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Fixed-Point Estimation by Iterative Strategies and Stability Analysis with Applications

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Abstract: In this study, we developed a new faster iterative scheme for approximate fixed points. This technique was applied to discuss some convergence and stability results for almost contraction mapping in a Banach space and for Suzuki generalized nonexpansive mapping in a uniformly convex Banach space. Moreover, some numerical experiments were investigated to illustrate the behavior and efficacy of our iterative scheme. The proposed method converges faster than symmetrical iterations of the S algorithm, Thakur algorithm and K^* algorithm. Eventually, as an application, the nonlinear Volterra integral equation with delay was solved using the suggested method.

Keywords: fixed-point methodology; convergence result; stability analysis; Volterra integral equation; delay term

MSC: 47H10; 47H09; 54H25

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1. Introduction

Many problems in mathematics and other fields of science can be modeled into an equation with a suitable operator. The existence of the solution to this equation equates to the existence of the fixed point (FP) of the appropriate operator. Due to the large number of recent, valuable studies that include the FP method, these points have become the mainstay for nonlinear analysis due to the ease and smoothness of this method, in addition to the numerous and exciting applications in economics, biology, chemistry, game theory, engineering, physics, etc. [1–6].

A very important branch is the involvement of FPs in approximation by symmetrical algorithms. Numerous problems such as convex feasibility problems, convex optimization problems, monotone variational inequalities and image restoration problems can be thought of as FP problems of nonexpansive mappings; hence, approximating them has a range of specialized applications.

In this paper, the symbols Ω, Θ, $\mathbb{R}^+$, $\mathbb{N}$, $\rightharpoonup$, $\longrightarrow$ and $\lambda(\Im)$ refer to the Banach space, a nonempty closed convex subset (CCS) of an Ω, the set of nonnegative real numbers, the set of natural numbers, weak convergence, strong convergence and the set of all FPs (the point $\theta \in \Theta$ so that an equation $\theta = \Im\theta$ is true).

There are two main categories that can be used to group the main concepts of FP theory. Finding the prerequisites and requirements necessary for an operator to own fixed points is the first step. Another option is to locate these fixed points using certain schematic methods. The first category is known formally as the existence part, while the second category is known as the computation or approximation part. Studying the behaviors of FPs, such as stability and data dependence, is an essential but less-well-known topic of FP theory.

The class of weak contractions that appropriately covers the class of Zamfirescu operators [7] was supplied by Berinde in [8]. Similarly, many authors also refer to this class of mappings as almost contraction mappings (ACMs).

Definition 1. *A mapping $\Im : \Theta \to \Theta$ is called ACM if the following inequality is true: :*

$$\|\Im\theta - \Im\vartheta\| \leq \ell_1\|\theta - \vartheta\| + \ell_2\|\theta - \Im\theta\|, \text{ for all } \theta, \vartheta \in \Theta, \tag{1}$$

where $0 < \ell_1 < 1$ *and* $\ell_2 \geq 0$.

In 2003, ACM (1) was generalized by Imoru and Olantiwo [9] by replacing the constant ℓ_2 with a strictly increasing continuous function $\varpi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varpi(0) = 0$ as follows:

Definition 2. *A mapping $\Im : \Theta \to \Theta$ is called contractive-like if there exist a constant $0 < \ell_1 < 1$ and a strictly increasing continuous function $\varpi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varpi(0) = 0$ such that*

$$\|\Im\theta - \Im\vartheta\| \leq \ell_1\|\theta - \vartheta\| + \varpi(\|\theta - \Im\theta\|), \text{ for all } \theta, \vartheta \in \Theta. \tag{2}$$

Clearly, Inequality (2) is symmetrical to Inequality (1) if $\varpi(t) = \ell_2(t)$.

The analysis of the performance and behavior of algorithms that make significant contributions to real-world applications is one of the key trends in FP techniques. Therefore, in order to enhance the functionality and convergence behavior of algorithms for nonexpansive mappings, several authors tended to develop numerous symmetrical iterative schemes for approximating FPs, for example, Mann [10], Ishikawa [11], Noor [12], Argawal et al. [13], Abbas and Nazir [14] and HR [15,16].

Let $\{\sigma_j\}$ and $\{\kappa_j\}$ be sequences in $[0,1]$; the following procedures are known as the S algorithm [13], Picard-S algorithm [17], Thakur algorithm [18] and K^* algorithm [19]:

$$\begin{cases} \xi_\circ \in \Theta, \\ \rho_j = (1-\sigma_j)\xi_j + \sigma_j\Im\xi_j, & \forall j \geq 1. \\ \xi_{j+1} = (1-\kappa_j)\xi_i + \gamma_i\Im\rho_i, \end{cases} \tag{3}$$

$$\begin{cases} \xi_\circ \in \Theta, \\ \rho_j = (1-\sigma_j)\xi_j + \sigma_j\Im\xi_j, \\ \Upsilon_j = (1-\kappa_j)\xi_j + \kappa_j\Im\rho_j, \\ \xi_{j+1} = \Im\Upsilon_j, \end{cases} \forall j \geq 1. \tag{4}$$

$$\begin{cases} \xi_\circ \in \Theta, \\ \rho_j = (1-\sigma_j)\xi_j + \sigma_j\Im\xi_j, \\ \Upsilon_j = \Im\big((1-\kappa_j)\xi_j + \kappa_j\rho_j\big), \\ \xi_{j+1} = \Im\Upsilon_j, \end{cases} \forall j \geq 1. \tag{5}$$

$$\begin{cases} \xi_\circ \in \Theta, \\ \rho_j = (1-\sigma_j)\xi_j + \sigma_j\Im\xi_j, \\ \Upsilon_j = \Im\big((1-\kappa_j)\xi_j + \kappa_j\xi_j\big), \\ \xi_{j+1} = \Im\Upsilon_j, \end{cases} \forall j \geq 1. \tag{6}$$

Analytically and numerically, for contractive-like mappings, the authors showed that the iterative technique (6) converges faster than those of Karakaya et al. [20] and Thakur et al. [18], respectively. Consequently, Iteration (6) converges faster than (3), (4) and (5).

On the other hand, nonlinear integral Equations (NIEs) are used to explain mathematical models that derive from mathematical physics, engineering, economics, biology, etc. In particular, NIEs are produced by boundary value problems and mathematical modeling of the spatiotemporal dynamics of the epidemic. Recently, many authors have used iterative methods to solve NIEs; for instance, see [21–26]. The efficiency and effectiveness of iterative methods are determined by several factors, the most important of which are speed, stability and reliability. Many researchers and writers have studied these factors using the fixed-point method. For more details, see [27,28].

Continuing on the same approach, in this paper, we introduce the convergence and stability results for ACMs and Suzuki generalized nonexpansive mappings (SGNMs) in a BS and a uniformly convex Banach space (UCBS) in the following faster iterative scheme:

$$\begin{cases} \xi_\circ \in \Theta, \\ \rho_j = (1-\sigma_j)\xi_j + \sigma_j \Im \xi_j, \\ \Upsilon_j = \Im\big((1-\kappa_j)\rho_j + \kappa_j \Im \rho_j\big), \quad \text{for all } j \in \mathbb{N}, \\ \Lambda_j = \Im\big((1-\tau_j)\Upsilon_j + \tau_j \Im \Lambda_j\big), \\ \xi_{j+1} = \Im \Lambda_j, \end{cases} \tag{7}$$

where σ_j, κ_j and τ_j are sequences in $[0,1]$. Some numerical examples are given to illustrate that the considered iteration converges faster than the iterations of the S algorithm, Thakur algorithm and K^* algorithm with appropriate parameters. Ultimately, the proposed method is implicated in finding the solution to a nonlinear Volterra integral equation with delay.

2. Preliminary Work

In this section, we provide some definitions and lemmas that will be helpful in the sequel.

Definition 3. *A mapping $\Im : \Omega \to \Omega$ is called an SGNM if*

$$\frac{1}{2}\|\theta - \Im\theta\| \leq \|\theta - \vartheta\| \Rightarrow \|\Im\theta - \Im\vartheta\| \leq \|\theta - \vartheta\|, \textit{ for all } \theta, \vartheta \in \Omega.$$

Definition 4. *A BS Ω is called a uniformly convex if for each $\epsilon \in (0,2]$ there exists $\delta > 0$, such that for $\theta, \vartheta \in \Omega$ satisfying $\|\theta\| \leq 1$, $\|\vartheta\| \leq 1$ and $\|\theta - \vartheta\| > \epsilon$, we have $\left\|\frac{\theta+\vartheta}{2}\right\| < 1 - \delta$.*

Definition 5. *A BS Ω is considered to satisfy Opial's condition if for any sequence $\{\theta_j\}$ in Ω such that $\theta_j \rightharpoonup \theta \in \Omega$ implies*

$$\limsup_{j\to\infty}\|\theta_j - \theta\| < \limsup_{j\to\infty}\|\theta_j - \vartheta\|$$

for all $\vartheta \in \Omega$, where $\theta \neq \vartheta$.

Definition 6. *Assume that $\{\theta_j\}$ is a bounded sequence in Ω. For $\theta \in \Omega$, we set*

$$\nabla(\theta, \{\theta_j\}) = \limsup_{j\to\infty}\|\theta_j - \vartheta\|.$$

The asymptotic radius and center of $\{\theta_j\}$ relative to Ω are described as

$$\nabla(\Omega, \{\theta_j\}) = \inf\{\nabla(\theta, \{\theta_j\}) : \theta \in \Omega\}.$$

The asymptotic center of $\{\theta_j\}$ relative to Ω is defined by

$$Z(\Omega, \{\theta_j\}) = \{\theta \in \Omega \textit{ such that } \nabla(\theta, \{\theta_j\})) = \nabla(\Omega, \{\theta_j\})\}.$$

Clearly, $Z(\Omega, \{\theta_j\})$ contains one single point in a UCBS.

Definition 7 ([29])**.** *Let $\{\sigma_j\}$ and $\{\kappa_j\}$ be nonnegative real sequences converging to σ and κ, respectively. If there exist $\zeta \in \mathbb{R}^+$ such that $\zeta = \lim_{i\to\infty}\frac{\|\sigma_j - \sigma\|}{\|\kappa_j - \kappa\|}$, then we have the following possibilities:*

- *If $\zeta = 0$, then $\{\sigma_j\}$ converges to σ faster than κ_j does to κ;*
- *If $\zeta \in (0,\infty)$, then the two sequences have the same rate of convergence.*

Definition 8 ([30]). *A mapping $\Im : \Omega \to \Omega$ is said to be satisfy Condition I, if the inequality*

$$\mho(d(\vartheta, \lambda(\Im))) \leq \|\vartheta - \Im\vartheta\|,$$

is true, for all $\vartheta \in \Omega$, where $d(\vartheta, \lambda(\Im)) = \inf\{\|\vartheta - \theta\| : \theta \in \lambda(\Im)\}$.

Proposition 1 ([31]). *For a self-mapping $\Im : \Omega \to \Omega$, we have*

(1) *$\Im$ is an SGNM if $\Im$ is nonexpansive.*
(2) *$\Im$ is a quasi-nonexpansive mapping if $\Im$ is an SGNM with a nonempty FP set.*
(3) *If $\Im$ is an SGNM, then the inequality below holds*

$$\|\theta - \Im\theta\| \leq 3\|\Im\theta - \theta\| + \|\theta - \vartheta\|, \text{ for all } \vartheta, \theta \in \Omega.$$

Lemma 1 ([31]). *Assume that Θ is any subset of a BS Ω, which satisfies Opial's condition. Let $\Im : \Theta \to \Theta$ be an SGNM. If $\{\theta_i\} \rightharpoonup \theta$ and $\lim_{j\to\infty}\|\Im\vartheta_j - \theta_j\| = 0$, then $I - \Im$ is demiclosed at zero and $\Im\theta = \theta$.*

Lemma 2 ([31]). *If $\Im : \Omega \to \Omega$ is an SGNM, then it owns a FP provided that Θ is a weakly compact convex subset of a BS Ω.*

Lemma 3 ([29]). *Let $\{\psi_i\}$ and $\{\psi_j^*\}$ be nonnegative real sequences such that*

$$\psi_{i+1} \leq (1 - \varkappa_j)\psi_j + \psi_j^*, \ \varkappa_j \in (0,1), \text{ for each } j \geq 1,$$

if $\sum_{j=0}^{\infty} \varkappa_j = \infty$ and $\lim_{i\to\infty} \frac{\psi_j^*}{\varkappa_j} = 0$, then $\lim_{j\to\infty} \psi_j = 0$.

Lemma 4 ([32]). *Let $\{\psi_j\}$ and $\{\psi_j^*\}$ be nonnegative real sequences such that*

$$\psi_{j+1} \leq (1 - \varkappa_j)\psi_j + \varkappa_j\psi_j^*, \ \varkappa_j \in (0,1), \text{ for each } j \geq 1.$$

if $\sum_{j=0}^{\infty} \varkappa_j = \infty$, and $\psi_j^* \geq 0$, then $\limsup_{j\to\infty} \psi_j^* \geq \limsup_{j\to\infty} \psi_j \geq 0$.

Lemma 5 ([33]). *Let Ω be a UCBS and $\{\varkappa_j\}$ be a sequence such that $0 < u \leq \varkappa_j \leq u^* < 1$, for all $j \geq 1$. Assume that $\{\theta_j\}$ and $\{\vartheta_j\}$ are two sequences in Ω such that for some $\mu \geq 0$,*

$$\limsup_{j\to\infty}\{\vartheta_j\} \leq \mu, \ \limsup_{j\to\infty}\{\theta_j\} \leq \mu \text{ and } \limsup_{i\to\infty}\|\varkappa_j\theta_j + (1 - \varkappa_j)\vartheta_j\| = \mu.$$

Then $\lim_{i\to\infty}\|\theta_j - \vartheta_j\| = 0$.

3. Rate of Convergence

In this part, we demonstrate analytically that for ACMs, our iterative method (7) converges faster than the iterative method in (6) in the sense of Berinde [29].

Theorem 1. *Let Θ be a nonempty CCS of a BS Ω and $\Im : \Theta \to \Theta$ be a ACM with $\lambda(\Im) \neq \emptyset$. If $\{\xi_j\}$ is the iterative sequence given by (7) with $\{\sigma_j\}, \{\kappa_j\}, \{\tau_j\} \in [0,1]$ and $\sum_{j=0}^{\infty} \tau_j = \infty$. Then $\{\xi_j\} \longrightarrow \theta \in \lambda(\Im)$.*

Proof. Let $\theta \in \lambda(\Im)$; using (7), one has

$$\begin{aligned}
\|\rho_j - \theta\| &= \|(1-\sigma_j)\xi_j + \sigma_j \Im \xi_j - \theta\| \\
&= \|(1-\sigma_j)(\xi_j - \theta) + \sigma_j(\Im \xi_j - \zeta)\| \\
&\leq (1-\sigma_j)\|\xi_j - \theta\| + \sigma_j\|\Im \xi_j - \theta\| \\
&\leq (1-\sigma_j)\|\xi_j - \zeta\| + \ell_1 \sigma_j \|\xi_j - \theta\| \\
&= (1-(1-\ell_1)\sigma_j)\|\xi_j - \theta\|. \qquad (8)
\end{aligned}$$

From (7) and (8), we have

$$\begin{aligned}
\|\Upsilon_j - \theta\| &= \|\Im((1-\kappa_j)\rho_j + \kappa_j \Im \rho_j) - \theta\| \\
&= \|\Im\theta - \Im((1-\kappa_j)\rho_j + \kappa_j \Im \rho_j)\| \\
&\leq \ell_1\|\theta - ((1-\kappa_j)\rho_j + \kappa_j \Im \rho_j)\| + \ell_2\|\theta - \Im\theta\| \\
&= \ell_1\|(1-\kappa_j)(\rho_j - \theta) + \kappa_j(\Im \rho_j - \theta)\| \\
&\leq \ell_1[(1-\kappa_j)\|\rho_j - \theta\| + \ell_1\kappa_j\|\rho_j - \theta\|] \\
&\leq \ell_1[1-(1-\ell_1)\kappa_j]\|\rho_j - \zeta\| \\
&\leq \ell_1(1-(1-\ell_1)\kappa_j)(1-(1-\ell_1)\sigma_j)\|\xi_j - \theta\|. \qquad (9)
\end{aligned}$$

Using (7) and (9), we get

$$\begin{aligned}
\|\Lambda_j - \theta\| &= \|\Im((1-\tau_j)\Upsilon_j + \tau_j \Im \Upsilon_j) - \theta\| \\
&\leq \ell_1\|(1-\tau_j)(\Upsilon_j - \theta) + \kappa_j(\Im \Upsilon_j - \theta)\| \\
&\leq \ell_1[1-(1-\ell_1)\tau_j]\|\Upsilon_j - \zeta\| \\
&\leq \ell_1^2(1-(1-\ell_1)\kappa_j)(1-(1-\ell_1)\sigma_j)(1-(1-\ell_1)\tau_j)\|\xi_j - \theta\|. \qquad (10)
\end{aligned}$$

Utilizing (7) and (10), we can write

$$\begin{aligned}
\|\xi_{j+1} - \theta\| &= \|\Im \Lambda_j - \theta\| \\
&\leq \ell_1\|\Lambda_j - \theta\| \\
&\leq \ell_1^3(1-(1-\ell_1)\tau_j)(1-(1-\ell_1)\kappa_j)(1-(1-\ell_1)\sigma_j)\|\xi_j - \theta\|. \qquad (11)
\end{aligned}$$

As $\theta < 1$ and $0 \leq \kappa_j, \sigma_j \leq 1$, for all $j \in \mathbb{N}$, then $(1-(1-\ell_1)\kappa_j)(1-(1-\ell_1)\sigma_j) < 1$. Thus, (11) takes the form

$$\|\xi_{j+1} - \theta\| \leq \ell_1^3(1-(1-\ell_1)\tau_j)\|\xi_j - \theta\|. \qquad (12)$$

From (12), we deduce that

$$\begin{aligned}
\|\xi_{j+1} - \theta\| &\leq \ell_1^3(1-(1-\ell_1)\tau_j)\|\xi_j - \theta\| \\
&\leq \ell_1^3(1-(1-\ell_1)\tau_{j-1})\|\xi_{j-1} - \theta\| \\
&\quad \vdots \\
&\leq \ell_1^3(1-(1-\ell_1)\tau_0)\|\xi_0 - \theta\|. \qquad (13)
\end{aligned}$$

It follows from (13) that

$$\|\xi_{j+1} - \theta\| \leq \ell_1^{3(j+1)}\|\xi_0 - \theta\| \prod_{u=0}^{j}(1-(1-\ell_1)\tau_u). \qquad (14)$$

From the definition of θ and τ, we have $(1-(1-\theta)\gamma_u) < 1$. Since $1-u \leq e^{-u}$ for all $u \in [0,1]$, the inequality (14) can be written as

$$\|\xi_{j+1}-\theta\| \leq \frac{\ell_1^{3(j+1)}}{e^{(1-\ell_1)\sum_{u=0}^{j}\tau_u}}\|\xi_0-\theta\|. \tag{15}$$

Letting $j \to \infty$ in (15), we get $\lim_{j\to\infty}\|\xi_j-\theta\| = 0$,, i.e., $\{\xi_j\} \longrightarrow \theta \in \lambda(\Im)$.

For uniqueness. Let $\theta, \theta^* \in \lambda(\Im)$ such that $\theta \neq \theta^*$, hence

$$\begin{aligned}
\|\theta-\theta^*\| &= \|\Im\theta-\Im\theta^*\| \\
&\leq \ell_1\|\theta-\theta^*\| + \ell_2\|\theta-\Im\theta\| \\
&= \ell_1\|\theta-\theta^*\| \\
&< \|\theta-\theta^*\|,
\end{aligned}$$

which is a contradiction, that is, $\theta \neq \theta^*$. □

Theorem 2. *Let Θ be a nonempty CCS of a BS Ω and $\Im : \Theta \to \Theta$ be a ACM with $\lambda(\Im) \neq \emptyset$. If $\{\xi_j\}$ is the iterative sequence considered by (7) with $\{\sigma_j\}, \{\kappa_j\}, \{\tau_j\} \in [0,1]$ and $0 < \tau \leq \tau_j \leq 1$, for all $i \geq 1$. Then, $\{\theta_j\}$ converges to θ faster than the iterative approach (6).*

Proof. It follows from (14) and the assumption $0 < \tau \leq \tau_j \leq 1$ that

$$\begin{aligned}
\|\xi_{j+1}-\theta\| &\leq \ell_1^{3(j+1)}\|\xi_0-\theta\|\prod_{u=0}^{j}(1-(1-\ell_1)\tau_u) \\
&= \ell_1^{3(j+1)}\|\xi_0-\theta\|(1-(1-\ell_1)\tau)^{j+1}.
\end{aligned}$$

Similarly, the iterative process (6) ([19], Theorem 3.2) takes the form:

$$\|m_{j+1}-\theta\| \leq \ell_1^{2(j+1)}\|m_0-\theta\|\prod_{u=0}^{j}(1-(1-\ell_1)\tau_u). \tag{16}$$

Since $0 < \tau \leq \tau_j \leq 1$, for some $\tau > 0$ and all $j \geq 1$, then (16) can be written as

$$\begin{aligned}
\|m_{j+1}-\theta\| &\leq \ell_1^{2(j+1)}\|m_0-\theta\|\prod_{u=0}^{j}(1-(1-\ell_1)\tau_u) \\
&= \ell_1^{2(j+1)}\|m_0-\theta\|(1-(1-\ell_1)\tau)^{j+1}.
\end{aligned}$$

Set

$$\zeta = \ell_1^{3(j+1)}\|\xi_0-\theta\|(1-(1-\ell_1)\tau)^{j+1},$$

and

$$\widehat{\zeta} = \ell_1^{2(j+1)}\|m_0-\theta\|(1-(1-\ell_1)\tau)^{j+1}.$$

Then

$$\Delta_j = \frac{\zeta}{\widehat{\zeta}} = \frac{\ell_1^{3(j+1)}\|\xi_0-\theta\|(1-(1-\ell_1)\tau)^{j+1}}{\ell_1^{2(j+1)}\|m_0-\theta\|(1-(1-\ell_1)\tau)^{j+1}} = \ell_1^{j+1}.$$

Letting $j \to \infty$, we get $\lim_{j\to\infty}\Delta_j = 0$. Hence, $\{\xi_j\}$ converges faster than $\{m_j\}$ to θ. □

4. Convergence Results

In this section, we provide some convergence results of our iteration scheme (7) for the SGNM in the setting of UCBSs. First, we prove the following lemmas:

Lemma 6. *Let Θ be a nonempty CCS of a BS Ω and $\Im : \Theta \to \Theta$ be an SGNM with $\lambda(\Im) \neq \emptyset$. If $\{\xi_j\}$ is the iterative sequence proposed by (7), then $\lim_{j\to\infty} \|\xi_j - \theta\|$ exists, for each $\theta \in \lambda(\Im)$.*

Proof. Let $\theta \in \lambda(\Im)$ and $\vartheta \in \Theta$. By Proposition 1 (2), we have

$$\frac{1}{2}\|\theta - \Im\theta\| = 0 \leq \|\theta - \vartheta\| \Rightarrow \|\Im\theta - \Im\vartheta\| \leq \|\theta - \vartheta\|.$$

From (7), one has

$$\begin{aligned} \|\rho_j - \theta\| &= \|(1-\sigma_j)\xi_j + \sigma_j \Im\xi_j - \theta\| \\ &\leq (1-\sigma_j)\|\xi_j - \theta\| + \sigma_j\|\Im\xi_j - \theta\| \\ &\leq (1-\sigma_j)\|\xi_j - \theta\| + \sigma_j\|\xi_j - \theta\| \\ &= \|\xi_j - \theta\|. \end{aligned} \tag{17}$$

Using (7) and (17), we obtain

$$\begin{aligned} \|\Upsilon_j - \theta\| &= \|\Im((1-\kappa_j)\rho_j + \kappa_j \Im\rho_j) - \theta\| \\ &\leq \|(1-\kappa_j)\rho_j + \eta_i \Xi\rho_j - \theta\| \\ &\leq (1-\kappa_j)\|\rho_j - \theta\| + \kappa_j\|\Xi\rho_j - \theta\| \\ &\leq (1-\kappa_j)\|\rho_j - \theta\| + \kappa_j\|\rho_j - \theta\| \\ &= \|\rho_j - \theta\| \\ &\leq \|\xi_j - \theta\|. \end{aligned} \tag{18}$$

Similarly, from (7) and (18), we get

$$\begin{aligned} \|\Lambda_j - \theta\| &= \|\Im((1-\tau_j)\Upsilon_j + \tau_j \Im\Upsilon_j) - \theta\| \\ &\leq (1-\tau_j)\|\Upsilon_j - \theta\| + \tau_j\|\Upsilon_j - \theta\| \\ &\leq \|\Upsilon_j - \theta\| \\ &\leq \|\xi_j - \theta\|. \end{aligned} \tag{19}$$

Finally, it follows from (7) and (19) that

$$\begin{aligned} \|\xi_{j+1} - \theta\| &= \|\Im\Lambda_j - \theta\| \\ &= \|\Lambda_j - \theta\| \\ &\leq \|\xi_j - \theta\|, \end{aligned}$$

which implies that $\{\|\xi_j - \theta\|\}$ is bounded and nondecreasing sequence. Hence $\lim_{j\to\infty} \|\xi_j - \theta\|$ exists for each $\theta \in \lambda(\Im)$. □

Lemma 7. *Let $\emptyset \neq \Theta$ be a nonempty CCS of a UCBs Ω and $\Im : \Theta \to \Theta$ be an SGNM If $\{\xi_j\}$ is the iterative sequence given by (7), then $\lambda(\Im) \neq \emptyset$ if and only if $\{\xi_j\}$ is bounded and* $\lim_{j\to\infty} \|\Im\xi_j - \xi_j\| = 0$.

Proof. Let $\lambda(\Im) \neq \emptyset$ and $\theta \in \lambda(\Im)$. Due to Lemma 6, $\{\xi_j\}$ is bounded and $\lim_{j\to\infty} \|\xi_j - \theta\|$ exists. Set

$$\lim_{j\to\infty} \|\xi_j - \theta\| = \omega. \tag{20}$$

From (20) in (17) and taking the lim sup, we have

$$\limsup_{j\to\infty} \|\rho_j - \theta\| \leq \limsup_{j\to\infty} \|\xi_j - \theta\| = \omega.$$

Based on Proposition 1 (2), one can write

$$\limsup_{j\to\infty}\|\Im\xi_j-\theta\|\leq\limsup_{j\to\infty}\|\xi_j-\theta\|=\omega. \tag{21}$$

From (7) and (17)–(19), we have

$$\begin{aligned}\|\xi_{j+1}-\theta\| &= \|\Im\Lambda_j-\theta\|\\ &\leq \|\Lambda_j-\theta\|\\ &= \|\Im((1-\tau_j)\Upsilon_j+\tau_j\Im\Upsilon_j)-\theta\|\\ &\leq (1-\tau_j)\|\Upsilon_j-\theta\|+\tau_j\|\Upsilon_j-\theta\|\\ &\leq \|\Upsilon_j-\theta\|\\ &= \|\Im((1-\kappa_j)\rho_j+\kappa_j\Im\rho_j)-\theta\|\\ &\leq (1-\kappa_j)\|\rho_j-\theta\|+\kappa_j\|\Im\rho_j-\theta\|\\ &\leq (1-\kappa_j)\|\xi_j-\theta\|+\kappa_j\|\rho_j-\theta\|\\ &= \|\xi_j-\theta\|-\kappa_j\|\xi_j-\theta\|+\kappa_j\|\rho_j-\theta\|.\end{aligned}$$

Hence,

$$\frac{\|\xi_{j+1}-\theta\|-\|\xi_j-\theta\|}{\kappa_j}\leq\|\rho_j-\theta\|-\|\xi_j-\theta\|. \tag{22}$$

As $\kappa_j\in[0,1]$, from (22), we get

$$\|\xi_{j+1}-\theta\|-\|\xi_j-\theta\|\leq\frac{\|\xi_{j+1}-\theta\|-\|\xi_j-\theta\|}{\kappa_j}\leq\|\rho_j-\theta\|-\|\xi_j-\theta\|,$$

which leads to $\|\xi_{j+1}-\theta\|\leq\|\rho_j-\theta\|$.

Applying (20), we have

$$\omega\leq\liminf_{j\to\infty}\|\rho_j-\theta\|. \tag{23}$$

From (21) and (23), we get

$$\begin{aligned}\omega &= \lim_{j\to\infty}\|\rho_j-\theta\|=\lim_{i\to\infty}\|(1-\sigma_j)\xi_j+\sigma_j\Im\xi_j-\theta\|\\ &= \lim_{j\to\infty}\|(1-\sigma_j)(\xi_j-\theta)+\sigma_j(\Im\xi_j-\theta)\|\\ &= \lim_{j\to\infty}\|\sigma_j(\Im\xi_j-\theta)+(1-\sigma_j)(\xi_j-\theta)\|.\end{aligned} \tag{24}$$

It follows from (20), (21), (24) and Lemma 5 that $\{\xi_j\}$ is bounded and $\lim_{j\to\infty}\|\Im\xi_j-\xi_j\|=0$.

Conversely, let $\{\xi_j\}$ be a bounded and $\lim_{j\to\infty}\|\Im\xi_j-\xi_j\|=0$. Consider $\Im\theta\in Z(\Omega,\{\xi_j\})$, then by Proposition 1 (3), and Definition 6, we have

$$\begin{aligned}\nabla(\Im\theta,\{\xi_j\}) &= \limsup_{j\to\infty}\|\xi_j-\Im\theta\|\\ &\leq \limsup_{j\to\infty}(3\|\Im\xi_j-\xi_j\|+\|\xi_j-\theta\|)\\ &= \limsup_{j\to\infty}\|\xi_j-\theta\|=\nabla(\theta,\{\xi_j\}),\end{aligned}$$

which implies that $\theta\in Z(\Omega,\{\xi_j\})$. Since Ω is a uniformly convex and $Z(\Lambda,\{\xi_j\})$ has exactly one point, then we have $\Im\theta=\theta$. □

Theorem 3. *Let $\{\xi_j\}$ be a sequence iterated by (7) and let Ω, Θ and $\Im$ be defined as in Lemma 7. Then, $\{\xi_j\} \rightharpoonup \theta \in \lambda(\Im)$ if Λ satisfies Opial's condition and $\lambda(\Im) \neq \emptyset$.*

Proof. Assume that $\theta \in \lambda(\Im)$, due to Lemma 6, $\lim_{j\to\infty} \|\xi_j - \theta\|$ exists.

Next, we show that $\{\xi_j\}$ has a weak sequential limit in $\lambda(\Im)$. In this regard, consider $\{\xi_{j_a}\}, \{\xi_{j_b}\} \subset \{\xi_j\}$ with $\{\xi_{j_a}\} \rightharpoonup \theta$ and $\{\xi_{j_b}\} \rightharpoonup \theta^*$ for all $\theta, \theta^* \in \Theta$. From Lemma 7, we get $\lim_{j\to\infty} \|\Im\xi_j - \xi_j\| = 0$. Using Lemma 1 and since $I - \Im$ is demiclosed at 0, we have $(I - \Im)\theta = 0$, which implies that $\Im\theta = \theta$. Similarly $\Im\theta^* = \theta^*$.

Now, if $\theta \neq \theta^*$, then by Opial's condition, we get

$$\begin{aligned} \lim_{j\to\infty} \|\xi_j - \theta\| &= \lim_{a\to\infty} \|\xi_{j_a} - \theta\| < \lim_{a\to\infty} \|\xi_{j_a} - \theta^*\| \\ &= \lim_{j\to\infty} \|\xi_j - \theta^*\| = \lim_{b\to\infty} \|\xi_{j_b} - \theta^*\| \\ &< \lim_{b\to\infty} \|\xi_{j_b} - \theta\| = \lim_{j\to\infty} \|\xi_j - \theta\|, \end{aligned}$$

which is a contradiction; hence, $\theta = \theta^*$ and $\{\xi_j\} \rightharpoonup \theta \in \lambda(\Im)$. □

Theorem 4. *Let $\{\xi_j\}$ be a sequence iterated by (7). Furthermore, let $\emptyset \neq \Theta$ be a nonempty CCS of a UCBS Ω and $\Im : \Theta \to \Theta$ be an SGNM. Then, $\{\xi_j\} \longrightarrow \theta \in \lambda(\Im)$.*

Proof. Due to Lemma 2 and 7, $\lambda(\Im) \neq \emptyset$ and $\lim_{j\to\infty} \|\Im\xi_j - \xi_j\| = 0$. Since Θ is compact, then there exists a subsequence $\{\xi_{j_a}\} \subset \{\xi_j\}$ so that $\xi_{j_a} \to \theta$ for any $\theta \in \Theta$.
From Proposition 1 (3), one has

$$\|\xi_{j_a} - \Im\theta\| \leq 3\|\xi_{j_a} - \Im\xi_{j_a}\| + \|\xi_{j_a} - \theta\|, \text{ for all } j \in \mathbb{N}.$$

Letting $a \to \infty$, we get $\Im\theta = \theta$, that is, $\theta \in \lambda(\Im)$. From Lemma 6, we conclude that $\lim_{j\to\infty} \|\xi_j - \theta\|$ exists for each $\theta \in \lambda(\Im)$, therefore $\{\xi_j\} \longrightarrow \theta$. □

Theorem 5. *Let $\{\xi_j\}$ be a sequence iterated by (7) and let Ω, Θ and $\Im$ be defined as in Lemma 7. Then, $\{\xi_j\} \longrightarrow \theta \in \lambda(\Im)$ if and only if $\liminf_{j\to\infty} d(\xi_j, \lambda(\Im)) = 0$, where $d(\theta, \lambda(\Im)) = \inf\{\|\theta - \vartheta\| : \vartheta \in \lambda(\Im)\}$.*

Proof. It is clear that the necessary condition is satisfied. Let

$$\liminf_{j\to\infty} d(\xi_j, \lambda(\Im)) = 0.$$

For Lemma 6, $\lim_{j\to\infty} \|\xi_j - \theta\|$ exists for each $\theta \in \lambda(\Im)$, which leads to $\liminf_{j\to\infty} d(\xi_j, \lambda(\Im))$ exists. Hence

$$\lim_{i\to\infty} d(\xi_j, \lambda(\Im)) = 0.$$

Now, we claim that $\{\xi_j\}$ is a Cauchy sequence in Θ. Since $\liminf_{j\to\infty} d(\xi_j, \lambda(\Im)) = 0$, for every $\epsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that

$$d(\xi_j, \lambda(\Im)) \leq \frac{\epsilon}{2} \text{ and } d(\xi_m, \lambda(\Im)) \leq \frac{\epsilon}{2}, \text{ for each } j, m \geq j_0.$$

Therefore

$$\begin{aligned}\|\xi_j - \xi_m\| &\leq \|\xi_j - \lambda(\Im)\| + \|\lambda(\Im) - \xi_m\| \\ &= d(\xi_j, \lambda(\Im)) + d(\xi_m, \lambda(\Im)) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\end{aligned}$$

Thus, $\{\xi_j\}$ is a Cauchy sequence in Θ. Since Θ is closed, then there exists $\widehat{\theta} \in \Theta$ such that $\lim_{j\to\infty} \xi_j = \widehat{\theta}$. As $\lim_{i\to\infty} d(\xi_j, \lambda(\Im)) = 0$, $\lim_{i\to\infty} d(\widehat{\theta}, \lambda(\Im)) = 0$. Therefore, $\widehat{\theta} \in \lambda(\Im)$ and this completes the proof. □

5. Stability Results

In 1987, Harder [34] rigorously examined the idea of stability of an FP iteration process in her Ph.D. thesis as follows:

Definition 9. *Let $\Im : \Theta \to \Theta$ be a self-mapping and $\varpi_{r+1} = g(\Im, \varpi_r)$ be a FP iteration so that $\{\varpi_r\}$ converges to $\varpi \in \Xi(\Im)$. For arbitrary sequence $\{q_r\}$ in Ω, define*

$$\varepsilon_r = \|q_r - g(\Im, q_r)\|, \text{ for all } r \in \mathbb{N}.$$

Then, the FP iteration method is called $\Im$-stable if the assertion below holds

$$\lim_{r\to\infty} \varepsilon_r = 0 \text{ if and only if } \lim_{r\to\infty} q_r = \varpi.$$

Several writers have lately examined the idea of stability in Definition 9 for various classes of contraction mappings; for example, see [35]. Since the sequence $\{q_r\}$ is arbitrarily chosen, Berinde pointed out in [1] that the concept of stability in Definition 9 is not precise. To overcome this restriction, the same author noted that if $\{q_r\}$ were approximate sequences of $\{\varpi_r\}$, then the definition would make sense. As a result, any iterative process will be weakly stable if it is stable, but the converse is not true in general.

Definition 10 ([1])**.** *A sequence $\{q_r\} \subset \Theta$ is called an approximate sequence of $\{\varpi_r\} \subset \Theta$, if for any $b \geq 1$, there is $\alpha = \alpha(b)$ so that*

$$\|\varpi_r - q_r\| \leq \alpha, \text{ for all } r \geq b.$$

Definition 11 ([1])**.** *Let $\{\varpi_r\}$ be an iterative process defined for $\varpi_0 \in \Theta$ as*

$$\varpi_{r+1} = g(\Im, \varpi_r),\ r \geq 0, \tag{25}$$

where $\Im : \Theta \to \Theta$ is a given mapping. Suppose that $\{\varpi_r\}$ converges to a FP ϖ^* of $\Im$ and for any approximate sequence $\{q_r\} \subset \Theta$ of $\{\varpi_r\}$

$$\lim_{r\to\infty} \varepsilon_r = \lim_{r\to\infty} \|q_{r+1} - g(\Im, q_r)\| = 0 \text{ implies } \lim_{r\to\infty} q_r = \varpi^*,$$

then, Equation (25) is called weakly stable with respect to $\Im$, or weakly $\Im$-stable.

The following theorem demonstrates the stability of our iteration approach (7).

Theorem 6. *Let Θ be a nonempty CCS of a BS Ω. Suppose that $\{\xi_j\}$ is a sequence iterated by (7) with $\{\sigma_j\}, \{\kappa_j\}, \{\tau_j\} \in [0,1]$ and $\sum_{j=0}^{\infty} \tau_j = \infty$. If the mapping $\Im : \Theta \to \Theta$ satisfies (1), then the proposed method is $\Im$-stable.*

Proof. Let $\{Y_j\} \subset \Theta$ be a chosen sequence and $\{\xi_j\}$ be a sequence generating by (7) such that $\xi_{j+1} = \hbar(\Im, \xi_j)$ with $\xi_j \to \zeta$ as $j \to \infty$. Consider

$$\varphi_j = \|Y_{j+1} - \hbar(\Im, Y_j)\|.$$

To prove $\Im$ is stable, it is sufficient to show that

$$\lim_{j\to\infty} \varphi_j = 0 \text{ if and only if } \lim_{j\to\infty} Y_j = \theta, \text{ wher } \theta \in \lambda(\Im).$$

Now, assume that $\lim_{j\to\infty} \varphi_j = 0$. Using (7) and (12), one has

$$\begin{aligned}
& \|Y_{j+1} - \theta\| \\
= & \|Y_{j+1} - \hbar(\Im, Y_j) + \hbar(\Im, Y_j) - \theta\| \\
\leq & \|Y_{j+1} - \hbar(\Im, Y_j)\| + \|\hbar(\Im, Y_j) - \theta\| \\
= & \varphi_j + \|\hbar(\Im, Y_j) - \theta\| \\
= & \varphi_j + \left\| \Im\left(\Im\left(\begin{array}{c} (1-\tau_j)\left[\Im\left(\begin{array}{c} (1-\kappa_j)\left[(1-\sigma_j)Y_j + \sigma_j \Im Y_j\right] \\ +\kappa_j \Im\left[(1-\sigma_j)Y_j + \sigma_j \Im Y_j\right] \end{array} \right)\right] \\ +\tau_j \Im\left[\Im\left(\begin{array}{c} (1-\kappa_j)\left[(1-\sigma_j)Y_j + \sigma_j \Im Y_j\right] \\ +\kappa_j \Im\left[(1-\sigma_j)Y_j + \sigma_j \Im Y_j\right] \end{array} \right)\right] \end{array} \right)\right) - \theta \right\| \\
= & \ell_1^3(1 - (1-\ell_1)\tau_j)\|Y_j - \theta\| + \varphi_j,
\end{aligned}$$

for $j \in \mathbb{N}$. Let

$$\psi_j = \|Y_j - \theta\|, \; e_j = (1-\ell_1)\tau_j \in (0,1) \text{ and } \psi_j^* = \varphi_j.$$

Since $\lim_{j\to\infty} \varphi_j = 0$, then $\lim_{i\to\infty} \frac{\psi_j^*}{e_j} = \lim_{i\to\infty} \frac{\varphi_j}{e_j} = 0$. Therefore, all assumptions of Lemma 3 hold, consequently $\lim_{j\to\infty} \|Y_j - \theta\| = 0$,, i.e., $\lim_{j\to\infty} Y_j = \theta$.

Conversely, let $\lim_{j\to\infty} Y_j = \theta$, then

$$\begin{aligned}
\varphi_j &= \|Y_{j+1} - \hbar(\Im, Y_j)\| \\
&= \|Y_{j+1} - \theta + \theta - \hbar(\Im, Y_j)\| \\
&\leq \|Y_{j+1} - \theta\| + \|\theta - \hbar(\Im, Y_j)\| \\
&\leq \|Y_{j+1} - \theta\| + \ell_1^3(1 - (1-\ell_1)\tau_j)\|Y_j - \theta\|,
\end{aligned}$$

passing $j \to \infty$, we obtain $\lim_{j\to\infty} \varphi_j = 0$. This finishes the proof. □

The following example supports Theorem 6.

Example 1. *Assume that* $\Theta = [0,1]$ *and* $(\mathbb{R}, \|.\|)$ *is a BS equipped with the usual norm. Define a mapping* $\Im : [0,1] \to [0,1]$ *by* $\Im\xi = \frac{\xi}{8}$. *Clearly,* 0 *is a unique FP of* $\Im$ *and* $\Im$ *fulfills (7) with* $\ell_1 = \frac{1}{4}$ *and* $\ell_2 \geq 0$.

Next, we show that the proposed algorithm (7) is $\Im$-stable. In this regard, assume that $\sigma_j = \kappa_j = \tau_j = \frac{1}{j+4}$ and $\xi_0 \in [0,1]$, then by (7), one has

$$
\begin{aligned}
\rho_j &= \left(1 - \frac{1}{j+4} + \frac{1}{8(j+4)}\right)\xi_i = \left(1 - \frac{7}{8(j+4)}\right)\xi_j, \\
\Upsilon_j &= \frac{1}{8}\left(1 - \frac{7}{4(j+4)} + \frac{49}{8^2(j+4)^2}\right)\xi_j, \\
\Lambda_j &= \frac{1}{8^3}\left(1 - \frac{7}{4(j+4)} + \frac{49}{8^2(j+4)^2}\right)\xi_j, \\
\xi_{j+1} &= \frac{1}{8^3}\left(1 - \frac{42}{8(j+4)} + \frac{49}{8^2(j+4)^2} + \frac{147}{2\times 8^2(j+4)^2} - \frac{343}{8^3(j+4)^3}\right)\xi_j \\
&= \left(\frac{1}{8^3} - \frac{42}{8^4(j+4)} + \frac{49}{8^5(j+4)^2} + \frac{147}{2\times 8^5(j+4)^2} - \frac{343}{8^6(j+4)^3}\right) \\
&= \left[1 - \left(\frac{511}{512} + \frac{42}{8^4(j+4)} - \frac{49}{8^5(j+4)^2} - \frac{147}{2\times 8^5(j+4)^2} + \frac{343}{8^6(j+4)^3}\right)\right]\xi_j.
\end{aligned}
$$

Put $\Upsilon_j = \frac{511}{512} + \frac{42}{8^4(j+4)} - \frac{49}{8^5(j+4)^2} - \frac{147}{2\times 8^5(j+4)^2} + \frac{343}{8^6(j+4)^3}$. Clearly, $\Upsilon_j \in (0,1)$ for each $j > 0$ and $\sum_{i=0}^{\infty} \Upsilon_j = 0$. Due to Lemma 3, $\lim_{j\to\infty} \xi_j = 0$. It is easy to see that $\lim_{j\to\infty}\|\xi_j\| = \|\lim_{j\to\infty}\xi_j\| = 0$. Now, consider $\Re_j = \frac{1}{j+5}$ for each $j > 0$, we have

$$
0 \le \lim_{j\to\infty}\|\xi_j - \Re_j\| \le \lim_{j\to\infty}\|\xi_j\| + \lim_{j\to\infty}\|\Re_j\| = 0,
$$

which implies that $\lim_{j\to\infty}\|\xi_j - \Re_j\| = 0$. Hence, the two sequences $\{\xi_j\}$ and $\{\Re_j\}$ are equivalent. Finally, assume that φ_j is the sequence associated with the iterative sequence $\{\xi_j\}$, then, we have

$$
\begin{aligned}
\varphi_j &= \left\|\Re_{j+1} - \left(\frac{511}{512} + \frac{42}{8^4(j+4)} - \frac{49}{8^5(j+4)^2} - \frac{147}{2\times 8^5(j+4)^2} + \frac{343}{8^6(j+4)^3}\right)\Re_j\right\| \\
&= \left\|\frac{1}{j+6} - \left(\frac{511}{512} + \frac{42}{8^4(j+4)} - \frac{49}{8^5(j+4)^2} - \frac{147}{2\times 8^5(j+4)^2} + \frac{343}{8^6(j+4)^3}\right)\Re_j\right\| \\
&\to 0, \text{ as } r \to \infty.
\end{aligned}
$$

Hence, the proposed algorithm (7) is $\Im$-stable.

6. Numerical Examples

In this section, we provide illustrative examples to assess the convergence of iteration (7) in comparison to some of the most popular iterative schemes in the literature.

Example 2. *Assume that $\Omega = \mathbb{R}^3$ and $\Theta = \{\varpi = (\varpi_1, \varpi_2, \varpi_3) : (\varpi_1, \varpi_2, \varpi_3) \in [0,6]^3\}$, where $([3,6]^3 = [3,6]\times[3,6]\times[3,6])$ is a subset of Ω equipped with the norm $\|\varpi\| = \|(\varpi_1,\varpi_2,\varpi_3)\| = |\varpi_1| + |\varpi_2| + |\varpi_3|$. Define a mapping $\Im : \Theta \to \Theta$ by*

$$
\Im\varpi = \begin{cases} \left(\frac{\varpi_1}{3}, \frac{\varpi_2}{3}, \frac{\varpi_3}{3}\right), & \text{if } (\varpi_1,\varpi_2,\varpi_3) \in [0,3)^3, \\ \left(\frac{\varpi_1}{6}, \frac{\varpi_2}{6}, \frac{\varpi_3}{6}\right), & \text{if } (\varpi_1,\varpi_2,\varpi_3) \in [3,6]^3. \end{cases}
$$

It is clear that $\Im$ owns a unique FP; it is $(0,0,0)$. Now, we shall show that $\Im$ is a contractive-like mapping and, hence, an ACM. In this regard, we define the function $\xi : [0,\infty) \to [0,\infty)$ by

$\xi(\varpi) = \frac{\varpi}{4}$. *Obviously,* ξ *is a strictly increasing continuous function with* $\xi(0) = 0$. *If* $\varpi \in [0,3)^3$, *we have*

$$\|\varpi - \Im\varpi\| = \left\|(\varpi_1, \varpi_2, \varpi_3) - \left(\frac{\varpi_1}{3}, \frac{\varpi_2}{3}, \frac{\varpi_3}{3}\right)\right\| = \left\|\left(\frac{2\varpi_1}{3}, \frac{2\varpi_2}{3}, \frac{2\varpi_3}{3}\right)\right\|,$$

and

$$\begin{aligned} \xi(\|\varpi - \Im\varpi\|) &= \xi\left(\left\|\left(\frac{2\varpi_1}{3}, \frac{2\varpi_2}{3}, \frac{2\varpi_3}{3}\right)\right\|\right) \\ &= \left\|\left(\frac{\varpi_1}{6}, \frac{\varpi_2}{6}, \frac{\varpi_3}{6}\right)\right\| = \left|\frac{\varpi_1}{6}\right| + \left|\frac{\varpi_2}{6}\right| + \left|\frac{\varpi_3}{6}\right|. \end{aligned} \tag{26}$$

Analogously, if $\varpi \in [3,6]^3$, *one has*

$$\|\varpi - \Im\varpi\| = \left\|(\varpi_1, \varpi_2, \varpi_3) - \left(\frac{\varpi_1}{6}, \frac{\varpi_2}{6}, \frac{\varpi_3}{6}\right)\right\| = \left\|\left(\frac{5\varpi_1}{6}, \frac{5\varpi_2}{6}, \frac{5\varpi_3}{6}\right)\right\|,$$

and

$$\begin{aligned} \xi(\|\varpi - \Im\varpi\|) &= \xi\left(\left\|\left(\frac{5\varpi_1}{6}, \frac{5\varpi_2}{6}, \frac{5\varpi_3}{6}\right)\right\|\right) \\ &= \left\|\left(\frac{5\varpi_1}{24}, \frac{5\varpi_2}{24}, \frac{5\varpi_3}{24}\right)\right\| = \left|\frac{5\varpi_1}{24}\right| + \left|\frac{5\varpi_2}{24}\right| + \left|\frac{5\varpi_3}{24}\right|. \end{aligned} \tag{27}$$

After that, we discuss the cases below:

(I) *If* $\varpi, v \in [0,3)^3$, *then by (26), we get*

$$\begin{aligned} \|\Im\varpi - \Im v\| &= \left\|\left(\frac{\varpi_1}{3}, \frac{\varpi_2}{3}, \frac{\varpi_3}{3}\right) - \left(\frac{v_1}{3}, \frac{v_2}{3}, \frac{v_3}{3}\right)\right\| \\ &= \left|\frac{\varpi_1}{3} - \frac{v_1}{3}\right| + \left|\frac{\varpi_2}{3} - \frac{v_2}{3}\right| + \left|\frac{\varpi_3}{3} - \frac{v_3}{3}\right| \\ &= \frac{1}{3}[|\varpi_1 - v_1| + |\varpi_2 - v_2| + |\varpi_3 - v_3|] \\ &= \frac{1}{3}\|(\varpi_1, \varpi_2, \varpi_3) - (v_1, v_2, v_3)\| = \frac{1}{3}\|\varpi - v\| \\ &\leq \frac{1}{3}\|\varpi - v\| + \left|\frac{\varpi_1}{6}\right| + \left|\frac{\varpi_2}{6}\right| + \left|\frac{\varpi_3}{6}\right| \\ &= \frac{1}{3}\|\varpi - v\| + \xi(\|\varpi - \Im\varpi\|). \end{aligned}$$

(II) *If* $\varpi, v \in [3,6]^3$, *then by (27), we have*

$$\begin{aligned} \|\Im\varpi - \Im v\| &= \left\|\left(\frac{\varpi_1}{6}, \frac{\varpi_2}{6}, \frac{\varpi_3}{6}\right) - \left(\frac{v_1}{6}, \frac{v_2}{6}, \frac{v_3}{6}\right)\right\| \\ &= \left|\frac{\varpi_1}{6} - \frac{v_1}{6}\right| + \left|\frac{\varpi_2}{6} - \frac{v_2}{6}\right| + \left|\frac{\varpi_3}{6} - \frac{v_3}{6}\right| \\ &= \frac{1}{6}[|\varpi_1 - v_1| + |\varpi_2 - v_2| + |\varpi_3 - v_3|] \\ &= \frac{1}{6}\|(\varpi_1, \varpi_2, \varpi_3) - (v_1, v_2, v_3)\| = \frac{1}{6}\|\varpi - v\| \\ &\leq \frac{1}{6}\|\varpi - v\| + \left|\frac{5\varpi_1}{24}\right| + \left|\frac{5\varpi_2}{24}\right| + \left|\frac{5\varpi_3}{24}\right| \\ &\leq \frac{1}{3}\|\varpi - v\| + \xi(\|\varpi - \Im\varpi\|). \end{aligned}$$

(III) If $\varpi \in [0,3)^3$ and $\upsilon \in [3,6]^3$, then by (26), we obtain that

$$
\begin{aligned}
\|\Im\varpi - \Im\upsilon\| &= \left\|\left(\frac{\varpi_1}{3}, \frac{\varpi_2}{3}, \frac{\varpi_3}{3}\right) - \left(\frac{\upsilon_1}{6}, \frac{\upsilon_2}{6}, \frac{\upsilon_3}{6}\right)\right\| \\
&= \left\|\left(\frac{\varpi_1}{3} - \frac{\upsilon_1}{6}\right), \left(\frac{\varpi_2}{3} - \frac{\upsilon_2}{6}\right), \left(\frac{\varpi_3}{3} - \frac{\upsilon_3}{6}\right)\right\| \\
&= \left\|\left(\frac{\varpi_1}{6} + \frac{\varpi_1}{6} - \frac{\upsilon_1}{6}\right), \left(\frac{\varpi_2}{6} + \frac{\varpi_2}{6} - \frac{\upsilon_2}{6}\right), \left(\frac{\varpi_3}{6} + \frac{\varpi_3}{6} - \frac{\upsilon_3}{6}\right)\right\| \\
&\leq \left|\frac{\varpi_1}{6} + \frac{\varpi_1}{6} - \frac{\upsilon_1}{6}\right| + \left|\frac{\varpi_2}{6} + \frac{\varpi_2}{6} - \frac{\upsilon_2}{6}\right| + \left|\frac{\varpi_3}{6} + \frac{\varpi_3}{6} - \frac{\upsilon_3}{6}\right| \\
&\leq \left|\frac{\varpi_1}{6}\right| + \left|\frac{\varpi_2}{6}\right| + \left|\frac{\varpi_3}{6}\right| + \left|\frac{\varpi_1}{6} - \frac{\upsilon_1}{6}\right| + \left|\frac{\varpi_2}{6} - \frac{\upsilon_2}{6}\right| + \left|\frac{\varpi_3}{6} - \frac{\upsilon_3}{6}\right| \\
&= \frac{1}{6}[|\varpi_1 - \upsilon_1| + |\varpi_2 - \upsilon_2| + |\varpi_3 - \upsilon_3|] + \xi(\|\varpi - \Im\varpi\|) \\
&\leq \frac{1}{3}\|(\varpi_1, \varpi_2, \varpi_3) - (\upsilon_1, \upsilon_2, \upsilon_3)\| + \xi(\|\varpi - \Im\varpi\|) \\
&= \frac{1}{3}\|\varpi - \upsilon\| + \xi(\|\varpi - \Im\varpi\|).
\end{aligned}
$$

(IV) If $\upsilon \in [0,3)^3$ and $\varpi \in [3,6]^3$, then by (26), one has

$$
\begin{aligned}
\|\Im\varpi - \Im\upsilon\| &= \left\|\left(\frac{\varpi_1}{6}, \frac{\varpi_2}{6}, \frac{\varpi_3}{6}\right) - \left(\frac{\upsilon_1}{3}, \frac{\upsilon_2}{3}, \frac{\upsilon_3}{3}\right)\right\| \\
&= \left\|\left(\frac{\varpi_1}{6} - \frac{\upsilon_1}{3}\right), \left(\frac{\varpi_2}{6} - \frac{\upsilon_2}{3}\right), \left(\frac{\varpi_3}{6} - \frac{\upsilon_3}{3}\right)\right\| \\
&= \left\|\left(\frac{\varpi_1}{3} - \frac{\varpi_1}{6} - \frac{\upsilon_1}{3}\right), \left(\frac{\varpi_2}{3} - \frac{\varpi_2}{6} - \frac{\upsilon_2}{3}\right), \left(\frac{\varpi_3}{3} - \frac{\varpi_3}{6} - \frac{\upsilon_3}{3}\right)\right\| \\
&\leq \left|\frac{\varpi_1}{3} - \frac{\varpi_1}{6} - \frac{\upsilon_1}{3}\right| + \left|\frac{\varpi_2}{3} - \frac{\varpi_2}{6} - \frac{\upsilon_2}{3}\right| + \left|\frac{\varpi_3}{3} - \frac{\varpi_3}{6} - \frac{\upsilon_3}{3}\right| \\
&\leq \left|\frac{\varpi_1}{6}\right| + \left|\frac{\varpi_2}{6}\right| + \left|\frac{\varpi_3}{6}\right| + \left|\frac{\varpi_1}{3} - \frac{\upsilon_1}{3}\right| + \left|\frac{\varpi_2}{3} - \frac{\upsilon_2}{3}\right| + \left|\frac{\varpi_3}{3} - \frac{\upsilon_3}{3}\right| \\
&= \frac{1}{3}[|\varpi_1 - \upsilon_1| + |\varpi_2 - \upsilon_2| + |\varpi_3 - \upsilon_3|] + \xi(\|\varpi - \Im\varpi\|) \\
&= \frac{1}{3}\|(\varpi_1, \varpi_2, \varpi_3) - (\upsilon_1, \upsilon_2, \upsilon_3)\| + \xi(\|\varpi - \Im\varpi\|) \\
&= \frac{1}{3}\|\varpi - \upsilon\| + \xi(\|\varpi - \Im\varpi\|).
\end{aligned}
$$

Based on the above cases, we conclude that condition (7) is satisfied. Hence, $\Im$ is a contractive-like mapping.

Example 3. *Assume that $(\mathbb{R}, \|.\|)$ is a BS equipped with the usual norm and $\Delta = [3,5]$. Define a mapping $\Im : \Theta \to \Theta$ by*

$$
\Im\varpi = \begin{cases} \frac{\varpi+6}{3}, & \text{if } \varpi < 5, \\ 2, & \text{if } \varpi = 5. \end{cases}
$$

To prove that $\Im$ does not satisfy Condition (C), we take $\varpi = 4$ and $\upsilon = 5$, then

$$
\frac{1}{2}|\varpi - \Im\varpi| = \frac{1}{2}|4 - \Im 4| = \frac{1}{3} < 1 = |\varpi - \upsilon|.
$$

But

$$
|\Im\varpi - \Im\upsilon| \leq |\Im 4 - \Im 5| = \left|\frac{10}{3} - \frac{6}{3}\right| = \frac{4}{3} > 1 = |\varpi - \upsilon|.
$$

Hence, $\Im$ does not satisfy Condition (C).

Now, to show that $\Im$ is an SGNM, we consider following cases:

(I) *If* $\varpi, v < 5$*, we get*

$$\begin{aligned} & \ell|\varpi - \Im\varpi| + \ell|v - \Im v| + (1-2\ell)|\varpi - v| \\ = & \frac{1}{2}\left|\varpi - \left(\frac{\varpi+6}{3}\right)\right| + \frac{1}{2}\left|v - \left(\frac{v+6}{3}\right)\right| \\ = & \frac{1}{2}\left|\frac{2\varpi-6}{3}\right| + \frac{1}{2}\left|\frac{2v-6}{3}\right| \\ \geq & \frac{1}{2}\left|\left(\frac{2\varpi-6}{3}\right) - \left(\frac{2v-6}{3}\right)\right| \\ = & \frac{1}{2}\left|\frac{2\varpi}{3} - \frac{2v}{3}\right| = \frac{1}{3}|\varpi - v| = |\Im\varpi - \Im v|. \end{aligned}$$

(II) *If* $\varpi < 5$ *and* $v = 5$*, we obtain*

$$\begin{aligned} & \ell|\varpi - \Im\varpi| + \ell|v - \Im v| + (1-2\ell)|\varpi - v| \\ = & \frac{1}{2}\left|\varpi - \left(\frac{\varpi+6}{3}\right)\right| + \frac{1}{2}|5-2| \\ = & \frac{1}{2}\left|\frac{2\varpi-6}{3}\right| + \frac{3}{2} = \left|\frac{\varpi}{3}\right| + \frac{1}{2} \\ \geq & \left|\frac{\varpi}{3}\right| = |\Im\varpi - \Im v|. \end{aligned}$$

(III) *If* $v < 5$ *and* $\varpi = 5$*, we have*

$$\begin{aligned} & \ell|\varpi - \Im\varpi| + \ell|v - \Im v| + (1-2\ell)|\varpi - v| \\ = & \frac{1}{2}|5-2| + \frac{1}{2}\left|v - \left(\frac{v+6}{3}\right)\right| \\ = & \frac{3}{2} + \frac{1}{2}\left|\frac{2v-6}{3}\right| = \frac{1}{2} + \left|\frac{v}{3}\right| \\ \geq & \left|\frac{v}{3}\right| = |\Im\varpi - \Im v|. \end{aligned}$$

- *If* $v = \varpi = 5$*, we can write*

$$\begin{aligned} & \ell|\varpi - \Im\varpi| + \ell|v - \Im v| + (1-2\ell)|\varpi - v| \\ = & 3 > 0 = |2 - \Im v| = |\Im\varpi - \Im v|. \end{aligned}$$

Hence, $\Im$ *is an SGNM and has a unique FP 3.*

Numerically, by using MATLAB R2015a, we show that our iterative scheme (7) converges faster than both iterations (5) and (6) as follows:

Let $\Omega = (-\infty, \infty)$*,* $\Theta = [0, 50]$*, and* $\Im : \Theta \to \Theta$ *be a mapping described as*

$$\Im(\xi) = \sqrt{\xi^2 - 9\xi + 54}.$$

Clearly, 6 is a unique FP of $\Im$*. Consider* $\sigma_j = \kappa_j = \tau_j = \frac{1}{5j+10}$*, with distinct starting points. Then, we get the following Tables 1 and 2 and Figures 1–6 for comparing the different iterative techniques.*

Table 1. Numerical comparison of results of Equations (5)–(7).

	Number of Iterations		
Initial Point (z_1)	**Equation (5)**	**Equation (6)**	**Equation (7)**
3.00	16	13	7
3.82	23	18	10
4.44	25	20	10

Table 2. Numerical comparison of results of Equations (5)–(7).

	Execution Time in Seconds		
Initial Point (z_1)	**Equation (5)**	**Equation (6)**	**Equation (7)**
3.00	0.004832900000000000	0.005957500000000000	0.000157200000000000
3.82	0.002367600000000000	0.007555200000000000	0.00779860000000000
4.44	0.007059300000000000	0.009460300000000000	0.00744400000000000

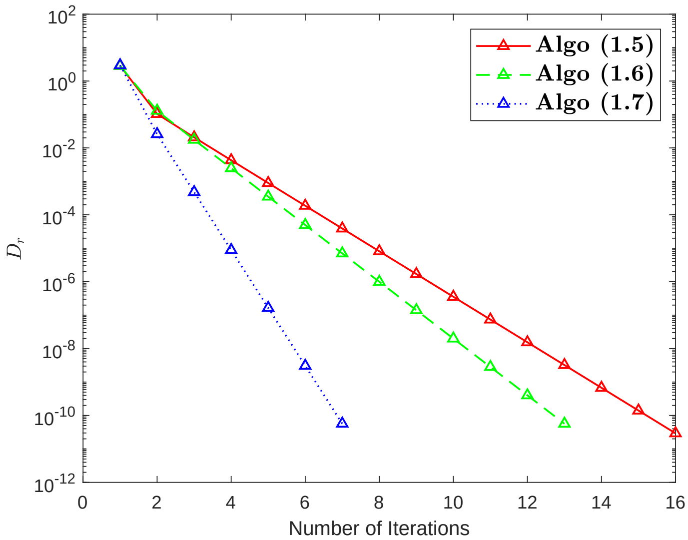

Figure 1. A graphical comparison of Equations (5)–(7) where $z_1 = 3.00$.

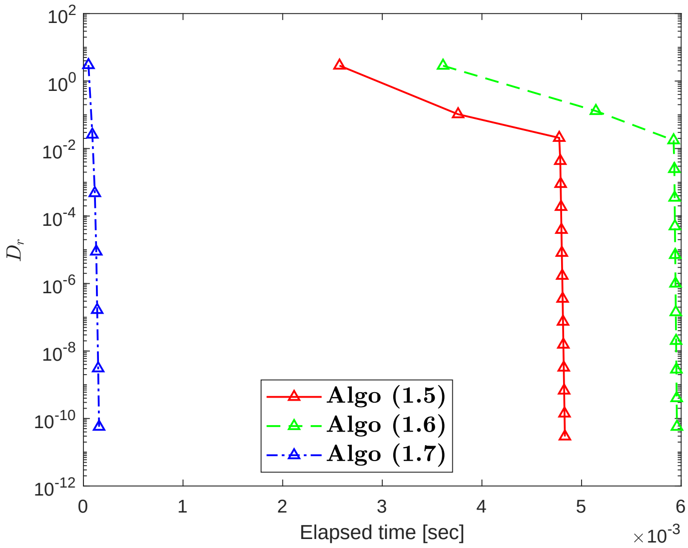

Figure 2. A graphical comparison of Equations (5)–(7) where $z_1 = 3.00$.

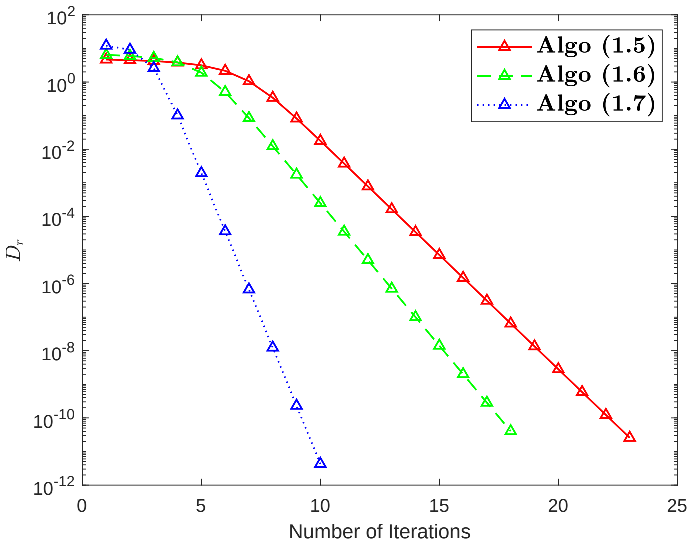

Figure 3. A graphical comparison of Equations (5)–(7) where $z_1 = 3.82$.

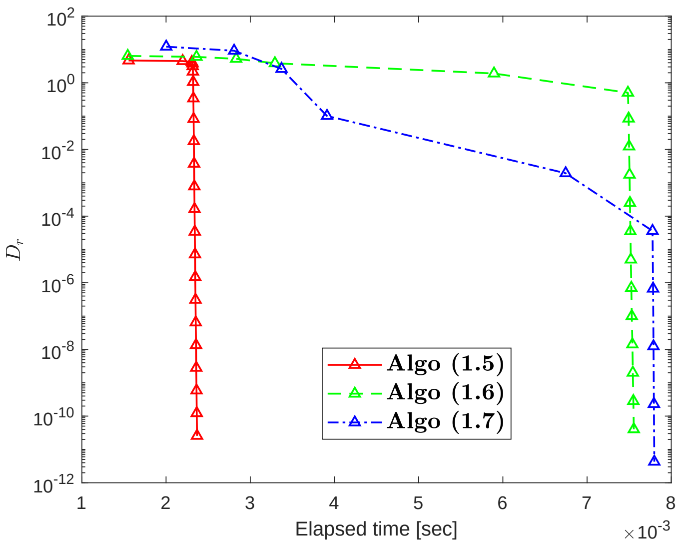

Figure 4. A graphical comparison of Equations (5)–(7) where $z_1 = 3.82$.

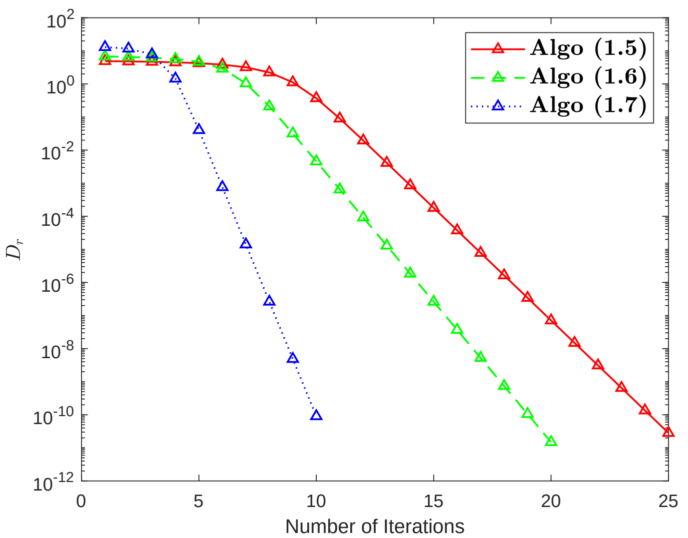

Figure 5. A graphical comparison of Equations (5)–(7) where $z_1 = 4.44$.

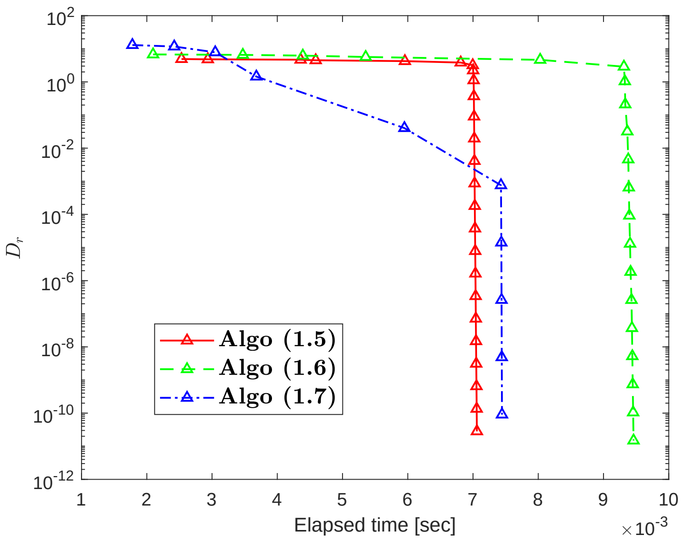

Figure 6. A graphical comparison of Equations (5)–(7) where $z_1 = 4.44$.

Remark 1. *The two main metrics used to evaluate the efficiency and success of the iterative process are time and the number of repetitions. saves time and effort when strong convergence is easily attained with the fewest repetitions in various optimization and variational inequality problems. The tables and figures above make it clear that our strategy is effective, and our algorithm acts properly when compared to other more sober iterations in this direction.*

7. Solving a Nonlinear Integral Equation with Delay

In this section, we apply iteration (7) to determine the existence of the solution to the following nonlinear Volterra equation with delay:

$$\xi(t) = \eta(t) + \daleth \int_k^t \Phi(t, \varsigma, \xi(\varsigma), \xi(\varsigma - \mu))d\varsigma,\ t \in J = [k, l]. \tag{28}$$

with the condition

$$\xi(t) = \Xi(t),\ t \in [k - \mu, k], \tag{29}$$

where $k, l \in \mathbb{R}$, $\Xi \in (C[k - \mu, k], \mathbb{R})$ and $\mu, \daleth > 0$. Clearly, the space $\beth = ((C[k, l], \mathbb{R}), \|.\|_\infty)$ is a BS, where the norm $\|.\|_\infty$ is described as $\|\xi - \vartheta\|_\infty = \max_{t \in J}\{|\xi(t) - \vartheta(t)|\}$ and $(C[k, l], \mathbb{R})$ is the set of all continuous functions defined on $[k, l]$.

Th following theorem is the main result in this part:

Theorem 7. *Suppose that Θ is a nonempty CCS of a BS $\beth$ and $\{\xi_j\}$ is a sequence generated by (7) with $\{\sigma_j\}, \{\kappa_j\}, \{\tau_j\} \in [0, 1]$. Let $\Im : \beth \to \beth$ be an operator described as*

$$\Im\xi(t) = \eta(t) + \daleth \int_k^t \Phi(t, \varsigma, \xi(\varsigma), \xi(\varsigma - \mu))d\varsigma,\ t \in J,$$

with $\Im\xi(t) = \Xi(t)$, $t \in [k - \mu, k]$. Assume that the following statements are true:

(s_i) *the functions $\eta : J \to \mathbb{R}$ and $\Phi : J \times J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous;*

(s_{ii}) *there exists a constant $A_\Phi > 0$ so that*

$$|\Phi(t, \varsigma, \xi_1, \xi_2)) - \Phi(t, \varsigma, \xi_1^*, \xi_2^*))| \leq A_\Phi(|\xi_1 - \xi_1^*| + |\xi_2 - \xi_2^*|),$$

for all $\xi_1, \xi_2, \xi_1^*, \xi_2^* \in \mathbb{R}_+$ and $t, \varsigma \in J$;

(s_{iii}) *for each $t, \varsigma \in J$, $2\daleth A_\Phi(t - k) < 1$.*

Then the integral Equation (28) with (29) has a unique solution $\widehat{\xi} \in C[k,l]$. Further, if $\Im$ is a mapping satisfying (2), then $\xi_j \longrightarrow \widehat{\xi}$.

Proof. First, we prove that $\Im$ has a FP by using the Banach contraction principle. Recall that

$$|\Im\xi(t) - \Im\xi^*(t)| = 0,\ \xi, \xi^* \in (C[k-\mu,k],\mathbb{R}),\ t \in [k-\mu,k].$$

Next, for each $t \in J$, we can write

$$\begin{aligned}
|\Im\xi(t) - \Im\xi^*(t)| &= \Bigg|\eta(t) + \daleth\int_k^t \Phi(t,\varsigma,\xi(\varsigma),\xi(\varsigma-\mu)))d\varsigma \\
&\quad - \eta(t) + \daleth\int_k^t \Phi(t,\varsigma,\xi^*(\varsigma),\xi^*(\varsigma-\mu)))d\varsigma\Bigg| \\
&\le \daleth\int_k^t |\Phi(t,\varsigma,\xi(\varsigma),\xi(\varsigma-\mu))) - \Phi(t,\varsigma,\xi^*(\varsigma),\xi^*(\varsigma-\mu)))|d\varsigma \\
&\le \daleth A_\Phi \int_k^t [|\xi(\varsigma) - \xi^*(\varsigma)| + |\xi(\varsigma-\mu) - \xi^*(\varsigma-\mu)|]d\varsigma \\
&\le \daleth A_\Phi \int_k^t \left[\max_{k-\mu\varsigma\le l}|\xi(\varsigma) - \xi^*(\varsigma)| + \max_{k-\mu\varsigma\le l}|\xi(\varsigma-\mu) - \xi^*(\varsigma-\mu)|\right]d\varsigma \\
&= \daleth A_\Phi \int_k^t [\|\xi - \xi^*\|_\infty + \|\xi - \xi^*\|_\infty]d\varsigma \\
&= 2\daleth A_\Phi \int_k^t \|\xi - \xi^*\|d\varsigma = 2\daleth A_\Phi(t-k)\|\xi - \xi^*\|_\infty.
\end{aligned}$$

Since $2\daleth A_\Phi(t-k) < 1$, we conclude that the operator $\Im$ has a unique FP $\lambda(\Im) = \{\widehat{\xi}\}$ because it is a contraction. Hence, the problem (28) with (29) has a unique solution $\widehat{\xi} \in C[k,l]$.

Finally, we show that $\xi_j \longrightarrow \widehat{\xi}$. For each $\xi, \xi^* \in \Theta$, one has

$$\begin{aligned}
&|\Im\xi(t) - \Im\xi^*(t)| \\
\le\ &|\Im\xi(t) - \xi(t)| + |\xi(t) - \Im\xi^*(t)| \\
\le\ &|\Im\xi(t) - \xi(t)| + \Bigg|\eta(t) + \daleth\int_k^t \Phi(t,\varsigma,\xi(\varsigma),\xi(\varsigma-\mu)))d\varsigma \\
&-\eta(t) + \daleth\int_k^t \Phi(t,\varsigma,\xi^*(\varsigma),\xi^*(\varsigma-\mu)))d\varsigma\Bigg| \\
\le\ &|\Im\xi(t) - \xi(t)| + \daleth A_\Phi \int_k^t [|\xi(\varsigma) - \xi^*(\varsigma)| + |\xi(\varsigma-\mu) - \xi^*(\varsigma-\mu)|]d\varsigma \\
\le\ &\max_{k-\mu\varsigma\le l}|\Im\xi(t) - \xi(t)| + \daleth A_\Phi \int_k^t \left[\max_{k-\mu\varsigma\le l}|\xi(\varsigma) - \xi^*(\varsigma)| + \max_{k-\mu\varsigma\le l}|\xi(\varsigma-\mu) - \xi^*(\varsigma-\mu)|\right]d\varsigma \\
=\ &\|\Im\xi - \xi\|_\infty + \daleth A_\Phi \int_k^t [\|\xi - \xi^*\|_\infty + \|\xi - \xi^*\|_\infty]d\varsigma \\
\le\ &\|\Im\xi - \xi\|_\infty + 2\daleth A_\Phi(t-k)\|\xi - \xi^*\|_\infty.
\end{aligned}$$

Hence,

$$\|\Im\xi - \Im\xi^*\| \le \|\Im\xi - \xi\|_\infty + 2\daleth A_\Phi(t-k)\|\xi - \xi^*\|_\infty.$$

It is clear that the mapping $\Im$ fulfills (1) with $\ell_1 = 2\daleth A_\Phi(t-k) < 1$ and $\ell_2 = 0$. Therefore, all requirements of Theorem 1 are satisfied. Then, the sequence $\{\xi_j\}$ established by the iterative scheme (7) strongly converges to the unique solution of Equation (28) with (29). $\square$

8. Conclusion and Open Discussions

The FPs of contractive-like mappings were approximated in this work using a four-step iterative method. It has been shown analytically that for contractive-like mappings, the new iterative technique converges faster than the iterative approaches (6). Furthermore, we have demonstrated numerically that our iterative method converges faster than numerous well-known iterative schemes, such as (5) and (6) for ACMs. Similarly, the stability result of the iterative scheme (7) was also obtained. Moreover, some weak and strong convergence results are proved for SGNMs in UCBSs. Further, illustrative examples were investigated to support our results. The considered iteration was applied to determine the existence of a solution to a nonlinear Volterra integral equation. Ultimately, we identified the following as potential future work:

- The variational inequality problem can be solved using our iteration (1) if we define the mapping $\Im$ in a Hilbert space Ω endowed with an inner product space. This problem can be described as: find $\wp^* \in \beth$ such that
$$\langle \Im\wp^*, \wp - \wp^* \rangle \geq 0 \text{ for all } \wp \in \Omega,$$
where $\Im : \Omega \to \Omega$ is a nonlinear mapping. In several disciplines, including engineering mechanics, transportation, economics and mathematical programming, variational inequalities are a crucial and indispensable modeling tool; see [36,37] for more details.
- Our methodology can be extended to include gradient and extra-gradient projection techniques, which are crucial for locating saddle points and resolving a variety of optimization-related issues; see [38].
- We can accelerate the convergence of the proposed algorithm by adding shrinking projection and CQ terms. These methods stimulate algorithms and improve their performance to obtain strong convergence; for more details, see [39–42].
- If we consider the mapping $\Im$ as an α-inverse strongly monotone and the inertial term is added to our algorithm, then we have the inertial proximal point algorithm. This algorithm is used in many applications such as monotone variational inequalities, image restoration problems, convex optimization problems and split convex feasibility problems, see [43,44]. For more accuracy, these problems can be expressed as mathematical models such as machine learning and the linear inverse problem.
- In addition, second-order differential equations and fractional differential equations, which Green's function can be used to transform into integral equations, can be solved using our approach. Therefore, they are simple to treat and resolve using the same method as in Section 7.

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Abbreviations

FPs	Fixed points
BSs	Banach spaces
CCS	Closed convex subset
$\rightharpoonup$	Weak convergence
$\longrightarrow$	Strong convergence
ACMs	Almost contraction mappings
NIEs	Nonlinear integral equations
SGNMs	Suzuki generalized nonexpansive mappings
UCBSs	Uniformly convex Banach spaces

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