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Fixed Point Theory on Triple Controlled Metric-like Spaces with a Numerical Iteration

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Abstract: Fixed point theory is a versatile mathematical theory that finds applications in a wide range of disciplines, including computer science, engineering, fractals, and even behavioral sciences. In this study, we propose triple controlled metric-like spaces as a generalization of controlled rectangular metric-like spaces. By examining the Θ-contraction mapping within these spaces, we extend and enhance the existing literature to establish significant fixed point results. Utilizing these findings, we demonstrate the existence of solutions to a Fredholm integral equation and provide an example of a numerical iteration method applicable to a specific case of this Fredholm integral equation.

Keywords: fixed point; triple controlled metric like-spaces; Θ-contraction; Fredholm integral equation; controlled functions; numerical iteration method; metric-like spaces

MSC: 47H10; 54H25

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1. Introduction

Fixed point theory, initially proposed by Banach in 1922 [1], has become a valuable tool in both theoretical mathematics and practical applications. Its versatility extends to variational inequalities, linear inequalities, approximation theory, integral and differential equations, dynamic systems, fractals, game theory, optimization problems, mathematical modeling, and image authentication schemes for secure communication and malicious modification detection [2–4]. The applications of fixed point theory in various disciplines have been comprehensively documented by Debnath et al. [5]. In recent years, there has been a growing interest in generalizing metric spaces. This has resulted in the development of various spaces such as b-metric spaces [6], extended b-metric spaces [7], controlled metric spaces [8,9], rectangular metric spaces [10], rectangular b-metric spaces [11,12], and controlled rectangular metric-like spaces [13]. These generalizations offer new perspectives and possibilities in the study of metric spaces, constituting a dynamic and evolving field characterized by continuous research endeavors (for examples, see [10–12,14–22]). These spaces present novel and intriguing approaches to metric space concepts, showing promise for diverse applications.

Furthermore, fixed point theory plays a fundamental role in solving existence and uniqueness problems, particularly in the context of differential and integral equations. It serves as a fundamental framework for addressing various problems, including integrodifferential equations [13,21,23,24]. Researchers have explored various metric-type spaces under different contraction conditions such as the F-contraction introduced by Wardowski [25] and the concept of Θ-contraction introduced by Jleli et al. [26,27] in the context of Branciari metric spaces. The Banach contraction principle provides a powerful tool in nonlinear analysis, enabling the establishment of existence and uniqueness theorems and the resolution of differential and integral equations.

Inspired by the notion of triple controlled metric-type spaces introduced by Tasneem et al. [20] and the work of Azmi [22], this article introduces the concept of triple controlled metric-like spaces. We utilize the Θ-contraction mapping on these spaces to establish a fixed point theorem and provide illustrative examples. In addition, we apply our main fixed point results to prove the existence of solutions for a Fredholm integral equation. To demonstrate practical relevance, we provide an example of a numerical iteration method applicable to a specific case of this Fredholm integral equation. Finally, we conclude the article with an open question for future research.

2. Preliminaries

In 2020, Mlaiki introduced the concept of double controlled metric-like spaces [28]. This novel extension expands the understanding and application of metric-like spaces.

Definition 1. *Let $\mathfrak{F}$ be a non-empty set, and consider the functions $\beta, \mu : \mathfrak{F} \times \mathfrak{F} \to [1, +\infty)$. A mapping $\sigma : \mathfrak{F} \times \mathfrak{F} \to [0, +\infty)$ is referred to as a double controlled metric-like space if it satisfies the following criteria:*
(q1) $\sigma(\xi_1, \xi_2) = 0$ implies that $\xi_1 = \xi_2$ for all $\xi_1, \xi_2 \in \mathfrak{F}$;
(q2) $\sigma(\xi_1, \xi_2) = \sigma(\xi_2, \xi_1)$, a symmetry condition for all $\xi_1, \xi_2 \in \mathfrak{F}$;
(q3) $\sigma(\xi_1, \xi_2) \leq \beta(\xi_1, \xi_3)\sigma(\xi_1, \xi_3) + \mu(\xi_3, \xi_2)\sigma(\xi_3, \xi_2)$, for all $\xi_1, \xi_2, \xi_3 \in \mathfrak{F}$.
The pair $(\mathfrak{F}, \sigma)$ is called a double controlled metric-like space.

Continuing with our exploration, we now introduce the concept of triple controlled rectangular metric-like spaces.

Definition 2. *Let $\mathfrak{F}$ be a non-empty set, and consider the functions $\beta, \mu, \gamma : \mathfrak{F} \times \mathfrak{F} \to [1, +\infty)$. A mapping $\sigma : \mathfrak{F} \times \mathfrak{F} \to [0, +\infty)$ is referred to as triple controlled rectangular metric-like space if it satisfies the following criteria:*
(q1) $\sigma(\xi_1, \xi_2) = 0$ implies that $\xi_1 = \xi_2$ for all $\xi_1, \xi_2 \in \mathfrak{F}$;
(q2) $\sigma(\xi_1, \xi_2) = \sigma(\xi_2, \xi_1)$, a symmetry condition for all $\xi_1, \xi_2 \in \mathfrak{F}$;
(q3) $\sigma(\xi_1, \xi_2) \leq \beta(\xi_1, \xi_3)\sigma(\xi_1, \xi_3) + \mu(\xi_3, \xi_4)\sigma(\xi_3, \xi_4) + \gamma(\xi_4, \xi_2)\sigma(\xi_4, \xi_2)$, for all $\xi_1, \xi_2 \in \mathfrak{F}$ and for all distinct points $\xi_3, \xi_4 \in \mathfrak{F}$, each distinct from ξ_1 and ξ_2.
The pair $(\mathfrak{F}, \sigma)$ is denoted as a triple controlled rectangular metric-like space and will be abbreviated to $\mathcal{TCRMLS}$ throughout this article.

Example 1. *Let $\mathfrak{F} = \{0, 1, 2, 3\}$. Consider the function $\sigma : \mathfrak{F} \times \mathfrak{F} \to [0, +\infty)$ defined by*

$$\sigma(0,0) = \sigma(1,1) = \sigma(0,1) = \sigma(1,0) = \sigma(1,3) = \sigma(3,1) = 1, \sigma(3,3) = 0,$$

and

$$\sigma(2,2) = \sigma(2,1) = \sigma(1,2) = 2, \sigma(0,3) = \sigma(3,0) = 3, \sigma(2,3) = \sigma(3,2) = 1/3,$$

and the three functions $\beta, \mu, \gamma : \mathfrak{F} \times \mathfrak{F} \to [1, +\infty)$ are defined by:

$$\beta(x,x) = 1, x \in \mathfrak{F}, \beta(1,2) = \beta(2,1) = \beta(2,0) = \beta(0,2) = \beta(0,1) = \beta(1,0) = 1,$$

$$\beta(0,3) = \beta(3,0) = 4/3, \beta(1,3) = \beta(3,1) = 3/2,\ \beta(2,3) = \beta(3,2) = 3.$$

$$\mu(x,x) = 1, x \in \mathfrak{F}, \mu(1,2) = \mu(2,1) = 3/2, \mu(2,0) = \mu(0,2) = 2, \mu(2,3) = \mu(3,2) = 4,$$

$$\mu(0,1) = \mu(1,0) = \mu(0,3) = \mu(3,0) = \mu(1,3) = \mu(3,1) = 1,$$

and

$$\gamma(x,x) = 1, x \in \mathfrak{F}, \gamma(1,2) = \gamma(2,1) = 2, \gamma(2,0) = \gamma(0,2) = 1, \gamma(2,3) = \gamma(3,2) = 3,$$

$$\gamma(0,1) = \gamma(1,0) = \gamma(0,3) = \gamma(3,0) = \gamma(1,3) = \gamma(3,1) = 1.$$

One can easily show that $(\mathfrak{F}, \sigma)$ is a triple controlled rectangular metric-like space rather than a triple controlled metric-type space since $\sigma(1,1) = 1 \neq 0$.

Example 2. *Let $\mathfrak{F} = A \cup B$, where A is the set of natural numbers and $B = \{\frac{1}{n} : n \in N\}$. We define $\sigma : \mathfrak{F}^2 \to [0, +\infty)$ by:*

$$\sigma(a,b) = \begin{cases} 0, & \textit{implies}\ a = b, \\ a+6, & \textit{if}\ a \in \{1,2,3\}, b \in B \ \ \textit{or} \ \ a \in B, b \in \{1,2,3\}, \\ 1 & \textit{otherwise}. \end{cases} \tag{1}$$

Let $\beta, \mu, \gamma : \mathfrak{F} \times \mathfrak{F} \to [1, +\infty)$ be defined as:

$$\beta(a,b) = \begin{cases} a, & \textit{if}\ a \in A, b \in B, \\ 1, & \textit{otherwise}. \end{cases}$$

$$\mu(a,b) = \begin{cases} \frac{1}{b}, & \textit{if}\ a \in A, b \in B, \\ 1.5, & \textit{otherwise}. \end{cases}$$

$$\gamma(a,b) = \begin{cases} a+b, & \textit{if}\ a \in A, b \in B, \\ 2, & \textit{otherwise}. \end{cases}$$

One can easily show that $(\mathfrak{F}, \sigma)$ is a triple controlled rectangular metric-like space rather than a triple controlled metric-type space since $\sigma(4,4) = 1 \neq 0$.
Observe that:

$$\sigma(1, 1/2) = 7 > \beta(1,4)\sigma(1,4) + \mu(4,1/2)\sigma(4,1/2) = 3.$$

Hence, $(\mathfrak{F}, \sigma)$ is not a double controlled metric-like space. In addition,

$$\sigma(1,1/2) = 7 > \beta(1,5)\sigma(1,5) + \beta(5,4)\sigma(5,4) + \beta(4,1/2)\sigma(4,1/2) = 4.$$

Hence, $(\mathfrak{F}, \sigma)$ is not a rectangular b-metric-like space.

Example 3. *Consider the space $\mathfrak{F} = C([0,1], \mathbb{R})$ consisting of all continuous functions defined on the interval $[0,1]$. Define $\sigma : \mathfrak{F} \times \mathfrak{F} \longrightarrow [0, +\infty)$ by:*

$$\sigma(f,g) = sup_{t \in [0,1]} |f(t) - g(t)|^4,$$

and $\sigma(1/5, 1/5) = 1/85$. The controlled functions $\beta, \mu, \gamma : \mathfrak{F} \times \mathfrak{F} \to [1, +\infty)$ are expressed as follows:

$$\beta(f,g) = \begin{cases} \sup_{t \in [0,1]} |f(t) - g(t)| + 85, & \textit{if}\ f(t) \neq g(t), \\ 80, & \textit{if}\ f(t) = g(t), \end{cases}$$
$$\mu(f,g) = \begin{cases} \sup_{t \in [0,1]} (f^2(t) + g^2(t)) + 85, & \textit{if}\ f(t) \neq g(t), \\ 85, & \textit{if}\ f(t) = g(t), \end{cases}$$
$$\gamma(f,g) = \begin{cases} max\{\sup_{t \in [0,1]} |f(t)|, \sup_{t \in [0,1]} |g(t)|\} + 85, & \textit{if}\ f(t) \neq g(t), \\ 82, & \textit{if}\ f(t) = g(t). \end{cases}$$

To show that $(\mathfrak{F}, \sigma)$ is a $\mathcal{TCRMLS}$, we verify the conditions from Definition 2. Observe that $\sigma(f,g) = 0$ implies $f = g$ and $\sigma(f,g) = \sigma(g,f)$ are straightforward. To show the third condition, we expand it as follows:

$$\begin{aligned} |f-g|^4 &= |f-h+h-k+k-g|^4. \\ &\leq |f-h|^4 + 4|f-h|^3|h-k| + 4|f-h|^3|k-g| + 6|f-h|^2|h-k|^2 \\ &+ 12|f-h|^2|h-k||k-g| + 6|f-h|^2|k-g|^2 + 4|f-h||h-k|^3 \\ &+ 12|f-h||h-k|^2|k-g| + 12|f-h||h-k||k-g|^2 + 4|f-h||k-g|^3 \\ &+ |h-k|^4 + 4|h-k|^3|k-g| + 6|h-k|^2|k-g|^2 + 4|h-k||k-g|^3 + |k-g|^4. \end{aligned} \tag{2}$$

Next, we consider other cases. For all $f, g \in \mathfrak{F}$ and $f, g \neq 1/5$.
Case 1, if $sup|k-g| \leq sup|h-k| \leq sup|f-h|$, we can apply these conditions to Equation (2) and demonstrate that:

$$|f-g|^4 \quad \leq \quad \beta(f,h)\sigma(f,h) + \mu(h,k)\sigma(h,k) + \gamma(k,g)\sigma(k,g),$$

which gives $\sigma(f,g) \leq \beta(f,h)\sigma(f,h) + \mu(h,k)\sigma(h,k) + \gamma(k,g)\sigma(k,g)$.

Case 2, if $sup|f-h| \leq sup|k-g| \leq sup|h-k|$, we can once again utilize the conditions in Equation (2) to readily demonstrate that:

$$\sigma(f,g) \leq \beta(f,h)\sigma(f,h) + \mu(h,k)\sigma(h,k) + \gamma(k,g)\sigma(k,g). \tag{3}$$

Similarly, for the other cases, if both $f = g = 1/5$, we can utilize Equation (2) to demonstrate, by considering various scenarios, that

$$\beta(1/5,h)\sigma(1/5,h) + \mu(h,k)\sigma(h,k) + \gamma(k,1/5)\sigma(k,1/5) \geq \sigma(1/5,1/5) = 1/85.$$

Hence, we have shown that $(\mathfrak{F}, \sigma)$ is a $\mathcal{TCRMLS}$.

In a triple controlled rectangular metric-like space, the following concepts are relevant: continuity, open ball, convergence, and Cauchy sequences.

Definition 3. *Let $(\mathfrak{F}, \sigma)$ be a $\mathcal{TCRMLS}$:*

- *The open ball with center ξ_0 and radius $r > 0$ is defined by:*

$$\mathbb{B}(\xi_0, r) = \{w \in \mathfrak{F}, |\sigma(\xi_0, w) - \sigma(\xi_0, \xi_0)| < r\}.$$

- *The mapping $S : \mathfrak{F} \to \mathfrak{F}$ is said to be continuous at ξ if for each $\varepsilon > 0$, there exists $\delta > 0$ such that:*

$$S(\mathbb{B}(\xi, \delta)) \subseteq \mathbb{B}(S(\xi), \varepsilon).$$

Definition 4. *Let $(\mathfrak{F}, \sigma)$ be a $\mathcal{TCRMLS}$, and let $\{\xi_n\}$ be any sequence in $\mathfrak{F}$.*

(1) A sequence $\{\xi_n\}$ in $\mathfrak{F}$ is said to be convergent to some w in $\mathfrak{F}$ if and only if

$$\lim_{n\to+\infty} \sigma(\xi_n, w) = \sigma(w, w).$$

(2) We say that $\{\xi_n\}$ is a Cauchy sequence if and only if the limit $\lim_{n,m\to+\infty} \sigma(\xi_n, \xi_m)$ exists and is finite.

(3) The space $(\mathfrak{F}, \sigma)$ is called complete if every Cauchy sequence in $\mathfrak{F}$ is convergent. In other words, for any Cauchy sequence $\{\xi_n\}$, there exists a $w \in \mathfrak{F}$ such that:

$$\lim_{n\to+\infty} \sigma(\xi_n, w) = \sigma(w, w) = \lim_{n,m\to+\infty} \sigma(\xi_n, \xi_m).$$

Remark 1. *Metric-like spaces differ from metric spaces as they do not possess certain topological and convergence properties. One such property is the uniqueness of the limit of a convergent sequence, which may not hold in metric-like spaces. For example, consider Example 1, where $\mathfrak{F} = \{0, 1, 2, 3\}$. Let $\{\xi_n = 1\}$ represent the constant sequence $\{1\}$. In this case, we observe that:*

$$\lim_{n\to+\infty} \sigma(\xi_n, 1) = \sigma(1, 1) = 1,$$

and

$$\lim_{n\to+\infty} \sigma(\xi_n, 2) = \sigma(1, 2) = 2 = \sigma(2, 2).$$

This demonstrates that the limit of a convergent sequence is not unique in a triple controlled rectangular metric-like space.

3. The Main Results

The main section of our study begins with the definition of the family of functions θ, as originally introduced in [26].

Definition 5. *Let Θ be the set of all θ functions $\theta : (0,+\infty) \to (1,+\infty)$ that satisfy the following requirements:*
(θ1) θ is non-decreasing.
(θ2) For any sequence $\{t_m\}$ of positive real numbers, it holds that:

$$\lim_{m\to+\infty} t_m = 0^+ \text{ if and only if } \lim_{m\to+\infty} \theta(t_m) = 1.$$

(θ3) There exist the constants k and M, where $0 < k < 1$ and $M \in (0,+\infty]$, such that:

$$\lim_{t\to 0^+} \frac{\theta(t)-1}{t^k} = M.$$

Next, we present our main theorem, which showcases our important contributions to the field of fixed point theory in the context of a complete$\mathcal{TCRMLS}$.

Theorem 1. *Consider a complete triple controlled rectangular metric-like space $(\mathfrak{F},\sigma)$, where $\mathfrak{F}$ is a non-empty set. Let $T : \mathfrak{F} \to \mathfrak{F}$ be a self-mapping such that:*

$$\xi, w \in \mathfrak{F},\ \sigma(T\xi, Tw) \neq 0 \text{ implies } \theta(\sigma(T\xi, Tw)) \leq [\theta(\sigma(\xi, w))]^r, \tag{4}$$

where $\theta \in \Theta$ and $r \in (0,1)$. Furthermore, for any $\xi_0 \in \mathfrak{F}$, we define the sequence $\{\xi_n\}$ by $\xi_n = T^n\xi_0$, such that:

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\beta(\xi_{2i+2},\xi_{2i+3})}{\beta(\xi_{2i},\xi_{2i+1})}\gamma(\xi_{2i+2},\xi_m) < 1, \tag{5}$$

and

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\mu(\xi_{2i+3},\xi_{2i+4})}{\mu(\xi_{2i+1},\xi_{2i+2})}\gamma(\xi_{2i+2},\xi_m) < 1. \tag{6}$$

In addition, for each $\xi \in \mathfrak{F}$

$$\lim_{n\to+\infty} \beta(\xi,\xi_n),\ \lim_{n\to+\infty} \mu(\xi_n,\xi), \text{ and } \lim_{n\to+\infty} \gamma(\xi_n,\xi) \text{ exist and are finite.} \tag{7}$$

Then, T admits a unique fixed point in $\mathfrak{F}$.

Proof. Let ξ_0 be an arbitrary point in $\mathfrak{F}$. We construct a sequence $\{\xi_n\}$ using the following iteration: $\xi_1 = T\xi_0$, $\xi_2 = T\xi_1 = T^2\xi_0$, and $\xi_3 = T\xi_2 = T^2\xi_1 = T^3\xi_0$, thus $\xi_n = T^n\xi_0$ for all $n \in \mathbb{N}$.

In the case where for some $m \in \mathbb{N}$ we have $T^m\xi_0 = T^{m+1}\xi_0$, it implies that $T^m\xi_0$ is a fixed point of T. Thus, we can assume without loss of generality that $\xi_n \neq \xi_{n+1}$, i.e., $\sigma(T^n\xi_0, T^{n+1}\xi_0) > 0$ for all $n \in \mathbb{N}$.

Applying (4) recursively, we arrive at

$$\begin{aligned}
\theta(\sigma(\xi_n,\xi_{n+1})) &= \theta(\sigma(T\xi_{n-1},T\xi_n)) \\
&\leq [\theta(\sigma(\xi_{n-1},\xi_n))]^r \\
&\leq [\theta(\sigma(\xi_{n-2},\xi_{n-1}))]^{r^2} \\
&\leq [\theta(\sigma(\xi_{n-3},\xi_{n-2}))]^{r^3} \leq \cdots \qquad \leq [\theta(\sigma(\xi_0,\xi_1))]^{r^n}.
\end{aligned} \tag{8}$$

Since $\theta(t) > 1$, we have

$$1 < \theta(\sigma(\xi_n,\xi_{n+1})) \leq [\theta(\sigma(\xi_0,\xi_1))]^{r^n}. \tag{9}$$

As $0 < r < 1$, inferring from the fact that n tends to infinity in Equation (9):

$$\lim_{n\to+\infty} \theta(\sigma(\xi_n, \xi_{n+1})) = 1.$$

By utilizing property $(\theta 2)$, we derive

$$\lim_{n\to+\infty} \sigma(\xi_n, \xi_{n+1}) = 0. \tag{10}$$

Analogous techniques can be employed to establish that:

$$\lim_{n\to+\infty} \sigma(\xi_n, \xi_{n+2}) = 0. \tag{11}$$

By $(\theta 3)$, there exist $k \in (0, 1)$ and $M \in (0, +\infty]$ so that:

$$\lim_{n\to+\infty} \frac{\theta(\sigma(\xi_n, \xi_{n+1})) - 1}{[\sigma(\xi_n, \xi_{n+1})]^k} = M. \tag{12}$$

Case I: Assume that $0 < M < +\infty$, and let $L = \frac{M}{2}$. From Equation (12), we can find some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we obtain:

$$\left| \frac{\theta(\sigma(\xi_n, \xi_{n+1})) - 1}{[\sigma(\xi_n, \xi_{n+1})]^k} - M \right| \leq L.$$

This implies that

$$L = M - L \leq \frac{\theta(\sigma(\xi_n, \xi_{n+1})) - 1}{[\sigma(\xi_n, \xi_{n+1})]^k}, \quad \text{for all } n \geq n_0.$$

Hence, for all $n \geq n_0$, we have

$$n[\sigma(\xi_n, \xi_{n+1})]^k \leq n[\frac{\theta(\sigma(\xi_n, \xi_{n+1})) - 1}{L}].$$

By employing Equation (9), we obtain

$$n[\sigma(\xi_n, \xi_{n+1})]^k \leq n[\frac{[\theta(\sigma(\xi_0, \xi_1))]^{r^n} - 1}{L}].$$

By making $n \to +\infty$ in the above inequality above, we obtain

$$\lim_{n\to+\infty} n[\sigma(\xi_n, \xi_{n+1})]^k = 0. \tag{13}$$

Case II: In the case where $M = +\infty$, we select an arbitrary number $L > 0$. By employing the definition of a limit, we can find an $n_1 \in \mathbb{N}$ such that

$$\frac{\theta(\sigma(\xi_n, \xi_{n+1})) - 1}{[\sigma(\xi_n, \xi_{n+1})]^k} \geq L, \quad \text{for all } n \geq n_1,$$

which gives

$$n[\sigma(\xi_n, \xi_{n+1})]^k \leq n[\frac{\theta(\sigma(\xi_n, \xi_{n+1})) - 1}{L}].$$

Again, by applying Equation (9) to the aforementioned inequality and taking the limit as $n \to +\infty$, we obtain:

$$\lim_{n\to+\infty} n[\sigma(\xi_n, \xi_{n+1})]^k = 0. \tag{14}$$

Thus, based on Equations (13) and (14), we can deduce that for any $M \in (0, +\infty]$ and $0 < k < 1$, there exists $\tilde{N} \in \mathbb{N}$, where $\tilde{N} = \max\{n_0, n_1\}$ such that:

$$\sigma(\xi_n, \xi_{n+1}) \leq \frac{1}{n^{1/k}}, \quad \text{for all } n \geq \hat{N}. \tag{15}$$

To demonstrate that the sequence $\{\xi_n\}$ is a Cauchy sequence, we consider two cases. For any $m, n \in \mathbb{N}$, we have the following:

Case 1: We begin by considering an odd number denoted as $p = 2m+1$, where $m \geq 1$. By utilizing the property (q3) from Definition 2, we obtain:

$$\begin{aligned}\sigma(\xi_n, \xi_{n+2m+1}) &\leq \beta(\xi_n, \xi_{n+1})\sigma(\xi_n, \xi_{n+1}) + \mu(\xi_{n+1}, \xi_{n+2})\sigma(\xi_{n+1}, \xi_{n+2}) \\ &+ \gamma(\xi_{n+2}, \xi_{n+2m+1})\sigma(\xi_{n+2}, \xi_{n+2m+1}) \\ &\leq \beta(\xi_n, \xi_{n+1})\sigma(\xi_n, \xi_{n+1}) + \mu(\xi_{n+1}, \xi_{n+2})\sigma(\xi_{n+1}, \xi_{n+2}) \\ &+ \gamma(\xi_{n+2}, \xi_m)[\beta(\xi_{n+2}, \xi_{n+3})\sigma(\xi_{n+2}, \xi_{n+3}) + \mu(\xi_{n+3}, \xi_{n+4})\sigma(\xi_{n+3}, \xi_{n+4}) \\ &+ \gamma(\xi_{n+4}, \xi_{n+2m+1})\sigma(\xi_{n+4}, \xi_{n+2m+1})]. \\ &\vdots\end{aligned}$$

$$\begin{aligned}\sigma(\xi_n, \xi_{n+2m+1}) &\leq \beta(\xi_n, \xi_{n+1})\sigma(\xi_n, \xi_{n+1}) + \mu(\xi_{n+1}, \xi_{n+2})\sigma(\xi_{n+1}, \xi_{n+2}) \\ &+ \sum_{i=\frac{n}{2}+1}^{\frac{n+2m-2}{2}} [\beta(\xi_{2i}, \xi_{2i+1})\sigma(\xi_{2i}, \xi_{2i+1}) + \mu(\xi_{2i+1}, \xi_{2i+2})\sigma(\xi_{2i+1}, \xi_{2i+2})] \\ &\left(\prod_{j=\frac{n}{2}+1}^{i} \gamma(\xi_{2j}, \xi_{n+2m+1})\right) + \prod_{i=\frac{n}{2}+1}^{\frac{n+2m}{2}} \gamma(\xi_{2i}, \xi_{n+2m+1})\sigma(\xi_{n+2m}, \xi_{n+2m+1}) \\ &\leq \beta(\xi_n, \xi_{n+1})\sigma(\xi_n, \xi_{n+1}) + \mu(\xi_{n+1}, \xi_{n+2})\sigma(\xi_{n+1}, \xi_{n+2}) \\ &+ \sum_{i=\frac{n}{2}+1}^{\frac{n+2m-2}{2}} \beta(\xi_{2i}, \xi_{2i+1})\sigma(\xi_{2i}, \xi_{2i+1}) \left(\prod_{j=\frac{n}{2}+1}^{i} \gamma(\xi_{2j}, \xi_{n+2m+1})\right) \\ &+ \prod_{i=\frac{n}{2}+1}^{\frac{n+2m}{2}} \gamma(\xi_{2i}, \xi_{n+2m+1})\sigma(\xi_{n+2m}, \xi_{n+2m+1}) \\ &+ \sum_{i=\frac{n}{2}+1}^{\frac{n+2m-2}{2}} \mu(\xi_{2i+1}, \xi_{2i+2})\sigma(\xi_{2i+1}, \xi_{2i+2}) \left(\prod_{j=\frac{n}{2}+1}^{i} \gamma(\xi_{2j}, \xi_{n+2m+1})\right).\end{aligned}$$

We can express the last inequality as follows:

$$\begin{aligned}\sigma(\xi_n, \xi_{n+2m+1}) &\leq \beta(\xi_n, \xi_{n+1})\sigma(\xi_n, \xi_{n+1}) + \mu(\xi_{n+1}, \xi_{n+2})\sigma(\xi_{n+1}, \xi_{n+2}) \\ &+ \sum_{i=\frac{n}{2}+1}^{\frac{n+2m}{2}} \beta(\xi_{2i}, \xi_{2i+1})\sigma(\xi_{2i}, \xi_{2i+1}) \left(\prod_{j=\frac{n}{2}+1}^{i} \gamma(\xi_{2j}, \xi_{n+2m+1})\right) \\ &+ \sum_{i=\frac{n}{2}+1}^{\frac{n+2m-2}{2}} \mu(\xi_{2i+1}, \xi_{2i+2})\sigma(\xi_{2i+1}, \xi_{2i+2}) \left(\prod_{j=\frac{n}{2}+1}^{i} \gamma(\xi_{2j}, \xi_{n+2m+1})\right).\end{aligned}$$

By utilizing Equation (15) in the last inequality, we can deduce that $\sigma(\xi_{2i}, \xi_{2i+1}) \leq \frac{1}{(2i)^{1/k}}$. Hence, we obtain:

$$\begin{aligned}\sigma(\xi_n, \xi_{n+2m+1}) &\leq \beta(\xi_n, \xi_{n+1})\frac{1}{(n)^{1/k}} + \mu(\xi_{n+1}, \xi_{n+2})\frac{1}{(n+1)^{1/k}} \\ &+ \sum_{i=\frac{n}{2}+1}^{\frac{n+2m}{2}} \beta(\xi_{2i}, \xi_{2i+1}) \left(\prod_{j=\frac{n}{2}+1}^{i} \gamma(\xi_{2j}, \xi_{n+2m+1})\right) \frac{1}{(2i)^{1/k}} \\ &+ \sum_{i=\frac{n}{2}+1}^{\frac{n+2m-2}{2}} \mu(\xi_{2i+1}, \xi_{2i+2}) \left(\prod_{j=\frac{n}{2}+1}^{i} \gamma(\xi_{2j}, \xi_{n+2m+1})\right) \frac{1}{(2i+1)^{1/k}}.\end{aligned}$$

We can express it as follows:

$$\sigma(\xi_n, \xi_{n+2m+1}) \leq \beta(\xi_n, \xi_{n+1}) \frac{1}{(n)^{1/k}} + \mu(\xi_{n+1}, \xi_{n+2}) \frac{1}{(n+1)^{1/k}} \tag{16}$$

$$+ [\Lambda_{n+2m/2} - \Lambda_{n/2}] + [\Omega_{n+2m-2/2} - \Omega_{n/2}]. \tag{17}$$

where

$$\Lambda_p = \sum_{i=1}^{p} \left(\prod_{j=1}^{i} \gamma(\xi_{2j}, \xi_{n+2m+1}) \right) \beta(\xi_{2i}, \xi_{2i+1}) \frac{1}{(2i)^{1/k}},$$

and

$$\Omega_q = \sum_{i=1}^{q} \left(\prod_{j=1}^{i} \gamma(\xi_{2j}, \xi_{n+2m+1}) \right) \mu(\xi_{2i+1}, \xi_{2i+2}) \frac{1}{(2i+1)^{1/k}}.$$

By employing the ratio test and utilizing (5) and (6), we can conclude that $\lim_{n,m\to+\infty}[\Lambda_{n+2m/2} - \Lambda_{n/2}] = 0$ and $\lim_{n,m\to+\infty}[\Omega_{n+2m-2/2} - \Omega_{n/2}] = 0$.

Moreover, (7) implies that $\lim_{n\to+\infty} \beta(\xi_n, \xi_{n+1})(\frac{1}{n^{1/k}}) = 0$ and $\lim_{n\to+\infty} \mu(\xi_{n+1}, \xi_{n+2}) \frac{1}{(n+1)^{1/k}} = 0$ since $k \in (0,1)$. Hence,

$$\lim_{n,m\to+\infty} \sigma(\xi_n, \xi_{n+2m+1}) = 0. \tag{18}$$

Case 2: Now, let us consider an even number denoted as $p = 2m$, where $m \geq 1$. By applying the property (q3) from Definition 2, we obtain:

$$\begin{aligned}\sigma(\xi_n, \xi_{n+2m}) &\leq \beta(\xi_n, \xi_{n+1})\sigma(\xi_n, \xi_{n+1}) + \mu(\xi_{n+1}, \xi_{n+2})\sigma(\xi_{n+1}, \xi_{n+2}) \\ &\quad + \gamma(\xi_{n+2}, \xi_{n+2m})\sigma(\xi_{n+2}, \xi_{n+2m}). \\ &\vdots\end{aligned}$$

By following the same steps as in case 1, we eventually arrive at the conclusion:

$$\begin{aligned}\sigma(\xi_n, \xi_{n+2m}) &\leq \beta(\xi_n, \xi_{n+1})\sigma(\xi_n, \xi_{n+1}) + \mu(\xi_{n+1}, \xi_{n+2})\sigma(\xi_{n+1}, \xi_{n+2}) \\ &\quad + \sum_{i=\frac{n}{2}+1}^{\frac{n+2m-1}{2}} \beta(\xi_{2i}, \xi_{2i+1})\sigma(\xi_{2i}, \xi_{2i+1}) \left(\prod_{j=\frac{n}{2}+1}^{i} \gamma(\xi_{2j}, \xi_{n+2m}) \right) \\ &\quad + \sum_{i=\frac{n}{2}+1}^{\frac{n+2m-3}{2}} \mu(\xi_{2i+1}, \xi_{2i+2})\sigma(\xi_{2i+1}, \xi_{2i+2}) \left(\prod_{j=\frac{n}{2}+1}^{i} \gamma(\xi_{2j}, \xi_{n+2m}) \right).\end{aligned}$$

By utilizing the fact that $\sigma(\xi_{2i}, \xi_{2i+1}) \leq \frac{1}{(2i)^{1/k}}$, we can express the above inequalities as follows:

$$\begin{aligned}\sigma(\xi_n, \xi_{n+2m}) &\leq \beta(\xi_n, \xi_{n+1}) \frac{1}{(n)^{1/k}} + \mu(\xi_{n+1}, \xi_{n+2}) \frac{1}{(n+1)^{1/k}} \\ &\quad + \sum_{i=\frac{n}{2}+1}^{\frac{n+2m-1}{2}} \beta(\xi_{2i}, \xi_{2i+1}) \left(\prod_{j=\frac{n}{2}+1}^{i} \gamma(\xi_{2j}, \xi_{n+2m}) \right) \frac{1}{(2i)^{1/k}} \\ &\quad + \sum_{i=\frac{n}{2}+1}^{\frac{n+2m-3}{2}} \mu(\xi_{2i+1}, \xi_{2i+2}) \left(\prod_{j=\frac{n}{2}+1}^{i} \gamma(\xi_{2j}, \xi_{n+2m}) \right) \frac{1}{(2i+1)^{1/k}}.\end{aligned}$$

By expressing it as in Equation (16):

$$\sigma(\xi_n,\xi_{n+2m}) \leq \beta(\xi_n,\xi_{n+1})\frac{1}{(n)^{1/k}} + \mu(\xi_{n+1},\xi_{n+2})\frac{1}{(n+1)^{1/k}}$$
$$+ [\Lambda_{n+2m/2} - \Lambda_{n/2}] + [\Omega_{n+2m-2/2} - \Omega_{n/2}].$$

where

$$\Lambda_p = \sum_{i=1}^{p}\left(\prod_{j=1}^{i}\gamma(\xi_{2j},\xi_{n+2m})\right)\beta(\xi_{2i},\xi_{2i})\frac{1}{(2i)^{1/k}},$$

and

$$\Omega_q = \sum_{i=1}^{q}\left(\prod_{j=1}^{i}\gamma(\xi_{2j},\xi_{n+2m})\right)\mu(\xi_{2i+1},\xi_{2i+2})\frac{1}{(2i+1)^{1/k}}.$$

By employing the ratio test and applying (5)–(7), we can proceed with the same procedures as before to establish that:

$$\lim_{n,m\to+\infty}\sigma(\xi_n,\xi_{n+2m}) = 0. \tag{19}$$

Hence, the sequence $\{\xi_n\}$ is a Cauchy sequence in a complete $\mathcal{TCRMLS}$, implying that it converges to some $\hat{\xi} \in \mathfrak{F}$, i.e.,

$$\lim_{n\to+\infty}\sigma(\xi_n,\hat{\xi}) = \sigma(\hat{\xi},\hat{\xi}) = \lim_{n,m\to+\infty}\sigma(\xi_n,\xi_{n+2m}) = 0.$$

Next, in order to demonstrate that the mapping T fixes $\hat{\xi}$, we observe that since T satisfies (4), we have the following:

$$ln(\theta(\sigma(T\xi,Tw))) \leq r(ln(\theta(\sigma(\xi,w)))) \leq ln(\theta(\sigma(\xi,w))).$$

Since θ is non-decreasing, we deduce that T is continuous. Hence, by utilizing Lemma 2 in [29], we obtain

$$\lim_{n\to+\infty}\sigma(T\xi_n,T\hat{\xi}) = \sigma(T\hat{\xi},T\hat{\xi}) = 0. \tag{20}$$

To illustrate that $T\hat{\xi} = \hat{\xi}$ by (20) we have

$$\sigma(\hat{\xi},T\hat{\xi}) = \lim_{n\to+\infty}\sigma(\xi_{n+1},T\hat{\xi}) = \lim_{n\to+\infty}\sigma(T\xi_n,T\hat{\xi}) = 0. \tag{21}$$

This implies that $\sigma(\hat{\xi},T\hat{\xi}) = 0$, i.e., $\hat{\xi}$ is fixed by T, $T\hat{\xi} = \hat{\xi}$.

Suppose that T has two fixed points, $\hat{a}$ and $\hat{b}$. To establish the uniqueness of the fixed points, assume that $\hat{a} \neq \hat{b}$. Then, by utilizing Equation (4), we can deduce the following:

$$\begin{aligned}\theta(\sigma(\hat{a},\hat{b})) &= \theta(\sigma(T\hat{a},T\hat{b}))\\ &\leq [\theta(\sigma(\hat{a},\hat{b}))]^r\\ &< \theta(\sigma(\hat{a},\hat{b})).\end{aligned}$$

This leads to a contradiction, implying that $\hat{a} = \hat{b}$. Therefore, T has a unique fixed point. □

Now, we provide an illustrative example that supports Theorem 1.

Example 4. *Let $\mathfrak{F} = [0,2]$. Define the mapping $\sigma : \mathfrak{F} \times \mathfrak{F} \to [0,+\infty)$ by:*

$$\sigma(\xi, w) = \begin{cases} 1 & \text{if } \xi = w = 2, \\ |\xi - w|^4 & \text{otherwise.} \end{cases}$$

Let $\beta, \mu, \gamma : \mathfrak{F} \times \mathfrak{F} \to [1,+\infty)$ be defined as:

$$\beta(\xi, w) = \begin{cases} max\{\xi, w\} + 1 & \text{if } 0 \leq \xi, w \leq 1, \\ 4 & \text{otherwise.} \end{cases}$$

$$\mu(\xi, w) = \begin{cases} \xi^2 + w^2 + 1 & \text{if } 0 \leq \xi, w \leq 1, \\ 1 & \text{otherwise.} \end{cases}$$

and

$$\gamma(\xi, w) = \begin{cases} \xi + w & \text{if } 0 \leq \xi \leq 1, \\ 2 & \text{otherwise.} \end{cases}$$

One can verify that $(\mathfrak{F}, \sigma)$ is a complete $\mathcal{TCRMLS}$.

Consider the contraction mapping $T : \mathfrak{F} \to \mathfrak{F}$ defined by $T(\xi) = \frac{\xi}{5}$, and let $\theta : (0,+\infty) \to (1,+\infty)$ be defined as $\theta(t) = 5^{\sqrt{t}}$. Then, $\theta \in \Theta$. The sequence $\{\xi_n\}$ is formed as follows: we start with $\xi_0 = 1$, and then by repeatedly applying the iteration $T\xi_{n-1} = \xi_n$, we obtain $\xi_1 = T\xi_0 = T(1) = \frac{1}{5}$, $\xi_2 = T(\frac{1}{5}) = \frac{1}{5^2}$, thus $\xi_n = T^n(1) = \frac{1}{5^n}$ for all $n \in \mathbb{N}$. So,

$$\frac{\beta(\xi_{2i+2}, \xi_{2i+3})}{\beta(\xi_{2i}, \xi_{2i+1})} \gamma(\xi_{2i+2}, \xi_m) = \frac{(\frac{1}{5^{2i+2}} + 1)}{(\frac{1}{5^{2i}} + 1)} (\frac{1}{5^{2i+2}} + \frac{1}{5^m}).$$

Hence, we obtain

$$\sup_{m \geq 1} \lim_{i \to +\infty} \frac{\beta(\xi_{2i+2}, \xi_{2i+3})}{\beta(\xi_{2i}, \xi_{2i+1})} \gamma(\xi_{2i+2}, \xi_m) < 1. \tag{22}$$

Similarly,

$$\sup_{m \geq 1} \lim_{i \to +\infty} \frac{\mu(\xi_{2i+3}, \xi_{2i+4})}{\mu(\xi_{2i+1}, \xi_{2i+2})} \gamma(\xi_{2i+2}, \xi_m) = \sup_{m \geq 1} \lim_{i \to +\infty} \frac{((\frac{1}{5^{2i+3}})^2 + (\frac{1}{5^{2i+4}})^2 + 1)}{((\frac{1}{5^{2i+1}})^2 + (\frac{1}{5^{2i+2}})^2 + 1)} (\frac{1}{5^{2i+2}} + \frac{1}{5^m}) < 1. \tag{23}$$

Furthermore, it can be easily shown that:

$$\lim_{n \to +\infty} \beta(\xi, \xi_n), \ \lim_{n \to +\infty} \mu(\xi_n, \xi), \ \text{and}, \ \lim_{n \to +\infty} \gamma(\xi_n, \xi) \ \text{are all finite.}$$

Taking $r = 1/5$, we determine if $\theta(\sigma(T\xi, Tw)) \leq [\theta(\sigma(\xi, w))]^{1/5}$. Take any $\xi, w \in \mathfrak{F}$ so that $\sigma(T\xi, Tw) \neq 0$. As $\sigma(T\xi, Tw) = |\frac{\xi}{5} - \frac{w}{5}|^4$, hence

$$\theta(\sigma(T\xi, Tw)) = 5^{\sqrt{|\frac{\xi}{5} - \frac{w}{5}|^4}} = 5^{\frac{1}{25}|\xi - w|^2} \leq [5^{\sqrt{|\xi - w|^4}}]^{1/5}.$$

Therefore, all conditions of Theorem 1 are satisfied. Hence, T has a unique fixed point $\xi = 0$ in $\mathfrak{F}$.

4. Application to Fredholm Type Integral Equation and Numerical Iteration

In this section, we examine the application of our main theorem. Let us consider Example 3, where $\mathfrak{F} = C([0,1], \mathbb{R})$, and $\sigma : \mathfrak{F} \times \mathfrak{F} \longrightarrow [0,+\infty)$ is defined as before:

$$\sigma(f, g) = sup_{t \in [0,1]} |f(t) - g(t)|^4.$$

Furthermore, we have $\sigma(1/5, 1/5) = 1/85$. The controlled functions β, μ, and γ are defined as in Example 3. This establishes that $(\mathfrak{F}, \sigma)$ is a complete $\mathcal{TCRMLS}$.

Let $T : \mathfrak{F} \to \mathfrak{F}$, be the Fredholm-type integral operator defined as follows:

$$Tf(t) = \Xi(t) + \int_0^1 \lambda(t, s, f(s))ds \text{ for } t, s \in [0,1], \text{ and } f, \xi \in \mathfrak{F}, \tag{24}$$

where $\lambda(t, s, f(s)) : [0,1]^2 \longrightarrow \mathbb{R}$, and $\Xi : [0,1] \longrightarrow \mathbb{R}$ are continuous functions.

Theorem 2. *Consider the complete $\mathcal{TCRMLS}$ defined in Example 3. Assume that for any $f, g \in \mathfrak{F}$, there exists $\nu \in [1, +\infty)$ such that the following condition holds:*

$$|\lambda(t, s, f(s)) - \lambda(t, s, g(s))| \leq e^{\frac{-\nu}{4}} |f(s) - g(s)|. \tag{25}$$

Then, the integral Equation (24) has a unique solution.

Proof. For any $t \in [0,1]$ and $f, g \in \mathfrak{F}$, we have

$$\begin{aligned} \Big|Tf(t) - Tg(t)\Big|^4 &= \Big|(\Xi(t) + \int_0^1 \lambda(t, s, f(s))ds) - (\Xi(t) + \int_0^1 \lambda(t, s, g(s))ds)\Big|^4 \\ &\leq \int_0^1 \Big|\lambda(t, s, f(s)) - \lambda(t, s, g(s))\Big|^4 ds \\ &\leq \int_0^1 e^{-\nu} |f(s) - g(s)|^4 ds \\ &\leq e^{-\nu} \sigma(f, g). \end{aligned}$$

This gives

$$\sigma(Tf, Tg) \leq e^{-\nu} \sigma(f, g). \tag{26}$$

Let $\theta(t) = e^{\sqrt{t}}$; therefore, $\theta(t) \in \Theta$. To demonstrate that T satisfies Equation (4), let us take the square root of (26), which gives us the following inequality:

$$\sqrt{\sigma(Tf, Tg)} \leq \sqrt{e^{-\nu} \sigma(f, g)}.$$

Thus,

$$e^{\sqrt{\sigma(Tf,Tg)}} \leq (e^{\sqrt{\sigma(f,g)}})^r, \text{ where } r = \sqrt{e^{-\nu}} < 1, \text{since } \nu \geq 1.$$

Therefore, T satisfies (4) and accordingly, all the requirements of Theorem 1 are fulfilled, which suggests that the integral Equation (24) has a unique solution. □

Next, let us consider a special case of the Fredholm-type integral operator $T : \mathfrak{F} \to \mathfrak{F}$ in (24), where we set $\Xi(t) = \frac{4}{5}t$ and $\lambda(t, s, f(s)) = tf^4(s)$. Thus, the Fredholm-type integral operator T can be expressed as follows:

$$Tf(t) = \frac{4}{5}t + \int_0^1 tf^4(s)ds \text{ for } t, s \in [0,1], \text{ and } f \in \mathfrak{F} \tag{27}$$

where $\frac{4}{5}t$ and $tf^4(s)$ are continuous functions.

Theorem 3. *Consider $(\mathfrak{F}, \sigma)$ a complete $\mathcal{TCRMLS}$, as in Example 3, and let T be as in (27). Assume that for any $f, g \in \mathfrak{F}$, there exists $\nu \in [1, +\infty)$ such that*

$$|tf^4(s) - tg^4(s)| \leq e^{\frac{-\nu}{4}} |f(s) - g(s)|. \tag{28}$$

Then, the integral Equation (27) has a unique solution.

Proof. The proof follows a similar approach to the proof of Theorem 2.

Next, starting with the initial condition $f_0(t) = 0$ and performing numerical iterations on Equation (27), we obtain the following:

$$f_{n+1}(t) = Tf_n(t) = \frac{4}{5}t + \int_0^1 t f_n^4(s) ds, \; f \in \mathfrak{F}. \tag{29}$$

By substituting values for $n \in \mathbb{N}$ into Equation (29), we obtain the following:

$$f_1(t) = 0.8t,$$

$$f_2(t) = 0.888192t,$$

$$f_3(t) = 0.9209892437t,$$

$$f_4(t) = 0.9438958364t,$$

$$f_5(t) = 0.9587545851t,$$

$$f_6(t) = 0.9689895354t,$$

$$f_7(t) = 0.9763219350t,$$

$$f_8(t) = 0.9817197717t,$$

$$f_9(t) = 0.9857719529t,$$

$$f_{10}(t) = 0.9888581911t,$$

$$\vdots$$

$$f_{81}(t) = 0.99999999986t,$$

$$f_{82}(t) = 0.99999999989t,$$

$$f_{83}(t) = 0.99999999991t,$$

$$f_{84}(t) = 0.99999999993t,$$

$$f_{85}(t) = 0.99999999994t,$$

$$f_{86}(t) = 0.99999999995t,$$

$$f_{87}(t) = 0.99999999996t,$$

$$f_{88}(t) = 0.99999999997t,$$

$$f_{89}(t) = 0.99999999998t,$$

$$f_{90}(t) = 0.99999999998t,$$

$$f_{91}(t) = t.$$

All the requirements of Theorem 1 are satisfied, indicating that the integral Equation (27) has a unique solution, which is the function $f(t) = t$. Furthermore, the effectiveness of the proposed methodology has been confirmed by the numerical results. Figure 1 illustrates the convergence behavior of the iterations, with the t values depicted on the x-axis and the numerical iteration values $f_j(t)$ shown on the y-axis. $\square$

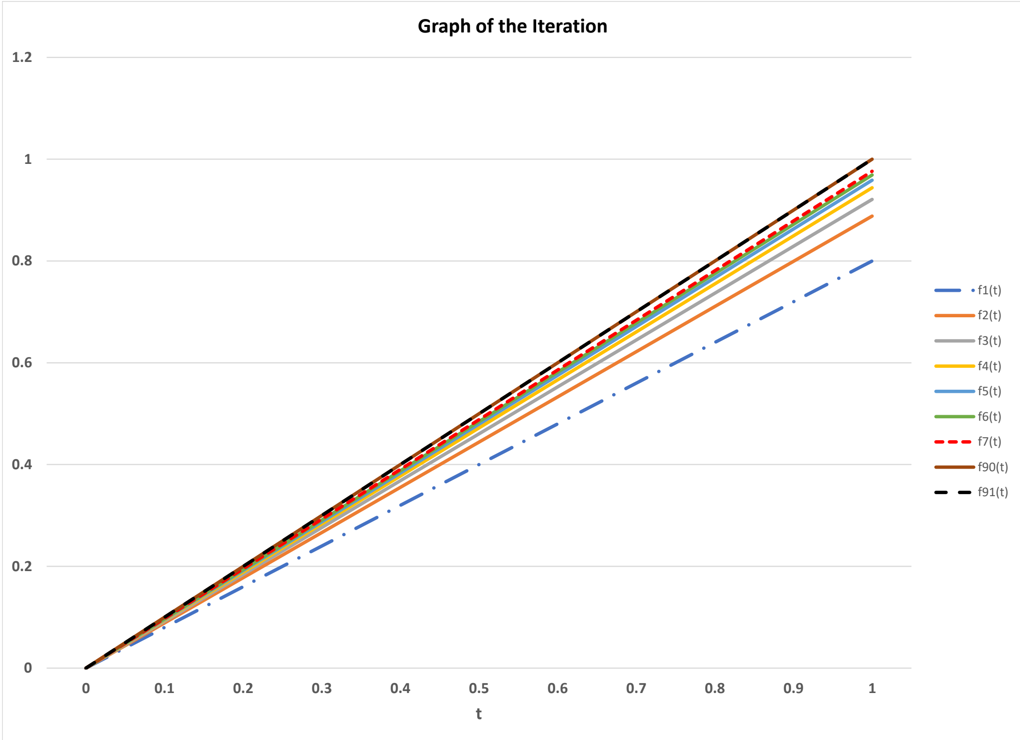

Figure 1. The graph displays the convergence of the iterations.

5. Conclusions

In this article, we have introduced the novel concept of triple controlled metric-like spaces, denoted as $\mathcal{TCRMLS}$. We have provided a comprehensive definition of these spaces and presented illustrative examples. We have also investigated the properties of Θ-contraction mappings within the class of complete $\mathcal{TCRMLS}$ and established a fixed point theorem in this setting. Furthermore, we have demonstrated the applicability of our theory by considering the existence of solutions to the Fredholm integral equation. The numerical analysis and results have provided strong support for our main findings, highlighting the significance of our approach.

Our study only considered Θ-contraction mappings in triple controlled metric-like spaces; however, there are many other types of contraction mappings that could be investigated. Additionally, our study focused on the theoretical development of triple controlled metric-like spaces and their application to fixed point theory. However, we did not explore the practical implications of our work in other areas of mathematics such as computer science, engineering, and behavioral sciences.

We conclude the article with the following question for future research: Wardowski [25] introduced a new type of contractions called $\mathcal{F}$-contractions, whereas Azmi [22] introduced $(\alpha - \mathcal{F})$-contractive mappings on triple controlled metric-type spaces and established fixed point results.

Question:

Let $(\mathfrak{F}, \sigma)$ be a complete triple controlled metric-like space, and let $T : \mathfrak{F} \to \mathfrak{F}$ be an $(\alpha - \mathcal{F})$-contractive mapping. This means that there exists a mapping $\alpha : \mathfrak{F} \times \mathfrak{F} \to [0, +\infty)$, $F \in \mathcal{F}$, and a constant $\tau > 0$ such that the following inequality holds:

$$\tau + \alpha(x, y)F(\sigma(Tx, Ty)) \leq F(\sigma(x, y)),$$

for all $x, y \in \mathfrak{F}$, where $\sigma(Tx, Ty) > 0$.

Under what other condition(s) does T have a unique fixed point in $(\mathfrak{F}, \sigma)$?

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