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Weakly Coupled System of Semi-Linear Fractional θ-Evolution Equations with Special Cauchy Conditions

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Abstract: In this paper, we consider a weakly system of fractional θ-evolution equations. Using the fixed-point theorem, a global-in-time existence of small data solutions to the Cauchy problem is proved for one single equation. Using these results, we prove the global existence for the system under some mixed symmetrical conditions that describe the interaction between the equations of the system.

Keywords: fractional derivatives; θ-evolution equation; weakly coupled system of equations; global existence

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1. Introduction

In this paper, we show the existence of the global (in time) solutions with small data to the weakly coupled system of fractional wave equations

$$\begin{aligned} &D^{1+\lambda_1}u + (-\Delta)^{\frac{\theta_1}{2}}u = |v|^p, \qquad J^{1-\lambda_1}u(0,x) = u_{\lambda_1}(x), D^{\lambda_1}u(0,x) = 0, \\ &D^{1+\lambda_2}v + (-\Delta)^{\frac{\theta_2}{2}}v = |u|^q, \qquad J^{1-\lambda_2}u(0,x) = u_{\lambda_2}(x), D^{\lambda_2}v(0,x) = 0, \end{aligned} \tag{1}$$

where $\lambda_1, \lambda_2 \in (0,1), \theta_1, \theta_2$ are real positive numbers and $D^{1+\lambda}$ is the Riemann–Liouville fractional derivative defined by

$$D^{1+\lambda}f(t) := \partial_t^2(J^{1-\lambda}f)(t) \tag{2}$$

with the Riemann–Liouville fractional integral operator

$$D^a f(t) := \frac{1}{\Gamma(a)}\int_0^t (t-s)^{a-1}f(s)ds, t > 0 \tag{3}$$

for $\Re(a) > 0$, and Γ is the Euler Gamma function.

Such mathematical models have promising applications in engineering and in other physical sciences, as well as in numerical simulations of some fractional nonlinear viscoelastic flow problems, and they impact the bioconvection on the free stream flow of a pseudoplastic nanofluid past a rotating cone.

At the outset, since the fractional equation interpolates the heat equation for $\lambda \to 0$ and the wave equation for $\lambda \to 1$ we will provide briefly some previous results of the wave equations and heat equation.

On the one hand, we consider the Cauchy problem for the semi-linear heat equation

$$u_t - \Delta u = |u|^p, \qquad u(0,x) = u_0(x).$$

Fujita in [1] proved that the exponent $p_{Fuj} := 1 + \frac{2}{n}$ is critical for the classical heat model, which means that we have the global (in time) existence of small data solutions for $p > p_{crit}$, and the blow up if we have the inverse $1 < p < p_{Fuj}$. In [2,3], the authors proved the blow-up for the critical case $p = p_{Fuj}$.

On the other hand, let us consider the Cauchy problem for the semi-linear wave equation

$$u_{tt} - \Delta u = |u|^p, \qquad u(0,x) = u_0(x), u_t(0,x) = u_1(x),$$

where the authors in [4] proved for $n = 3$ that the critical exponent is defined as a positive root of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

The defined exponent by the last equation is called the Strauss exponent and denoted by p_S for further considerations, which means that we have the global (in time) existence of small data weak solutions for the above p_S , whereas the local (in time) existence for $p > 1$ and large data can be only expected. In [5,6], the author proved in $\mathbb{R}^2$ that the Strauss exponent p_S is critical. After that, the global existence for $n = 2, 3$ was treated in [7] and for $n \geq 4$ in [8,9]. The nonexistence of solutions for data compactly supported was studied in [10] for $1 < p < \frac{n+1}{n-1}$. For $n = 3$, the authors proved some optimal results in [11] for $p = 1 + \sqrt{2}$. For $n > 3$, a nonexistence result with small data proved in [12] for $1 < p < p_S$.

In 2017, D'Abbicco et al. [13] considered the semi-linear fractional wave equation

$$\partial_t^{1+\lambda} u - \Delta u = |u|^p, \qquad u(0,x) = u_0(x), u_t(0,x) = 0, \tag{4}$$

where $\lambda \in (0,1)$ with the fractional Riemann–Liouville fractional derivative. They proved the critical exponent for the global existence of a small data solution in a low space dimension. The Caputo fractional order and the existence of non-null Cauchy data was studied in [14].

In [15], the authors proved the global (in time) existence of small data solutions to semi-linear fraction θ-evolution equations with mass or power nonlinearity. A similar problem was treated in [16] by considering a memory term instead of the power nonlinearity.

In the first part of our main results, we show the global existence of a small data solution to the fractional Riemann–Liouville order to the semi-linear θ-evolution problem (7).

For the systems, let us first consider the weakly coupled system of damped wave equations semi-linear heat equations

$$\begin{aligned} u_t - \Delta u &= |v|^p, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \\ v_t - \Delta v &= |u|^q, \;\; v(0,x) = v_0(x), \;\; v_t(0,x) = v_1(x), \end{aligned}$$

where $t \in [0,\infty), x \in \mathbb{R}^n, p, q > 1$ and $pq > 1$. The authors of [17] showed that the exponents p and q satisfying

$$\frac{n}{2} = \frac{\max\{p,q\}+1}{pq-1}$$

are critical, which means that the solutions exist globally for $\frac{n}{2} > \frac{\max\{p,q\}+1}{pq-1}$ and blow-up for the inverse case. For more details about the system of damped wave equations semi-linear heat equations, the reader can also see [18–21].

Some papers are considered for the weakly coupled systems of semilinear classical damped wave equations with power non-linearities. The problem we have in mind is

$$\begin{aligned} u_{tt} - \Delta u + u_t &= |v|^p, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \\ v_{tt} - \Delta v + v_t &= |u|^q, \;\; v(0,x) = v_0(x), \;\; v_t(0,x) = v_1(x), \end{aligned} \tag{5}$$

where $t \in [0,\infty), x \in \mathbb{R}^n$. In 2007, Sun and Wang proved in [22] that if

$$\lambda := \frac{\max\{p;q\}+1}{pq-1} < \frac{n}{2}. \tag{6}$$

for $n = 1$ or 3, then the solution exists globally in time for small initial data, while, if $\lambda \geq \frac{n}{2}$, then every solution having positive average value does not exist globally. In [23], the authors generalized the previous results to the case where $n = 1,2,3$ and improved the time decay estimates for $n = 2$. In 2014, using the weighted energy method, Nishihara and Wakasugi proved, in [24], the critical exponent for any space dimensions. Considering the time-dependent dissipation terms, the authors of [25–27] proved the global (in time) existence of small data solutions under a plan condition , which presents the interplay between the exponents of power nonlinearities.

In our paper, we consider first the single equation from system (1) where we proved the global existence for some range of the exponent p under conditions related to the regularity of the data and the dimension. After that, we apply the results of the single equation to study the weakly coupled systems (1). We proved the global existence for the system with a loss of decay if one of the exponents of power nonlinearities did not satisfy the condition of the single equation.

The paper is organized as follows. In Section 2, we will show our main results of global (in time) existence with examples. Moreover, we mention some remarks of the interpolated cases of wave and heat equations. Next, in Section 3, we prove the existence of solution by applying Banach's fixed point. Appendix A concludes the paper.

2. Main Results

2.1. Single Equation of Fractional Integral Equation

In this section, we will show our main results where we start with the global (in time) existence of solutions to the single equation of the Cauchy problem. Using the formal representation of the solution to our equation, we obtain the estimates of the solutions, and finally we prove the existence using fixed-point theorem explained in the Appendix A.

$$D^{\lambda+1}u + (-\Delta)^{\frac{\theta}{2}}u = |u|^p, \qquad J^{1-\lambda}u(0,x) = u_\lambda(x), D^\lambda u(0,x) = 0 \tag{7}$$

where $\lambda \in (0,1), \theta > 0$.

Theorem 1. *Let $n \geq 1$, and the data u_λ are supposed to belong to $L^1 \cap L^p$. The following conditions are satisfied for the exponent p :*

$$p > 1 + \frac{1+\lambda}{\frac{n}{\theta}(1+\lambda) - \lambda}, \tag{8}$$

and

$$p < 1 + \frac{\theta}{n-\theta} \quad \text{if} \quad n > \theta. \tag{9}$$

Then, a small constant ϵ exists such that, if $\|u_\lambda\|_{L^1\cap L^p} \leq \epsilon$, then there is a uniquely determined globally (in time) energy solution to (7) in $\mathcal{C}([0,\infty), L^1 \cap L^p)$.
Furthermore, the solution satisfies the estimates:

$$\|u\|_{L^q} \lesssim (1+t)^{\lambda - \frac{n}{\theta}(1+\lambda)\left(1-\frac{1}{q}\right)} \|u_\lambda\|_{L^1\cap L^p},$$

where $q \in [1, p]$.

The new type of date has a strong influence in the representation of the solution of (1) after [28], which leads to a quite different admissible range of the exponent p compared with the classical equations presented in [14].

Remark 1. *If $\lambda \to 0$, then the admissible range for the global (in time) existence corresponds with a Fujita like exponent $1 + \frac{\theta}{n}$. On the contrary for $\lambda \to 1$, we obtain a gap of continuity with respect to the Strauss exponent, which appeared in previous results as a critical exponent for the classical wave equation.*

Remark 2. *One can obtain the optimal for the exponent p in (8) using the scaling argument similarity to prove of the critical exponent to (4) illustrated in [14].*

Example 1. *We consider a concrete example by giving values to the parameters appearing in the theorem. Let us consider in* $\mathbb{R}^3$ *the following model:*

$$D^{\frac{3}{2}}u + (-\Delta)^{\frac{3}{4}}u = |u|^p, \qquad J^{\frac{1}{2}}u(0,x) = u_\lambda(x), D^{\frac{1}{2}}u(0,x) = 0.$$

Then, using Theorem 1, the admissible range for the global existence is $\frac{8}{5} < p < 2$.

2.2. Weakly Coupled System of Fractional Integral Equations

In this section, we apply the results of the previous theorem to study systems of weakly coupled fractional θ-evolution equations.

Theorem 2. *Let* $n \geq 1$*, and the data* $u_{\lambda_1}, u_{\lambda_2}$ *is supposed to belong to* $(L^1 \cap L^p) \times (L^1 \cap L^q)$*. The following conditions are satisfied for the exponent p and q:*

$$p < 1 + \frac{1+\lambda_2}{\frac{n}{\theta_2}(1+\lambda_2) - \lambda_2}, \quad q > 1 + \frac{1+\lambda_1}{\frac{n}{\theta_1}(1+\lambda_1) - \lambda_1}, \tag{10}$$

$$p < 1 + \frac{\theta}{n-\theta_1}, q < 1 + \frac{\theta}{n-\theta_2} \quad \text{if} \quad n > \min\{\theta_1; \theta_2\} \tag{11}$$

and

$$Q(\lambda_1, \lambda_2, \theta_1, \theta_2, q) > o. \tag{12}$$

Then, a small constant ϵ *exists such that, if* $\|u_{\lambda_1}\|_{L^1 \cap L^p} + \|v_{\lambda_2}\|_{L^1 \cap L^q} \leq \epsilon$*, then there is a uniquely determined globally (in time) energy solution to (1) in* $\mathcal{C}([0,\infty), L^1 \cap L^p) \times \mathcal{C}([0,\infty), L^1 \cap L^q)$*. Furthermore, the solution satisfies the estimates:*

$$\|u\|_{L^{r_1}} \lesssim (1+t)^{\lambda + L(p) - \frac{n}{\theta_1}(1+\lambda_1)\left(1 - \frac{1}{r_1}\right)} \|u_\lambda\|_{L^1 \cap L^p},$$

$$\|v\|_{L^{r_2}} \lesssim (1+t)^{\lambda_2 - \frac{n}{\theta_2}(1+\lambda_2)\left(1 - \frac{1}{r_2}\right)} \|v_\lambda\|_{L^1 \cap L^q},$$

where $L(p) = -\frac{n}{\theta_2}(1+\lambda_2)(p-1) + p\lambda_2, Q(\lambda_1, \lambda_2, \theta_1, \theta_2, q) = \left(\frac{n}{\theta_2}(1+\lambda_2) - \lambda_2\right)q^2 - \left(\frac{n}{\theta_1}(1+\lambda_1) - \frac{n}{\theta_2}(1+\lambda_2) - \lambda_1\right)q - \frac{n}{\theta_1}(1+\lambda_1)$ *and* $r_2 \in [1,p], r_2 \in [1,q]$.

Remark 3. *If we take in Theorem 2 the condition* $p > 1 + \frac{1+\lambda_2}{\frac{n}{\theta_2}(1+\lambda_2)-\lambda_2}$*, then we cannot feel any interplay between the equations of the system since it will behave as a single equation.*

Remark 4. *If we consider* $p = 1 + \frac{1+\lambda_2}{\frac{n}{\theta_2}(1+\lambda_2)-\lambda_2}$ *then, after using Proposition A1, we obtain a new decay generated by the* log *term appearing in the estimate of u, exactly,* $(1+t)^{-1}\log(1+t) \approx (1+t)^{-1+\varepsilon}$.

Example 2. *Let us consider* $\theta_1 = \theta_2 = 2$ *in* $\mathbb{R}^2$ *and the parameter of the fractional derivative of the first equation* $\lambda_1 \to 0$ *and the second* $\lambda_2 \to 1$*. Then, with the Cauchy condition the model, we obtain*

$$\begin{aligned} \partial_t u + -\Delta u &= |v|^p, \\ \partial_{tt} v + -\Delta v &= |u|^q. \end{aligned}$$

Applying Theorem 2, we obtain the global (in time) existence of the solution for $p < 3$ *and* $q > 2$ *.*

Remark 5. *The reader can apply the last theorem for several examples. Giving values to some parameters such as the dimension or the order of the fractional derivative, we obtain the mixed condition that leads to the global existence .*

3. Philosophy of Our Approach

In this section, we will prove results for the Cauchy problems (1) and (7). Our main interest is to prove the global (in time) existence of small data solutions, which means the global existence after the perturbation of the null Cauchy condition $\|u_\lambda\|_{L^1\cap L^p} \le \epsilon$. Such results imply immediate stability results for the zero solution .

3.1. Proof of Theorem 1

In this section, we deal with the following single equation:

$$\partial_t^{\lambda+1}u + (-\Delta)^{\frac{\theta}{2}}u = |u|^p, \qquad J^{1-\lambda}u(0,x) = u_\lambda(x), D^\lambda u(0,x) = 0. \tag{13}$$

We define the norm of the solution space $X(t)$, which we will propose in all of the proofs of the above theorems by

$$\|u\|_{X(t)} = \sup_{\tau\in[0,t]} (1+t)^{-\lambda}\left\{\|u(\tau,\cdot)\|_{L^1} + (1+t)^{\frac{n}{\theta}(1+\lambda)\left(1-\frac{1}{p}\right)}\|u(\tau,\cdot)\|_{L^p}\right\}, \tag{14}$$

We introduce the operator N by

$$N : u \in X(t) \to Nu = Nu(t,x) := u^{ln}(t,x) + u^{nl}(t,x),$$

where u^{ln} is a Sobolev solution to the Cauchy problem

$$\partial_t^{\lambda+1}u + (-\Delta)^{\frac{\theta}{2}}u = 0, \qquad J^{1-\lambda}u(0,x) = u_\lambda(x), D^\lambda u(0,x) = 0,$$

and u^{nl} is a Sobolev solution to the Cauchy problem

$$\partial_t^{\lambda+1}u + (-\Delta)^{\frac{\theta}{2}}u = |u|^p, \qquad J^{1-\lambda}u(0,x) = u_\lambda(x), D^\lambda u(0,x) = 0.$$

Using Fourier analysis together with Theorem A1 from Appendix A, we can show that the solutions of the previous problems can be presented by $u(t,x) = u^{ln}(t,x) + u^{nl}(t,x)$ as follows:

$$u^{ln}(t,x) = t^{\lambda-1}\mathcal{F}^{-1}\left(E_{1+\lambda,\lambda}\left(-t^{1+\lambda}|\xi|^\theta\right)\right)(t,x) *_{(x)} u_\lambda(x), \tag{15}$$

and

$$u^{nl}(t,x) = \int_0^t (t-s)^\lambda \mathcal{F}^{-1}\left(E_{1+\lambda,1+\lambda}\left(-t^{1+\lambda}|\xi|^\theta\right)\right)(t-s,x) *_{(x)} |u(s,x)|^p ds. \tag{16}$$

Following Proposition A2, our aim is to prove the following inequalities:

$$\|Nu\|_{X(t)} \lesssim \|u_\lambda\|_{L^1\cap L^p} + \|u\|^p_{X(t)}, \tag{17}$$

$$\|Nu - Nv\|_{X(t)} \lesssim \|u-v\|_{X(t)}\left(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\right). \tag{18}$$

After proving these both inequalities, we apply Banach's fixed-point theorem. In this way, we obtain the local (in time) existence of large data Sobolev solutions and the global (in time) existence of small data Sobolev solutions as well.

We split the prove of the first inequality (17) into the following inequalities:

$$\left\|u^{ln}\right\|_{X(t)} \lesssim \|u_\lambda\|_{L^1\cap L^p}, \tag{19}$$

and

$$\|u^{nl}\|_{X(t)} \lesssim \|u\|^{p}_{X(t)}. \tag{20}$$

To prove inequality (19) we have to derive the estimate of $\|\mathcal{F}^{-1}(E_{1+\lambda,\lambda}(-t^{1+\lambda}|\xi|^{\theta}))\|_{L^p}$ in order to use Young's inequality. Using the scaling property, we obtain

$$\left\|\mathcal{F}^{-1}\left(E_{1+\lambda,\lambda}\left(-t^{1+\lambda}|\xi|^{\theta}\right)\right)\right\|_{L^p} = t^{-\frac{n}{\theta}(1+\lambda)\left(1-\frac{1}{p}\right)}\left\|\mathcal{F}^{-1}\left(E_{1+\lambda,\lambda}\left(-|\xi|^{\theta}\right)\right)\right\|_{L^p}. \tag{21}$$

Indeed, after change of variable $\xi_1 = t^{1+\lambda}|\xi|$ we obtain

$$\begin{aligned}
\mathcal{F}^{-1}\left(G\left(t^{1+\lambda}|\xi|^{\theta}\right)\right) &= t^{-\frac{n}{\theta}(1+\lambda)}\int_{\mathbb{R}^n} e^{it^{-\frac{1+\lambda}{\theta}}x\xi_1}G(|\xi_1|^{\theta})d\xi_1 \\
&= t^{-\frac{n}{\theta}(1+\lambda)}\mathcal{F}^{-1}\left(G(|\xi|^{\theta})\right)(t^{-\frac{1+\lambda}{\theta}}x).
\end{aligned}$$

Using the last equality, we obtain

$$\begin{aligned}
\left\|\mathcal{F}^{-1}\left(G\left(t^{1+\lambda}|\cdot|^{\theta}\right)\right)\right\|^{p}_{L^p} &= t^{-\frac{n}{\theta}(1+\lambda)p}\left\|\mathcal{F}^{-1}\left(G(|\cdot|^{\theta})\right)(t^{-\frac{1+\lambda}{\theta}}x)\right\|_{L^p} \\
&= t^{-\frac{n}{\theta}(1+\lambda)p}\int_{\mathbb{R}^n}\left|\mathcal{F}^{-1}\left(G(|\cdot|^{\theta})\right)(t^{-\frac{1+\lambda}{\theta}}x)\right|^{p}dx.
\end{aligned}$$

The change of variable $y = t^{-\frac{1+\lambda}{\theta}}x$ leads to

$$\left\|\mathcal{F}^{-1}\left(G\left(t^{1+\lambda}|\cdot|^{\theta}\right)\right)\right\|^{p}_{L^p} = t^{-\frac{n}{\theta}(1+\lambda)p+\frac{n}{\theta}(1+\lambda)}\left\|\mathcal{F}^{-1}\left(G\left(|\cdot|^{\theta}\right)\right)\right\|^{p}_{L^p},$$

which completes the proof of 21.
Then, we restrict ourselves to the estimates of $\|\mathcal{F}^{-1}(E_{1+\lambda,\lambda}(-|\xi|^{\theta}))\|_{L^p}$. After applying Theorem A2 from the Appendix A, we obtain

$$\begin{aligned}
E_{1+\lambda,\lambda}\left(-|\xi|^{\theta}\right) &= \frac{2}{1+\lambda}|\xi|^{-\theta\left(1-\frac{2}{1+\lambda}\right)}e^{\frac{\theta}{1+\lambda}\cos\left(\frac{\pi}{1+\lambda}\right)}cos\left(|\xi|^{\frac{\theta}{1+\lambda}}sin\frac{\pi}{1+\lambda}\right) \\
&+\pi^{-1}|\xi|^{-\theta\left(1-\frac{2}{1+\lambda}\right)}\int_0^{\infty}\frac{s^{2+\lambda}}{s^{2(1+\lambda)}+2cos(\pi(1+\lambda))+1}e^{-s|\xi|^{\frac{\theta}{1+\lambda}}}ds\,\sin(\lambda\pi),
\end{aligned}$$

which leads to

$$\mathcal{F}^{-1}\left(E_{1+\lambda,\lambda}\left(-|\xi|^{\theta}\right)\right) = \frac{2}{1+\lambda}A(s,x)+\pi^{-1}\sin(\lambda\pi)\int_0^{\infty}\frac{s^{2+\lambda}}{s^{2(1+\lambda)}+2cos(\pi(1+\lambda))+1}B(s,x)ds,$$

where

$$\begin{aligned}
A(s,x) &= \mathcal{F}^{-1}\left(|\xi|^{-\theta\left(1-\frac{2}{1+\lambda}\right)}e^{\frac{\theta}{1+\lambda}\cos\left(\frac{\pi}{1+\lambda}\right)}cos\left(|\xi|^{\frac{\theta}{1+\lambda}}sin\frac{\pi}{1+\lambda}\right)\right), \\
B(s,x) &= \mathcal{F}^{-1}\left(|\xi|^{-\theta\left(1-\frac{2}{1+\lambda}\right)}e^{-s|\xi|^{\frac{\theta}{1+\lambda}}}\right).
\end{aligned}$$

First, we consider $B(s,x)$. Similarly to (21), we have

$$\begin{aligned}
\|B(s,\cdot)\|_{L^p} &= \left\|\mathcal{F}^{-1}\left(|\xi|^{-\theta\left(1-\frac{2}{1+\lambda}\right)}e^{-s|\xi|^{\frac{\theta}{1+\lambda}}}\right)\right\|_{L^p} \\
&= \left\|\mathcal{F}^{-1}\left(s^{(1+\lambda)\left(1-\frac{2}{1+\lambda}\right)}(s^{1+\lambda}|\xi|^{\theta})^{-\left(1-\frac{2}{1+\lambda}\right)}e^{-(s^{1+\lambda}|\xi|^{\theta})^{\frac{1}{1+\lambda}}}\right)\right\|_{L^p} \\
&= s^{\lambda-1-\frac{n}{\theta}(1+\lambda)\left(1-\frac{1}{p}\right)}\left\|\mathcal{F}^{-1}\left(|\xi|^{-\theta\left(1-\frac{2}{1+\lambda}\right)}e^{-|\xi|^{\frac{\theta}{1+\lambda}}}\right)\right\|_{L^p} \\
&= s^{\lambda-1-\frac{n}{\theta}(1+\lambda)\left(1-\frac{1}{p}\right)}\|B(1,\cdot)\|_{L^p}.
\end{aligned}$$

Then,

$$\|B(s,\cdot)\|_{L^p} = s^{\lambda-1-\frac{n}{\theta}(1+\lambda)\left(1-\frac{1}{p}\right)}\|B(1,\cdot)\|_{L^p}. \tag{22}$$

Then,

$$\left\|\mathcal{F}^{-1}\left(E_{1+\lambda,\lambda}\left(-|\cdot|^{\theta}\right)\right)\right\|_{L^p} = \frac{2}{1+\lambda}\|A(s,\cdot)\|_{L^p} + \pi^{-1}\sin(\lambda\pi)\int_0^{\infty}\frac{s^{1+2\lambda-\frac{n}{\theta}(1+\lambda)\left(1-\frac{1}{p}\right)}}{s^{2(1+\lambda)}+2cos(\pi(1+\lambda))+1}\|B(1,\cdot)\|_{L^p}ds,$$

Using the last estimate together with (A5) from Remark A1, one can obtain the following estimate from Lemma 2.1 in [14] for $d = -\theta\left(1-\frac{2}{1+\lambda}\right)$:

$$\mathcal{F}^{-1}\left(E_{1+\lambda,\lambda}\left(-|\xi|^{\theta}\right)\right) \in L^p \quad \text{if} \quad \frac{n}{\theta}\left(1-\frac{1}{p}\right) < 2, \tag{23}$$

which satisfied (9) .

From (15) with (21), and after using Young's inequality, we obtain

$$\begin{aligned}
\left\|u^{ln}(t,x)\right\|_{L^1} &\lesssim (1+t)^{\lambda-1}\|u_\lambda\|_{L^1}, &(24)\\
\left\|u^{ln}(t,x)\right\|_{L^p} &\lesssim (1+t)^{\lambda-1-\frac{n}{\theta}(1+\lambda)\left(1-\frac{1}{p}\right)}(\|u_\lambda\|_{L^1}+\|u_\lambda\|_{L^p}). &(25)
\end{aligned}$$

Replacing last estimates in the definition of the norm of solution space (14) leads to the desired estimate (19).

For the second estimate (20), under the same conditions requested for (23) we have

$$\left\|\mathcal{F}^{-1}\left(E_{1+\lambda,1+\lambda}\left(-t^{1+\lambda}|\xi|^{\theta}\right)\right)\right\|_{L^p} \lesssim (1+t)^{-\frac{n}{\theta}(1+\lambda)\left(1-\frac{1}{p}\right)}.$$

From (16), we obtain

$$\left\|u^{nl}(t,x)\right\|_{L^1} \lesssim \int_0^t (t-s)^{\lambda}\||u(s,x)|^p\|_{L^1}ds, \tag{26}$$

and

$$\left\|u^{nl}(t,x)\right\|_{L^p} \lesssim \int_0^t (t-s)^{\lambda-\frac{n}{\theta}(1+\lambda)\left(1-\frac{1}{p}\right)}\||u(s,x)|^p\|_{L^1}ds. \tag{27}$$

Using the definition of solution space from (14), we obtain

$$\begin{aligned}
\left\|u^{nl}(t,x)\right\|_{L^1} &\lesssim \|u\|^p_{X(t)}\int_0^t (t-s)^{\lambda}(1+s)^{-\frac{n}{\theta}(1+\lambda)(p-1)+p\lambda}ds, \\
\left\|u^{nl}(t,x)\right\|_{L^p} &\lesssim \|u\|^p_{X(t)}\int_0^t (t-s)^{\lambda-\frac{n}{\theta}(1+\lambda)\left(1-\frac{1}{p}\right)}(1+s)^{-\frac{n}{\theta}(1+\lambda)(p-1)+p\lambda}ds.
\end{aligned}$$

Using Proposition A1, we obtain

$$\left\|u^{nl}(t,x)\right\|_{L^1} \lesssim (1+t)^{\lambda}\|u\|^p_{X(t)}, \tag{28}$$

$$\left\|u^{nl}(t,x)\right\|_{L^p} \lesssim (1+t)^{\lambda-\frac{n}{\theta}(1+\lambda)\left(1-\frac{1}{p}\right)}\|u\|^p_{X(t)}, \tag{29}$$

provided that $\frac{n}{\theta}(1+\lambda)\left(1-\frac{1}{p}\right)-\lambda<1$ and $\frac{n}{\theta}(1+\lambda)(p-1)-p\lambda>1$, which are equivalent to (8) and (9), respectively.

Replacing the last estimates in the norm of solution space, we obtain (20), which complete, together with (19), the proof of the first inequality (17).

For the second condition (18), we assume that u and v belong to $X(t)$. Then,

$$Nu-Nv=\int_0^t (t-s)^{\lambda}\mathcal{F}^{-1}\left(E_{1+\lambda,1+\lambda}\left(-t^{1+\lambda}|\xi|^{\theta}\right)\right)(t-s,x)*_{(x)}\left(|u(s,x)|^p-|v(s,x)|^p\right)ds.$$

We control all norms appearing in $\|Nu-Nv\|_{X(t)}$. These are the norms $\|Nu-Nv\|_{L^1}$ and $\||D|^s(Nu-Nv)\|_{L^p}$.

Similarly to (26), we have

$$\|Nu-Nv\|_{L^1}\lesssim\int_0^t (t-s)^{\lambda}\left\|\left(|u(s,x)|^p-|v(s,x)|^p\right)\right\|_{L^1}ds.$$

Hölder's inequality implies

$$\left\||u(s,\cdot)|^p-|v(s,\cdot)|^p\right\|_{L^1}\lesssim\left\|u(s,\cdot)-v(s,\cdot)\right\|_{L^p}\left(\|u(s,\cdot)\|^{p-1}_{L^p}+\|v(s,\cdot)\|^{p-1}_{L^p}\right), \tag{30}$$

Using the norm of the solution space $X(t)$, we obtain

$$\left\|u(s,\cdot)-v(s,\cdot)\right\|_{L^p} \lesssim (1+s)^{-\frac{n}{\theta}(1+\lambda)(1-\frac{1}{p})+\lambda}\left\|u(s,\cdot)-v(s,\cdot)\right\|_{X(t)},$$

$$\|u(s,\cdot)\|^{p-1}_{L^p} \lesssim (1+s)^{\left(-\frac{n}{\theta}(1+\lambda)(1-\frac{1}{p})+\lambda\right)(p-1)}\|v(s,\cdot)\|^{(p-1)}_{X(t)},$$

$$\|u(s,\cdot)\|^{p-1}_{L^p} \lesssim (1+s)^{\left(-\frac{n}{\theta}(1+\lambda)(1-\frac{1}{p})+\lambda\right)(p-1)}\|v(s,\cdot)\|^{(p-1)}_{X(t)}.$$

Using the last estimates, we can obtain similarly to (28) and (29)

$$\|Nu-Nv\|_{L^1}\lesssim(1+t)^{\lambda}\|u-v\|_{X(t)}\left(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\right),$$

and

$$\|Nu-Nv\|_{L^p}\lesssim(1+t)^{\lambda-\frac{n}{\theta}(1+\lambda)\left(1-\frac{1}{p}\right)}\|u-v\|_{X(t)}\left(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\right).$$

Then, the proof of the second condition and the theorem is completed.

3.2. Proof of Theorem 2

We define the norm of the solution space $X(t)$ by

$$\|(u,v)\|_{X(t)}=\sup_{\tau\in[0,t]}\left\{M(\tau,u)+M(\tau,v)\right\} \tag{31}$$

where

$$M(\tau,u) = (1+t)^{-\lambda_1-L(p)}\left[\|u(\tau,\cdot)\|_{L^1}+(1+t)^{\frac{n}{\theta_1}(1+\lambda_1)\left(1-\frac{1}{p}\right)}\|u(\tau,\cdot)\|_{L^p}\right], \tag{32}$$

$$M(\tau,v) = (1+t)^{-\lambda_2}\left[\|v(\tau,\cdot)\|_{L^1}+(1+t)^{\frac{n}{\theta_2}(1+\lambda_2)\left(1-\frac{1}{q}\right)}\|v(\tau,\cdot)\|_{L^q}\right]. \tag{33}$$

Then, we introduce the operator N by

$$N : (u,v) \in X(t) \to N(u,v) = (u^{ln} + u^{nl}, v^{ln} + v^{nl}),$$

where

$$\begin{aligned}
u^{ln}(t,x) &:= t^{\lambda_1 - 1}\mathcal{F}^{-1}\Big(E_{1+\lambda_1,\lambda_1}\Big(-t^{1+\lambda_1}|\xi|^{\theta_1}\Big)\Big)(t,x) *_{(x)} u_{\lambda_1}(x),\\
u^{nl}(t,x) &:= \int_0^t (t-s)^{\lambda_1}\mathcal{F}^{-1}\Big(E_{1+\lambda_1,1+\lambda_1}\Big(-t^{1+\lambda_1}|\xi|^{\theta_1}\Big)\Big)(t-s,x) *_{(x)} |v(s,x)|^p ds,\\
v^{ln}(t,x) &:= t^{\lambda_2 - 1}\mathcal{F}^{-1}\Big(E_{1+\lambda_2,\lambda_1}\Big(-t^{1+\lambda_2}|\xi|^{\theta_2}\Big)\Big)(t,x) *_{(x)} v_{\lambda_2}(x),\\
v^{nl}(t,x) &:= \int_0^t (t-s)^{\lambda_2}\mathcal{F}^{-1}\Big(E_{1+\lambda_2,1+\lambda_2}\Big(-t^{1+\lambda_2}|\xi|^{\theta_2}\Big)\Big)(t-s,x) *_{(x)} |u(s,x)|^q ds.
\end{aligned}$$

If we consider the results Proposition A3, then our aim is to prove the following inequalities, which imply, among other things, the global existence of small data solutions:

$$\|N(u,v)\|_{X(t)} \lesssim \|u_{\lambda_1}\|_{L^1\cap L^p} + \|v_{\lambda_2}\|_{L^1\cap L^q} + \|(u,v)\|^p_{X(t)} + \|(u,v)\|^q_{X(t)}, \tag{34}$$

$$\begin{aligned}
\|N(u,v) - N(\widetilde{u},\widetilde{v})\|_{X(t)} &\lesssim \|(u,v) - (\widetilde{u},\widetilde{v})\|_{X(t)}\big(\|(u,v)\|^{p-1}_{X(t)} + \|(\widetilde{u},\widetilde{v})\|^{p-1}_{X(t)}\\
&\quad + \|(u,v)\|^{q-1}_{X(t)} + \|(\widetilde{u},\widetilde{v})\|^{q-1}_{X(t)}\big).
\end{aligned} \tag{35}$$

Let us start by the first condition. Similarly to (24) and (25), we obtain

$$\begin{aligned}
\Big\|u^{ln}(t,x)\Big\|_{L^1} &\lesssim (1+t)^{\lambda_1 - 1}\|u_{\lambda_1}\|_{L^1},\\
\Big\|u^{ln}(t,x)\Big\|_{L^p} &\lesssim (1+t)^{\lambda_1 - 1 - \frac{n}{\theta_1}(1+\lambda_1)\left(1-\frac{1}{p}\right)}\big(\|u_{\lambda_1}\|_{L^1} + \|u_{\lambda_1}\|_{L^p}\big),\\
\Big\|v^{ln}(t,x)\Big\|_{L^1} &\lesssim (1+t)^{\lambda_2 - 1}\|v_{\lambda_2}\|_{L^1},\\
\Big\|v^{ln}(t,x)\Big\|_{L^q} &\lesssim (1+t)^{\lambda_2 - 1 - \frac{n}{\theta_2}(1+\lambda_2)\left(1-\frac{1}{q}\right)}\big(\|v_{\lambda_2}\|_{L^1} + \|u_{\lambda_2}\|_{L^q}\big).
\end{aligned}$$

The last estimates, together with the definition of the norm in (31), lead to

$$\|(u^{ln}, v^{ln})\|_{X(t)} \lesssim \|u_{\lambda_1}\|_{L^1\cap L^p} + \|v_{\lambda_2}\|_{L^1\cap L^q}. \tag{36}$$

Then, we complete the proof by showing the inequality

$$\|(u^{nl}, v^{nl})\|_{X(t)} \lesssim \|(u,v)\|^p_{X(t)} + \|(u,v)\|^q_{X(t)}. \tag{37}$$

For u^{nl}, we have

$$\Big\|u^{nl}(t,x)\Big\|_{L^1} \lesssim \int_0^t (t-s)^{\lambda_1}\||v(s,x)|^p\|_{L^1} ds,$$

and

$$\Big\|u^{nl}(t,x)\Big\|_{L^p} \lesssim \int_0^t (t-s)^{\lambda_1 - \frac{n}{\theta_1}(1+\lambda_1)\left(1-\frac{1}{p}\right)}\||v(s,x)|^p\|_{L^1} ds.$$

Using the definition of solution space from (31), we obtain

$$\begin{aligned}
\Big\|u^{nl}(t,x)\Big\|_{L^1} &\lesssim \|(u,v)\|^p_{X(t)}\int_0^t (t-s)^{\lambda_1}(1+s)^{-\frac{n}{\theta_2}(1+\lambda_2)(p-1)+p\lambda_2} ds,\\
\Big\|u^{nl}(t,x)\Big\|_{L^p} &\lesssim \|(u,v)\|^p_{X(t)}\int_0^t (t-s)^{\lambda_1 - \frac{n}{\theta_1}(1+\lambda_1)\left(1-\frac{1}{p}\right)}(1+s)^{-\frac{n}{\theta_2}(1+\lambda_2)(p-1)+p\lambda_2} ds.
\end{aligned}$$

From Proposition A1, one can obtain

$$\left\|u^{nl}(t,x)\right\|_{L^1} \lesssim \|(u,v)\|^p_{X(t)}(1+t)^{\lambda_1-\frac{n}{\theta_2}(1+\lambda_2)(p-1)+p\lambda_2} = \|(u,v)\|^p_{X(t)}(1+t)^{\lambda_1+L(p)}, \tag{38}$$

$$\left\|u^{nl}(t,x)\right\|_{L^p} \lesssim \|(u,v)\|^p_{X(t)}(1+t)^{\lambda_1-\frac{n}{\theta_1}(1+\lambda_1)\left(1-\frac{1}{p}\right)-\frac{n}{\theta_2}(1+\lambda_2)(p-1)+p\lambda_2} = \|(u,v)\|^p_{X(t)}(1+t)^{\lambda_1-\frac{n}{\theta_1}(1+\lambda_1)\left(1-\frac{1}{p}\right)+L(p)}, \tag{39}$$

provided that $\frac{n}{\theta}(1+\lambda_1)\left(1-\frac{1}{p}\right)-\lambda_1<1$, which is equivalent to (11).

For u^{nl}, we have

$$\left\|v^{nl}(t,x)\right\|_{L^1} \lesssim \int_0^t (t-s)^{\lambda_2}\||u(s,x)|^q\|_{L^1}ds,$$

and

$$\left\|v^{nl}(t,x)\right\|_{L^q} \lesssim \int_0^t (t-s)^{\lambda_2-\frac{n}{\theta_2}(1+\lambda_2)\left(1-\frac{1}{q}\right)}\||u(s,x)|^q\|_{L^1}ds.$$

Using the norm of the solution space, we obtain

$$\left\|v^{nl}(t,x)\right\|_{L^1} \lesssim \|(u,v)\|^p_{X(t)}\int_0^t (t-s)^{\lambda_2}(1+t)^{-Q(\lambda_1,\lambda_2,\theta_1,\theta_2,q)}ds,$$

and

$$\left\|v^{nl}(t,x)\right\|_{L^q} \lesssim \|(u,v)\|^p_{X(t)}\int_0^t (t-s)^{\lambda_2-\frac{n}{\theta_2}(1+\lambda_2)\left(1-\frac{1}{q}\right)}(1+t)^{-Q(\lambda_1,\lambda_2,\theta_1,\theta_2,q)}ds.$$

Proposition A1, together with (12), leads to

$$\left\|v^{nl}(t,x)\right\|_{L^1} \lesssim \|(u,v)\|^p_{X(t)}(1+t)^{\lambda_2}, \tag{40}$$

and

$$\left\|v^{nl}(t,x)\right\|_{L^q} \lesssim \|(u,v)\|^p_{X(t)}(1+t)^{\lambda_2-\frac{n}{\theta_2}(1+\lambda_2)\left(1-\frac{1}{q}\right)}ds, \tag{41}$$

provided that $\frac{n}{\theta}(1+\lambda_2)\left(1-\frac{1}{q}\right)-\lambda_2<1$, which is equivalent to (11).

From (38) to (41), we obtain (37), which implies, together with (36), the first condition (34).

To prove (35), we assume that (u,v) and $(\tilde{u},\tilde{v})$ are two elements from the function space $X(t)$. Then, we have

$$\begin{aligned} N(u,v)-N(\widetilde{u},\widetilde{v}) &= \left(u^{nl}(t,x)-\widetilde{u}^{nl}(t,x), v^{nl}(t,x)-\widetilde{v}^{nl}(t,x)\right) \\ &= \Big(\int_0^t \mathcal{F}^{-1}\left(E_{1+\lambda_1,1+\lambda_1}\left(-t^{1+\lambda_1}|\xi|^{\theta_1}\right)\right)(t-s,x) *_{(x)} \left(|v(s,x)|^p-|\widetilde{v}(s,x)|^p\right)ds, \qquad (42) \\ &\quad \int_0^t \mathcal{F}^{-1}\left(E_{1+\lambda_2,1+\lambda_2}\left(-t^{1+\lambda_2}|\xi|^{\theta_2}\right)\right)(t-s,x) *_{(x)} \left(|u(s,x)|^q-|\widetilde{u}(s,x)|^q\right)ds\Big). \qquad (43) \end{aligned}$$

Similarly to the proof of the estimates (30), we can derive the following estimates for $0\le\tau\le t$:

$$\left\||v(\tau,\cdot)|^p-|\widetilde{v}(\tau,\cdot)|^p\right\|_{L^1} \lesssim (1+t)^{-\frac{n}{\theta_2}(1+\lambda_2)(p-1)+p\lambda_2}\|v-\widetilde{v}\|_{X(t)}\left(\|v\|^{p-1}_{X(t)}+\|\widetilde{v}\|^{p-1}_{X(t)}\right), \tag{44}$$

$$\left\||u(\tau,\cdot)|^q-|\widetilde{u}(\tau,\cdot)|^q\right\|_{L^1} \lesssim (1+t)^{-Q(\lambda_1,\lambda_2,\theta_1,\theta_2,q)}\|u-\widetilde{u}\|_{X(t)}\left(\|u\|^{q-1}_{X(t)}+\|\widetilde{u}\|^{q-1}_{X(t)}\right). \tag{45}$$

Using the last estimates, one may finally conclude, similarly to (38) to (41), the following estimates:

$$\left\|\int_0^t \mathcal{F}^{-1}\Big(E_{1+\lambda_1,1+\lambda_1}\Big(-t^{1+\lambda_1}|\xi|^{\theta_1}\Big)\Big)(t-s,x) *_{(x)} \big(|v(s,x)|^p - |\widetilde{v}(s,x)|^p\big)\,ds\right\|_{L^1}$$
$$\lesssim (1+t)^{\lambda_1+L(p)}\|v-\widetilde{v}\|_{X(t)}\big(\|v\|^{p-1}_{X(t)} + \|\widetilde{v}\|^{p-1}_{X(t)}\big),$$
$$\left\|\int_0^t \mathcal{F}^{-1}\Big(E_{1+\lambda_1,1+\lambda_1}\Big(-t^{1+\lambda_1}|\xi|^{\theta_1}\Big)\Big)(t-s,x) *_{(x)} \big(|v(s,x)|^p - |\widetilde{v}(s,x)|^p\big)\,ds\right\|_{L^p}$$
$$\lesssim (1+t)^{\lambda_1-\frac{n}{\theta_1}(1+\lambda_1)\left(1-\frac{1}{p}\right)+L(p)}\|v-\widetilde{v}\|_{X(t)}\big(\|v\|^{p-1}_{X(t)} + \|\widetilde{v}\|^{p-1}_{X(t)}\big),$$
$$\left\|\int_0^t \mathcal{F}^{-1}\Big(E_{1+\lambda_2,1+\lambda_2}\Big(-t^{1+\lambda_2}|\xi|^{\theta_2}\Big)\Big)(t-s,x) *_{(x)} \big(|u(s,x)|^q - |\widetilde{u}(s,x)|^q\big)\,ds\Big)\right\|_{L^1}$$
$$\lesssim (1+t)^{\lambda_2}\|v-\widetilde{v}\|_{X(t)}\big(\|v\|^{p-1}_{X(t)} + \|\widetilde{v}\|^{p-1}_{X(t)}\big),$$
$$\left\|\int_0^t \mathcal{F}^{-1}\Big(E_{1+\lambda_2,1+\lambda_2}\Big(-t^{1+\lambda_2}|\xi|^{\theta_2}\Big)\Big)(t-s,x) *_{(x)} \big(|u(s,x)|^q - |\widetilde{u}(s,x)|^q\big)\,ds\Big)\right\|_{L^q}$$
$$\lesssim (1+t)^{\lambda_2-\frac{n}{\theta_2}(1+\lambda_2)\left(1-\frac{1}{q}\right)}\|v-\widetilde{v}\|_{X(t)}\big(\|v\|^{p-1}_{X(t)} + \|\widetilde{v}\|^{p-1}_{X(t)}\big),$$

In this way, we can conclude the proof of the last condition (35) and the theorem.

4. Concluding Remarks

- We need to prove the blow-up for the system an interaction between the exponents of both equations. However, the method of scaling is not suitable to prove the blow-up result for the system since we have no interactions between the exponents. Moreover, the influence of each equation to the other one generated a condition presented by several parameters, fractional derivatives, dimensions, and others. For this reason, we will devote the blow-up problem in a forthcoming project using another approach.
- The applications of our results in real world problems and phenomena can be investigated after mathematical modeling by choosing the suitable parameters involved in our problem, such as dimension, and by taking the experimental values into consideration.

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Appendix A

Theorem A1. *Let $\lambda \in (0,1), a_\lambda \in \mathbb{R}$. Then, the unique solution solution to*

$$\partial_t^{\lambda+1} f + |\xi|^\theta f = g(t), \qquad J^{1-\lambda} f(0) = a_\lambda, D^\lambda g(0) = 0. \tag{A1}$$

is given by

$$f(t) = t^{\lambda-1}E_{1+\lambda,\lambda}\Big(-t^{1+\lambda}|\xi|^\theta\Big)a_\lambda + \int_0^t (t-s)^\lambda E_{1+\lambda,1+\lambda}\Big(-t^{1+\lambda}|\xi|^\theta\Big)(t-s,\cdot)g(t)ds, \tag{A2}$$

where $E_{1+\lambda,\mu}$ are the Mittag–Leffler functions defined by

$$E_{1+\lambda,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+\lambda k+\mu)}.$$

For the proof, see [28].

Theorem A2. *Let $0 < \lambda < 2, \mu \in \mathbb{R}$, and $m \in \mathbb{N}$, with $m \geq \frac{\mu}{1+\lambda} - 1$. Then, for the real number $z > 0$, the following holds :*

$$E_{1+\lambda,\mu}(z^{1+\lambda}) = \frac{2}{1+\lambda} z^{1-\mu} e^{z\cos(\frac{\pi}{1+\lambda})} \cos(z\sin(\frac{\pi}{1+\lambda}) - \frac{\pi}{1+\lambda}(\mu-1)) \tag{A3}$$

$$+ \sum_{k=1}^{m} \frac{(-1)^{k-1}}{\Gamma(\mu - k(1+\lambda))} z^{k(1+\lambda)} + \Omega_m(z), \tag{A4}$$

where

$$\Omega_m(z) = \frac{(-1)^m z^{1-\mu}}{\pi} (I_{1,m}(z)\sin(\pi(\mu - (m+1)(1+\lambda)) + I_{2,m}(z)\sin(\pi(\mu - m(1+\lambda)))),$$

and

$$I_{j,m}(z) = \int_0^\infty \frac{s^{(m+j)(1+\lambda)-\mu}}{s^{2(1+\lambda)} 2\cos(\pi(1+\lambda)) s^{1+\lambda} + 1} e^{sz} ds.$$

Remark A1. *The integral $I_{j,m}(z)$ is uniformly bounded if*

$$-1 < m + j - 1 + \frac{1-\mu}{1+\lambda} < 1. \tag{A5}$$

For the proof, see [29].

Proposition A1. *Let $a \in \mathbb{R} < 1$ and $b \in \mathbb{R}$. Then,*

$$\int_0^t (t-s)^{-a}(1+s)^{-b} ds \lesssim \begin{cases} (1+t)^{-a} & \textit{if} \quad a < 1 < b; \\ (1+t)^{-1}\log(1+t) & \textit{if} \quad a < 1 = b; \\ (1+t)^{1-a-b} & \textit{if} \quad a, b < 1. \end{cases} \tag{A6}$$

The reader can find the proof of Proposition A1 in [14].

Proposition A2. *The operator N maps $X(t)$ into itself and has one and only one fixed point $u \in X(t)$ if the following inequalities hold:*

$$\|Nu\|_{X(t)} \leq C_0(t)\|(u_0, u_1)\|_{\mathcal{A}_{m,s}} + C_1(t)\|u\|^p_{X(t)}, \tag{A7}$$

$$\|Nu - Nv\|_{X(t)} \leq C_2(t)\|u - v\|_{X(t)}(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}), \tag{A8}$$

where $C_1(t), C_2(t) \longrightarrow 0$ for $t \longrightarrow +0$ and $C_0(t), C_1(t), C_2(t) \leq C$ for all $t \in [0, \infty)$.

For the proof, see [30].

Proposition A3. *Let us suppose that for any $(u_0, u_1), (v_0, v_1) \in \mathcal{A}_{m,s_1} \times \mathcal{A}_{m,s_2}$, the mapping N satisfies the following estimates:*

$$\begin{aligned} \|N(u,v)\|_{X(t)} \leq C_0(t)\Big(\|(u_0,u_1)\|_{\mathcal{A}_{m_1,s_1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m_2,s_2}}\Big) \\ + C_1(t)\Big(\|(u,v)\|^p_{X(t)} + \|(u,v)\|^q_{X(t)}\Big), \end{aligned} \tag{A9}$$

$$\begin{aligned} \|N(u,v) - N(\tilde{u},\tilde{v})\|_{X(t)} \leq C_2(t)\|(u,v) - (\tilde{u},\tilde{v})\|_{X(t)} \\ \times (\|(u,v)\|^{p-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{p-1}_{X(t)} + \|(u,v)\|^{q-1}_{X(t)} + \|(\tilde{u},\tilde{v})\|^{q-1}_{X(t)}), \end{aligned} \tag{A10}$$

where $C_1(t), C_2(t) \longrightarrow 0$ for $t \longrightarrow +0$ and $C_0(t), C_1(t), C_2(t) \leq C$ for all $t \in [0, \infty)$. Then, N maps $X(t)$ into itself and has one and only one fixed point $(u, v) \in X(t)$.

For the proof, see [26].

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