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Averaging Principle for ψ-Capuo Fractional Stochastic Delay Differential Equations with Poisson Jumps

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Abstract: In this paper, we study the averaging principle for ψ-Capuo fractional stochastic delay differential equations (FSDDEs) with Poisson jumps. Based on fractional calculus, Burkholder-Davis-Gundy's inequality, Doob's martingale inequality, and the Hölder inequality, we prove that the solution of the averaged FSDDEs converges to that of the standard FSDDEs in the sense of L^p. Our result extends some known results in the literature. Finally, an example and simulation is performed to show the effectiveness of our result.

Keywords: averaging principle; ψ-Capuo fractional stochastic delay differential equations; Poisson jumps; L^p convergence

MSC: 34K50; 26A33; 60J75

1. Introduction

Many systems exhibit natural symmetry, such as chemical, physical, and biological systems. It is well known that stochastic differential equations play an important role in explaining some symmetry phenomena (see [1–3]). Additionally, we know that stochastic differential equations are mathematical tools widely used to simulate and model stochastic processes. Recently, more in-depth research has been conducted on the theory and application aspects of these equations to adapt to more complex systems, such as chemical reaction networks, atmospheric environments, and financial markets; readers can refer to the papers [4–7] for more information.

In 1968, Khasminskii [8] extended the averaging principles for ODEs to the case of stochastic differential equations (SDEs). Since then, the averaging principles for SDEs have found applications in many areas of science and engineering, including fluid dynamics, control theory, and climate modeling. Many people have devoted their efforts to the study of averaging principles for SDEs, for example, see [9–11].

As we all know, compared with integer-order derivatives, fractional-order derivatives provide a magnificent approach to describe the memory and hereditary properties of various processes. Thus, fractional differential equations are more accurate and convenient than integer-order ones. The numerical solution of fractional-order nonlinear systems is an active research area with ongoing developments and improvements in the different numerical algorithms and techniques used [12–14].

With the development of fractional calculus, the averaging principles for fractional stochastic differential equations (FSDEs) have become a widespread concern [15–17]. One notable approach of research is the fractional averaging principle, which extends the classical averaging principle to FSDEs. Another approach of research is the stochastic averaging principle, which combines averaging methods with stochastic calculus. Overall, research into averaging principles for FSDEs is still ongoing, and there is much to be explored in terms of developing new methods and exploring their applications.

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Recently, Wang and Lin [18] extended the averaging principle of the following fractional stochastic differential equations (FSDEs)

$$\begin{cases} {}^{C}D_0^{\alpha}[x(t) - h(t,x(t))] = f(t,x(t)) + g(t,x(t))\frac{dB_t}{dt}, \quad t \in J = [0,T], \\ x(0) = x_0, \end{cases} \tag{1}$$

in the sense of mean square (L^2 convergence) to L^p convergence ($p \geq 2$), which generated some works on the averaging principle for FSDES [19–21].

The periodic averaging method for impulsive conformable fractional stochastic differential equations with Poisson jumps are discussed in [22] by Ahmed. For some recent works on Hilfer fractional order stochastic differential systems, we refer to [23–26]. In [27], Ahmed and Zhu investigated the averaging principle for the following Hilfer fractional stochastic delay differential equation with Poisson jumps in the sense of mean square

$$\begin{cases} D_0^{\aleph,\hbar}x(t) = \Re(t,x(t),x(t-\tau)) + \sigma(t,x(t),x(t-\tau))\frac{dB}{dt}, \\ \qquad\qquad + \int_V h(t,x(t),x(t-\tau),v)\bar{N}(dt,dv), \quad t \in J = (0,T], \\ x(t) = \phi(t), \quad -\tau \leq t \leq 0, \\ I_{0^+}^{(1-\aleph)(1-\hbar)}x(0) = \phi(0). \end{cases} \tag{2}$$

In [28], Almeida generalized the definition of the Caputo fractional derivative by considering the Caputo fractional derivative of a function with respect to another function ψ. Since then, there have been so many papers involving the ψ-Caputo fractional derivative, see [29–32]. Recently, there have been many works on SDEs with Poisson jumps, see, for example, [33–35] and the references therein. However, to the best of our knowledge, the averaging principle for the ψ-Capuo fractional stochastic delay differential equation with Poisson jumps in the sense of L^p convergence has not yet been researched in the literature. In the present paper, motivated by the above-mentioned works, we study the following ψ-Caputo fractional stochastic delay differential equation with Poisson jumps

$$\begin{cases} {}^{C}D_0^{\alpha,\psi}[x(t) - h(t,x(t))] = f(t,x(t),x(t-\tau)) + \sigma(t,x(t),x(t-\tau))\frac{dB_t}{dt}, \\ \qquad\qquad + \int_V g(t,x(t),x(t-\tau),v)\bar{N}(dt,dv), \quad t \in J = (0,T], \\ x(t) = \phi(t), \quad -\tau \leq t \leq 0, \end{cases} \tag{3}$$

where ${}^{C}D_0^{\alpha,\psi}$ is the left ψ-Caputo fractional derivative with $0 < \alpha < 1$ and $\psi \in C^1([a,b])$ is an increasing function with $\psi'(t) \neq 0$ for all $t \in [0,T]$, $J = (0,T]$, $x \in \mathbb{R}^n$ is a stochastic process, $h, f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $g : J \times \mathbb{R}^n \times \mathbb{R}^n \times V \to \mathbb{R}^n$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual condition. Here, B_t is an m-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$ adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Let $(V, \Phi, \lambda(dv))$ be a σ-finite measurable space, given the stationary Poisson point process $(p_t)_{t\geq 0}$, which is defined on $(\Omega, \mathcal{F}, P)$ with values in V and with the characteristic measure λ. We denote by $N(t,dv)$ the counting measure of p_t such that $\bar{N}(t,\Theta) := \mathbb{E}(N(t,\Theta)) = t\lambda(\Theta)$ for $\Theta \in \Phi$. Define $\bar{N}(t,dv) := N(t,dv) - t\lambda(dv)$, and the Poisson martingale measure is generated by p_t .

In this paper, we prove that the solution of the averaged neutral SFDDEs with Poisson random measure converges to that of the standard one in L^p sense. The main contributions and advantages of this paper are as follows:

(i) For the first time in the literature, the averaging principle for ψ-Capuo fractional stochastic delay differential equations with Poisson jumps is investigated.

(ii) The fractional calculus, stochastic inequality, and Hölder inequality are effectively used to establish our result.

(iii) Our work in this paper is novel and more technical. Our result has greatly promoted and extended the main result of [18].

This paper will be organized as follows. In Section 2, we will briefly recall some definitions and preliminaries. Section 3 is devoted to obtaining an averaging principle for the solution of the considered system (3). Additionally, a numerical simulation example is provided to illustrate our main result. Finally, the paper is concluded in Section 4.

2. Preliminaries

In this section, we recall some basic definitions and lemmas, which are used in the sequel.

Definition 1 ([36])**.** *Let $\alpha > 0$, f be an integrable function defined on $[a,b]$ and $\psi \in C^1([a,b])$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in [a,b]$. The left ψ-Riemann-Liouville fractional integral operator of order α of a function f is defined by*

$$ {}_aI_t^{\alpha,\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s)ds. \tag{4}$$

Definition 2 ([28,36])**.** *Let $n-1 < \alpha < n$, $f \in C^n([a,b])$ and $\psi \in C^n([a,b])$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in [a,b]$. The left ψ-Caputo fractional derivative of order α of a function f is defined by*

$$\begin{aligned} {}_a^CD_t^{\alpha,\psi} f(t) &= ({}_aI_t^{n-\alpha,\psi} f^{[n]})(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (\psi(t)-\psi(s))^{n-\alpha-1} f^{[n]}(s)\psi'(s)ds, \end{aligned} \tag{5}$$

where $n = [\alpha]+1$ and $f^{[n]}(t) := \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n f(t)$ on $[a,b]$.

In the following, we will give some properties of the combinations of the fractional integral and the fractional derivatives of a function with respect to another function.

Lemma 1 ([28])**.** *Let $f \in C^n([a,b])$ and $n-1 < \alpha < n$. Then, we have*

(1) ${}_a^CD_t^{\alpha,\psi}\,{}_aI_t^{\alpha,\psi} f(t) = f(t)$;

(2) $I_t^{\alpha,\psi}\,{}_a^CD_t^{\alpha,\psi} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{[k]}(a^+)}{\Gamma(k-\alpha)}(\psi(t)-\psi(a))^k.$

In particular, given $\alpha \in (0,1)$, one has

$$I_t^{\alpha,\psi}\,{}_a^CD_t^{\alpha,\psi} = f(t) - f(a).$$

To study the averaging method of Equation (3), we impose the following conditions on data of the problem.

(H1) If $|h(0,\phi(0))| < \infty$, $t \in [0,T]$ and for all $x, y \in R^n$, a constant $C_1 \in (0,1)$ exists such that

$$|h(t,x) - h(t,y)| \leq C_1|x-y|.$$

(H2) For any $x_1, x_2, y_1, y_2 \in R^n$ and $t \in J$, two constants $C_2, C_3 > 0$ exist such that

$$\begin{aligned} &|f(t,x_1,y_1) - f(t,x_2,y_2)|^p \vee |\sigma(t,x_1,y_1) - \sigma(t,x_2,y_2)|^p \\ &\vee \int_V |g(t,x_1,y_1,v) - g(t,x_2,y_2,v)|^p \lambda(dv) \leq C_2^p(|x_1-x_2|^p + |y_1-y_2|^p), \end{aligned}$$

and

$$|f(t,x_1,y_1)|^p \vee |\sigma(t,x_1,y_1)|^p \vee \int_V |g(t,x_1,y_1,v)|^p \lambda(dv) \leq C_3^p(1+|x_1|^p+|y_1|^p).$$

According to Lemma 1 and [37], an R^n-value stochastic process $\{x(t), -\tau \leq t \leq T\}$ is called a unique solution of Equation (3) if $x(t)$ satisfies the following :

$$x(t)=\begin{cases}\phi_0-h(0,\phi_0)+h(t,x(t))+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)f(s,x(s),x(s-\tau))ds\\ \quad+\frac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\sigma(s,x(s),x(s-\tau))dB_s\\ \quad+\frac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\int_V g(s,x(s),x(s-\tau),v)\bar{N}(ds,dv),\quad t\in J,\\ \phi(t),\quad t\in[-r,0],\end{cases}\tag{6}$$

where $\phi_0=\phi(0)$.

For each $t\in J$, we consider the standard form of Equation (6)

$$\begin{aligned}x_\epsilon(t)&=\phi_0-h(0,\phi_0)+h(t,x_\epsilon(t))+\frac{\epsilon}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)f(s,x_\epsilon(s),x_\epsilon(s-\tau))ds\\&+\frac{\sqrt{\epsilon}}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\sigma(s,x_\epsilon(s),x_\epsilon(s-\tau))dB_s\\&+\frac{\sqrt{\epsilon}}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\int_V g(s,x_\epsilon(s),x_\epsilon(s-\tau),v)\bar{N}(ds,dv),\quad t\in J,\end{aligned}\tag{7}$$

where $\epsilon\in(0,\epsilon_0]$ is a positive small parameter with ϵ_0 being a fixed number.

Consider the averaged form, which corresponds to the standard form (7) as follows:

$$\begin{aligned}y_\epsilon(t)&=\phi_0-h(0,\phi_0)+h(t,y_\epsilon(t))+\frac{\epsilon}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\bar{f}(y_\epsilon(s),y_\epsilon(s-\tau))ds\\&+\frac{\sqrt{\epsilon}}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\bar{\sigma}(y_\epsilon(s),y_\epsilon(s-\tau))dB_s\\&+\frac{\sqrt{\epsilon}}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\int_V\bar{g}(y_\epsilon(s),y_\epsilon(s-\tau),v)\bar{N}(ds,dv),\quad t\in J,\end{aligned}\tag{8}$$

where $\bar{f}:R^n\times R^n\to R^n$, $\bar{\sigma}:R^n\times R^n\to R^{n\times m}$, and $\bar{g}:R^n\times R^n\times V\to R^n$ satisfying the following averaging condition :

(H3) For any $T_1\in[0,T]$, $x,y\in R^n$ and $p\geq 2$, a positive bounded function $\beta(\cdot)$ exists such that

$$\begin{aligned}&\frac{1}{T_1}\int_0^{T_1}|f(t,x,y)-\bar{f}(x,y)|^p dt\vee\frac{1}{T_1}\int_0^{T_1}|\sigma(t,x,y)-\bar{\sigma}(x,y)|^p dt\\&\quad\vee\frac{1}{T_1}\int_0^{T_1}\left(\int_V|g(t,x,y,v)-\bar{g}(x,y,v)|^p\lambda(dv)\right)dt\leq\beta(T_1)(1+|x|^p+|y|^p),\end{aligned}$$

and $\lim_{T_1\to\infty}\beta(T_1)=0$.

Lemma 2. Suppose that (H2) and (H3) hold. Then, for $T_1\in(0,T]$ we have

$$|\bar{\sigma}(x,y)|^p\leq C_4(1+|x|^p+|y|^p)\quad\text{and}\quad\int_V|\bar{g}(x,y,v)|^p\lambda(dv)\leq C_4(1+|x|^p+|y|^p),$$

where $C_4=2^{p-1}(\beta(T_1)+C_3^p)$.

Proof. Using (H2), (H3) and Jensen's inequality, we obtain

$$\begin{aligned}|\bar{\sigma}(x,y)|^p&\leq\frac{2^{p-1}}{T_1}\int_0^{T_1}|\bar{\sigma}(x,y)-\sigma(t,x,y)|^p dt+\frac{2^{p-1}}{T_1}\int_0^{T_1}|\sigma(t,x,y)|^p dt\\&\leq 2^{p-1}\beta(T_1)(1+|x|^p+|y|^p)+2^{p-1}C_3^p(1+|x|^p+|y|^p)\end{aligned}$$

$$= 2^{p-1}(\beta(T_1) + C_3^p)(1 + |x|^p + |y|^p).$$

Similarly, we can prove that

$$\int_V |\bar{g}(x, y, v)|^p \lambda(dv) \le 2^{p-1}(\beta(T_1) + C_3^p)(1 + |x|^p + |y|^p).$$

□

Lemma 3 ([38])**.** If $p \ge 2$ and $a, b \in \mathbb{R}^n$, then for any $k \in (0, 1)$, one has

$$|a + b|^p \le \frac{|a|^p}{k^{p-1}} + \frac{|b|^p}{(1-k)^{p-1}}.$$

Lemma 4 ([39,40])**.** Let $\phi : R_+ \times V \to R^n$ and assume that

$$\int_0^t \int_V |\phi(s, v)|^p \lambda(dv) ds < \infty, \quad p \ge 2.$$

Then, $D_p > 0$ exists such that

$$\mathbb{E}\left(\sup_{0 \le t \le u} \left| \int_0^t \int_V \phi(s, v) \bar{N}(ds, dv) \right|^p\right)$$
$$\le D_p \left\{ \mathbb{E}\left(\int_0^u \int_V |\phi(s, v)|^2 \lambda(dv) ds \right)^{\frac{p}{2}} + \mathbb{E} \int_0^u \int_V |\phi(s, v)|^p \lambda(dv) ds \right\}.$$

Lemma 5 ([41])**.** *Let u, v be two integrable functions and g be continuous defined on domain $[a, b]$. Let $\psi \in C^1[a, b]$ be an increasing function such that $\psi'(t) \neq 0$, $\forall t \in [a, b]$. Moreover, assume that*

(1) u and v are nonnegative, and v is nondecreasing;

(2) g is nonnegative and nondecreasing.

If

$$u(t) \le v(t) + g(t) \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} u(\tau) d\tau,$$

then

$$u(t) \le v(t) E_\alpha(g(t)\Gamma(\alpha)(\psi(t) - \psi(a))^\alpha), \quad \forall t \in [a, b],$$

where E_α is the Mittag–Leffler function.

3. Main Results

Theorem 1. *Assume that (H1)–(H3) are satisfied. Then, for a given arbitrary small number $\delta > 0$, $p = 2$, $\frac{1}{2} < \alpha < 1$, or $p > 2$ and $\max\left\{\frac{p-1}{p}, \frac{p+2}{2p}\right\} < \alpha < 1$, $M > 0$, $\epsilon_1 \in (0, \epsilon_0]$ and $\gamma \in (0, 1)$ exist such that*

$$\mathbb{E}\left(\sup_{t \in [-\tau, M\epsilon^{-\gamma}]} |x_\epsilon(t) - y_\epsilon(t)|^p\right) \le \delta, \tag{9}$$

for all $\epsilon \in (0, \epsilon_1]$.

Proof. If $p = 2$, it is easy to prove that (9) holds by using the similar method as in [27]. In the following, we will only consider the case $p > 2$. From Equations (7) and (8), we obtain

$$x_\epsilon(t) - y_\epsilon(t) = h(t, x_\epsilon(t)) - h(t, y_\epsilon(t))$$

$$+\frac{\epsilon}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)[f(s,x_\epsilon(s),x_\epsilon(s-\tau))-\bar{f}(y_\epsilon(s),y_\epsilon(s-\tau))]ds$$
$$+\frac{\sqrt{\epsilon}}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)[\sigma(s,x_\epsilon(s),x_\epsilon(s-\tau))-\bar{\sigma}(y_\epsilon(s),y_\epsilon(s-\tau))]dB_s$$
$$+\frac{\sqrt{\epsilon}}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\int_V[g(s,x_\epsilon(s),x_\epsilon(s-\tau),v))$$
$$-\bar{g}(x_\epsilon(s),x_\epsilon(s-\tau),v))]\bar{N}(ds,dv).$$

Choosing $k=C_1$ in Lemma 3, using (H1) and the following elementary inequalities:

$$|a+b|^p\le 2^{p-1}(|a|^p+|b|^p),\quad |a+b+c|^p\le 3^{p-1}(|a|^p+|b|^p+|c|^p),\quad (10)$$

we obtain

$$|x_\epsilon(t)-y_\epsilon(t)|^p\le C_1|x_\epsilon(t)-y_\epsilon(t)|^p$$
$$+\frac{3^{p-1}\epsilon^p}{(1-C_1)^{p-1}\Gamma(\alpha)^p}\left|\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)[f(s,x_\epsilon(s),x_\epsilon(s-\tau))-\bar{f}(y_\epsilon(s),y_\epsilon(s-\tau))]ds\right|^p$$
$$+\frac{3^{p-1}\epsilon^{\frac{p}{2}}}{(1-C_1)^{p-1}\Gamma(\alpha)^p}\left|\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)[\sigma(s,x_\epsilon(s),x_\epsilon(s-\tau))-\bar{\sigma}(y_\epsilon(s),y_\epsilon(s-\tau))]dB_s\right|^p$$
$$+\frac{3^{p-1}\epsilon^{\frac{p}{2}}}{(1-C_1)^{p-1}\Gamma(\alpha)^p}\left|\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\int_V[g(s,x_\epsilon(s),x_\epsilon(s-\tau),v))\right.$$
$$\left.-\bar{g}(x_\epsilon(s),x_\epsilon(s-\tau),v))]\bar{N}(ds,dv)\right|^p.\quad (11)$$

For any $t\in[0,u]\subset[0,T]$, taking the expectation on both sides Equation (11), we have

$$\mathbb{E}\left(\sup_{0\le t\le u}|x_\epsilon(t)-y_\epsilon(t)|^p\right)$$
$$\le\frac{3^{p-1}\epsilon^p}{(1-C_1)^p\Gamma(\alpha)^p}\mathbb{E}\left(\sup_{0\le t\le u}\left|\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)[f(s,x_\epsilon(s),x_\epsilon(s-\tau))-\bar{f}(y_\epsilon(s),y_\epsilon(s-\tau))]ds\right|^p\right)$$
$$+\frac{3^{p-1}\epsilon^{\frac{p}{2}}}{(1-C_1)^p\Gamma(\alpha)^p}\mathbb{E}\left(\sup_{0\le t\le u}\left|\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)[\sigma(s,x_\epsilon(s),x_\epsilon(s-\tau))-\bar{\sigma}(y_\epsilon(s),y_\epsilon(s-\tau))]dB_s\right|^p\right)$$
$$+\frac{3^{p-1}\epsilon^{\frac{p}{2}}}{(1-C_1)^p\Gamma(\alpha)^p}\mathbb{E}\left(\sup_{0\le t\le u}\left|\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\int_V[g(s,x_\epsilon(s),x_\epsilon(s-\tau),v))\right.\right.$$
$$\left.\left.-\bar{g}(x_\epsilon(s),x_\epsilon(s-\tau),v))]\bar{N}(ds,dv)\right|^p\right).$$
$$=I_1+I_2+I_3.\quad (12)$$

Applying Jensen inequality, we obtain

$$I_1\le\frac{6^{p-1}\epsilon^p}{(1-C_1)^p\Gamma(\alpha)^p}\mathbb{E}\left(\sup_{0\le t\le u}\left|\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)[f(s,x_\epsilon(s),x_\epsilon(s-\tau))-f(s,y_\epsilon(s),y_\epsilon(s-\tau))]ds\right|^p\right)$$
$$+\frac{6^{p-1}\epsilon^p}{(1-C_1)^p\Gamma(\alpha)^p}\mathbb{E}\left(\sup_{0\le t\le u}\left|\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)[f(s,y_\epsilon(s),y_\epsilon(s-\tau))-\bar{f}(y_\epsilon(s),y_\epsilon(s-\tau))]ds\right|^p\right)$$

$$= I_{11} + I_{12}. \tag{13}$$

Thanks to the Hölder inequality and (H2), we obtain

$$\begin{aligned}
I_{11} \leq{}& \frac{6^{p-1}\epsilon^p}{(1-C_1)^p\Gamma(\alpha)^p}\left(\int_0^u 1ds\right)^{p-1} \\
&\cdot\mathbb{E}\left(\sup_{0\leq t\leq u}\int_0^t (\psi(u)-\psi(s))^{p(\alpha-1)}\psi'(s)^p|f(s,x_\epsilon(s),x_\epsilon(s-\tau))-f(s,y_\epsilon(s),y_\epsilon(s-\tau))|^p ds\right) \\
\leq{}& \frac{6^{p-1}\epsilon^p}{(1-C_1)^p\Gamma(\alpha)^p}u^{p-1}K^{p-1}C_2^p \\
&\cdot\mathbb{E}\left(\sup_{0\leq t\leq u}\int_0^t (\psi(u)-\psi(s))^{p(\alpha-1)}\psi'(s)[|x_\epsilon(s)-y_\epsilon(s)|^p+|x_\epsilon(s-\tau))-y_\epsilon(s-\tau))|^p]ds\right) \\
\leq{}& A_{11}\epsilon^p u^{p-1}\int_0^u (\psi(u)-\psi(s))^{p(\alpha-1)}\psi'(s)\left[\mathbb{E}\left(\sup_{0\leq\theta\leq s}|x_\epsilon(\theta)-y_\epsilon(\theta)|^p\right)\right. \\
&\left.+\mathbb{E}\left(\sup_{0\leq\theta\leq s}|x_\epsilon(\theta-\tau)-y_\epsilon(\theta-\tau)|^p\right)\right]ds,
\end{aligned} \tag{14}$$

where $A_{11} = \frac{6^{p-1}C_2^pK^{p-1}}{(1-C_1)^p\Gamma(\alpha)^p}$ and $K = \sup_{t\in[0,T]}\psi'(t)$.

Applying the Hölder inequality, we obtain

$$\begin{aligned}
I_{12} \leq{}& \frac{6^{p-1}\epsilon^p}{(1-C_1)^p\Gamma(\alpha)^p}\left(\int_0^u (\psi(u)-\psi(s))^{\frac{(\alpha-1)p}{p-1}}\psi'(s)^{\frac{p}{p-1}}ds\right)^{p-1} \\
&\cdot\mathbb{E}\left(\sup_{0\leq t\leq u}\int_0^t |f(s,y_\epsilon(s),y_\epsilon(s-\tau))-\bar{f}(y_\epsilon(s),y_\epsilon(s-\tau))|^p ds\right).
\end{aligned} \tag{15}$$

Since

$$\begin{aligned}
\int_0^u (\psi(u)-\psi(s))^{\frac{(\alpha-1)p}{p-1}}\psi'(s)^{\frac{p}{p-1}}ds ={}& \int_0^u (\psi(u)-\psi(s))^{\frac{(\alpha-1)p}{p-1}}\psi'(s)\cdot\psi'(s)^{\frac{1}{p-1}}ds \\
\leq{}& K^{\frac{1}{p-1}}\int_0^u (\psi(u)-\psi(s))^{\frac{(\alpha-1)p}{p-1}}\psi'(s)ds \\
={}& K^{\frac{1}{p-1}}\frac{p-1}{\alpha p-1}(\psi(u)-\psi(0))^{\frac{\alpha p-1}{p-1}},
\end{aligned} \tag{16}$$

we have by (15), (16), and (H3) that

$$I_{12} \leq A_{12}\epsilon^p(\psi(u)-\psi(0))^{\alpha p-1}u, \tag{17}$$

where,

$$A_{12} = \frac{6^{p-1}K}{(1-C_1)^p\Gamma(\alpha)^p}\left(\frac{p-1}{\alpha p-1}\right)^{p-1}\|\beta\|_{L^\infty([0,u])}\left[1+\mathbb{E}\left(\sup_{0\leq t\leq u}|y_\epsilon(t)|^p\right)+\mathbb{E}\left(\sup_{0\leq t\leq u}|y_\epsilon(t-\tau)|^p\right)\right],$$

here, $\|\beta\|_{L^\infty([0,u])} = \sup_{t\in[0,u]}|\beta(t)|$.

For the second term I_2, we have

$$I_2 \leq \frac{6^{p-1}\epsilon^{\frac{p}{2}}}{(1-C_1)^p\Gamma(\alpha)^p}\mathbb{E}\left(\sup_{0\leq t\leq u}\left|\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)[\sigma(s,x_\epsilon(s),x_\epsilon(s-\tau))-\sigma(s,y_\epsilon(s),y_\epsilon(s-\tau))]dB_s\right|^p\right)$$
$$+\frac{6^{p-1}\epsilon^{\frac{p}{2}}}{(1-C_1)^p\Gamma(\alpha)^p}\mathbb{E}\left(\sup_{0\leq t\leq u}\left|\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)[\sigma(s,y_\epsilon(s),y_\epsilon(s-\tau))-\bar{\sigma}(y_\epsilon(s),y_\epsilon(s-\tau))]dB_s\right|^p\right)$$
$$= I_{21}+I_{22}. \tag{18}$$

In view of the Burkholder–Davis–Gundy's inequality, Hölder's inequality, and Doob's martingale inequality, a constant $C_p > 0$ exists such that

$$I_{21} \leq \frac{6^{p-1}\epsilon^{\frac{p}{2}}C_p}{(1-C_1)^p\Gamma(\alpha)^p}\mathbb{E}\left(\int_0^u(\psi(u)-\psi(s))^{2\alpha-2}\psi'(s)^2|\sigma(s,x_\epsilon(s),x_\epsilon(s-\tau))-\sigma(s,y_\epsilon(s),y_\epsilon(s-\tau))|^2ds\right)^{\frac{p}{2}}$$
$$\leq \frac{6^{p-1}C_p}{(1-C_1)^p\Gamma(\alpha)^p}\epsilon^{\frac{p}{2}}u^{\frac{p}{2}-1}\mathbb{E}\left(\int_0^u(\psi(u)-\psi(s))^{(\alpha-1)p}\psi'(s)^p\right.$$
$$\left.\cdot|\sigma(s,x_\epsilon(s),x_\epsilon(s-\tau))-\sigma(s,y_\epsilon(s),y_\epsilon(s-\tau))|^pds\right)$$
$$\leq \frac{6^{p-1}C_p}{(1-C_1)^p\Gamma(\alpha)^p}\epsilon^{\frac{p}{2}}u^{\frac{p}{2}-1}K^{p-1}C_2^p\cdot\mathbb{E}\left(\int_0^u(\psi(u)-\psi(s))^{(\alpha-1)p}\psi'(s)\right.$$
$$\left.\cdot[|x_\epsilon(s)-y_\epsilon(s)|^p+|x_\epsilon(s-\tau)-y_\epsilon(s-\tau)|^p]ds\right)$$
$$\leq A_{21}\epsilon^{\frac{p}{2}}u^{\frac{p}{2}-1}\int_0^u(\psi(u)-\psi(s))^{(\alpha-1)p}\psi'(s)\left[\mathbb{E}\left(\sup_{0\leq\theta\leq s}|x_\epsilon(\theta)-y_\epsilon(\theta)|^p\right)\right.$$
$$\left.+\mathbb{E}\left(\sup_{0\leq\theta\leq s}|x_\epsilon(\theta-\tau)-y_\epsilon(\theta-\tau)|^p\right)\right]ds, \tag{19}$$

where $A_{21}=\frac{6^{p-1}C_pK^{p-1}C_2^p}{(1-C_1)^p\Gamma(\alpha)^p}$.

Since $\alpha > \frac{p-1}{p}$, we have $\alpha p-p+1>0$. Applying Lemma 2 and an estimation method similar to Equation (19), we obtain

$$I_{22} \leq \frac{12^{p-1}C_pK^{p-1}}{(1-C_1)^p\Gamma(\alpha)^p}\epsilon^{\frac{p}{2}}u^{\frac{p}{2}-1}\cdot\mathbb{E}\left(\int_0^u(\psi(u)-\psi(s))^{(\alpha-1)p}\psi'(s)\right.$$
$$\left.\cdot(|\sigma(s,y_\epsilon(s),y_\epsilon(s-\tau))|^p+|\bar{\sigma}(y_\epsilon(s),y_\epsilon(s-\tau))|^p)ds\right)$$
$$\leq A_{22}\epsilon^{\frac{p}{2}}u^{\frac{p}{2}-1}(\psi(u)-\psi(0))^{(\alpha-1)p+1}, \tag{20}$$

where

$$A_{22}=\frac{12^{p-1}C_pK^{p-1}(C_3^p+C_4)}{(1-C_1)^p\Gamma(\alpha)^p(\alpha p-p+1)}\left[1+\mathbb{E}\left(\sup_{0\leq t\leq u}|y_\epsilon(t)|^p\right)+\mathbb{E}\left(\sup_{0\leq t\leq u}|y_\epsilon(t-\tau)|^p\right)\right].$$

Next, we deal with the third term. Similar to the method used in (18), we have

$$I_3 \leq \frac{6^{p-1}\epsilon^{\frac{p}{2}}}{(1-C_1)^p\Gamma(\alpha)^p}\mathbb{E}\left(\sup_{0\leq t\leq u}\left|\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\int_V[g(s,x_\epsilon(s),x_\epsilon(s-\tau),v)\right.\right.$$

$$-g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)]\bar{N}(ds,dv)\bigg|^p\Bigg)$$

$$+\frac{6^{p-1}\epsilon^{\frac{p}{2}}}{(1-C_1)^p\Gamma(\alpha)^p}\mathbb{E}\Bigg(\sup_{0\le t\le u}\bigg|\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\int_V[g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)$$

$$-\bar{g}(y_\epsilon(s),y_\epsilon(s-\tau),v)]\bar{N}(ds,dv)\bigg|^p\Bigg)$$

$$=I_{31}+I_{32}. \tag{21}$$

From Lemma 4, one has

$$I_{31}\le\frac{6^{p-1}\epsilon^{\frac{p}{2}}}{(1-C_1)^p\Gamma(\alpha)^p}D_p\Bigg\{\mathbb{E}\Bigg(\int_0^u(\psi(u)-\psi(s))^{2\alpha-2}\psi'(s)^2\int_V|g(s,x_\epsilon(s),x_\epsilon(s-\tau),v)$$

$$-g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)|^2\lambda(dv)ds\Bigg)^{\frac{p}{2}}$$

$$+\mathbb{E}\Bigg(\int_0^u(\psi(u)-\psi(s))^{p(\alpha-1)}\psi'(s)^p\int_V|g(s,x_\epsilon(s),x_\epsilon(s-\tau),v)$$

$$-g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)|^p\lambda(dv)ds\Bigg)\Bigg\}. \tag{22}$$

By using the Hölder inequality and (H2), we obtain

$$\mathbb{E}\Bigg(\int_0^u(\psi(u)-\psi(s))^{2\alpha-2}\psi'(s)^2\int_V|g(s,x_\epsilon(s),x_\epsilon(s-\tau),v)-g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)|^2\lambda(dv)ds\Bigg)^{\frac{p}{2}}$$

$$\le(u\lambda(V))^{\frac{p-2}{2}}\mathbb{E}\Bigg(\int_0^u\int_V(\psi(u)-\psi(s))^{p(\alpha-1)}\psi'(s)^p|g(s,x_\epsilon(s),x_\epsilon(s-\tau),v)$$

$$-g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)|^p\lambda(dv)ds\Bigg)$$

$$\le(u\lambda(V))^{\frac{p-2}{2}}K^{p-1}C_2^p\mathbb{E}\Bigg(\int_0^u(\psi(u)-\psi(s))^{p(\alpha-1)}\psi'(s)[|x_\epsilon(s)-y_\epsilon(s)|^p+|x_\epsilon(s-\tau)-y_\epsilon(s-\tau)|^p]ds\Bigg)$$

$$\le K^{p-1}C_2^p\lambda(V)^{\frac{p-2}{2}}u^{\frac{p-2}{2}}\int_0^u(\psi(u)-\psi(s))^{p(\alpha-1)}\psi'(s)\Bigg[\mathbb{E}\Bigg(\sup_{0\le\theta\le s}|x_\epsilon(\theta)-y_\epsilon(\theta)|^p\Bigg)$$

$$+\mathbb{E}\Bigg(\sup_{0\le\theta\le s}|x_\epsilon(\theta-\tau)-y_\epsilon(\theta-\tau)|^p\Bigg)\Bigg]ds, \tag{23}$$

and

$$\mathbb{E}\Bigg(\int_0^u(\psi(u)-\psi(s))^{p(\alpha-1)}\psi'(s)^p\int_V|g(s,x_\epsilon(s),x_\epsilon(s-\tau),v)-g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)|^p\lambda(dv)ds\Bigg)$$

$$\le C_2^p\mathbb{E}\Bigg(\int_0^u(\psi(u)-\psi(s))^{p(\alpha-1)}\psi'(s)^p[|x_\epsilon(s)-y_\epsilon(s)|^p+|x_\epsilon(s-\tau)-y_\epsilon(s-\tau)|^p]ds\Bigg)$$

$$\le C_2^pK^{p-1}\int_0^u(\psi(u)-\psi(s))^{p(\alpha-1)}\psi'(s)\Bigg[\mathbb{E}\Bigg(\sup_{0\le\theta\le s}|x_\epsilon(\theta)-y_\epsilon(\theta)|^p\Bigg)$$

$$+\mathbb{E}\Bigg(\sup_{0\le\theta\le s}|x_\epsilon(\theta-\tau)-y_\epsilon(\theta-\tau)|^p\Bigg)\Bigg]ds. \tag{24}$$

Plugging (23) and (24) into (22), we obtain

$$I_{31} \le A_{31}\epsilon^{\frac{p}{2}}\left(1+\lambda(V)^{\frac{p-2}{2}}u^{\frac{p-2}{2}}\right)\int_0^u(\psi(u)-\psi(s))^{p(\alpha-1)}\psi'(s)\left[\mathbb{E}\left(\sup_{0\le\theta\le s}|x_\epsilon(\theta)-y_\epsilon(\theta)|^p\right)\right.$$
$$\left.+\mathbb{E}\left(\sup_{0\le\theta\le s}|x_\epsilon(\theta-\tau)-y_\epsilon(\theta-\tau)|^p\right)\right]ds, \tag{25}$$

where $A_{31}=\frac{6^{p-1}}{(1-C_1)^p\Gamma(\alpha)^p}D_pC_2^pK^{p-1}$. We also have

$$I_{32} \le \frac{6^{p-1}\epsilon^{\frac{p}{2}}}{(1-C_1)^p\Gamma(\alpha)^p}D_p\cdot\left\{\mathbb{E}\left(\int_0^u(\psi(u)-\psi(s))^{2\alpha-2}\psi'(s)^2\right.\right.$$
$$\left.\cdot\int_V|g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)-\bar{g}(y_\epsilon(s),y_\epsilon(s-\tau),v)|^2\lambda(dv)ds\right)^{\frac{p}{2}}$$
$$\left.+\mathbb{E}\left(\int_0^u(\psi(u)-\psi(s))^{p(\alpha-1)}\psi'(s)^p\int_V|g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)-\bar{g}(y_\epsilon(s),y_\epsilon(s-\tau),v)|^p\lambda(dv)ds\right)\right\}. \tag{26}$$

Since $\alpha>\frac{p+2}{2p}$, we have $2p\alpha-p-2>0$. By using the Hölder inequality, (10), (H2), and (H3), we obtain

$$\mathbb{E}\left(\int_0^u(\psi(u)-\psi(s))^{p(\alpha-1)}\psi'(s)^p\int_V|g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)-\bar{g}(y_\epsilon(s),y_\epsilon(s-\tau),v)|^p\lambda(dv)ds\right)$$
$$\le 2^{p-1}\mathbb{E}\left(\int_0^u(\psi(u)-\psi(s))^{p(\alpha-1)}\psi'(s)^p\left[\int_V(|g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)|^p\right.\right.$$
$$\left.\left.+|\bar{g}(y_\epsilon(s),y_\epsilon(s-\tau),v))|^p)\lambda(dv)ds\right]\right)$$
$$\le 2^{p-1}(C_3^p+C_4)K^{p-1}\mathbb{E}\left(\int_0^u(\psi(u)-\psi(s))^{p(\alpha-1)}\psi'(s)(1+|y_\epsilon(s)|^p+|y_\epsilon(s-\tau)|^p)ds\right)$$
$$\le \frac{2^{p-1}(C_3^p+C_4)K^{p-1}}{p(\alpha-1)+1}(\psi(u)-\psi(0))^{p(\alpha-1)+1}\left[1+\mathbb{E}\left(\sup_{0\le t\le u}|y_\epsilon(t)|^p\right)+\mathbb{E}\left(\sup_{0\le t\le u}|y_\epsilon(t-\tau)|^p\right)\right], \tag{27}$$

and

$$\mathbb{E}\left(\int_0^u(\psi(u)-\psi(s))^{2\alpha-2}\psi'(s)^2\int_V|g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)-\bar{g}(y_\epsilon(s),y_\epsilon(s-\tau),v)|^2\lambda(dv)ds\right)^{\frac{p}{2}}$$
$$\le \mathbb{E}\left[\left(\int_0^u\int_V(\psi(u)-\psi(s))^{\frac{2p(\alpha-1)}{p-2}}\psi'(s)^{\frac{2p}{p-2}}\lambda(dv)ds\right)^{\frac{p-2}{2}}\right.$$
$$\left.\cdot\left(\int_0^u\int_V|g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)-\bar{g}(y_\epsilon(s),y_\epsilon(s-\tau),v)|^p\lambda(dv)ds\lambda(dv)ds\right)\right]$$
$$\le K^{\frac{p+2}{2}}\lambda(V)^{\frac{p-2}{2}}\left(\frac{p-2}{2p\alpha-p-2}\right)^{\frac{p-2}{2}}(\psi(u)-\psi(0))^{\frac{2p\alpha-p-2}{2}}$$
$$\cdot u\mathbb{E}\left(\frac{1}{u}\int_0^u\int_V|g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)-\bar{g}(y_\epsilon(s),y_\epsilon(s-\tau),v)|^p\lambda(dv)ds\right)$$

$$\leq K^{\frac{p+2}{2}}\lambda(V)^{\frac{p-2}{2}}\left(\frac{p-2}{2p\alpha-p-2}\right)^{\frac{p-2}{2}}\beta(u)u(\psi(u)-\psi(0))^{\frac{2p\alpha-p-2}{2}}$$

$$\cdot\left[1+\mathbb{E}\left(\sup_{0\leq t\leq u}|y_\epsilon(t)|^p\right)+\mathbb{E}\left(\sup_{0\leq t\leq u}|y_\epsilon(t-\tau)|^p\right)\right]. \tag{28}$$

Substituting (27) and (28) into (26), we obtain

$$I_{32}\leq A_{321}\epsilon^{\frac{p}{2}}(\psi(u)-\psi(0))^{p(\alpha-1)+1}+A_{322}\epsilon^{\frac{p}{2}}\beta(u)u(\psi(u)-\psi(0))^{\frac{2p\alpha-p-2}{2}}, \tag{29}$$

where

$$A_{321}=\frac{12^{p-1}D_p}{(1-C_1)^p\Gamma(\alpha)^p}\frac{(C_3^p+C_4)K^{p-1}}{p(\alpha-1)+1}\left[1+\mathbb{E}\left(\sup_{0\leq t\leq u}|y_\epsilon(t)|^p\right)+\mathbb{E}\left(\sup_{0\leq t\leq u}|y_\epsilon(t-\tau)|^p\right)\right],$$

$$A_{322}=\frac{6^{p-1}}{(1-C_1)^p\Gamma(\alpha)^p}D_pK^{\frac{p+2}{2}}\lambda(V)^{\frac{p-2}{2}}\left(\frac{p-2}{2p\alpha-p-2}\right)^{\frac{p-2}{2}}$$

$$\cdot\left[1+\mathbb{E}\left(\sup_{0\leq t\leq u}|y_\epsilon(t)|^p\right)+\mathbb{E}\left(\sup_{0\leq t\leq u}|y_\epsilon(t-\tau)|^p\right)\right].$$

Combining (13), (14), (17)–(21), (25), with (29), for $u\in(0,T]$ we obtain

$$\mathbb{E}\left(\sup_{0\leq t\leq u}|x_\epsilon(t)-y_\epsilon(t)|^p\right)$$

$$\leq A(u)+B(u)\int_0^u(\psi(u)-\psi(s))^{p(\alpha-1)}\psi'(s)$$

$$\cdot\left[\mathbb{E}\left(\sup_{0\leq\theta\leq s}|x_\epsilon(\theta)-y_\epsilon(\theta)|^p\right)+\mathbb{E}\left(\sup_{0\leq\theta\leq s}|x_\epsilon(\theta-\tau)-y_\epsilon(\theta-\tau)|^p]\right)\right]ds, \tag{30}$$

where

$$A(u)=A_{12}\epsilon^p(\psi(u)-\psi(0))^{\alpha p-1}u+A_{22}\epsilon^{\frac{p}{2}}u^{\frac{p}{2}-1}(\psi(u)-\psi(0))^{(\alpha-1)p+1}$$

$$+A_{321}\epsilon^{\frac{p}{2}}(\psi(u)-\psi(0))^{p(\alpha-1)+1}+A_{322}\epsilon^{\frac{p}{2}}\beta(u)u(\psi(u)-\psi(0))^{\frac{2p\alpha-p-2}{2}},$$

and

$$B(u)=A_{11}\epsilon^pu^{p-1}+A_{21}\epsilon^{\frac{p}{2}}u^{\frac{p}{2}-1}+A_{31}\epsilon^{\frac{p}{2}}\left(1+\lambda(V)^{\frac{p-2}{2}}u^{\frac{p-2}{2}}\right).$$

Set

$$\Sigma(u):=\mathbb{E}\left(\sup_{0\leq\theta\leq u}|x_\epsilon(\theta)-y_\epsilon(\theta)|^p\right).$$

Noting that $\mathbb{E}\left(\sup_{-\tau\leq\theta<0}|x_\epsilon(\theta)-y_\epsilon(\theta)|^p\right)=0$, then

$$\mathbb{E}\left(\sup_{0\leq\theta\leq s}|x_\epsilon(\theta-\tau)-y_\epsilon(\theta-\tau)|^p\right)=\Sigma(s-\tau).$$

Hence, it follows from (30) that

$$\Sigma(u) \le A(u) + B(u) \int_0^u (\psi(u) - \psi(s))^{p(\alpha-1)} \psi'(s)(\Sigma(s) + \Sigma(s-\tau))ds.$$

For each $u \in [0, T]$, denote $\Phi(u) = \sup_{-\tau \le t \le u} \Sigma(t)$. Then,

$$\Sigma(s) \le \Phi(s) \quad \text{and} \quad \Sigma(s-\tau) \le \Phi(s).$$

Thus, one has

$$\Phi(u) = \sup_{-\tau \le t \le u} \Sigma(u) \le A(u) + 2B(u) \int_0^u (\psi(u) - \psi(s))^{p(\alpha-1)} \psi'(s)\Phi(s)ds.$$

By using Lemma 5, we obtain

$$\Phi(u) \le A(u) E_{p(\alpha-1)+1}\Big(2B(u)\Gamma(p(\alpha-1)+1)(\psi(u) - \psi(0))^{p(\alpha-1)+1}\Big).$$

Moreover, we have

$$\mathbb{E}\left(\sup_{0 \le t \le u} |x_\epsilon(t) - y_\epsilon(t)|^p\right) \le A(u) E_{p(\alpha-1)+1}\Big(2B(u)\Gamma(p(\alpha-1)+1)(\psi(u) - \psi(0))^{p(\alpha-1)+1}\Big).$$

Choose $M > 0$ and $\beta \in (0,1)$ such that for all $t \in (0, M\epsilon^{-\beta}] \subset (0, T]$

$$\mathbb{E}\left(\sup_{0 < t \le M\epsilon^{-\beta}} |x_\epsilon(t) - y_\epsilon(t)|^p\right) \le \bar{A} E_{p(\alpha-1)+1}\Big(2\bar{B}\Gamma(p(\alpha-1)+1)(\psi(T) - \psi(0))^{p(\alpha-1)+1}\Big)\epsilon^{1-\beta},$$

where

$$\bar{A} = A_{12} M\epsilon^{p-1}(\psi(T) - \psi(0))^{\alpha p - 1} + A_{22} M^{\frac{p}{2}-1}\epsilon^{(\frac{p}{2}-1)(1-\beta)+\beta}(\psi(T) - \psi(0))^{(\alpha-1)p+1}$$
$$+ A_{321}\epsilon^{\frac{p}{2}-(1-\beta)}(\psi(T) - \psi(0))^{p(\alpha-1)+1} + A_{322} M m \epsilon^{\frac{p}{2}-1}(\psi(T) - \psi(0))^{\frac{2p\alpha - p - 2}{2}},$$

here, m is a positive bounded of function $\beta(\cdot)$, and

$$\bar{B} = A_{11} M^{p-1}\epsilon^{p-(p-1)\beta} + A_{21} M^{\frac{p}{2}-1}\epsilon^{\frac{p}{2}(1-\beta)+\beta} + A_{31}\epsilon^{\frac{p}{2}} + A_{31}\lambda(V)^{\frac{p-2}{2}} M^{\frac{p-2}{2}}\epsilon^{\frac{p}{2}(1-\beta)+\beta},$$

are two constants. Thus, for any given number $\delta > 0$, $\epsilon_1 \in (0, \epsilon_0]$ exists such that for each $\epsilon \in (0, \epsilon_1]$ and $t \in [-\tau, M\epsilon^{-\beta}]$,

$$\mathbb{E}\left(\sup_{t \in [-\tau, M\epsilon^{-\beta}]} |x_\epsilon(t) - y_\epsilon(t)|^p\right) \le \delta.$$

□

Remark 1. *If $\psi(t) \equiv t$, $g \equiv 0$, and $\tau = 0$, then FSDDEs (3) reduces to FSDEs (1) in [18]. Therefore, Theorem 1 generalizes the main result of [18].*

Example 1. *Consider the following ψ-Caputo fractional stochastic delay differential equation (FSDDEs) with Poisson jumps :*

$$\begin{cases} {}^{C}D_0^{0.9,\sqrt{t}}\Big[x_\varepsilon(t) - \Big(\frac{1}{8}t^{\frac{1}{5}} + \frac{1}{9}\sin(x_\varepsilon(t))\Big)\Big] = \frac{1}{2}\varepsilon x_\varepsilon(t-\tau) + \frac{3\pi}{4}\sqrt{\varepsilon}\sin^3 t x_\varepsilon(t)\frac{dB_t}{dt} \\ \qquad + \sqrt{\varepsilon}\int_V 3\bar{N}(dt, dv), \quad t \in [0, 25], \\ x_\varepsilon(t) = 0.5, \quad -0.25 \le t \le 0, \end{cases} \tag{31}$$

where $\alpha = 0.9$, $\psi(t) = \sqrt{t}$, $T = 25$, $\tau = 0.25$, and

$$h(t, x_\varepsilon(t)) = \frac{1}{8}t^{\frac{1}{5}} + \frac{1}{9}\sin(x_\varepsilon(t)), \quad f(t, x_\varepsilon(t), x_\varepsilon(t-\tau)) = \frac{1}{2}x_\varepsilon(t-\tau),$$

$$\sigma(t, x_\varepsilon(t), x_\varepsilon(t-\tau)) = \frac{3\pi}{4}\sin^3 t \cdot x_\varepsilon(t), \quad g(t, x_\varepsilon(t), x_\varepsilon(t-\tau), v) = 3.$$

Then,

$$\bar{f}(y_\varepsilon(t), y_\varepsilon(t-\tau)) = \frac{1}{\pi}\int_0^\pi f(t, y_\varepsilon(t), y_\varepsilon(t-\tau))dt = \frac{1}{2}y_\varepsilon(t-\tau),$$

$$\bar{\sigma}(y_\varepsilon(t), y_\varepsilon(t-\tau)) = \frac{1}{\pi}\int_0^\pi \sigma(t, y_\varepsilon(t), y_\varepsilon(t-\tau))dt = y_\varepsilon(t),$$

$$\bar{g}(y_\varepsilon(t), y_\varepsilon(t-\tau), v) = \frac{1}{\pi}\int_0^\pi g(t, y_\varepsilon(t), y_\varepsilon(t-\tau), v)dt = 3.$$

Thus, we have the corresponding averaged FSDDEs with Poisson jumps :

$$\begin{cases} {}^C D_0^{0.9,\sqrt{t}}\left[y_\varepsilon(t) - \left(\frac{1}{8}t^{\frac{1}{5}} + \frac{1}{9}\sin(y_\varepsilon(t))\right)\right] = \frac{1}{2}\varepsilon y_\varepsilon(t-\tau) + \sqrt{\varepsilon}y_\varepsilon(t)\frac{dB_t}{dt} \\ \quad +\sqrt{\varepsilon}\int_V 3\bar{N}(dt, dv), \quad t \in [0, 25], \\ y_\varepsilon(t) = 0.5, \quad -0.25 \le t \le 0. \end{cases} \tag{32}$$

It is easy to check that the conditions of Theorem 1 are satisfied. So, as $\varepsilon \to 0$, the original solution x_ε and the average solution y_ε are equivalent in the sense of L^p ($p = 2$ or $p > 2$ with $\max\left\{\frac{p-1}{p}, \frac{p+2}{2p}\right\} < 0.9$). To test this, Equations (31) and (32) are calculated numerically and error $Err = |x_\varepsilon(t) - y_\varepsilon(t)|^3$ are given in Figures 1 and 2. So, the averaging principle for the ψ-Capuo FSDDE with Poisson jumps is successfully established.

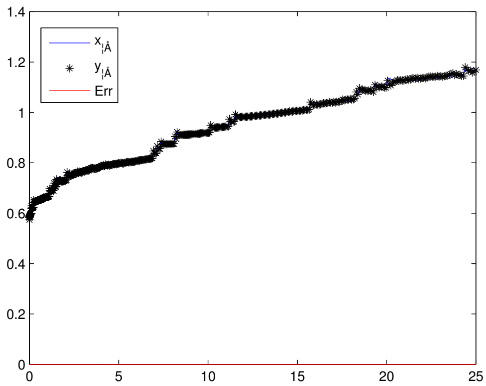

Figure 1. Comparison of x_ε and y_ε for Equations (31) and (32) with $\alpha = 0.9$ and $\varepsilon = 0.1$.

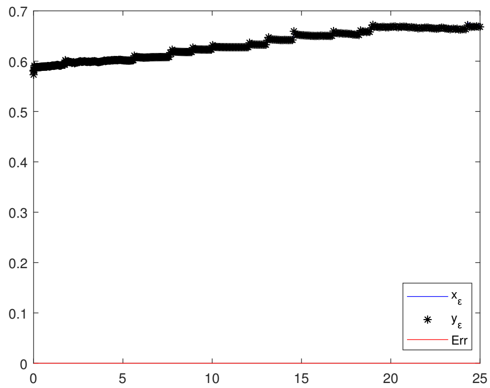

Figure 2. Comparison of x_ε and y_ε for Equations (31) and (32) with $\alpha = 0.9$ and $\varepsilon = 0.01$.

4. Conclusions

In this article, the averaging principle for FSDDEs in the sense of L^p has been proved. Hölders inequality, Jensen's inequality, Burkholder-Davis-Gundys inequality, Doobs martingale inequality, and fractional Gronwall's inequality are applied in the estimation. To the best of our knowledge, this is the first work dealing with the averaging principle for ψ-Capuo fractional stochastic delay differential equations with Poisson jumps. The obtained results generalize the two cases of $p = 2$ and the classical Caputo fractional derivative. For future research, the averaging principle for fractional stochastic neutral functional differential equations driven by the Rosenblatt process with delay and Poisson jumps is both interesting and important. It is worth further investigation in the future.

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