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An Evolving Spacetime Metric Induced by a 'Static' Source

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Abstract: In a series of recent papers we developed a formulation of general relativity in which spacetime and the dynamics of matter evolve with a Poincaré invariant parameter τ. In this paper, we apply the formalism to derive the metric induced by a 'static' event evolving uniformly along its t-axis at the spatial origin $\mathbf{x} = 0$. The metric is shown to vary with t and τ, as well as spatial distance r, taking its maximum value for a test particle at the retarded time $\tau = t - r/c$. In the resulting picture, an event localized in space and time produces a metric field similarly localized, where both evolve in τ. We first derive this metric as a solution to the wave equation in linearized field theory, and discuss its limitations by studying the geodesic motion it produces for an evolving event. By then examining this solution in the 4+1 formalism, which poses an initial value problem for the metric under τ-evolution, we clarify these limitations and indicate how they may be overcome in a solution to the full nonlinear field equations.

Keywords: general relativity; the problem of time; Stueckelberg–Horwitz–Piron theory; parameterized relativistic mechanics

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Citation: Land, M. An Evolving Spacetime Metric Induced by a 'Static' Source. *Symmetry* **2023**, *15*, 1381. https://doi.org/10.3390/sym15071381

Academic Editors: Tomohiro Inagaki and Olga Kodolova

Received: 22 June 2023
Revised: 4 July 2023
Accepted: 5 July 2023
Published: 7 July 2023

1. Introduction

The 4+1 formalism [1–5] in general relativity (GR) poses an initial value for the spacetime metric in which evolution of fields and matter is parameterized by a Poincaré invariant chronological time τ. Parameterization in proper time was introduced in 1937 by Fock [6] in their manifestly covariant electrodynamics. However, in 1941, Stueckelberg [7,8] showed that neither coordinate time t nor the proper time $ds = \sqrt{-dx^\mu dx_\mu}$ can be used as a chronological evolution parameter in an electrodynamics that accounts for pair creation/annihilation processes. Instead, to describe antiparticles as particles whose trajectory reverses direction in coordinate time t, he introduced a strictly monotonic evolution parameter τ, independent of phase space and external to the spacetime manifold.

Piron and Horwitz [9] generalized Stueckelberg's formalism, constructing a relativistic canonical many-body theory [10–14] with Lorentz scalar Hamiltonian. By including τ in the U(1) gauge freedom (but not the spacetime manifold), the Stueckelberg–Horwitz–Piron (SHP) formalism in flat spacetime [15–18] provides an electrodynamic theory of events interacting through five gauge potentials. The evolution of a localized spacetime event induces a field acting on a localized remote event, through an interaction synchronized by the chronological time τ, and recovering Maxwell electrodynamics in τ-equilibrium.

The structure of these interactions suggests a higher symmetry such as O(3,2) or O(4,1) for free fields, but the observed Lorentz invariance of spacetime requires that any 5D symmetry break to 4D tensor and scalar representations of O(3,1) in the presence of matter. A similar conflict of symmetries is familiar from classical acoustics, where the pressure wave equation appears invariant under Lorentz-like transformations, but no relativistic effects are expected for observers approaching the speed of sound. These considerations are a guiding principle in extension of the formalism to general relativity.

Horwitz has extended the SHP framework to curved spacetime [19,20], developing a classical and quantum theory of interacting event evolution in a background metric $g_{\mu\nu}(x)$. As a many-body theory with τ-evolution, the scalar event density $\rho(x,\tau)$ and energy-momentum tensor $T_{\mu\nu}(x,\tau)$ naturally become explicitly τ-dependent. In keeping with Wheeler's characterization [21] of Einstein gravitation as "spacetime tells matter how to move; matter tells spacetime how to curve", the τ-dependent matter distribution must be reflected in a τ-dependent local metric $\gamma_{\mu\nu}(x,\tau)$. Particle dynamics in such a metric spacetime may differ from standard GR—some details are indicated in Section 2.2. As in flat space electrodynamics, the free fields of GR—the geometrical structures—enjoy 5D spacetime and gauge symmetries, but the spacetime symmetry must break to O(3,1) in the presence of matter. Because the metric evolution is parameterized by the external parameter τ and the matter evolution is determined by an O(3,1) scalar Hamiltonian, there is no conflict with the diffeomorphism invariance of general relativity. Some details of the 4+1 method are reviewed in Sections 2.3 and 2.4.

Several simple examples of the 4+1 formalism were given in previous papers, but these did not involve a source event evolving along a localized trajectory. In this paper, we study the field induced by a localized event, with the goal of describing a τ-localized metric and the gravitational field it produces on a remote localized event. We proceed in analogy to SHP electrodynamics where a particle is modeled as an ensemble of events [18] located at $\mathbf{x} = 0$ in space, but narrowly distributed along the t-axis. The 5D wave equation leads to the Coulomb potential [18] in the form

$$a_0(x,\tau) = -\frac{\varphi(t - r/c - \tau)}{r} + o\left(\frac{1}{r^2}\right), \tag{1}$$

where $\varphi(s)$ is the distribution on the t-axis, with maximum at $\varphi(0)$ and normalized as $\int d\tau\, \varphi(\tau) = 1$. At long distances, the higher order term may be neglected. A test event at spatial distance r will thus experience a potential localized around $\tau = t_R = t - r/c$, the retarded time at which the source event produced the field, where c is the speed of light. The general Liénard–Wiechert potentials induced by an event on an arbitrary trajectory appear in their usual form [18], but multiplied by $\varphi(t - r/c - \tau)$.

For the gravitational field, we similarly consider the metric induced by a 'static' event evolving uniformly along the t-axis in its rest frame, fixed at the spatial origin $\mathbf{x} = 0$, leading to an event current and mass-energy-momentum tensor $T^{\mu\nu}(x,\tau)$. In Section 3, we use this tensor as the source of a wave equation in linearized GR, and derive a metric that varies with t and τ, as well as spatial distance r. Neglecting the higher order contribution as in electrodynamics, a test particle with coordinates $x = \left(x^0(\tau), \mathbf{x}(\tau)\right)$ experiences a metric that takes its maximum at $\tau = t - |\mathbf{x}|/c$. However, unlike the flat space motion of an event under the electrodynamic Lorentz force, the geodesic equations for an event moving in this metric differ from our expectations, suggesting that localization along the t-axis may cause the gravitational force to change sign. We show that this issue follows from the structure of the Green's function for the wave equation in linearized GR and will obtain for any t-dependent event density.

In Section 4, we examine this solution in the 4+1 formalism, which poses an exact initial value problem for the metric under τ-evolution. In this context, neglecting the higher-order term is seen to contradict the assumption of an evolving metric, clarifying the limitations of the linearized method. We pose the problem of an evolving metric produced by an evolving source narrowly distributed in spacetime in terms of the full nonlinear field equations and discuss the additional complexities associated with this system. Finally, Section 5 is devoted to conclusions and discussion. In a subsequent paper, numerical solutions to the initial value problem will be discussed.

2. Review of General Relativity with Invariant Evolution

2.1. Gauge and Spacetime Symmetries

In a flat Minkowski spacetime with $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$, the free particle action

$$S = \int d\tau\, \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu, \tag{2}$$

is made maximally U(1) gauge invariant [15] by introducing five gauge fields as

$$S_{\text{SHP}} = \int d\tau \, \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e}{c} \dot{x}^\mu a_\mu(x,\tau) + \frac{e}{c} c_5 a_5(x,\tau), \tag{3}$$

$$= \int d\tau \, \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e}{c} \dot{x}^\beta a_\beta(x,\tau), \tag{4}$$

where we introduce $x^5 = c_5 \tau$ in analogy to $x^0 = ct$ and partition Greek indices such that

$$\alpha,\beta,\gamma,\delta = 0,1,2,3,5 \qquad \lambda,\mu,\nu,\rho \ldots = 0,1,2,3. \tag{5}$$

This action enjoys the 5D gauge invariance $a_\alpha(x,\tau) \longrightarrow a_\alpha(x,\tau) + \partial_\alpha \Lambda(x,\tau)$, but because $\dot{x}^\mu \dot{x}_\mu$, $\dot{x}^\mu a_\mu$, and a_5 are O(3,1) scalars, its spacetime symmetry is restricted to 4D. As a guide to posing field equations for a τ-dependent metric in curved spacetime, we may consider Equation (4) as a standard 5D action in which we break the symmetry of the matter term by imposing the constraint $\dot{x}^5 \equiv c_5$ and making the replacement $\dot{x}^\alpha \dot{x}_\alpha \longrightarrow \dot{x}^\mu \dot{x}_\mu$ in the kinetic term, restricting the phase space to $(x^\mu, \dot{x}^\mu)$. The electrodynamics associated with the symmetry-broken action differ in significant ways from standard Maxwell theory in 5D. In particular, the Lorentz force [18]

$$M\ddot{x}_\mu = \frac{e}{c} \dot{x}^\beta f_{\mu\beta} \qquad \frac{d}{d\tau}\left(-\frac{1}{2} M \dot{x}^\mu \dot{x}_\mu\right) = c_5 \frac{e}{c} \dot{x}^\mu f_{5\mu} \qquad f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha, \tag{6}$$

permits mass exchange between particles and fields, while leaving the total mass, energy, and momentum of particles and fields conserved. Compatibility with standard electrodynamic phenomenology places restrictions on $c_5/c \ll 1$ but the strict limit $c_5 \longrightarrow 0$ produces a τ-equilibrium [18] that recovers standard Maxwell theory.

Field dynamics are determined by a kinetic term of the type

$$S_{\text{field}} = \int d\tau \, d^4x \, f^{\alpha\beta}(x,\tau) f_{\alpha\beta}(x,\tau), \tag{7}$$

where $f_{\mu\nu}$ is a second rank tensor, while $f_{5\mu}$ is a vector field strength, because the 5-index signifies an O(3,1) scalar quantity. Raising the 5-index in (7) suggests a 5D flat space metric

$$\eta_{\alpha\beta} = \text{diag}(-1,1,1,1,\sigma), \tag{8}$$

where $\sigma = \pm 1$. But expanding

$$f^{\alpha\beta}(x,\tau) f_{\alpha\beta}(x,\tau) = f^{\mu\nu}(x,\tau) f_{\mu\nu}(x,\tau) + 2\sigma f^\mu_{\ 5}(x,\tau) f_{\mu 5}(x,\tau), \tag{9}$$

we may regard σ as the choice of sign for the vector-vector interaction, with no inherent significance for the geometry of spacetime.

Following these considerations, we approach the construction of a τ-dependent GR by embedding 4D spacetime $\mathcal{M}$ in a 5D pseudo-spacetime $\mathcal{M}_5 = \mathcal{M} \times R$ with coordinates $X^\alpha = (x^\mu, c_5\tau)$ and a metric $g_{\alpha\beta}(x,\tau)$ determined by the standard 5D Einstein field equations on $\mathcal{M}_5$. By performing the embedding in a vielbein frame [22], we may specify the metric as (8) and break the 5D spacetime symmetry to O(3,1) at the source, by correcting the flat space metric for the quintrad for the matter terms under the replacement

$$\eta_{ab} = \text{diag}(-1,1,1,1,\sigma) \longrightarrow \widehat{\eta}_{ab} = \text{diag}(-1,1,1,1,0), \tag{10}$$

as in electrodynamics. The symmetry-broken field equations are transformed into the coordinate frame as

$$R_{\alpha\beta} = \frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{1}{2}\widehat{g}_{\alpha\beta}\widehat{T}\right), \tag{11}$$

using the known vielbein field. The LHS of Equation (11) enjoys 5D gauge and spacetime symmetries, while the RHS is O(3,1) covariant. Generalizing the 3+1 formalism in geometrodynamics [23–26] to 4+1, we take advantage of the natural foliation of $\mathcal{M}_5$ into 4D equal-τ spacetimes homeomorphic to $\mathcal{M}$. Standard techniques in the theory of embedded surfaces enable us to extract an initial value problem in the 4D spacetime sector, describing the τ-evolution of metric $\gamma_{\mu\nu}(x,\tau)$ and the extrinsic curvature $K_{\mu\nu}(x,\tau)$ that accounts for aspects of the 5D connection $\Gamma^{\gamma}{}_{\alpha\beta}$ not contained in the 4D metric.

2.2. Event Dynamics in Curved Spacetime

Applying the Euler–Lagrange equations to the Lagrangian

$$L = \frac{1}{2} m g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta, \tag{12}$$

we obtain the 5D geodesic equations for an event $x^\gamma(\tau)$

$$\frac{D\dot{x}^\gamma}{D\tau} = \ddot{x}^\gamma + \Gamma^\gamma_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta, \tag{13}$$

with Christoffel symbols

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\frac{\partial g_{\delta\beta}}{\partial x^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta} \right). \tag{14}$$

We break the 5D symmetry to O(3,1) by asserting

$$x^5 = c_5 \tau \quad \longrightarrow \quad \dot{x}^5 = c_5 \quad \longrightarrow \quad \ddot{x}^5 = 0, \tag{15}$$

as an a priori constraint. The dynamical system is now described by the equations

$$\frac{D\dot{x}^\mu}{D\tau} = \ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = \ddot{x}^\mu + \Gamma^\mu_{\nu\sigma} \dot{x}^\nu \dot{x}^\sigma + 2c_5 \Gamma^\mu_{5\nu} \dot{x}^\nu + c_5^2 \Gamma^\mu_{55} = 0, \tag{16}$$

$$\frac{D\dot{x}^5}{D\tau} = \ddot{x}^5 \equiv 0, \tag{17}$$

which under an appropriate metric field is consistent with the symmetries of matter, and recovers standard GR when $g_{5\alpha} = 0$ and $\partial_\tau g_{\mu\nu} = 0$. Defining the canonical momentum

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = m\left(g_{\mu\nu} \dot{x}^\nu + c_5 g_{\mu 5}\right), \tag{18}$$

the Hamiltonian is

$$K = \frac{1}{2M} p^2 + \frac{1}{2} c_5 g_{55} g^{5\mu} p_\mu - \frac{1}{2} c_5 g_{5\mu} g^{\mu\lambda} p_\lambda + \frac{1}{2} M c_5^2 g_{5\mu} g^{\mu\lambda} g_{\lambda 5} + \frac{1}{2} M c_5^2 g_{55}, \tag{19}$$

which takes the recognizable form

$$K = \frac{1}{2m} p^\mu p_\mu + \frac{1}{2} m c_5^2 \, g_{55}, \tag{20}$$

if $g^{5\mu} = 0$, with $g_{55}(x,\tau)$ playing the role of a τ-dependent potential on 4D spacetime. The canonical equations of motion are

$$\dot{x}^\mu = \frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu} \qquad\qquad \dot{p}_\mu = \frac{dp_\mu}{d\tau} = -\frac{\partial K}{\partial x^\mu}, \tag{21}$$

and since $p_5 \equiv 0$, the Poisson bracket is

$$\{F,G\} = \frac{\partial F}{\partial x^\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial x^\alpha} = \frac{\partial F}{\partial x^\mu}\frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu}\frac{\partial G}{\partial x^\mu}, \tag{22}$$

so for any scalar function $F(x,p,\tau)$ on phase space

$$\frac{dF}{d\tau} = \{F,K\} + \frac{\partial F}{\partial \tau}, \tag{23}$$

generalizing the nonrelativistic result. Therefore, the Hamiltonian is conserved unless K depends explicitly on τ through $g_{\alpha\beta}(x,\tau)$. We note that even when K is a constant of the motion, the 4D mass $p^\mu p_\mu/2m$ may vary under g_{55}.

As we showed in [2], mass variation can appear in the Newtonian approximation through τ-dependence of the metric. Expanding the geodesic equations as

$$0 = \ddot{x}^\mu + \Gamma^\mu_{00}\dot{x}^0\dot{x}^0 + 2\Gamma^\mu_{i0}\dot{x}^i\dot{x}^0 + \Gamma^\mu_{ij}\dot{x}^i\dot{x}^j + 2c_5\Gamma^\mu_{50}\dot{x}^0 + 2c_5\Gamma^\mu_{5i}\dot{x}^i + c_5^2\Gamma^\mu_{55}, \tag{24}$$

we take $\partial_0 g_{\alpha\beta} = 0$ and neglect terms containing $\dot{x}^i/c \ll 1$ for $i = 1,2,3$, so that inserting the nonzero Christoffel symbols

$$\begin{aligned} \Gamma^\mu_{00} &= -\frac{1}{2}\eta^{\mu\nu}\partial_\nu g_{00} & \Gamma^\mu_{ij} &= \frac{1}{2}\eta^{\mu k}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\nu}\right), \\ \Gamma^\mu_{50} &= \frac{1}{2}\eta^{\mu\nu}\frac{\partial g_{\nu 0}}{\partial x^5} & \Gamma^\mu_{55} &= -\frac{1}{2}\eta^{\mu\nu}\frac{\partial g_{55}}{\partial x^\nu}, \end{aligned} \tag{25}$$

the equations of motion reduce to

$$\frac{d^2t}{d\tau^2} = \frac{dt}{d\tau}\partial_\tau g_{00} \qquad \ddot{\mathbf{x}} = \frac{1}{2}c^2\left(\frac{dt}{d\tau}\right)^2\nabla g_{00} + \frac{1}{2}c_5^2\nabla g_{55}. \tag{26}$$

Compared to Newtonian gravity, which must obtain exactly when $\partial_\tau g_{\mu\nu}$ and ∇g_{55} vanish, these become

$$\frac{d^2t}{d\tau^2} = \frac{dt}{d\tau}\partial_\tau\left(\frac{2GM}{rc^2}\right) \qquad \frac{d^2\mathbf{x}}{d\tau^2} = -\left(\frac{dt}{d\tau}\right)^2\frac{GM}{r^2}\hat{\mathbf{r}} + \frac{1}{2}c_5^2\nabla g_{55}, \tag{27}$$

where M is a mass parameter associated with the source and G is the gravitational constant. Writing a perturbed source mass parameter $M = M_0 + \delta M(\tau)$ for the source and again neglecting $\dot{r}/c$ we may solve the t equation as

$$\frac{dt}{d\tau} = \exp\left(\frac{2G}{rc^2}\delta M\right), \tag{28}$$

so that in spherical coordinates, the radial equation takes the form

$$\ddot{r} - \frac{L^2}{m^2r^3} + \exp\left(\frac{4G}{rc^2}\delta M\right)\frac{GM_0}{r^2} = 0, \tag{29}$$

where we take $\nabla g_{55} = 0$ and introduce the conserved angular momentum

$$L = mr^2\dot{\phi}. \tag{30}$$

As required, Equations (28) and (29) recover Newtonian gravitation in the absence of the τ-dependent source mass δM. The Hamiltonian in this coordinate system is

$$K = \frac{1}{2}mg_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = -\frac{1}{2}mc^2\left(1 - \frac{2GM_0}{rc^2}\right)\exp\left(\frac{4G}{rc^2}\delta M\right) + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{L^2}{mr^2}, \tag{31}$$

with time derivative

$$\frac{d}{d\tau}K = \exp\left(\frac{4G}{rc^2}\delta M\right)\left(-\frac{Gm}{r} + \frac{4G^2 m M_0}{r^2 c^2}\right)\frac{d}{d\tau}\delta M, \tag{32}$$

and as expected, the Hamiltonian for the motion of this test particle is not conserved in the presence of a variable mass gravitational source. This may be interpreted as a transfer of mass across spacetime mediated by the metric.

For non-thermodynamic dust (a distribution of geodesically evolving events without mutual interaction), we define $\rho(x,\tau)$ as the number of events per spacetime volume, and write a 5-component event current with mass parameter M

$$j^\alpha(x,\tau) = M\rho(x,\tau)\dot{x}^\alpha(\tau), \tag{33}$$

with continuity equation

$$\nabla_\alpha j^\alpha = \frac{\partial j^\alpha}{\partial x^\alpha} + j^\gamma \Gamma^\alpha_{\gamma\alpha} = \frac{\partial \rho}{\partial \tau} + \nabla_\mu j^\mu = 0, \tag{34}$$

where the second equality holds because $j^5 = Mc_5\rho(x,\tau)$ is an O(3,1) scalar and not the 5-component of a vector with 5D symmetry. Generalizing the 4D energy-momentum tensor to 5D, the mass-energy-momentum tensor

$$T^{\alpha\beta} = M\rho\dot{x}^\alpha\dot{x}^\beta \longrightarrow \begin{cases} T^{\mu\nu} = M\rho\dot{x}^\mu\dot{x}^\nu, \\ T^{5\beta} = \dot{x}^5\dot{x}^\beta M\rho = c_5 j^\beta, \end{cases} \tag{35}$$

is conserved as

$$\nabla_\alpha T^{\alpha\beta} = 0, \tag{36}$$

by virtue of the continuity and geodesic equations. The mass-energy-momentum tensor is thus a suitable O(3,1) covariant source for a 5D field equation.

2.3. Evolution of the Local Metric

As indicated in Section 2.1, the derivation of field equations for $g_{\mu\nu}(x,\tau)$ possessing the desired 5D gauge symmetries and 4D spacetime symmetries relies heavily on the theory of embedded surfaces [23–25], the 3+1 ADM formalism [26], and the generalization of these techniques to 4+1 [2–5]. Here we provide a brief overview and refer the reader to the references for details.

We approach the construction of GR with a τ-dependent metric by embedding 4D spacetime $\mathcal{M}$ in a 5D pseudo-spacetime $\mathcal{M}_5 = \mathcal{M} \times R$ with coordinates $X^\alpha = (x^\mu, c_5\tau)$. Because the Bianchi identity

$$\nabla_\alpha G^{\alpha\beta} = \nabla_\alpha\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right) = 0 \qquad \nabla_\alpha X^\beta = \partial_\alpha X^\beta + X^\gamma\Gamma^\beta_{\gamma\alpha}, \tag{37}$$

is independent of dimension [27] and the mass-energy-momentum tensor (35) is conserved, we may combine the Einstein tensor $G^{\alpha\beta}$ and $T_{\alpha\beta}$ to write the field equations

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = k_G T_{\alpha\beta}, \tag{38}$$

for the metric $g_{\alpha\beta}(x,\tau)$ on $\mathcal{M}_5$.

To break the spacetime symmetry of the field equations to O(3,1), we first transform from a coordinate frame tangent to the manifold $\mathcal{M}_5$

$$\mathbf{g}_\alpha = \partial_\alpha \qquad \mathbf{g}^\alpha = \mathbf{d}X^\alpha \qquad \mathbf{g}_\alpha \cdot \mathbf{g}_\beta = g_{\alpha\beta} \qquad \mathbf{g}^\alpha \cdot \mathbf{g}^\beta = g^{\alpha\beta}, \tag{39}$$

to the constant quintrad frame

$$\mathbf{e}_a \cdot \mathbf{e}_b = \eta_{ab} \qquad \mathbf{e}^a \cdot \mathbf{e}^b = \eta^{ab} \qquad \partial_a \mathbf{e}_b = \partial_a \mathbf{e}^b = 0, \tag{40}$$

where by convention Latin letters indicate a reference to the quintrad. To facilitate foliation of $\mathcal{M}_5$ into spacetime hypersurfaces Σ_τ of equal-τ, we extend the partition of coordinate indices to the quintrad indices, leading to the combined index convention

$$\begin{aligned} &\alpha,\beta,\gamma,\delta = 0,1,2,3,5 && \lambda,\mu,\nu,\rho\ldots = 0,1,2,3, \\ &a,b,c,d, = 0,1,2,3,5 && k,l,m,n,\ldots = 0,1,2,3, \end{aligned} \tag{41}$$

where the five indices with respect to the quintrad frame are denoted $\bar{5}$. The transformation between frames is provided by the vielbein field

$$\begin{aligned} \mathbf{g}_\mu &= E_\mu{}^k \mathbf{e}_k + E_\mu{}^{\bar{5}} \mathbf{e}_5 \qquad & \mathbf{e}_k &= e^\mu{}_k \mathbf{g}_\mu + e^5{}_k \mathbf{g}_5, \\ \mathbf{g}_5 &= E_5{}^k \mathbf{e}_k + E_5{}^{\bar{5}} \mathbf{e}_5 \qquad & \mathbf{e}_5 &= e^\mu{}_{\bar{5}} \mathbf{g}_\mu + e^5{}_{\bar{5}} \mathbf{g}_5, \end{aligned} \tag{42}$$

where the spacetime hypersurface (quatrad) is spanned by the $\mathbf{e}_k$ while $\mathbf{e}_5$ points in the direction of τ-evolution orthogonal to Σ_τ. Introducing the ADM parameterization

$$\mathbf{g}_5 = N^\mu \mathbf{g}_\mu + Nn, \tag{43}$$

where N^μ generalizes the shift 3-vector, N is the lapse function with respect to τ, and $n = \mathbf{e}_5$ is the unit normal, the vielbein field becomes

$$\begin{aligned} E_\alpha{}^a &= \delta^\mu_\alpha \delta^a_k E_\mu{}^k + \delta^5_\alpha \left(E_\mu{}^k N^\mu \delta^a_k + N\delta^a_{\bar{5}} \right), \\ e^\alpha{}_a &= \delta^k_a \delta^\alpha_\mu e^\mu{}_k - \delta^{\bar{5}}_a \delta^\alpha_\mu \frac{1}{N} N^\mu + \delta^{\bar{5}}_a \delta^\alpha_5 \frac{1}{N}, \end{aligned} \tag{44}$$

leading to the coordinated metric

$$g_{\alpha\beta} = \begin{bmatrix} \gamma_{\mu\nu} & N_\mu \\ N_\mu & \sigma N^2 + \gamma_{\mu\nu} N^\mu N^\nu \end{bmatrix}, \tag{45}$$

which generalizes the ADM decomposition. Since N and N^μ are arbitrary functions acting as Lagrange multipliers whose choice is comparable to gauge freedom [26], the dynamical content in the vielbein field is contained entirely in the spacetime vierbein field $E_\mu{}^k$.

In the quintrad frame, the Einstein equations take the form

$$R_{ab} - \frac{1}{2}\eta_{ab} R = k_G T_{ab}, \tag{46}$$

and the spacetime symmetry may be broken to O(3,1) by making the replacement (10) as

$$\eta_{ab} \longrightarrow \widehat{\eta}_{ab} = \delta^k_a \delta^l_b \, \eta_{kl}, \tag{47}$$

in the matter terms, leading us to

$$R_{ab} = k_G \left(T_{ab} - \frac{1}{2} \widehat{\eta}_{ab} \widehat{T} \right), \tag{48}$$

where $\widehat{T} = \widehat{\eta}^{ab} T_{ab} = \eta^{kl} T_{kl}$. Using (44), this expression transforms back to the coordinate frame to provide the O(3,1) symmetric field equations

$$R_{\alpha\beta} = k_G \left(T_{\alpha\beta} - \frac{1}{2} P_{\alpha\beta} \widehat{T} \right), \tag{49}$$

with the transformed metric

$$\widehat{g}_{\alpha\beta} = E_\alpha{}^a E_\beta{}^b \widehat{\eta}_{ab} = g_{\alpha\beta} - \sigma n_\alpha n_\beta = P_{\alpha\beta}, \tag{50}$$

that acts as a projection operator from $\mathcal{M}_5$ onto the 4D spacetime hypersurface Σ_τ and thus breaks any higher symmetry to O(3,1).

Given the foliation of the pseudo-spacetime into equal-τ spacetimes, the initial value problem is found using $P_{\alpha\beta}$ to project the geometrical structures from $\mathcal{M}_5$ onto Σ_τ:

1. The covariant derivative D_α on Σ_τ is found using $P_{\alpha\beta}$ to project the covariant derivative ∇_α on $\mathcal{M}_5$,
2. The extrinsic curvature $K_{\alpha\beta}$ is defined by projecting the covariant derivative of the unit normal n_α,
3. The projected curvature $\bar{R}^\delta_{\gamma\alpha\beta}$ on Σ_τ is defined through the non-commutation of projected covariant derivatives D_α and D_β,
4. The Gauss relation is found by decomposing the 5D curvature $R^\delta_{\gamma\alpha\beta}$ in terms of $\bar{R}^\delta_{\gamma\alpha\beta}$ and $K_{\alpha\beta}$,
5. The mass-energy-momentum tensor is decomposed through the projections
$$\kappa = n_\alpha n_\beta T^{\alpha\beta} \qquad p_\beta = -n_{\alpha'} P_{\beta\beta'} T^{\alpha'\beta'} \qquad S_{\alpha\beta} = P_{\alpha\alpha'} P_{\beta\beta'} T^{\alpha'\beta'} \qquad S = P^{\alpha\beta} S_{\alpha\beta},$$
6. Projecting the 5D curvature $R^\delta_{\gamma\alpha\beta}$ on the unit normal n_α leads to the Codazzi relation providing a relationship between $K_{\alpha\beta}$ and p_α,
7. Lie derivatives of $P_{\alpha\beta}$ and $K_{\alpha\beta}$ along the direction of τ evolution, given by the unit normal n_α in the coordinate frame, are combined with these ingredients, along with the O(3,1) symmetric field Equation (49) to obtain τ-evolution equations for $\gamma_{\mu\nu}$ and $K_{\mu\nu}$ and a pair of constraints on the initial conditions.

The evolution equations are

$$\frac{1}{c_5} \partial_\tau \gamma_{\mu\nu} = \mathcal{L}_{\mathbf{N}} \gamma_{\mu\nu} - 2N K_{\mu\nu}, \tag{51}$$

$$\begin{aligned} \frac{1}{c_5} \partial_\tau K_{\mu\nu} = {} & -D_\mu D_\nu N + \mathcal{L}_{\mathbf{N}} K_{\mu\nu}, \\ & + N \left\{ -\sigma \bar{R}_{\mu\nu} + K K_{\mu\nu} - 2 K^\lambda_\mu K_{\nu\lambda} + \sigma k_G \left(S_{\mu\nu} - \frac{1}{2} P_{\mu\nu} S \right) \right\}, \end{aligned} \tag{52}$$

where $\mathcal{L}_{\mathbf{N}}$ is the Lie derivative along N^μ. Their solutions must satisfy the Hamiltonian constraint

$$\bar{R} - \sigma \left(K^2 - K^{\mu\nu} K_{\mu\nu} \right) = -k_G (S + \sigma\kappa), \tag{53}$$

and the momentum constraint

$$D_\mu K^\mu_\nu - D_\nu K = k_G p_\nu. \tag{54}$$

Expressions (52) and (53) differ slightly from those found in the standard 5D Einstein Equations (38), expressing the breaking of spacetime symmetry to O(3,1). The differences are

$$g_{\mu\nu} (S + \sigma\kappa) \to P_{\mu\nu} S, \tag{55}$$

in (52) and

$$-\sigma k_G \kappa \longrightarrow -k_G(S + \sigma\kappa), \tag{56}$$

in (53).

In some cases, such as a diagonal metric in Cartesian coordinates, it is possible to formulate the evolution equations directly in the quatrad frame [5] in the simplified form

$$\frac{1}{c_5}\partial_\tau E_\mu{}^k = -E_\mu{}^l K_l^k, \tag{57}$$

$$\frac{1}{c_5}\partial_\tau K_{kl} = -\sigma \bar{R}_{kl} + K K_{kl} + \sigma k_G \left(S_{kl} - \frac{1}{2}\eta_{kl} S \right), \tag{58}$$

with constraints

$$\bar{R} - \sigma\left(K^2 - K^{kl}K_{kl}\right) = -k_G(S + \sigma\kappa), \tag{59}$$

$$D_k K_m^k - D_m K = k_G p_m, \tag{60}$$

providing an initial value problem for $E_\mu{}^k$ and K_{kl}, with the metric obtained from the vierbein field.

2.4. Weak Field Approximation

As in standard GR, the weak field approximation [4,18] poses the local metric as a small perturbation $h_{\alpha\beta}$ of the flat metric

$$\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1, \sigma), \tag{61}$$

so that

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \longrightarrow \partial_\gamma g_{\alpha\beta} = \partial_\gamma h_{\alpha\beta} \qquad \left(h_{\alpha\beta}\right)^2 \simeq 0. \tag{62}$$

In this approximation, the 5D Ricci tensor takes the form

$$R_{\alpha\beta} \simeq \frac{1}{2}\left(\partial_\beta\partial_\gamma h_\alpha^\gamma + \partial_\alpha\partial_\gamma h_\beta^\gamma - \partial^\gamma\partial_\gamma h_{\alpha\beta} - \partial_\alpha\partial_\beta h\right) = -\frac{1}{2}\partial^\gamma\partial_\gamma h_{\alpha\beta}, \tag{63}$$

where $h = \eta^{\alpha\beta}h_{\alpha\beta}$ and we imposed the Lorenz gauge condition

$$\partial^\beta\left(h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h\right) = 0, \tag{64}$$

permitted by invariance of the metric under a 5D translation $x^\alpha \longrightarrow x^\alpha + \Lambda^\alpha(x,\tau)$. Conveniently exploiting the Ricci tensor in this form, the SHP field Equation (49) becomes the wave equation

$$-\partial^\gamma\partial_\gamma h_{\alpha\beta} = -\left(\partial^\mu\partial_\mu + \sigma\frac{1}{c_5^2}\partial_\tau^2\right)h_{\alpha\beta} = 2k_G\left(T_{\alpha\beta} - \frac{1}{2}P_{\alpha\beta}S\right), \tag{65}$$

which admits the principal part Green's function [18]

$$G(x,\tau) = \frac{1}{2\pi}\delta(x^2)\delta(\tau) + \frac{c_5}{2\pi^2}\frac{\partial}{\partial x^2}\theta(-\sigma g_{\alpha\beta}x^\alpha x^\beta)\frac{1}{\sqrt{-\sigma g_{\alpha\beta}x^\alpha x^\beta}}, \tag{66}$$

in which the leading term, denoted G_{Maxwell}, has lightlike support at equal-τ and is dominant at long distances. The second term, denoted $G_{\text{correlation}}$, drops off as $1/\text{distance}^2$ and has spacelike support for $\sigma = -1$ or timelike support for $\sigma = +1$.

For a general trajectory, we may consider an event distribution moving in tandem in the neighborhood of a point with 5D coordinates

$$X^\alpha(\tau) = (X^\mu(\tau), c_5\tau), \tag{67}$$

and for the shared velocity we introduce the notation

$$\xi^\alpha(\tau) = \frac{1}{c}u^\alpha(\tau) = \frac{1}{c}\frac{dX^\alpha}{d\tau}. \tag{68}$$

The spacetime event density is

$$\rho(x,\tau) = \rho(x - X(\tau)), \tag{69}$$

leading to the mass-energy-momentum tensor

$$T^{\alpha\beta} = M\rho(x,\tau)\dot{X}^\alpha \dot{X}^\beta = M\rho(x,\tau)u^\alpha u^\beta = Mc^2\rho(x,\tau)\xi^\alpha(\tau)\xi^\beta(\tau), \tag{70}$$

which is seen to be conserved by simply noting that $\partial_\tau\rho(x,\tau) = -\xi^\mu\partial_\mu\rho(x,\tau)$. The generic solution for the metric perturbation is thus

$$h^{\alpha\beta}(x,\tau) = 2k_G \int d^4x' d\tau' G(x - x', \tau - \tau')\left(\xi^\alpha\xi^\beta - \frac{1}{2}P^{\alpha\beta}\hat{\xi}^2\right)\rho(x',\tau'), \tag{71}$$

where $\xi^\alpha = \xi^\alpha(\tau')$ and $\hat{\xi}^2 = \hat{\eta}^{\mu\nu}\xi_\mu\xi_\nu$.

3. The Metric as Solution to a 5D Wave Equation

We are interested in the metric induced by 'static' events narrowly distributed along their t-axis at the spatial origin $\mathbf{x} = 0$ and evolving uniformly. As a preliminary case, we consider a source distributed evenly along the t-axis in its rest frame, as is typically posed in standard relativity. The center of the event trajectory is then

$$X(\tau) = (c\tau, \mathbf{0}, c_5\tau) \longrightarrow \xi(\tau) = \left(1, \mathbf{0}, \frac{c_5}{c}\right), \tag{72}$$

and the event density

$$\rho(x,\tau) = \rho(x - X(\tau)) = \rho(ct - c\tau)\delta^{(3)}(\mathbf{x}) = \delta^{(3)}(\mathbf{x}), \tag{73}$$

is independent of t and τ. Because $\xi(\tau)$ is constant, the generic solution to the wave equation becomes

$$h_{\alpha\beta}(x,\tau) = 2k_G Mc^2\, Z_{\alpha\beta}\mathcal{G}[\rho(x,\tau)], \tag{74}$$

where we denote

$$Z_{\alpha\beta} = \xi_\alpha\xi_\beta - \frac{1}{2}\hat{\eta}_{\alpha\beta}\xi^\mu\xi_\mu, \tag{75}$$

$$\mathcal{G}[\rho(x,\tau)] = \int d^4x'\, d\tau' G(x - x', \tau - \tau')\rho(x',\tau'), \tag{76}$$

as kinematic and dynamic factors. Integration of the event density (73) with the Green's function (66) leads to

$$\mathcal{G}_{\text{Maxwell}}[\rho(x,\tau)] = \int d^4x' d\tau' \frac{1}{2\pi}\delta((x - x')^2)\delta(\tau - \tau')\delta^{(3)}(\mathbf{x}') = \frac{1}{4\pi|\mathbf{x}|}, \tag{77}$$

$$\mathcal{G}_{\text{correlation}}[\rho(x,\tau)] = \frac{c_5}{2\pi^2}\int d^4x' d\tau' \frac{\partial}{\partial x^2}\frac{\theta(-\sigma(x - x')^\alpha(x - x')_\alpha)}{\sqrt{-\sigma(x - x')^\alpha(x - x')_\alpha}}\delta^{(3)}(\mathbf{x}') = 0, \tag{78}$$

and so taking

$$k_G = \frac{8\pi G}{c^4}, \tag{79}$$

the spacetime part of the metric becomes

$$g_{\mu\nu} = \text{diag}\left(-1 + \frac{2GM}{c^2 r}, \left(1 + \frac{2GM}{c^2 r}\right)\delta_{ij}\right) \simeq \text{diag}\left(-U, U^{-1}\delta_{ij}\right), \tag{80}$$

where

$$U = \left(1 - \frac{2GM}{c^2 r}\right). \tag{81}$$

Naturally, this metric is spatially isotropic, and is t-independent because the event density is spread evenly along the t-axis. Transforming to spherical coordinates (80) becomes

$$g_{\mu\nu} = \text{diag}\left(-U, U^{-1}, U^{-1}r^2, U^{-1}r^2 \sin^2\theta\right), \tag{82}$$

which for weak fields is recognized as the Schwarzschild metric

$$g_{\mu\nu} = \text{diag}\left(-U, U^{-1}, R^2, R^2 \sin^2\theta\right), \tag{83}$$

when expressed in the isotropic coordinates [28] defined through

$$R = r\left(1 + \frac{k}{2r}\right)^2. \tag{84}$$

The Schwarzschild metric is well-known to be Ricci-flat, $R_{\mu\nu} = 0$, a consequence of its R-dependence and t-independence.

To study the field induced by an event localized in both space and time, we again consider an event distribution centered on the t-axis around the trajectory (72), but write an event density

$$\rho(x,\tau) = \varphi(t-\tau)\delta^{(3)}(\mathbf{x}) \qquad\qquad \varphi_{\max} = \varphi(0), \tag{85}$$

with support in a neighborhood around $t = \tau$. Writing $i, j = 1,2,3$, the kinematic factors are

$$\begin{aligned} Z_{00} &= \frac{1}{2} & \qquad Z_{05} &= Z_{50} = -\sigma\frac{c_5}{c} \\ Z_{ij} &= \frac{1}{2}\delta_{ij} & \qquad Z_{55} &= \frac{c_5^2}{c^2} \end{aligned} \tag{86}$$

and the dynamic factors are

$$\mathcal{G}_{\text{Maxwell}} = \int d^4x' d\tau' \frac{1}{2\pi}\delta((x-x')^2)\delta(\tau-\tau')\varphi(t'-\tau')\delta^{(3)}(\mathbf{x}'), \tag{87}$$

$$\mathcal{G}_{\text{correlation}} = \frac{c_5}{2\pi^2}\int d^4x' d\tau' \frac{\partial}{\partial x^2}\frac{\theta(-\sigma(x-x')^\alpha(x-x')_\alpha)}{\sqrt{-\sigma(x-x')^\alpha(x-x')_\alpha}}\varphi(t'-\tau')\delta^{(3)}(\mathbf{x}')\,. \tag{88}$$

The leading term is easily evaluated as

$$\mathcal{G}_{\text{Maxwell}} = \frac{\varphi(t - |\mathbf{x}|/c - \tau)}{4\pi|\mathbf{x}|}, \tag{89}$$

producing a gravitational field that is maximized at $\tau = t - |\mathbf{x}|/c$. Since the source is centered at $t_{\text{source}} = \tau$, a test event evolving along its t-axis, at a constant spatial distance r from the source, will feel the strongest gravitational force if it is located at $t = t_{\text{source}} + |\mathbf{x}|/c$, placing the test event on the lightcone of the source and accounting for the propagation time of the gravitational field. This part of the solution is comparable to the Coulomb force given in (1) in electrodynamics.

The evaluation for $\mathcal{G}_{\text{correlation}}$ will depend upon the choice of σ and the details of the distribution $\varphi(s)$, in nearly all cases leading to numerical integration. Taking the derivative in $\mathcal{G}_{\text{correlation}}$ we have

$$G_{\text{Correlation}}(x,\tau) = \frac{c_5}{2\pi^2}\left(\frac{1}{2}\frac{\theta(-\sigma x^2 - c_5^2\tau^2)}{\left(-\sigma x^2 - c_5^2\tau^2\right)^{3/2}} - \frac{\delta\left(-\sigma x^2 - c_5^2\tau^2\right)}{\left(-\sigma x^2 - c_5^2\tau^2\right)^{1/2}}\right), \tag{90}$$

and writing

$$-\sigma x^2 - c_5^2\tau^2 = c^2\left[-\sigma\left(\frac{\mathbf{x}^2}{c^2} - t^2\right) - \frac{c_5^2}{c^2}\tau^2\right], \tag{91}$$

we might consider neglecting $c_5^2/c^2 \ll 1$. However, doing so makes this part of Green's function, independent of τ, so that the τ integration in (88) becomes

$$\int d\tau'\ \varphi(t' - \tau') = 1, \tag{92}$$

and the remaining integral is

$$\mathcal{G}_{\text{correlation}} = \frac{c_5}{2\pi^2}\frac{\partial}{\partial x^2}\int d^4x' \frac{\theta(-\sigma(x-x')^\alpha(x-x')_\alpha)}{\sqrt{-\sigma(x-x')^\alpha(x-x')_\alpha}}\delta^{(3)}(\mathbf{x}') = 0, \tag{93}$$

leaving no contribution from $G_{\text{Correlation}}$. In this sense, neglecting the contribution from this term is equivalent to the τ-equilibrium condition, a point we will examine again in Section 4.

To obtain a sense of $\mathcal{G}_{\text{correlation}}$ we choose the infinitely narrow distribution $\varphi(t-\tau) = \delta(t-\tau)$ so that

$$\mathcal{G}_{\text{correlation}} = \frac{c_5}{2\pi^2}\int ds\left(\frac{1}{2}\frac{\theta(g(s))}{(g(s))^{3/2}} - \frac{\delta(g(s))}{(g(s))^{1/2}}\right), \tag{94}$$

where

$$g(s) = c^2\left(\sigma(t-s)^2 - \sigma\frac{\mathbf{x}^2}{c^2} - \frac{c_5^2}{c^2}(\tau - s)^2\right), \tag{95}$$

in which the singularities of the two integrands cancel each other out when handled carefully. For $\sigma = -1$, describing spacelike support, $g(s) > 0$ between the roots of $g(s) = 0$ and cancellation of singularities causes the integral to vanish. Taking $\sigma = 1$, describing timelike support, $g(s) > 0$ above the upper root and so the integral takes its value as $s \longrightarrow \infty$ giving

$$\mathcal{G}_{\text{Correlation}} = \frac{c_5}{2\pi^2}\frac{1}{\mathbf{x}^2 - c_5^2\tau(2t-\tau)}\ . \tag{96}$$

Since this terms drops off as $1/\mathbf{x}^2$ the contribution from $\mathcal{G}_{\text{Maxwell}}$ will be dominant at long distance.

In summary, the perturbed metric is

$$g_{00} = -U = -1 + k_G Mc^2 \mathcal{G}\Big[\varphi(t-\tau)\delta^{(3)}(\mathbf{x})\Big], \tag{97}$$

$$g_{ij} = V\delta_{ij} = \Big(1 + k_G Mc^2 \mathcal{G}\Big[\varphi(t-\tau)\delta^{(3)}(\mathbf{x})\Big]\Big)\delta_{ij} \qquad i,j = 1,2,3 \tag{98}$$

$$g_{05} = g_{50} = -2\sigma\frac{c_5}{c}k_G \mathcal{G}\Big[\varphi(t-\tau)\delta^{(3)}(\mathbf{x})\Big], \tag{99}$$

$$g_{55} = 2\frac{c_5^2}{c^2}k_G \mathcal{G}\Big[\varphi(t-\tau)\delta^{(3)}(\mathbf{x})\Big]. \tag{100}$$

Using (24) we may write the equations of motion for a nonrelativistic test event as

$$0 = \ddot{x}^\mu + c^2\left(\Gamma^\mu_{00}\dot{t}^2 + 2\Gamma^\mu_{i0}\frac{\dot{x}^i}{c}\dot{t} + \Gamma^\mu_{ij}\frac{\dot{x}^i}{c}\frac{\dot{x}^j}{c} + 2\frac{c_5}{c}\Gamma^\mu_{50}\dot{t} + 2\frac{c_5}{c}\Gamma^\mu_{5i}\frac{\dot{x}^i}{c} + \frac{c_5^2}{c^2}\Gamma^\mu_{55}\right), \tag{101}$$

and from

$$\Gamma^\mu_{\beta\gamma} = \frac{1}{2}\left(\eta^{\mu\alpha}\frac{\partial h_{\alpha\beta}}{\partial x^\gamma} + \eta^{\mu\alpha}\frac{\partial h_{\alpha\gamma}}{\partial x^\beta} - \eta^{\mu\alpha}\frac{\partial h_{\beta\gamma}}{\partial x^\alpha}\right), \tag{102}$$

evaluate the nonzero Christoffel symbols

$$\Gamma^\mu_{00} = -\frac{1}{2c}\delta^{\mu 0}\frac{\partial h_{00}}{\partial t} - \frac{1}{2}\delta^{\mu k}\frac{\partial h_{00}}{\partial x^k} \qquad \Gamma^\mu_{i0} = \frac{1}{2c}\delta^{\mu j}\frac{\partial h_{ji}}{\partial t} - \frac{1}{2}\delta^{\mu 0}\frac{\partial h_{00}}{\partial x^i}, \tag{103}$$

$$\Gamma^\mu_{ij} = \frac{1}{2}\delta^{\mu k}\left(\frac{\partial h_{ki}}{\partial x^j} + \frac{\partial h_{kj}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^k}\right) + \frac{1}{2c}\delta^{\mu 0}\frac{\partial h_{ij}}{\partial t}, \tag{104}$$

$$\Gamma^\mu_{50} = -\frac{1}{2c_5}\delta^{\mu 0}\frac{\partial h_{00}}{\partial \tau} \qquad \Gamma^\mu_{5i} = \frac{1}{2c_5}\delta^{\mu k}\frac{\partial h_{ki}}{\partial \tau}, \tag{105}$$

where we used $h_{0i} = 0,\ i = 1,2,3$ and dropped $h_{5\alpha} \propto c_5/c \ll 1$. Similarly neglecting terms containing $\dot{x}^i/c \ll 1$ the equations of motion split into

$$0 = \ddot{t} - \frac{1}{2}\frac{\partial h_{00}}{\partial t}\dot{t}^2 - \left(\frac{\partial h_{00}}{\partial \tau} + \dot{\mathbf{x}}\cdot\nabla h_{00}\right)\dot{t} \qquad 0 = \ddot{\mathbf{x}} - \frac{1}{2}c^2\dot{t}^2\,\nabla h_{00}, \tag{106}$$

which differ from (26) in the t-dependence of h_{00}. In spherical coordinates, using

$$h_{00} = h_{00}(t,r,\tau) \ \longrightarrow\ \dot{\mathbf{x}}\cdot\nabla h_{00} = \dot{r}\partial_r h_{00}, \tag{107}$$

the equations of motion become

$$\ddot{t} = \frac{1}{2}(\partial_t h_{00})\dot{t}^2 + (\partial_\tau h_{00} + \dot{r}\partial_r h_{00})\dot{t} \qquad \ddot{r} = \frac{1}{2}c^2(\partial_r h_{00})\dot{t}^2 + \frac{L^2}{m^2 r^3}, \tag{108}$$

where we again introduce the conserved angular momentum $L = mr^2\dot{\phi}$.

To obtain a sense of this result, we localize the source in t by taking the Gaussian distribution

$$\varphi(s) = \frac{1}{\sqrt{2\pi}\lambda_0}e^{-s^2/\lambda_0^2}, \tag{109}$$

where λ_0 is a time scale representing the width of the event distribution along the t-axis, and consider only the leading term $\mathcal{G}_{\text{Maxwell}}$. From (97) the metric takes the form

$$h_{00} = \frac{k_G Mc^2}{4\pi r}\frac{1}{\sqrt{2\pi}\lambda_0}\exp\left[-\frac{(t - r/c - \tau)^2}{\lambda_0^2}\right] = \frac{k_G Mc^2}{4\pi}\frac{1}{\sqrt{2\pi}\lambda_0}\frac{1}{r}\hat{\varphi}, \tag{110}$$

where for convenience we notate

$$\hat{\varphi} = \exp\left[-\frac{(t-r/c-\tau)^2}{\lambda_0^2}\right]. \tag{111}$$

The partial derivatives are

$$\partial_t h_{00} = -\frac{k_G M c^2}{4\pi}\frac{1}{\sqrt{2\pi}\lambda_0}\frac{2(t-r/c-\tau)}{\lambda_0^2}\frac{1}{r}\hat{\varphi}, \tag{112}$$

$$\partial_\tau h_{00} = \frac{k_G M c^2}{4\pi}\frac{1}{\sqrt{2\pi}\lambda_0}\frac{2(t-r/c-\tau)}{\lambda_0^2}\frac{1}{r}\hat{\varphi}, \tag{113}$$

$$\partial_r h_{00} = \frac{k_G M c^2}{4\pi}\frac{1}{\sqrt{2\pi}\lambda_0}\left[-\frac{1}{r}+\frac{2(t-r/c-\tau)}{\lambda_0^2}\right]\frac{1}{r}\hat{\varphi}, \tag{114}$$

leading to the equations of motion

$$\ddot{t} = \frac{k_G M c^2}{4\pi}\frac{1}{\sqrt{2\pi}\lambda_0}\left[\frac{2(t-r/c-\tau)}{\lambda_0^2}\left(-\frac{1}{2}\dot{t}^2+\dot{t}+\frac{\dot{r}}{c}\right)\frac{1}{r}-\frac{\dot{r}}{r^2}\right]\hat{\varphi}, \tag{115}$$

$$\ddot{r} = \frac{1}{2}\left(\frac{k_G M c^2}{4\pi}\frac{c}{\sqrt{2\pi}\lambda_0}\left[-\frac{1}{r}+\frac{2(t-r/c-\tau)}{\lambda_0^2}\right]\frac{1}{r}\hat{\varphi}\right)\dot{t}^2+\frac{L^2}{m^2r^3}\,. \tag{116}$$

Locating the test event on the lightcone of the source event

$$t-\frac{r}{c}-\tau = 0 \longrightarrow \hat{\varphi}=1, \tag{117}$$

the equations of motion reduce to

$$\ddot{t} = -\frac{k_G M c^2}{4\pi r^2}\frac{1}{\sqrt{2\pi}\lambda_0}\frac{\dot{r}}{c}, \tag{118}$$

$$\ddot{r} = -\frac{1}{2}\frac{k_G M c^2}{4\pi}\frac{c}{\sqrt{2\pi}\lambda_0}\frac{1}{r^2}\dot{t}^2+\frac{L^2}{m^2r^3}\,. \tag{119}$$

Since we must have $\dot{r}/c \longrightarrow 0$ in (118) we may write $\dot{t}=1$ which recovers Newtonian gravitation in (119) by putting

$$\frac{1}{2}\frac{k_G M c^2}{4\pi}\frac{c}{\sqrt{2\pi}\lambda_0} = GM \qquad \longrightarrow \qquad k_G = \sqrt{2\pi}\,\frac{8\pi G}{c^2}\frac{\lambda_0}{c}, \tag{120}$$

in which the inverse length λ_0/c compensates for the dimensions $1/\text{length}^4$ of the spacetime event density, in relation to the usual $1/\text{length}^3$ dimensions of particle density. For an arbitrary position of the test event on the t-axis with $\dot{r}/c \ll 1$, the metric perturbation and equations of motion are

$$h_{00} = \frac{2GM}{c^2 r}\hat{\varphi} \longrightarrow g_{00} = -\left(1-\frac{2GM}{c^2 r}\hat{\varphi}\right), \tag{121}$$

$$\ddot{t} = \frac{2GM}{c^2}\left[\frac{2(t-r/c-\tau)}{\lambda_0^2}\left(-\frac{1}{2}\dot{t}^2+\dot{t}\right)\frac{1}{r}-\frac{\dot{r}}{r^2}\right]\hat{\varphi}, \tag{122}$$

$$\ddot{r} = -\frac{GM}{r^2}\left[1-\frac{r}{\lambda_0 c}\frac{2(t-r/c-\tau)}{\lambda_0}\right]\dot{t}^2\,\hat{\varphi}+\frac{L^2}{m^2r^3}\,. \tag{123}$$

For the nonrelativistic event, the t equation can be approximated

$$\ddot{t} \simeq \frac{2GM}{c^2 r}\left[\frac{t - r/c - \tau}{\lambda_0^2} - \frac{\dot{r}}{r}\right]\hat{\varphi}, \tag{124}$$

which is a product of small factors, so that acceleration in time will remain negligible. But the radial equation depends on the ratio $r/\lambda_0 c$ of the radial distance, taken to be large, and the width of the t distribution. Equation (123) approximates Newtonian gravitation for $t - r/c - \tau = 0$, but as the test event accelerates in the radial direction under the resulting force, the distance will decrease as $r \longrightarrow r - \delta r$ and so the acceleration becomes

$$\ddot{r} \simeq -\frac{GM}{r^2}\left[1 - 2\frac{r}{\lambda_0 c}\frac{\delta r}{\lambda_0 c}\right]\hat{\varphi} + \frac{L^2}{m^2 r^3}\ . \tag{125}$$

This shows that the width λ_0 of the source event distribution along the t-axis must be much larger than r/c, or else the gravitational force will weaken and possibly change sign from attraction to repulsion. In the limit $\lambda_0 \longrightarrow \infty$, we have $\hat{\varphi} = 1$ and the metric (97) and (98) recover the t-independent metric (80), losing the t-localization.

The resulting model, which is less than adequate, follows from a series of approximations, in particular using the linearized 5D theory and neglecting $\mathcal{G}_{\text{correlation}}$. Although we used a particular distribution $\varphi(t, r, \tau)$ to arrive at the equations of motion, any solution to the 5D wave equation found from the Green's function will have the form

$$h_{00} \propto \frac{1}{r}\hat{\varphi}(t, r, \tau), \tag{126}$$

as its leading term. As a result, the gravitational force appearing in the radial Equation (108) will take the form

$$\partial_r h_{00} \propto -\frac{1}{r^2}\left[\hat{\varphi}(t, r, \tau) - r\partial_r \hat{\varphi}(t, r, \tau)\right], \tag{127}$$

which may change the sign for any narrow distribution with $\partial_r \hat{\varphi}$ sufficient large at some value of its argument. For example, using the distribution

$$\hat{\varphi}(x, \tau) = \frac{1}{2}e^{-|t - |\mathbf{x}|/c - \tau|/\lambda_0} \qquad \longrightarrow \qquad r\partial_r \hat{\varphi} = \frac{r}{\lambda_0}\operatorname{sgn}(t - |\mathbf{x}|/c - \tau), \tag{128}$$

the gravitational force may change sign sharply for a small shift in the τ-synchronization of the test particle around $\tau = t - |\mathbf{x}|/c$. It thus appears that linearized GR will not provide an adequate model for the localized metric produced by a localized event. In Section 4, we analyze this question further in the context of the 4+1 method, and show that the initial value problem in full nonlinear GR involves a more complex structure than revealed in the 5D wave equation approach.

4. The Metric as Solution to 4+1 Evolution Equations

In this section, we apply the 4+1 method to study the solution to the linearized field equations found in Section 3. As mentioned in Section 2.3, the initial value problem may be posed in the quatrad frame using the simplified Equations (57)–(60) because the spacetime part of the metric is diagonal. It is easily seen that for

$$\gamma_{\mu\nu} = \operatorname{diag}(-U, V, V, V), \tag{129}$$

the vierbein field takes the form

$$E_\mu{}^k = \sqrt{U}\delta_\mu{}^0\delta_0{}^k + \sqrt{V}\delta_\mu{}^s\delta_s{}^k \qquad\qquad e^\mu{}_k = \frac{1}{\sqrt{U}}\delta^\mu{}_0\delta^0{}_k + \frac{1}{\sqrt{V}}\delta^\mu{}_s\delta^s{}_k, \tag{130}$$

where $s,t=1,2,3$. For the event distribution along the t-axis, the mass-energy-momentum tensor

$$T^{\alpha\beta}=Mc^2\varphi(t-\tau)\delta^{(3)}(\mathbf{x})\left(\delta_0^\alpha+\frac{c_5}{c}\delta_5^\alpha\right)\left(\delta_0^\beta+\frac{c_5}{c}\delta_5^\beta\right), \tag{131}$$

decomposes to

$$\kappa=T^{55}=\frac{c_5^2}{c^2}mc^2\varphi(t-\tau)\delta^{(3)}(\mathbf{x}), \tag{132}$$

$$p_k=-\sigma\eta_{kk'}E_\mu{}^{k'}T^{5\mu},=\frac{c_5}{c}\sigma\sqrt{U}mc^2\varphi(t-\tau)\delta^{(3)}(\mathbf{x}), \tag{133}$$

$$S_{kl}=\eta_{kk'}\eta_{ll'}E_\mu{}^{k'}E_{\mu'}{}^{l'}T^{\mu\mu'}=\eta_{k0}\eta_{l0}U\,mc^2\varphi(t-\tau)\delta^{(3)}(\mathbf{x}), \tag{134}$$

$$S=\eta^{kl}S_{kl}=-Umc^2\varphi(t-\tau)\delta^{(3)}(\mathbf{x}), \tag{135}$$

so that the source for (58)

$$S_{kl}-\frac{1}{2}\eta_{kl}S=\left(\eta_{k0}\eta_{l0}+\frac{1}{2}\eta_{kl}\right)Umc^2\varphi(t-\tau)\delta^{(3)}(\mathbf{x})=\frac{1}{2}\delta_{kl}S_{00}, \tag{136}$$

is diagonal and identical in each component.

In the weak field approximation, assuming a metric of the type obtained by perturbation

$$U=1-\Phi \qquad V=1+\Phi, \tag{137}$$

entails

$$\sqrt{U}=\sqrt{1-\Phi}\simeq 1-\frac{1}{2}\Phi \qquad \sqrt{V}=\sqrt{1+\Phi}\simeq 1+\frac{1}{2}\Phi\,. \tag{138}$$

Now the vierbein field can be written

$$E_\mu{}^k=\left(1-\frac{1}{2}\Phi\right)\delta_\mu{}^0\delta_0{}^k+\left(1+\frac{1}{2}\Phi\right)\delta_\mu{}^s\delta_s{}^k=\delta_\mu{}^k+\frac{1}{2}\left(-\delta_\mu{}^0\delta_0{}^k+\delta_\mu{}^s\delta_s{}^k\right)\Phi, \tag{139}$$

with derivatives

$$\partial_\alpha E_\mu{}^k=\frac{1}{2}\left(-\delta_\mu{}^0\delta_0{}^k+\delta_\mu{}^s\delta_s{}^k\right)\partial_\alpha\Phi\,. \tag{140}$$

Since the extrinsic curvature K_{kl} must also arise as a perturbation, we discard terms of the type $\Phi K_{kl}\simeq 0$ and the first evolution Equation (57) reduces to

$$\frac{1}{2c_5}\left(-\delta_\mu{}^0\delta_0{}^k+\delta_\mu{}^s\delta_s{}^k\right)\partial_\tau\Phi=-\delta_\mu{}^lK_l^k, \tag{141}$$

so that lowering the k index provides

$$\frac{1}{2c_5}\delta_{kl}\,\partial_\tau\Phi=-K_{kl}\,. \tag{142}$$

Similarly discarding the term $KK_{kl}\simeq 0$, the second evolution Equation (58) reduces to

$$\frac{1}{c_5}\partial_\tau K_{kl}=-\sigma\bar{R}_{kl}+\frac{1}{2}\sigma k_G\delta_{kl}S_{00}, \tag{143}$$

where we used (136) as the source. These expressions provide a pair of coupled first-order equations for $\Phi(x,\tau)$ and $K_{kl}(x,\tau)$, given initial conditions $\Phi(x,0)$ and $K_{kl}(x,0)$. It was shown in [4] that for weak fields, the constraints (59) and (60) are equivalent to the wave equation for $h_{5\alpha}$ and so, given the product structure of (74), these will be satisfied for any solution to the 5D wave equation for h_{00}.

Using the convenient linearized form

$$\bar{R}_{kl}=-\frac{1}{2}\delta_{kl}\,\partial^\mu\partial_\mu h_{00}=-\frac{1}{2}\delta_{kl}\,\partial^\mu\partial_\mu\Phi, \tag{144}$$

to evaluate the Ricci tensor, we see that each term in the evolution equations is diagonal, reducing the system to an initial value problem for $\Phi(x,\tau)$. By solving (142) for

$$K_{kl} = -\frac{1}{2c_5}\delta_{kl}\,\partial_\tau\Phi, \tag{145}$$

and inserting this into (143) we obtain

$$\partial^\mu\partial_\mu\Phi + \sigma\frac{1}{c_5^2}\partial_\tau^2\Phi + k_G S_{00} = 0, \tag{146}$$

which simply recovers the 5D wave Equation (65) we analyzed in Section 3. Since no complete closed-form solution is readily available, we studied the leading term

$$\Phi(x,\tau) = \mathcal{G}_{\text{Maxwell}} = \frac{\varphi(t-|\mathbf{x}|/c-\tau)}{4\pi|\mathbf{x}|}, \tag{147}$$

as a partial solution, and found that the resulting geodesic equations for a test event placed an unreasonable condition on the event density φ. We also showed that neglecting the subdominant terms $\mathcal{G}_{\text{correlation}}$ is equivalent to taking the limit $c_5/c \longrightarrow 0$. To see this another way, we rewrite the evolution Equation (143) as

$$\partial_\tau K_{kl} = \frac{1}{2}\sigma\delta_{kl}\;\; c_5\left[\partial^\mu\partial_\mu\Phi + mc^2\varphi(t-\tau)\delta^{(3)}(\mathbf{x})\right]. \tag{148}$$

leading to the equilibrium condition $\partial_\tau K_{kl} = 0$ either in the limit $c_5 \longrightarrow 0$, or by setting $\Phi = \mathcal{G}_{\text{Maxwell}}$ for which the expression in parentheses gives zero by the 4D wave equation. As seen from (127), these problems will be present in any solution to the linearized field equations for the source (136).

While the leading term G_{Maxwell} of the Green's function provides adequate solutions in SHP electromagnetism, this appears not to be the case in GR. It appears that the model of localized events interacting through a localized metric must be posed in the full nonlinear field theory, which admits structures not captured by the linearized equations. That is, we write the exact evolution equations and constraints (57)–(60) to find a diagonal metric (129) derived from the vierbein field (130). But in the absence of linearization, the convenient expression (144) for $R_{\mu\nu}$ is no longer applicable, adding significant complexity to the problem.

As mentioned in Section 3, the Ricci flatness $R_{\mu\nu} = 0$ of the Schwarzschild solution depends on the metric being a function of the three spatial coordinates (through $R = |\mathbf{x}|$) but t-independent. However, the Ricci tensor for a general diagonal metric with functional dependence on all spacetime coordinates x^μ will necessarily have nonzero off-diagonal components [29]. Thus, we may specify the initial vierbein field ${E_\mu}^k(x,0)$ and extrinsic curvature $K_{kl}(x,0)$ to be diagonal, as is the source (136), but $K_{kl}(x,\tau)$ will acquire off-diagonal terms from $\bar{R}_{kl}$ under the evolution described by Equation (58). Therefore, the vierbein field ${E_\mu}^k(x,\tau)$ will acquire off-diagonal terms from the extrinsic curvature via Equation (57). As a result, the metric $\gamma_{\mu\nu} = \eta_{kl}{E_\mu}^k{E_\nu}^l$ may acquire off-diagonal terms, in which case, (57) and (58) will no longer be valid, forcing us to use the coordinate frame expressions (51) and (52) as the evolution equations for the metric.

In summary, we require a metric that reproduces Newtonian gravitation for a nonrelativistic test event at large distance, falls off to $\eta_{\mu\nu}$ as $1/r$, is localized around $\tau = t - |\mathbf{x}|/c$, but is not of the separable form (147). In addition, the initial conditions for $\gamma_{\mu\nu}$ and $K_{\mu\nu}$ must be chosen carefully to satisfy the constraints (53) and (54). This list of requirements is complex and perhaps cannot be satisfied. A subsequent paper will discuss these issues at greater length.

5. Conclusions and Discussion

After reviewing the basic structure of the Stueckelberg–Horwitz–Piron formalism in relativity and its extension to GR, we constructed a model in which a localized spacetime event evolving with the invariant parameter τ induces a metric that similarly evolves with τ. Extending developments in SHP electrodynamics, we fixed the event at the spatial origin of its rest frame, in a narrow distribution moving uniformly along its t-axis, and using the Green's function $G(x, \tau)$, solved the 5D wave equation describing weak gravitation in linearized GR. The resulting solution, dominated by the leading term in $G(x, \tau)$, was shown to be analogous to the electromagnetic Coulomb force, falling off to the flat metric as $1/r$ and localized around the retarded time $\tau = t - r/c$ for a test event with coordinates $x = (ct, \mathbf{x})$. In this picture, a localized event produces a localized field that acts on a remote localized event, with the interaction synchronized by τ. However, unlike the electrodynamic Lorentz force, the effect of the metric through the geodesic equations of motion leads to a possible reversal of the gravitational force, because the functional dependence of the metric is a separable product of $1/r$ and the localized distribution $\hat{\varphi}(t - r/c - \tau)$. This issue was shown to be a necessary feature of any solution for weak gravitation, produced by the leading term in $G(x, \tau)$ for any source distribution.

We conclude that while the leading term G_{Maxwell} of the Green's function provides adequate solutions in SHP electromagnetism, additional work will be required to extend the model of a source localized in spacetime to GR. The required metric must reproduce Newtonian gravitation for a nonrelativistic test event at large distance, fall off to $\eta_{\mu\nu}$ as $r \to \infty$, be localized around $\tau = t_{\text{retarded}}$, but not be separable. Such a metric will likely include off-diagonal components and must be approached in the 4+1 formalism, which poses an initial value problem for the metric. This list of requirements is complex and whether they can be satisfied remains an open question. Candidates for such a metric will be discussed in a subsequent paper.

Funding: This research received no external funding.

Data Availability Statement: No data was produced or employed in this research, other than the equations presented and the physical concepts they represent.

Conflicts of Interest: The author declares no conflicts of interest.

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