bmatrix} -0.08 & 0.01 & 0.6 \\ 0.01 & -0.08 & 0.1 \\ 0.01 & 0.06 & -0.3 \end{bmatrix},$$

$$S = \begin{bmatrix} -0.006 & 0 & 0 \\ 0 & -0.006 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The coupling coefficient is $k = 1.5$. The activation function is $\Upsilon(\cdot) = \sin(\cdot)$ and satisfies that $||\Upsilon(h) - \Upsilon(p)|| \leq ||l - p||$ holds, so that means $\Phi = 1$. The initial value of the three-nodes and isolated node is $l_1 = (1, -1, 0)^T$, $l_2 = (-1, 1, 0)^T$, $l_3 = (1, -1, 1)^T$ and $\chi(t) = (-0.3, 0.3, 0)^T$. When $\mathfrak{L} = 2$, $\mathfrak{D} = 2.2$, the error system (71) is ESt.

Choose $\Delta = 0.4$, $\vartheta = 0$, *through calculation* $\mathcal{L}_3 = 0.6658$, $\mathcal{L}_4 = 0.0236$. *Hence, from (10), we can obtain* $\breve{\varrho} = 0.0604$, *and solve the following transcendental equation by MATLAB:*

$$2\exp(-2.2\times(0.4-\breve{\varrho})) + 0.0236\exp(2\times 0.4\times 0.6658) = 1. \tag{73}$$

Thus, by Theorem 1, when $\tilde{\varrho} < \min(\Delta/2, \breve{\varrho})$, *the system (71) is ESt. In Figure 1, we take* $\breve{\varrho} = 0.0280 < 0.0604$; *the state in Figure 1 is ESt, That is, systems (1) and (2) are ESy. In addition, Figure 2 shows the state of the system without delay interference under a linear controller.*

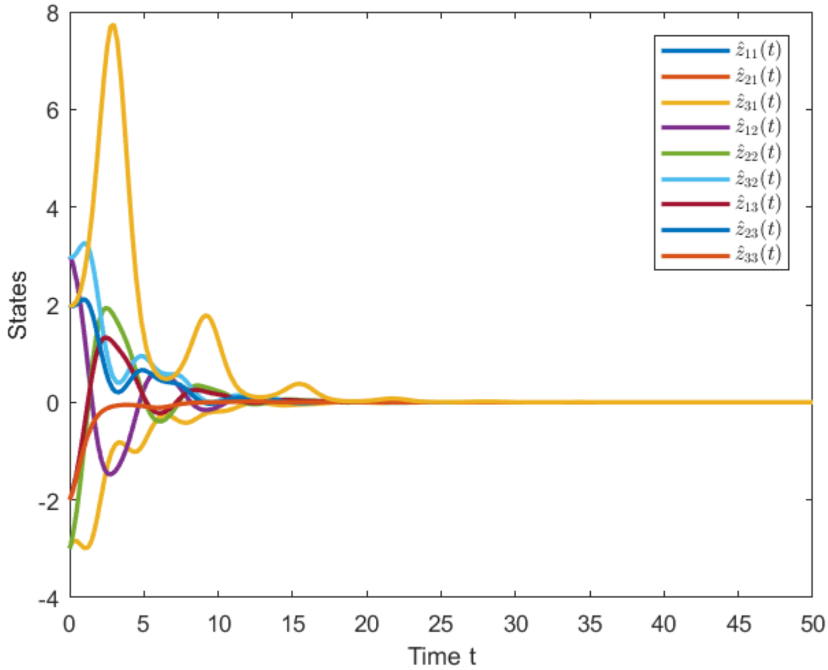

Figure 1. Convergence state of CDN (71) $\breve{\varrho} = 0.0280$ under linear controller.

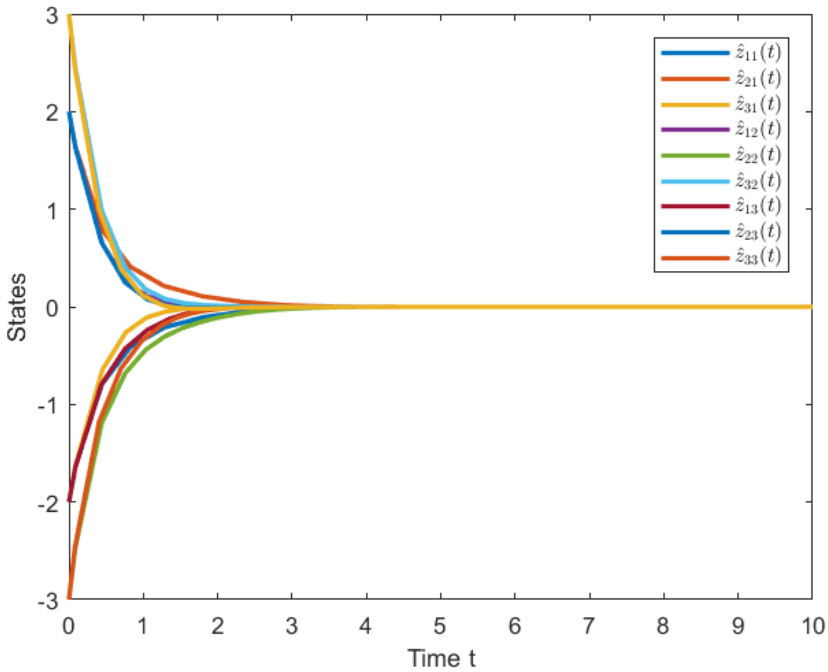

Figure 2. The state of CDN (72) without time delays under the linear controller.

In Figure 3, we select $\breve{\varrho} = 0.0350 < 0.0604$*; the state in Figure 3 is not ESt, that is, systems (1) and (2) are not ESy.*

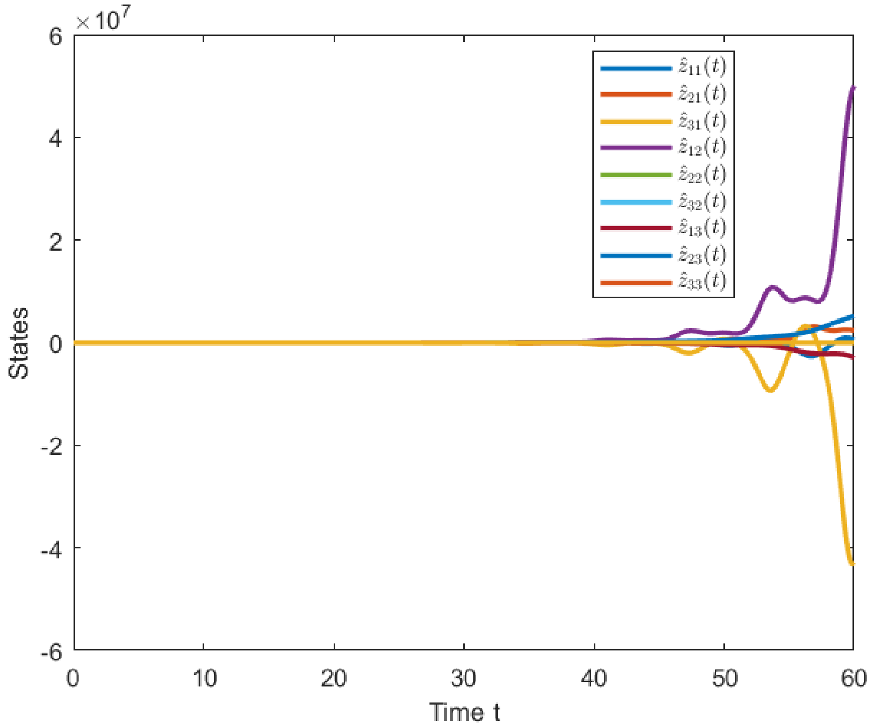

Figure 3. The state of CDN (71) at $\breve{\varrho} = 0.0350$ under linear controller.

Example 2. *A CDN consisting of three nodes with time delays under the nonlinear controller is considered:*

$$\begin{cases} \dot{\hat{z}}_i(t) = \Theta(\hat{z}_i(t), t) + k \sum_{j=1}^{3} a_{ij} \hat{z}_j(t) + k \sum_{j=1}^{3} b_{ij} \hat{z}_j(t - \varrho(t)) + c_i(t), \\ \hat{z}_i(t_0) = \hat{z}_{i0} \in \mathfrak{R}^{3\times 3}, \\ c_i(t) = -\Theta(l_i(t), t) + \Theta(\chi(t), t) + \breve{O} l_i(t), \\ \dot{\chi}(t) = \Theta(\chi(t), t). \end{cases} \tag{74}$$

Without delay interference, system (74) becomes

$$\begin{cases} \dot{\hat{z}}_i(t) = \Theta(\hat{z}_i(t), t) + k \sum_{j=1}^{3} a_{ij} \hat{z}_j(t) + k \sum_{j=1}^{3} b_{ij} \hat{z}_j(t) + c_i(t), \\ \hat{z}_i(t_0) = \hat{z}_{i0} \in \mathfrak{R}^{3\times 3}, \\ c_i(t) = -\Theta(l_i(t), t) + \Theta(\chi(t), t) + \breve{O} l_i(t), \\ \dot{\chi}(t) = \Theta(\chi(t), t), \end{cases} \tag{75}$$

where $i \in 1, 2, 3$*, let the synchronization error be*

$$l_i(t) = \hat{z}_i(t) - \chi(t),$$

and $l_i(t) = (l_{i1}(t), l_{i2}(t), l_{i3}(t))^T \in \mathfrak{R}^{3\times 3}$ *is the i state vector of the CDN.* $\varrho(t)$ *is the time delays and satisfies* $\varrho(t) : [t_0, +\infty), \varrho'(t) \leq \vartheta < 1, \psi = \{\psi(s) : -\dot{\varrho} \leq s \leq 0\} \in C([-\tilde{\varrho}, 0], \mathfrak{R}^m)$*.*

The error system is obtained from system (74) and system (75) and expressed in matrix form as

$$\begin{cases} \dot{l}(t) = kAl(t) + kBh(t - \varrho(t)) + Ol(t), \\ l(t_0) = l_0 \in \Re^{3\times 3}, \end{cases} \tag{76}$$

and

$$\begin{cases} \dot{l}(t) = kAl(t) + kBl(t) + Ol(t), \\ l(t_0) = l_0 \in \Re^{3\times 3}. \end{cases} \tag{77}$$

Suppose the coupling coefficient is $k = 1.5$. *The coupling matrix is*

$$A = \begin{bmatrix} -0.750 & 0.001 & 0 \\ 0.001 & 0.750 & 0.1 \\ 0 & -0.8 & 0.01 \end{bmatrix}, \; B = \begin{bmatrix} -0.8 & 0.02 & 0 \\ 0.02 & -0.8 & 0.1 \\ 0 & 0 & -0.8 \end{bmatrix},$$

$$O = \begin{bmatrix} -1.105 & 0 & 0 \\ 0 & -1.105 & 0 \\ 0 & 0 & -1.105 \end{bmatrix}.$$

The activation function is $\Upsilon(\cdot) = \sin(\cdot)$. *The initial value of the three nodes and isolated node is* $l_1 = (1, 0, -1)^T$, $l_2 = (-1, 1, 0)^T$, $l_3 = (-1, 1, 1)^T$ *and* $\chi(t) = (-0.1, 0.1, 0.1)^T$. *When* $\mathfrak{L} = 1.4$, $\mathfrak{D} = 1.2$, *the system (77) is ESt.*

Choose $\Delta = 0.3$, $\vartheta = 0$. *By calculation* $\mathcal{T}_3 = 0.3559$, $\mathcal{T}_4 = 0.1147$. *Hence, from (29), we have* $\breve{\varrho} = 0.2804$, *and solve the following transcendental equation by MATLAB:*

$$1.4 \exp(-1.2 \times (0.3 - \breve{\varrho})) + 0.1147 \exp(3 \times 0.3 \times 0.3559) = 1. \tag{78}$$

According to Theorem 2, when $\tilde{\varrho} < \min(\Delta/2, \breve{\varrho})$, *the system (77) is ESt. In Figure 4, we take* $\breve{\varrho} = 0.2250 < 0.2804$; *the state in Figure 4 is ESt, which means that systems (1) and (2) are ESy. In addition, Figure 5 shows the state of the system (77) without delay interference under a nonlinear controller.*

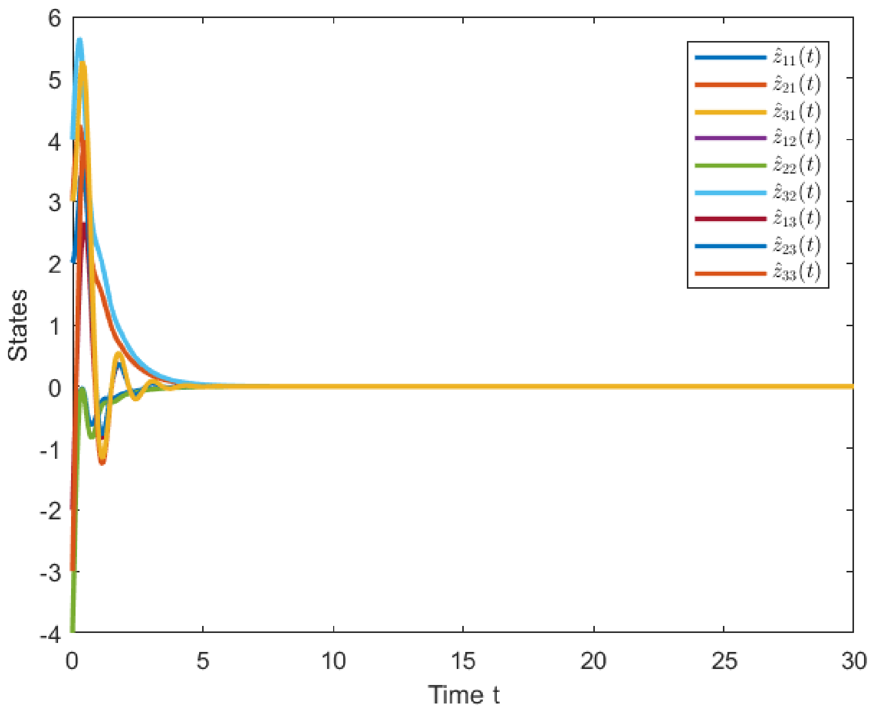

Figure 4. Convergence state of CDN (76) $\breve{\varrho} = 0.2250$ under linear controller.

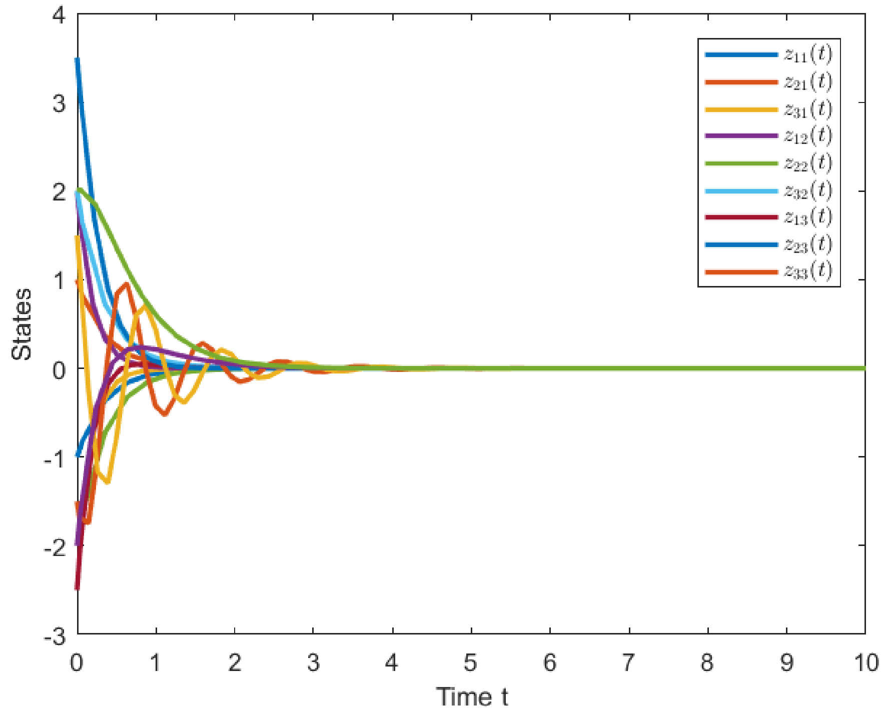

Figure 5. The state of CDN (77) without time delays under the linear controller.

In Figure 6, we select $\check{\varrho} = 0.0350 < 0.0604$*; the state in Figure 6 is not ESt, which means that systems (1) and (2) are not ESy.*

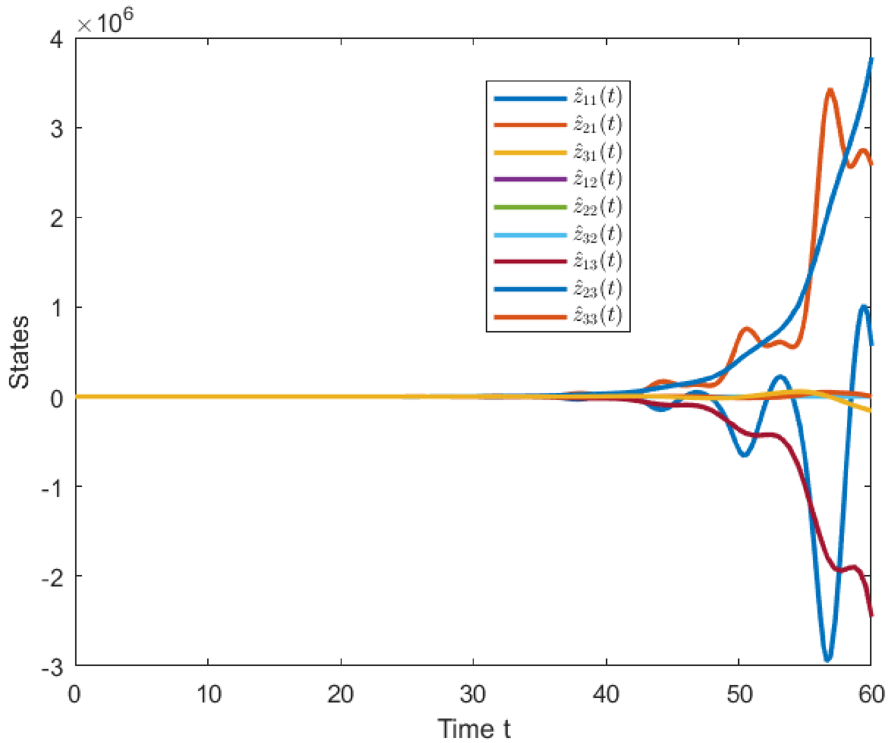

Figure 6. The state of CDN (76) at $\check{\varrho} = 0.3300$ under linear controller.

Example 3. *Consider CDN model consisting of three nodes with time delays and random disturbances under the linear controller*

$$\begin{cases} dz_i(t) = \Big[\Theta(z_i(t),t) + k\sum_{j=1}^{3} a_{ij}z_j(t) + k\sum_{j=1}^{3} b_{ij}z_j(t-\varrho(t)) + c_i(t)\Big]dt \\ \qquad\qquad + \delta\sum_{j=1}^{3} v_{ij}z_j(t)d\mathsf{B}(t), \\ z_i(t_0) = z_{i0} \in \mathfrak{R}^{3\times 3}, \\ c_i(t) = \check{S}l_i(t), \\ \dot{\chi}(t) = \Theta(\chi(t),t), \end{cases} \tag{79}$$

where $i \in 1,2,3$*, let the synchronization error be*

$$l_i(t) = \hat{z}_i(t) - \chi(t),$$

and $l_i(t) = (l_{i1}(t), l_{i2}(t), l_{i3}(t))^T \in \mathfrak{R}^{3\times 3}$ *be the i-th state vector of the CDN.* $\varrho(t)$ *is time delay, and satisfies* $\varrho(t) : [t_0, +\infty), \varrho'(t) \le \vartheta < 1, \psi = \{\psi(s) : -\dot{\varrho} \le s \le 0\} \in C([-\tilde{\varrho}, 0], \mathfrak{R}^m)$*.*

Without delay and random disturbance, system (79) becomes

$$\begin{cases} \dot{\hat{z}}_i(t) = \Theta(\hat{z}_i(t),t) + k\sum_{j=1}^{3} a_{ij}\hat{z}_j(t) + k\sum_{j=1}^{3} b_{ij}\hat{z}_j(t) + c_i(t)dt \\ \hat{z}_i(t_0) = z_{i0} \in \mathfrak{R}^{3\times 3}, \\ c_i(t) = \check{S}l_i(t), \\ \dot{\chi}(t) = \Theta(\chi(t),t). \end{cases} \tag{80}$$

The matrix forms of systems (79) and (80) are, respectively,

$$\begin{cases} dq(t) = \big[\Upsilon(q(t),t) + kAq(t) + kBq(t-\varrho(t)) \\ \qquad\qquad + Sq(t)\big]dt + \delta Vq(t)d\mathsf{B}(t), \\ q(t_0) = q_0 \in \mathfrak{R}^{3\times 3}, \end{cases} \tag{81}$$

and

$$\begin{cases} \dot{l}(t) = \Upsilon(l(t),t) + kAl(t) + kBl(t) + Sl(t), \\ l(t_0) = l_0 \in \mathfrak{R}^{3\times 3}. \end{cases} \tag{82}$$

Suppose the coupling matrix

$$A = \begin{bmatrix} -1 & 0.01 & -0.01 \\ -0.05 & -0.5 & 1 \\ 0.1 & -0.1 & -1 \end{bmatrix}, B = \begin{bmatrix} 0.2 & -0.1 & \\ -0.1 & 0.2 & \\ -0.1 & 0.2 & \end{bmatrix},$$

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The coupling coefficient is $k = 1.2$*. The activation function is* $\Upsilon(\cdot) = \sin(\cdot)$*. Select* $\Phi = 1$*, that is,* $||\Upsilon(h) - \Upsilon(p)|| \le ||l - p||$*. The initial value of the three nodes and isolated node is* $l_1 = (-2,2,2)^T$*,* $l_2 = (-1.6,0,1.6)^T$*,* $l_3 = (-1.6,1.6,0)^T$ *and* $\chi(t) = (-0.1,0.1,0.1)^T$*. For* $\mathfrak{L} = 0.8$*,* $\mathfrak{D} = 0.5$*, system (82) is ESt.*

Let $\Delta = 0.3$*,* $\vartheta = 0$*. From Theorem 3, we obtain the following two equations:*

$$1.28\exp(-0.3) + 0.0512\exp(10.073088 + 0.024\tilde{\delta}) = 1, \tag{83}$$

and

$$2\mathcal{N}_2 \exp(0.6\mathcal{N}_1) + 1.28 \exp(-(0.3 - \breve{\varrho})) = 1. \tag{84}$$

Combine Theorem 3 in $\mathcal{N}_1$, $\mathcal{N}_2$. *We have* $\tilde{\delta} = 0.0024, \breve{\varrho} = 0.0403$. *From Theorem 3, if the time delays and random disturbance are less than the above-derived bound, which is* $|\delta| \leq \tilde{\delta}, \tilde{\varrho} < \min(\Delta/2, \breve{\varrho})$, *then CDN (39) disturbed by noise is MSESt. Thus, CDN (39) and (2) are MSESy.*

In Figure 7, said CDNs under linear controller (39) and (2) have been implemented as ESy. Meanwhile, Figure 7 shows CDN (39) and (2) under the linear controller $\breve{\varrho} = 0.0303, \tilde{\delta} = 0.002$. *Because both the time delays and the strength of random disturbances are smaller than the bound derived from Theorem 3, error system (81) is stable. Therefore, CDNs (39) and (2) are MSESy and ASESy. In addition, Figure 8 shows the state of the system (81) without delay and random perturbations under a nonlinear controller.*

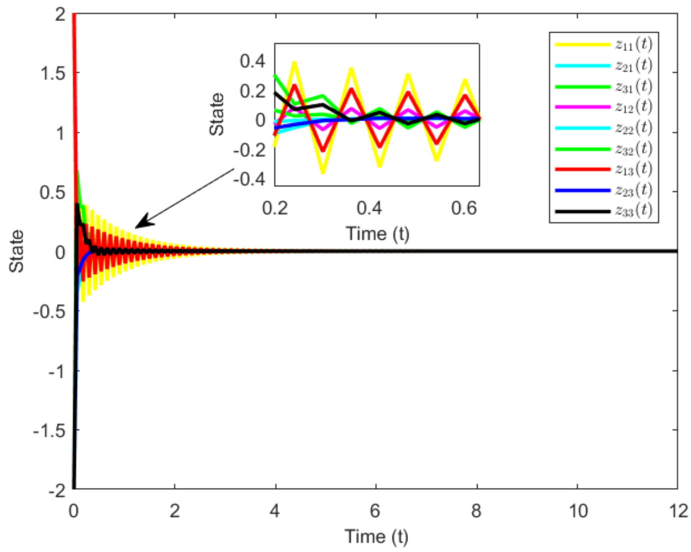

Figure 7. The state of CDN (81) with $\breve{\varrho} = 0.0303$ and $\tilde{\delta} = 0.002$ under linear controller.

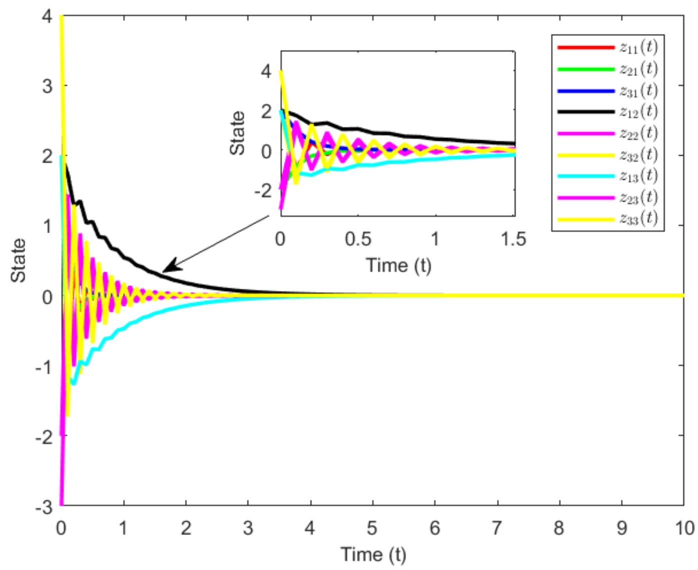

Figure 8. The state of CDN (81) without delay and disturbance under linear controller.

In Figure 9 describes the state of time delays and excessive noise intensity, i.e., $\breve{\varrho} = 0.403$, $\tilde{\delta} = 0.024$. *It is easy to obtain that the limits of the time delays and noise intensity are greater than* *the theoretical results derived from Theorem 3. Therefore, error system (82) is unstable, that is, error system (81) is unstable, so CDNs (39) and (2) are not ESy.*

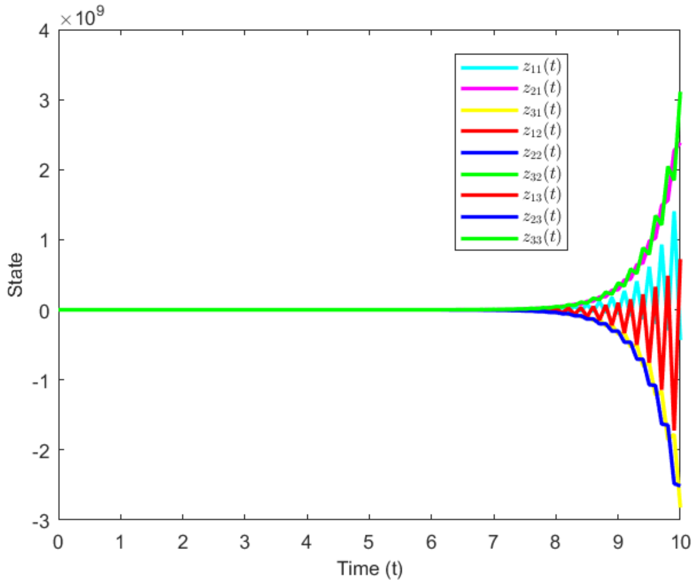

Figure 9. The state of CDN (82) with $\breve{\varrho} = 0.403$ and $\tilde{\delta} = 0.024$ under linear controller.

Example 4. *Consider a CDN with three nodes with time delays and random disturbances under the nonlinear controller*

$$\begin{cases} dz_i(t) = \Big[\Theta(z_i(t),t) + k\sum_{j=1}^{3} a_{ij}z_j(t) + k\sum_{j=1}^{3} b_{ij}z_j(t-\varrho(t)) + c_i(t)\Big]dt \\ \qquad + \delta\sum_{j=1}^{m} v_{ij}z_j(t)d\mathsf{B}(t), \\ z_i(t_0) = z_{i0} \in \mathfrak{R}^{3\times 3}, \\ c_i(t) = -\Theta(l_i(t),t) + \Theta(\chi(t),t) + \breve{O}l_i(t), \\ \dot{\chi}(t) = \Theta(\chi(t),t), \end{cases} \tag{85}$$

where $i \in 1,2,3$, the synchronization error is

$$l_i(t) = \hat{z}_i(t) - \chi(t),$$

and $l_i(t) = (l_{i1}(t), l_{i2}(t), l_{i3}(t))^T \in \mathfrak{R}^{3\times 3}$ is the i-th state vector of the CDN. $\varrho(t)$ is a time delay that satisfies $\varrho(t) : [t_0, +\infty), \varrho'(t) \leq \vartheta < 1, \psi = \{\psi(s) : -\dot{\varrho} \leq s \leq 0\} \in C([-\tilde{\varrho}, 0], \mathfrak{R}^m)$.

Without time delays and random disturbances, system (85) becomes

$$\begin{cases} \dot{\hat{z}}_i(t) = \Theta(\hat{z}_i(t),t) + k\sum_{j=1}^{3} a_{ij}\hat{z}_j(t) + k\sum_{j=1}^{3} b_{ij}\hat{z}_j(t) + c_i(t)dt \\ \hat{z}_i(t_0) = z_{i0} \in \mathfrak{R}^{3\times 3}, \\ c_i(t) = -\Theta(l_i(t),t) + \Theta(\chi(t),t) + \breve{O}l_i(t), \\ \dot{\chi}(t) = \Theta(\chi(t),t). \end{cases} \tag{86}$$

The matrix forms of systems (85) and (86) are, respectively,

$$\begin{cases} dq(t) = \big[kAq(t) + kBq(t-\varrho(t)) + Oq(t)\big]dt + \delta Vq(t)d\mathsf{B}(t), \\ q(t_0) = q_0 \in \mathfrak{R}^{3\times 3}, \end{cases} \tag{87}$$

and

$$\begin{cases} \dot{l}(t) = kAl(t) + kBl(t) + Ol(t), \\ l(t_0) = l_0 \in \Re^{3\times 3}. \end{cases} \tag{88}$$

Suppose the coupling matrix

$$A = \begin{bmatrix} -0.2 & -0.01 & 0 \\ -0.01 & -0.2 & 0 \\ 0.2 & -0.1 & -0.01 \end{bmatrix}, B = \begin{bmatrix} -0.12 & -0.1 & 0 \\ -0.12 & 0.1 & 0 \\ 0 & 0 & -0.2 \end{bmatrix},$$

$$O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, V = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The coupling coefficient is $k = 1.1$*. The activation function is* $\Upsilon(\cdot) = \sin(\cdot)$*, the initial value of the three nodes and isolated node is* $l_1 = (-2, 2, -2)^T$*,* $l_2 = (-1, 1, 1)^T$*,* $l_3 = (-2, 2, -2)^T$ *and* $\chi(t) = (-0.2, 0.2, -0.2)^T$*. When* $\mathfrak{L} = 1.1$*,* $\mathfrak{D} = 1$*, system (88) is ESt.*

Select $\Delta = 0.5$*,* $\vartheta = 0$*. From Theorem 4, we can obtain the following two equations.*

$$2.42\exp(-1) + 0.069696\tilde{\delta}^2 \exp(5.3246 + 0.0432\tilde{\delta}) = 1, \tag{89}$$

and

$$2\mathcal{N}_6 \exp(1.5\mathcal{N}_5) + 2.42\exp(-2(0.5 - \breve{\varrho})) = 1. \tag{90}$$

According to Theorem 4, $\mathcal{N}_5$*,* $\mathcal{N}_6$*, we can obtain* $\tilde{\delta} = 0.0876, \breve{\varrho} = 0.0351$*. According to Theorem 4, if the time delays and random disturbances are less than the bounds derived above, that is,* $|\delta| \leq \tilde{\delta}, \breve{\varrho} < \min(\Delta/2, \breve{\varrho})$*, then the CDN (39) disturbed by noise is MSESt. Thus, CDNs (39) and (2) are MSESy.*

In Figure 10 shows the state of CDN (39) and (2) under nonlinear controller $\breve{\varrho} = 0.0303$*,* $\tilde{\delta} = 0.002$*. In addition, Figure 11 shows the state of the system (88) without delay and random perturbations under a nonlinear controller. Because both the time delays and the strength of random disturbances are smaller than the bound derived from Theorem 3. Hence, error system (88) is stable and error system (87) is stable, Therefore, CDNs (39) and (2) are MSESy and ASESy under the nonlinear controller.*

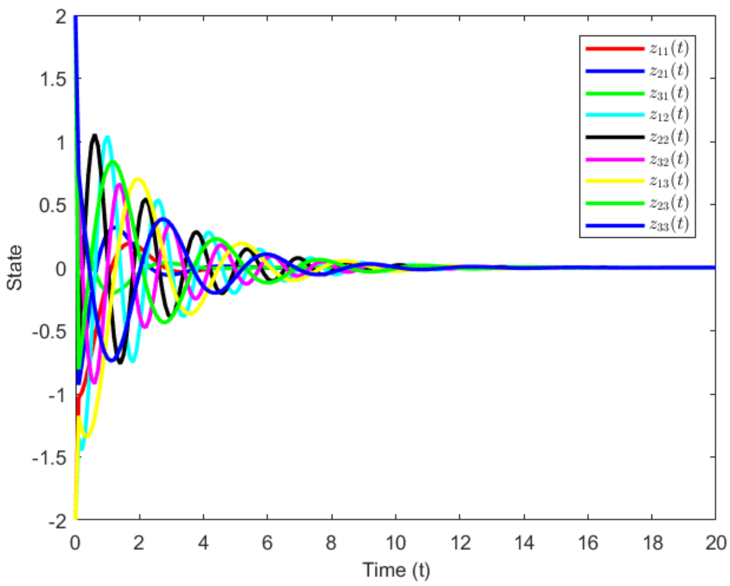

Figure 10. The state of CDN (87) with $\breve{\varrho} = 0.0351$ and $\tilde{\delta} = 0.0876$ under nonlinear controller.

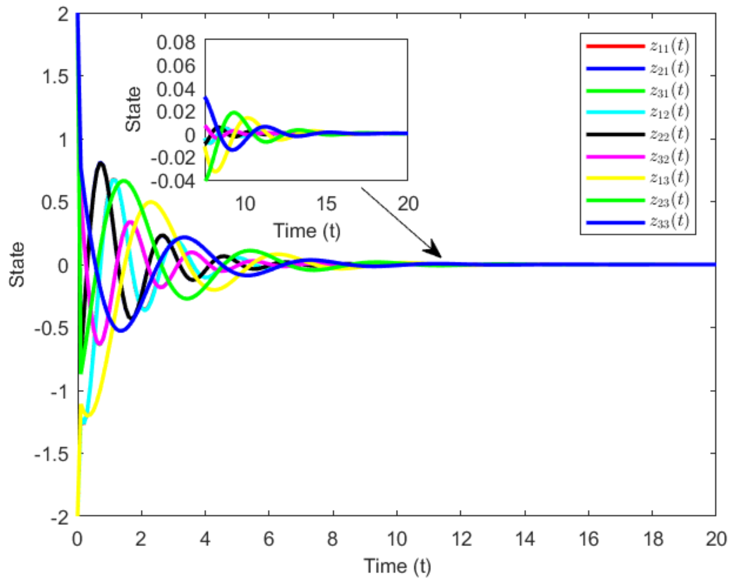

Figure 11. The state of CDN (88) without delay and disturbance under nonlinear controller.

Figure 12 shows the status of $\breve{\varrho} = 0.5, \tilde{\delta} = 0.094$. Obviously, the time delays and noise intensity are greater than the bounds derived from Theorem 4. Therefore, error system (88) is unstable, that is, error system (87) is unstable, so CDN (38) and (2) are not ESy.

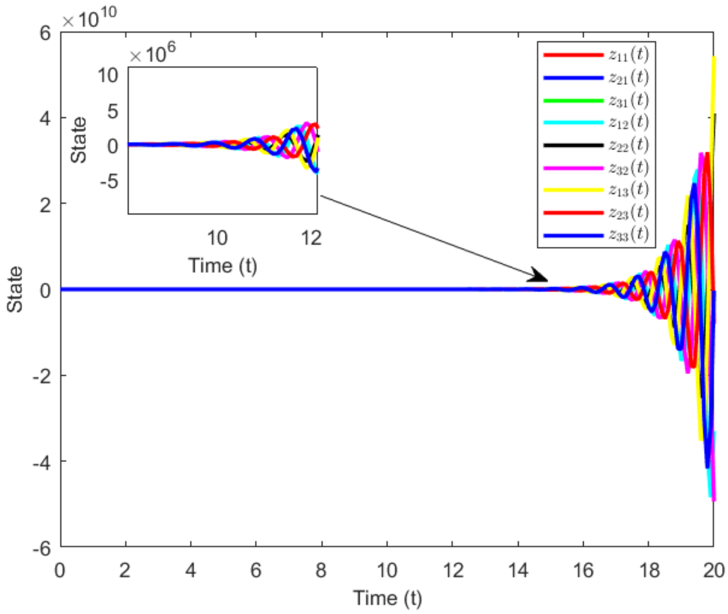

Figure 12. The state of CDN (87) with $\breve{\varrho} = 0.5$ and $\tilde{\delta} = 0.094$ under nonlinear controller.

Remark 8. *The proof and calculation process involved in this paper is complex. For the proof process, the Gronwall–Bellman lemma and some inequality techniques are used to analyze the results of this paper. The calculation process was performed by MATLAB calculation of the involved matrices and numerical simulations.*

5. Conclusions

This paper explores the robustness of ESy with time delays and random disturbances in the CDNs that the have realized the ESy under the controllers. However, the information of the CDN is inevitably disturbed by external time delay and the noise in the process of transmission between nodes. Hence, to what extent CDN can withstand external time delays and noise interference without losing synchronization has become the research hotspot of this paper. By applying the Gronwall–Bellman lemma and some inequality methods, the maximum intensities of CDNS with external time delays and noise interference are estimated by calculating transcendental equations, and sufficient conditions to ensure the ESy of CDNs with time delays and random disturbances are obtained. It is shown that (9), (28), (44) and (59) are ESt when the time delay limits and the size of random disturbances are less than the upper bound we derive, and therefore, systems (1), (39) and (2) are ESy. The results of this paper provide theoretical support for the analysis and design of CDN. In the future, we try to use less conservative inequality techniques to expand the upper limit of the delays and random disturbance strength. In view of the methods and techniques utilized in this article, we will further study the inequality techniques, which are less conservative, to analyze the upper bounds of delays and random interference in the next stage, so that the obtained results are less conservative, so that the synchronization of CDN is easier, or consider other systems, for example, generalized synchronization of delayed CDNs.

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