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Convex Families of q-Derivative Meromorphic Functions Involving the Polylogarithm Function

Khadeejah Rasheed Alhindi

Department of Computer Science, College of Computer Science and Information Technology, Imam Abdulrahman Bin Faisal University, P.O. Box 1982, Dammam 31441, Saudi Arabia; kralhindi@iau.edu.sa

Abstract: The aim of this research study is to establish a novel subclass of meromorphic functions in the mean of q-derivatives in combination with the well-known polylogarithm function. Two additional subfamilies for this class are also defined. Furthermore, the coefficient inequality and distortion bounds are highlighted. Finally, the convex families and related set structures are thoroughly investigated.

Keywords: polylogarithm function (or de Jonquière's function); meromorphic functions; hadamard product; q-derivatives; convex families; connected sets; coefficient inequality

1. Introduction

The evolution of polylogarithm function, also known as Jonquiere's function, was started in 1696 by two eminent mathematicians, Leibniz and Bernoulli [1]. In their work, the polylogarithm function was defined using an absolute convergent series. The development of this function was so significant that it was utilized in the research work of other prominent mathematicians such as Euler, Spence, Abel, Lobachevsky, Rogers, Ramanujan, etc., allowing them to discover various functional identities of great importance as a result [2]. It should come as no surprise that the increased utilization of the polylogarithm function appears to be related to its importance in a number of key areas of mathematics and physics such as topology, algebra, geometry, complex analysis quantum field theory, and mathematical physics [3–6]. In order to better understand their features, several new subclasses of meromorphic functions associated with polylogarithm functions and its analogues were developed, and their properties were investigated using various operators [7–9].

From another perspective, many special functions in analytic number theory and mathematical physics were used to derive new operators of a convolution attitude with numerous applications. This evidently enriched the geometric function theory and prompted many researchers to pursue their research in this area. One of the most important operators is derived from a well-known branch of mathematics, namely quantum calculus (or simply q-calculus), which has served as a bridge between mathematics and physics through its magnificent applications in many fields including number theory, quantum theory, differential equations, combinatorics, orthogonal polynomials, hypergeometric functions, electronics and most recently in quantum computing [10,11].

The methodical invention of q-calculus, and hence the stemmed q-derivative operator, is credited to the initial founder Jackson [12,13], who established it in the early twentieth century. Ever since and up until three decades ago, the appreciation of q-calculus applications by researchers in many domains of mathematics and physics was not discernible. In 1990, Ismail et al. [14] reported on a first attempt to use q-calculus in geometric function theory. They employed q-derivatives to define a generalized version of what is known as q-starlike functions and then analyzed its properties. Since then, the implementation of q-calculus, notably in geometric function theory, has risen substantially. For instance, Sirivastava has demonstrated a great interest in q-derivatives by extensively researching their significance in geometric function theory, as seen in his research effort [15].

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Of late, a new research direction has taken a detour towards uncovering the significance of a q-derivative operator by applying it in the creation of new subclasses of special functions based on bi-univalent, univalent and meromorphic functions and then examining its characteristics [16–24]. Nonetheless, the involvement of q-derivatives in the development and examination of novel subclasses of meromorphic functions and their features, particularly those linked with polylogarithm functions, has not been reported before. Thus, our current investigation is deemed the first of its type, focusing on the implementation of a q-derivative operator to the previously established polylogarithm function based on a meromorphic function. This genuine piece of work could pave the way for future researchers to utilize this approach to introduce and explore the properties of new subclasses of meromorphic functions with potential applications in geometric function theory.

Let Σ denotes the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k; \qquad a_k \geq 0, \tag{1}$$

which are analytic in the punctured open unit disk

$$\mathbb{U}^* := \{z : z \in \mathbb{C}, 0 < |z| < 1\} =: \mathbb{U} \backslash \{0\}.$$

A function $f(z)$ in Σ is said to be meromorphically starlike of order η if and only if

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \eta; \qquad (z \in \mathbb{U}^*), \tag{2}$$

for some $\eta(0 \leq \eta < 1)$. We denote by $\Sigma^*(\eta)$ the class of all meromorphically starlike functions of order η. The Hadamard product of two functions has been widely used in factorizing a newborn function (see, [25–27]). For functions $f \in \Sigma$ given by (1), and $g \in \Sigma$ given by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k; \qquad b_k \geq 0,$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k,$$

which is used in introducing the following function.

Let $Li_c(z)$ denote the well-known polylogarithm function, which was invented in 1696 by Leibniz and Bernoulli, as mentioned in [1].

$$Li_c(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^c}; \qquad (|z| < 1, c \geq 2).$$

In 2014, Alhindi and Darus [28] defined the new operator $\Omega_c f(z)$ in conjunction with the meromorphic functions as follows:

$$\begin{aligned} \Omega_c f(z) &= \Phi_c(z) * f(z) \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{1}{(k+2)^c} a_k z^k; \qquad (z \in \mathbb{U}^*, c \geq 2), \end{aligned} \tag{3}$$

where the function $\Phi_c(z)$ is given by

$$\Phi_c(z) = z^{-2} Li_c(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{1}{(k+2)^c} z^k,$$

Gasper and Rahman [29] defined the q–derivative $(0 < q < 1)$ of a meromorphic function f of the form (1), by

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}; \qquad (z \in \mathbb{U}^*). \tag{4}$$

From (3) and (4) we obtain:

$$D_q \Omega_c f(z) \quad = \quad -\frac{1}{qz^2} + \sum_{k=1}^{\infty} \frac{1}{(k+2)^c} a_k z^{k-1} [k]_q; \qquad (z \in \mathbb{U}^*, c \geq 2); \tag{5}$$

where

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + q^2 + q^3 + ... + q^{k-1}.$$

Simple calculation yields to the following equation:

$$\lim_{q \to 1^-} D_q \Omega_c f(z) \quad = \quad \Omega_c^{'} f(z)$$

Definition 1. *Let $\Sigma_q(\alpha, \beta, \eta)$ denote the subclass of Σ of functions f of the form (1) that satisfies the condition:*

$$\left| \frac{z^3 (D_q \Omega_c f(z))^{'} - \frac{2}{q}}{\eta z^2 (D_q \Omega_c f(z)) + (1+\eta)\frac{\alpha}{q} - \frac{1}{q}} \right| < \beta, \tag{6}$$

where $D_q \Omega_c f(z)$ is given in (5), $0 < q < 1$, $0 < \alpha < 1$, $0 \leq \eta \leq 1$ and $0 < \beta < 1$.

2. Main Results

In this section, we state coefficient estimates for functions that belong to the class $\Sigma_q(\alpha, \beta, \eta)$, then we discuss some characteristics of sub-classes of Σ, followed by studying the convexity and connectedness.

Firstly, we determine the coefficient estimates for functions belonging to the class $\Sigma_q(\alpha, \beta, \eta)$ in the following theorem.

Theorem 1. *Suppose $f(z) \in \Sigma$, then $f(z) \in \Sigma_q(\alpha, \beta, \eta)$ if and only if:*

$$\sum_{k=1}^{\infty} \frac{[k]_q (k - 1 + \eta\beta) q}{\beta(\eta+1)(1-\alpha)} a_k \leq 1. \tag{7}$$

The result is sharp for G(z) given by

$$G(z) = \frac{1}{z} + \frac{\beta(\eta+1)q^{-1}(1-\alpha)}{[k]_q(k-1+\eta\beta)} z^2. \tag{8}$$

Proof. Let $f(z) \in \Sigma_q(\alpha, \beta, \eta)$, then (6) holds true. By replacing (5) in (6), we obtain

$$\left| \frac{\sum_{k=1}^{\infty} (k-1)[k]_q a_k z^2}{(\eta+1)q^{-1}(\alpha-1) + \sum_{k=1}^{\infty} \eta [k]_q a_k z^2} \right| < \beta. \tag{9}$$

It is known that $Re\{z\} \leq |z|$, for all z, therefore,

$$Re\left\{ \frac{\sum_{k=1}^{\infty} (k-1)[k]_q a_k z^2}{(\eta+1)q^{-1}(1-\alpha) - \sum_{k=1}^{\infty} \eta [k]_q a_k z^2} \right\} < \beta. \tag{10}$$

By letting $z \to 1^-$ through real values, one can obtain

$$\sum_{k=1}^{\infty} [k]_q (k-1+\eta\beta) a_k \leq \beta(\eta+1) q^{-1}(1-\alpha). \tag{11}$$

On the other hand, suppose (7)is satisfied. It is enough to show that

$$H = \left| z^3 (D_q \Omega_c f(z))' - 2q^{-1} \right| - \beta \left| \eta z^2 (D_q \Omega_c f(z)) - q^{-1} + (1+\eta)\alpha q^{-1} \right| < 0. \tag{12}$$

For $0 < |z| = r < 1$, we have

$$\begin{aligned} H &= \left| \sum_{k=1}^{\infty} [k]_q (k-1) a_k z^2 \right| - \beta \left| (\eta+1) q^{-1}(1-\alpha) - \sum_{k=1}^{\infty} \eta [k]_q a_k z^2 \right| \\ &\leq \sum_{k=1}^{\infty} [k]_q (k-1) |a_k| r^2 - \beta(\eta+1) q^{-1}(1-\alpha) + \sum_{k=1}^{\infty} \eta [k]_q |a_k| r^2 \\ &\leq \sum_{k=1}^{\infty} [k]_q (k-1+\eta\beta) |a_k| r^2 - \beta(\eta+1) q^{-1}(1-\alpha). \end{aligned}$$

Since the above inequality holds true for all $0 < r < 1$ and by letting $r \to 1^-$ and applying (7), we conclude that $H \leq 0$ and the proof is complete. □

2.1. Characteristics of Subclasses of Σ

We introduce two subfamilies of Σ and derive some important characteristics of them. For $y \in \mathbb{R}$ such that $0 < y < 1$, let Σ^1 be a subclass of Σ satisfying the condition

$$y f(y) = 1, \tag{13}$$

and Σ^2 be a subclass of Σ satisfying the condition

$$-y^2 f'(y) = 1, \tag{14}$$

and

$$\Sigma_q^i(\alpha, \beta, \eta, y) = \Sigma_q(\alpha, \beta, \eta) \cap \Sigma^i; \qquad i = 1, 2, \tag{15}$$

Theorem 2. *Suppose $f(z) \in \Sigma$, then $f(z) \in \Sigma_q^1(\alpha, \beta, \eta, y)$ if and only if*

$$\sum_{k=1}^{\infty} \left(\frac{[k]_q (k-1+\eta\beta) q}{\beta(\eta+1)(1-\alpha)} + y^2 \right) a_k \leq 1. \tag{16}$$

Proof. Since $f(z) \in \Sigma_q^1(\alpha, \beta, \eta, y)$, we have

$$y f(y) = y \left(\frac{1}{y} + \sum_{k=1}^{\infty} a_k y^k \right) = 1, \tag{17}$$

by the definition of Σ^1, we have

$$1 = 1 - \sum_{k=1}^{\infty} a_k y^{k+1}. \tag{18}$$

By substituting (18) in (7), we obtain

$$\sum_{k=1}^{\infty} [k]_q (k-1+\eta\beta) q a_k \leq \beta(\eta+1)(1-\alpha) \Big(1 - \sum_{k=1}^{\infty} a_k y^{k+1} \Big),$$

which yields to

$$\sum_{k=1}^{\infty}[k]_q(k-1+\eta\beta)qa_k+\sum_{k=1}^{\infty}\beta(\eta+1)(1-\alpha)a_k y^{k+1}\le\beta(\eta+1)(1-\alpha),$$

and

$$\sum_{k=1}^{\infty}\left(\frac{[k]_q(k-1+\eta\beta)q}{\beta(\eta+1)(1-\alpha)}+y^2\right)a_k\le 1,$$

which completes the proof of the theorem. □

Theorem 3. *Suppose $f(z)\in\Sigma$, then $f(z)\in\Sigma_q^2(\alpha,\beta,\eta,y)$ if and only if*

$$\sum_{k=1}^{\infty}\left(\frac{[k]_q(k-1+\eta\beta)q}{\beta(\eta+1)(1-\alpha)}-y^2\right)a_k\le 1. \tag{19}$$

Proof. Since $-y^2f'(y)=1$, we have

$$1=1+\sum_{k=1}^{\infty}a_k y^{k+1}. \tag{20}$$

By substituting (18) in (7), we obtain the required result. □

Corollary 1. *Let $f(z)$ be a function of the form (1) and $f(z)\in\Sigma_q^1(\alpha,\beta,\eta,y)$, then*

$$a_k\le\frac{\beta(\eta+1)(1-\alpha)}{[k]_q(k-1+\eta\beta)q+\beta(\eta+1)(1-\alpha)y^2}.$$

Corollary 2. *Let $f(z)$ be a function of the form (1) and $f(z)\in\Sigma_q^2(\alpha,\beta,\eta,y)$, then*

$$a_k\le\frac{\beta(\eta+1)(1-\alpha)}{[k]_q(k-1+\eta\beta)q-\beta(\eta+1)(1-\alpha)y^2}.$$

Next, we obtain distortion bounds of the classes $\Sigma_q^i(\alpha,\beta,\eta,y)$ for $i=1,2$.

Theorem 4. *Suppose $f(z)\in\Sigma_q^1(\alpha,\beta,\eta,y)$, then for $0<|z|=r<1$*

$$|f(z)|\ge\frac{\eta\beta q-\beta(\eta+1)(1-\alpha)r^2}{r(\eta\beta q+\beta(\eta+1)(1-\alpha)y^2)}.$$

The result is sharp for

$$L_1(z)=\frac{\eta\beta q-\beta(\eta+1)(1-\alpha)z^2}{z(\eta\beta q+\beta(\eta+1)(1-\alpha)y^2)}.$$

Proof. Since $f(z)\in\Sigma_q^1(\alpha,\beta,\eta,y)$, so by (16), we have

$$\sum_{k=1}^{\infty}a_k\le\frac{\beta(\eta+1)(1-\alpha)}{\eta\beta q+\beta(\eta+1)(1-\alpha)y^2}. \tag{21}$$

By using (18), we obtain

$$1=1-\sum_{k=1}^{\infty}a_k y^{k+1}\ge\frac{\beta\eta q}{\eta\beta q+\beta(\eta+1)(1-\alpha)y^2}. \tag{22}$$

Therefore,

$$\begin{aligned}|f(z)| &= \left|\frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k\right| \\ &\geq \frac{1}{r} - r\sum_{k=1}^{\infty} a_k \\ &\geq \frac{\eta\beta q}{(\eta\beta q) + \beta(\eta+1)(1-\alpha)y^3} r^{-1} \\ &- r\left(\frac{\beta(\eta+1)(1-\alpha)}{\eta\beta q + \beta(\eta+1)(1-\alpha)y^2}\right) \\ &= \frac{\eta\beta q - \beta(\eta+1)(1-\alpha)r^2}{r\left(\eta\beta q + \beta(\eta+1)(1-\alpha)y^2\right)},\end{aligned}$$

which completes the proof. □

Theorem 5. *Suppose* $f(z) \in \Sigma_q^2(\alpha, \beta, \eta, y)$*, then for* $0 < |z| = r < 1$

$$|f(z)| \leq \frac{\eta\beta q + \beta(\eta+1)(1-\alpha)r^2}{r(\eta\beta q - \beta(\eta+1)(1-\alpha)y^2)}.$$

The result is sharp for

$$L_2(z) = \frac{\eta\beta q + \beta(\eta+1)(1-\alpha)z^2}{z(\eta\beta q - \beta(\eta+1)(1-\alpha)y^2)}.$$

Proof. Since $f(z) \in \Sigma_q^2(\alpha, \beta, \eta, y)$, so by (19), we have

$$\sum_{k=1}^{\infty} a_k \leq \frac{\beta(\eta+1)(1-\alpha)}{\eta\beta q - \beta(\eta+1)(1-\alpha)y^2}. \tag{23}$$

From (20), we have

$$1 = 1 + \sum_{k=1}^{\infty} a_k y^{k+1} \leq \frac{\beta\eta q}{\eta\beta q - \beta(\eta+1)(1-\alpha)y^2}. \tag{24}$$

Thus,

$$\begin{aligned}|f(z)| &= \left|\frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k\right| \\ &\leq \frac{1}{r} + r\sum_{k=1}^{\infty} a_k \\ &\leq \frac{\eta\beta q}{(\eta\beta q) - \beta(\eta+1)(1-\alpha)y^3} r^{-1} \\ &+ r\left(\frac{\beta(\eta+1)(1-\alpha)}{\eta\beta q - \beta(\eta+1)(1-\alpha)y^2}\right) \\ &= \frac{\eta\beta q + \beta(\eta+1)(1-\alpha)r^2}{r\left(\eta\beta q - \beta(\eta+1)(1-\alpha)y^2\right)}.\end{aligned}$$

Hence, the proof is complete. □

2.2. Convexity and Connectedness

In this section, firstly, we investigate the convexity of $\Sigma_q^i(\alpha,\beta,\eta,y)$ for $i=1,2$.

Theorem 6. *The classes $\Sigma_q^i(\alpha,\beta,\eta,y)$ for $i=1,2$ are convex sets.*

Proof. Let $f_t(z)$ be in the class $\Sigma_q^1(\alpha,\beta,\eta,y)$ and have the form

$$f_t(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_{k,t}z^k; \qquad t=0,1,2,\ldots m. \tag{25}$$

It is enough to prove that $F(z)$, which is of the form

$$F(z)=\sum_{t=0}^{m}d_tf_t(z); \qquad (d_t\geq 0), \tag{26}$$

is also in the class $\Sigma_q^1(\alpha,\beta,\eta,y)$, where $\sum_{t=0}^{m}d_t=1$.

By substituting (25) in (26), we obtain

$$\begin{aligned}F(z) &= \sum_{t=0}^{m}d_t\Big(\frac{1}{z}+\sum_{k=0}^{\infty}a_{k,t}z^k\Big)\\ &= \frac{1}{z}+\sum_{k=1}^{\infty}\Big(\sum_{t=0}^{m}d_ta_{k,t}\Big)z^k\\ &= \frac{1}{z}+\sum_{k=1}^{\infty}W_kz^k,\end{aligned}$$

where $W_k=\sum_{t=0}^{m}d_ta_{k,t}$.

Since $f_t(z)\in\Sigma_q^1(\alpha,\beta,\eta,y)$ for $t=0,1,\ldots,m$, then (16) holds true:

$$\sum_{k=1}^{\infty}\left(\frac{[k]_q(k-1+\eta\beta)q}{\beta(\eta+1)(1-\alpha)}+y^2\right)a_k\leq 1. \tag{27}$$

Moreover,

$$\begin{aligned}\sum_{k=1}^{\infty}\left(\frac{[k]_q(k-1+\eta\beta)q}{\beta(\eta+1)(1-\alpha)}+y^2\right)W_k &= \sum_{k=1}^{\infty}\left(\frac{[k]_q(k-1+\eta\beta)q}{\beta(\eta+1)(1-\alpha)}+y^2\right)\left(\sum_{t=0}^{m}d_ta_{k,t}\right)\\ &= \sum_{t=0}^{m}d_t\left(\sum_{k=1}^{\infty}\left(\frac{[k]_q(k-1+\eta\beta)q}{\beta(\eta+1)(1-\alpha)}+y^2\right)a_{k,t}\right)\\ &\leq \sum_{t=0}^{m}d_t=1.\end{aligned}$$

The proof is complete. □

Following the same technique, we can prove the same characteristic for the class $\Sigma_q^2(\alpha,\beta,\eta,y)$.

Next, we discuss the connectedness of $\Sigma_q^i(\alpha,\beta,\eta,y)$ for $i=1,2$.

Definition 2. *Let V be a non empty subset of $[0,1]$, then*

$$\Sigma_q^1(\alpha,\beta,\eta,V)=\bigcup_{y_1\in V}\Sigma_q^1(\alpha,\beta,\eta,y_1) \tag{28}$$

Note that if V has only one element, then $\Sigma_q^1(\alpha,\beta,\eta,V)$ is known to be a convex family by Theorem 6.

In order to prove the main result, the following lemma is required.

Lemma 1. *If $f(z)\in\Sigma_q^1(\alpha,\beta,\eta,y_1)\cap\Sigma_q^1(\alpha,\beta,\eta,y_2)$, where y_1 and y_2 are positive numbers with $y_1\neq y_2$, then $f(z)=\frac{1}{z}$.*

Proof. If $f(z)\in\Sigma_q^1(\alpha,\beta,\eta,y_1)\cap\Sigma_q^1(\alpha,\beta,\eta,y_2)$ and $f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_kz^k$, then

$$1=1-\sum_{k=1}^{\infty}a_ky_1^2=1-\sum_{k=1}^{\infty}a_ky_2^2.$$

In another word:

$$\sum_{k=1}^{\infty}a_k(y_1^2-y_2^2)=0.$$

But $a_k\geq 0$, $y_1>0$ and $y_2>0$, which yields $a_k=0$ for each $k\geq 0$ and so $f(z)=\frac{1}{z}$. □

Theorem 7. *If V is contained in [0,1], then $\Sigma_q^1(\alpha,\beta,\eta,V)$ is a convex family if and only if V is connected.*

Proof. Suppose V is connected and $y_1,y_2\in V$ with $y_1\leq y_2$. It suffices to prove that for $h(z)$ and $l(z)$ given by

$$h(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_kz^k\in\Sigma_q^1(\alpha,\beta,\eta,y_1), \tag{29}$$

$$l(z)=\frac{1}{z}+\sum_{k=1}^{\infty}b_kz^k\in\Sigma_q^1(\alpha,\beta,\eta,y_2), \tag{30}$$

and $0\leq\zeta\leq 1$, there exists a $y_1\leq x\leq y_2$ such that

$$m(z)=\zeta h(z)+(1-\zeta)l(z)\in\Sigma_q^1(\alpha,\beta,\eta,x). \tag{31}$$

From (18), we have

$$1=1-\sum_{k=1}^{\infty}a_ky_1^{k+1}\quad \textit{and}\quad 1=1-\sum_{k=1}^{\infty}a_ky_2^{k+1} \tag{32}$$

Thus, we obtain

$$\begin{aligned}
M(z) &= zm(z)=z\big(\zeta h(z)+(1-\zeta)l(z)\big)\\
&= \zeta+\sum_{k=1}^{\infty}\zeta a_kz^{k+1}+\big(1-\zeta\big)+\sum_{k=1}^{\infty}(1-\zeta)b_kz^{k+1}\\
&= \zeta-\sum_{k=1}^{\infty}\zeta a_ky_1^{k+1}+\sum_{k=1}^{\infty}\zeta a_kz^{k+1}\\
&+ (1-\zeta)-\sum_{k=1}^{\infty}\zeta b_ky_2^{k+1}+\sum_{k=1}^{\infty}\zeta b_kz^{k+1}\\
&= 1+\zeta\sum_{k=1}^{\infty}\big(z^{k+1}-y_1^{k+1}\big)a_k+(1-\zeta)\sum_{k=1}^{\infty}\big(z^{k+1}-y_2^{k+1}\big)b_k.
\end{aligned} \tag{33}$$

Since it is trivial that $M(y_1)\leq 1$ and $M(y_2)\geq 1$, then there exists $x\in[y_1,y_2]$ such that $M(x)=1$. Thus,

$$xm(x)=1. \tag{34}$$

Therefore, $m(z) \in \Sigma^1$. From another perspective by (33), (34) and (16), we have

$$\begin{aligned}
&\sum_{k=1}^{\infty}\left(\frac{[k]_q(k-1+\eta\beta)q}{\beta(\eta+1)(1-\alpha)}+y^{k+1}\right)(\zeta a_k+(1-\zeta)b_k) \\
&= \zeta\sum_{k=1}^{\infty}\left(\frac{[k]_q(k-1+\eta\beta)q}{\beta(\eta+1)(1-\alpha)}+x_1^{k+1}\right)a_k \\
&+ (1-\zeta)\sum_{k=1}^{\infty}\left(\frac{[k]_q(k-1+\eta\beta)q}{\beta(\eta+1)(1-\alpha)}+x_2^{k+1}\right)b_k \\
&\leq \zeta+(1-\zeta)=1.
\end{aligned}$$

Thus, $m(z) \in \Sigma_q^1(\alpha,\beta,\eta,x)$. Since y_1, y_2 and x are arbitrary, the family $m(z) \in \Sigma_q^1(\alpha,\beta,\eta,J)$ is convex.

Conversely, if V is not connected, then there exists y_1, y_2 and x such that $y_1 < x < y_2$ and $y_1, y_2 \in V$ but $x \notin V$. If $h(z) \in \Sigma_q^1(\alpha,\beta,\eta,y_1)$ and $l(z) \in \Sigma_q^1(\alpha,\beta,\eta,y_2)$, then by Lemma 1, $h(z)$ and $l(z)$ are not both equal to $\frac{1}{z}$, then for fixed x and $0 \leq \zeta \leq 1$ by (33), we obtain

$$M(x,\zeta) = 1+\zeta+\sum_{k=1}^{\infty}\left(z^{k+1}-y_1^{k+1}\right)a_k+(1-\zeta)\sum_{k=1}^{\infty}\left(z^{k+1}-y_2^{k+1}\right)b_k. \tag{35}$$

But $M(x,0) < 1$ and $M(x,1) > 1$, thus there exists ζ^*; $0 < \zeta^* < 1$, such that $M(x,\zeta^*) = 1$ or $xm(x) = 1$, where $m(z) = \zeta^* h(z) + (1-\zeta^*)l(z)$.

Therefore, $m(z) \in \Sigma_q^1(\alpha,\beta,\eta,x)$. By Lemma 1, we have $m(z) \notin \Sigma_q^1(\alpha,\beta,\eta,V)$.

Since $x \in V$ and $m(z) \neq \frac{1}{z}$, this implies that the family $\Sigma_q^1(\alpha,\beta,\eta,V)$ is not convex. This contradiction completes the proof of the theorem. □

Following the same technique, we can prove the same characteristic for the class $\Sigma_q^2(\alpha,\beta,\eta,V)$.

3. Conclusions

In this research paper, the q-derivative operator was applied on the meromorphic polylogarithm function to obtain the new operator $D_q\Omega_c f(z) = -\frac{1}{qz^2} + \sum_{k=1}^{\infty}\frac{1}{(k+2)^c}a_k z^{k-1}[k]_q$. The class $\Sigma_q(\alpha,\beta,\eta)$ was then introduced containing $D_q\Omega_c\, f(z)$ along with the coefficient estimate of the functions belonging to it. Moreover, two subclasses of $\Sigma_q(\alpha,\beta,\eta)$ i.e., $\Sigma_q^1(\alpha,\beta,\eta,y)$ and $\Sigma_q^2(\alpha,\beta,\eta,y)$ were defined and the necessary and sufficient conditions for a function to be in these two classes were proved. the distortion bounds for the two classes were derived to be $|f(z)| \geq \frac{\eta\beta q-\beta(\eta+1)(1-\alpha)r^2}{r(\eta\beta q+\beta(\eta+1)(1-\alpha)y^2)}$ and $|f(z)| \leq \frac{\eta\beta q+\beta(\eta+1)(1-\alpha)r^2}{r(\eta\beta q-\beta(\eta+1)(1-\alpha)y^2)}$, respectively. Eventually, we have proved in detail that these two subclasses are convex and connected sets.

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