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A Study on the Centroid of a Class of Solvable Lie Algebras

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Abstract: The centroid of Lie algebra is a basic concept and a necessary tool for studying the structure of Lie algebraic structure. The extended Heisenberg algebra is an important class of solvable Lie algebras. In any Lie algebra, the anti symmetry of Lie operations is an important property of Lie algebra. This article investigates the centroids and structures of $2n+2$ dimensional extended Heisenberg algebras, where all invertible elements form a group and all elements form a ring. Then, its main research results are extended to infinite dimensional extended Heisenberg algebras.

Keywords: lie algebra; centroid; ring; ideal

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1. Introduction

Centroid is a basic concept in Lie algebra, and it is also an important tool to study the theory of Lie algebraic structure, see [1] for details. In references [2], Chen, P. and Gao, S. studied the centroid of the extended Schrodinger-Virasoro Lie algebra. In ref. [3], the authors studied the centroids of n-Lie superalgebras. Zhou, J. and Cao, Y. studied the centroid of Jordan Lie algebra, see [4] for details. In refs. [5–7], scholars have studied the relevant properties of the centroid of n-Lie algebra, Lie triple system and Leibniz triple system algebra respectively. However, there are few centroids and their structures on finite dimensional Lie algebras. This article discusses the centroids and structures of finite and infinite dimensional Lie algebras.

The solvable extension Lie algebra of Heisenberg algebra is a class of solvable Lie algebras, which we call extension Heisenberg algebra. Solvable Lie algebras hold an important position in finite dimensional Lie algebras. Therefore, further research on the algebraic properties of the extended Heisenberg algebra has theoretical significance. Reference [8] studied the derived subalgebra and automorphism group of the extended Heisenberg algebra and the real submanifold of the two-dimensional complex projective space. In ref. [9], scholars studied two types of non weighted modules of twisted Heisenberg Virasoro. The centroid belongs to the structure and intersection problem of Lie algebra, and the structure and representation problem of Lie algebraic structure has always been a hot topic in the research of Lie algebra, see [10,11] for details. The author has also studied the structure and representation of Lie algebraic structure, see [12–14] for details.

This article investigates the centroids and structures of $2n+2$ dimensional extended Heisenberg algebras, where all invertible elements form a group and all elements form a ring H. H has only three ideals, and it is a principal ideal ring. The focus is on studying the structure of groups and rings. Then, its main research results are extended to infinite dimensional extended Heisenberg algebras. The Lie operations of these two extended Heisenberg algebras both have anti symmetry. This paper adds a new method to the study of the centroid of finite and infinite dimensional Lie algebra, which is helpful to the study of the centroid structure of general Lie algebra. The relatively important theorems in the text are Theorems 1 and 2. Symbol description: $\mathcal{C}$ represents a complex field, $\mathcal{Z}_+$ represents a set of positive integers.

2. Main Results

Definition 1. *Let g be a Lie algebra, where the centroid of g is a linear transformation on g and satisfies the following conditions:*

$$Cent(g) = \{X : g \to g | X([x,y]) = [x, X(y)], \forall x, y \in g\}.$$

According to the definition, $\forall \phi_1, \phi_2 \in Cent(g)$,

$$\phi_2\phi_1([x,y]) = \phi_2(\phi_1([x,y])) = \phi_2([x,\phi_1(y)]) = [x,\phi_2(\phi_1(y))], \forall x, y \in g,$$

so $\phi_2\phi_1 \in Cent(g)$. And because

$$(\phi_2 + \phi_1)([x,y]) = (\phi_2([x,y])) + (\phi_1([x,y])) = [x,\phi_2(y)] + [x,\phi_1(y)] = [x,(\phi_2+\phi_1)(y))],$$

$$\forall x, y \in g,$$

so $\phi_2 + \phi_1 \in Cent(g)$.

Obviously, identity mapping I belongs to $Cent(g)$, zero mapping 0 belongs to $Cent(g)$. In summary, all centroids on the Lie algebra g can form a loop by ordinary addition and multiplication of linear mappings.

Lemma 1. *Let g be a finite dimensional Lie algebra, and if a set of bases of g is $e_1, e_2, \cdots, e_n$, then $\phi \in Cent(g)$ holds if and only if*

$$\phi([e_i,e_j]) = [e_i,\phi(e_j)], \forall i, j \in \{1,2,\ldots,n\}.$$

Proof. Necessity: Assuming ϕ is the centroid of g.

Because $\phi([x,y]) = [x,\phi(y)], \forall x, y \in g$, so

$$\phi([e_i,e_j]) = [e_i,\phi(e_j)], \forall i, j \in \{1,2,3,\cdots n\}.$$

Adequacy: Because $\phi([e_i,e_j]) = [e_i,\phi(e_j)]$. $\forall x, y \in g, x = \sum_{i=1}^{n_1} k_i e_i, y = \sum_{j=1}^{n_2} l_j e_j$, so

$$\phi([x,y]) = \phi(\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} k_i l_j [e_i,e_j]) = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} k_i l_j \phi([e_i,e_j]) =$$

$$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} k_i l_j [e_i,\phi(e_j)] = [\sum_{i=1}^{n_1} k_i e_i, \phi(\sum_{j=1}^{n_2} l_j e_j)] = [x,\phi(y)].$$

□

Definition 2. *Let $g = span\{t, e_1, \cdots, e_n, \xi_1, \cdots, \xi_n, c\}$, define the Poisson product in g as follows:*

$$[t,e_i] = e_i, [t,\xi_i] = -\xi_i, [e_i,\xi_i] = c, [t,c] = 0, [e_i,c] = 0, [\xi_i,c] = 0 (i = 1,\cdots,n).$$

It is easy to verify that g is a solvable Lie algebra, which is called a $2n+2$ dimensional extended Heisenberg algebra.

Theorem 1. *g is a 2n+2 dimensional extension Heisenberg algebra, and $\varphi : g \to g$ is a linear mapping,*

$$\varphi\begin{pmatrix} t \\ e_1 \\ \vdots \\ e_n \\ \xi_1 \\ \vdots \\ \xi_n \\ c \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,2n+1} & c_{1,2n+2} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,2n+1} & c_{2,2n+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{2n+2,1} & c_{2n+2,2} & \cdots & c_{2n+2,,2n+1} & c_{2n+2,,2n+2} \end{pmatrix}\begin{pmatrix} t \\ e_1 \\ \vdots \\ e_n \\ \xi_1 \\ \vdots \\ \xi_n \\ c \end{pmatrix} = C\begin{pmatrix} t \\ e_1 \\ \vdots \\ e_n \\ \xi_1 \\ \vdots \\ \xi_n \\ c \end{pmatrix},$$

So φ is the centroid if and only if

$$C = \begin{pmatrix} c_{1,1} & 0 & \cdots & 0 & c_{1,2n+2} \\ 0 & c_{1,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{1,1} & 0 \\ 0 & 0 & \cdots & 0 & c_{1,1} \end{pmatrix}.$$

Proof. Necessity: (1) Because $\varphi[t,t] = [t, \varphi(t)] = 0$,

$$\varphi(t) = [t, c_{1,1}t + c_{1,2}e_1 + \cdots + c_{1,n+1}e_n + c_{1,n+2}\xi_1 + \cdots + c_{1,2n+1}\xi_n + c_{1,2n+2}c]$$

$$= c_{1,2}e_1 + \cdots + c_{1,n+1}e_n - c_{1,n+2}\xi_1 - \cdots - c_{1,2n+1}\xi_n = 0,$$

so $c_{1,i} = 0 (i = 2, \cdots, 2n+1)$.

Similarly, the following formula holds for $\forall i \in 1, \cdots, n$:
(2) $c_{i+1,1} = \cdots = c_{i+1,i} = c_{i+1,i+2} = \cdots = c_{i+1,2n+2} = 0$.
(3) $c_{i+n+1,i+n+1} = c_{1,1}$,
$c_{i+n+1,1} = \cdots = c_{i+n+1,i+n} = c_{i+n+1,i+n+2} = \cdots = c_{i+n+1,2n+2} = 0$.
(4) $c_{2n+2,2n+2} = c_{1,1}, c_{2n+2,1} = \cdots = c_{2n+2,2n+1} = 0$.
Adequacy: If the matrix of the linear transformation φ under a set of basis $t, e_1, e_2 \cdots e_n, \xi_1, \xi_2 \cdots \xi_n, c$ is

$$C = \begin{pmatrix} c_{1,1} & 0 & \cdots & 0 & c_{1,2n+2} \\ 0 & c_{1,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{1,1} & 0 \\ 0 & 0 & \cdots & 0 & c_{1,1} \end{pmatrix}.$$

Let $V = \{t, e_1, e_2 \cdots e_n, \xi_1, \xi_2 \cdots \xi_n, c\}, \forall v_i, v_j \in V$, it can be verified one by one that $\varphi([v_i, v_j]) = [v_i, \varphi(v_j)]$ is valid.

Therefore, according to Lemma 1: φ is the center of mass. □

Let G be the set of all reversible linear transformations in $Cent(g)$, that is,

$$G = \{\begin{pmatrix} c_{1,1} & 0 & \cdots & 0 & c_{1,2n+2} \\ 0 & c_{1,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{1,1} & 0 \\ 0 & 0 & \cdots & 0 & c_{1,1} \end{pmatrix} | \forall c_{11}, c_{1,2n+2} \in \mathcal{C}, c_{11} \neq 0\}.$$

Lemma 2. *Let*

$$G_1 = \{ \begin{pmatrix} 1 & 0 & \cdots & 0 & m_1 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} | \forall m_1 \in \mathcal{C} \},$$

$$G_2 = \{ \begin{pmatrix} m_1 & 0 & \cdots & 0 & 0 \\ 0 & m_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & m_1 & 0 \\ 0 & 0 & \cdots & 0 & m_1 \end{pmatrix} | \forall m_1 \in \mathcal{C}, m_1 \neq 0 \},$$

then G_1 and G_2 are all commutative subgroups of G.

Theorem 2. *If g is a $2n+2$ dimensional extended Heisenberg algebra, then we have $G = G_1 G_2$.*

Proof. For $\forall \varphi \in G, \forall c_{1,1}, c_{1,2n+2} \in \mathcal{C}, c_{1,1} \neq 0$, let

$$\phi_2 = \begin{pmatrix} 1 & 0 & \cdots & 0 & \frac{-c_{1,2n+2}}{c_{1,1}} \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in G_1.$$

Because

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \frac{-c_{1,2n+2}}{c_{1,1}} \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{1,1} & 0 & \cdots & 0 & c_{1,2n+2} \\ 0 & c_{1,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{1,1} & 0 \\ 0 & 0 & \cdots & 0 & c_{1,1} \end{pmatrix} =$$

$$\begin{pmatrix} c_{1,1} & 0 & \cdots & 0 & 0 \\ 0 & c_{1,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{1,1} & 0 \\ 0 & 0 & \cdots & 0 & c_{1,1} \end{pmatrix},$$

so

$$\phi_2 \varphi = \begin{pmatrix} c_{1,1} & 0 & \cdots & 0 & 0 \\ 0 & c_{1,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{1,1} & 0 \\ 0 & 0 & \cdots & 0 & c_{1,1} \end{pmatrix} \in G_2,$$

In summary, it can be seen that $\varphi \in G_1 G_2$,that is, $G \subseteq G_1 G_2$.

It is obvious that $G_1 G_2 \subseteq G$, so $G = G_1 G_2$. $\square$

Let

$$H = \{ \begin{pmatrix} a & 0 & \cdots & 0 & b \\ 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a & 0 \\ 0 & 0 & \cdots & 0 & a \end{pmatrix} | \forall a, b \in \mathcal{C} \}.$$

Obviously, H is a commutative ring with identity element, and H has zero divisor.

Lemma 3. *Let H_1 be the set of identity element and all zero divisor in H,*

$$H_1 = \{ \begin{pmatrix} 0 & 0 & \cdots & 0 & n \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \forall n \in \mathcal{C} \}.$$

Then H_1 is an ideal of H.

Theorem 3. *H has only three ideals: $0, H_1, H$. And H is a principal ideal ring.*

Proof. Obviously, $\{0\}, H_1$ and H are ideals for H.

Assuming that $U \subsetneq H$ is an ideal of H and satisfies: $U \neq 0, U \neq H_1$, then there is a non zero element $\gamma_1 \in U$. It would be well if

$$\gamma_1 = \begin{pmatrix} a_1 & 0 & \cdots & 0 & a_2 \\ 0 & a_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_1 & 0 \\ 0 & 0 & \cdots & 0 & a_1 \end{pmatrix},$$

where a_1 and a_2 are not equal to 0 at the same time.

(1) When $a_1 = 0, a_2 \neq 0$, there is $\gamma_1 \in H_1$. So $U = (\gamma_1) = H_1$. This contradicts the hypothesis.

(2) When $a_1 \neq 0, a_2 = 0$, there is

$$\frac{1}{a_1}\gamma_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

So $U = (\gamma_1) = H$. This contradicts the hypothesis.

(3) When $a_1 \neq 0, a_2 \neq 0$, there is

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 & a_2 \\ 0 & a_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_1 & 0 \\ 0 & 0 & \cdots & 0 & a_1 \end{pmatrix} \begin{pmatrix} \frac{b_1}{a_1} & 0 & \cdots & 0 & \frac{b_2}{a_1} - \frac{a_2 b_1}{a_1^2} \\ 0 & \frac{b_1}{a_1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{b_1}{a_1} & 0 \\ 0 & 0 & \cdots & 0 & \frac{b_1}{a_1} \end{pmatrix} = \begin{pmatrix} b_1 & 0 & \cdots & 0 & b_2 \\ 0 & b_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_1 & 0 \\ 0 & 0 & \cdots & 0 & b_1 \end{pmatrix},$$

any b_1 and b_2 belong to $\mathcal{C}$. So $U = (\gamma_1) = H$. This contradicts the hypothesis.

□

Definition 3. *Let $g^{'} = span\{t, e_1, \cdots, e_i, \cdots, \xi_1, \cdots, \xi_i, \cdots, c\}$,define the Lie operation in $g^{'}$ as follows:*

$$[t, e_i] = e_i, [t, \xi_i] = -\xi_i, [e_i, \xi_i] = c, [t, c] = 0, [e_i, c] = 0, [\xi_i, c] = 0 (i \in \mathcal{Z}_+).$$

$g^{'}$ is called an infinite dimensional extension Heisenberg algebra.

Lemma 4. *Let $g^{'}$ be an infinite dimensional Lie algebra, and if a set of bases of $g^{'}$ is $e_1, e_2, \cdots, e_n, \cdots$, then $\phi \in Cent(g^{'})$ holds if and only if*

$$\phi([e_i, e_j]) = [e_i, \phi(e_j)], \forall i, j \in \{1, 2, \cdots, n, \cdots\}.$$

Lemma 5. *Let $\varphi \in Cent(g^{'})$, and satisfy the following equation:*

$$\varphi(t) = a_{11}t + b_{11}c, \varphi(e_i) = a_{11}e_i, \varphi(\xi_i) = a_{11}\xi_i, \varphi(c) = a_{11}c,$$

so φ is a reversible centroid if and only if $a_{11} \neq 0$.

Let $G^{'}$ be the set of all reversible centroids on $Cent(g^{'})$.

Lemma 6. *Let $G_3 = \{\varphi | \varphi(t) = t + b_{11}c, \varphi(e_i) = e_i, \varphi(\xi_i) = \xi_i, \varphi(c) = c\}$. So $G_3 \subset G^{'}$, and G_3 is a subgroup of $G^{'}$.*

Lemma 7. *Let $G_4 = \{\varphi | \varphi(t) = a_{11}t, \varphi(e_i) = a_{11}e_i, \varphi(\xi_i) = a_{11}\xi_i, \varphi(c) = a_{11}c$. So $G_4 \subset G^{'}$, and G_4 is a subgroup of $G^{'}$.*

Theorem 4. $G^{'} = G_3G_4$.

The above Lemmas 4–7 and Theorem 4 are similar to the results related to $2n + 2$ dimensional extended Heisenberg algebra, and will not be repeated to save space.

Let R be the set of all centroids on g', and the linear operations on R are ordinary addition and multiplication The structure of R is similar to the centroid structure of a $2n + 2$ dimensional extended Heisenberg algebra, and will not be repeated to save space.

3. Conclusions

In this paper, the centroid structure of $2n + 2$ dimensional extended Heisenberg Lie algebra was studied ingeniously by using the elementary transformation of matrix, and the necessary and sufficient conditions for its centroid were obtained. It also characterized all reversible centroids forming a group, clearly showed the structure of the group, and extended its results to the infinite dimensional extended Heisenberg Lie algebra. This paper added a new method to the study of the centroid of finite and infinite dimensional Lie algebra, which was helpful to the study of the centroid structure of general Lie algebra.

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