symmetry

MDPI

Article

The Unit Alpha-Power Kum-Modified Size-Biased Lehmann Type II Distribution: Theory, Simulation, and Applications

Rabab S. Gomaa [1], Alia M. Magar [1], Najwan Alsadat [2], Ehab M. Almetwally [3] and Ahlam H. Tolba [1,*]

[1] Mathematics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt; dr.rsg12@yahoo.com (R.S.G.); aliamohammed@mans.edu.eg (A.M.M.)
[2] Department of Quantitative Analysis, College of Business Administration, King Saud University, P.O. Box 71115, Riyadh 11587, Saudi Arabia; nalsadat@ksu.edu.sa
[3] Department of Statistics, Faculty of Business Administration, Delta University for Science and Technology, Gamasa 11152, Egypt; ehab.metwaly@deltauniv.edu.eg
* Correspondence: Dr_ahamdy156@mans.edu.eg

Abstract: In order to represent the data with non-monotonic failure rates and produce a better fit, a novel distribution is created in this study using the alpha power family of distributions. This distribution is called the alpha-power Kum-modified size-biased Lehmann type II or, in short, the AP-Kum-MSBL-II distribution. This distribution is established for modeling bounded data in the interval $(0, 1)$. The proposed distribution's moment-generating function, mode, quantiles, moments, and stress–strength reliability function are obtained, among other attributes. To estimate the parameters of the proposed distribution, estimation methods such as the maximum likelihood method and Bayesian method are employed to estimate the unknown parameters for the AP-Kum-MSBL-II distribution. Moreover, the confidence intervals, credible intervals, and coverage probability are calculated for all parameters. The symmetric and asymmetric loss functions are used to find the Bayesian estimators using the Markov chain Monte Carlo (MCMC) method. Furthermore, the proposed distribution's usefulness is demonstrated using three real data sets. One of them is a medical data set dealing with COVID-19 patients' mortality rate, the second is a trade share data set, and the third is from the engineering area, as well as extensive simulated data, which were applied to assess the performance of the estimators of the proposed distribution.

Keywords: Lehmann type II distribution; power distribution; hazard rate function; asymmetric loss function; moments; maximum likelihood estimation

Citation: Gomaa, R.S.; Magar, A.M.; Alsadat, N.; Almetwally, E.M.; Tolba, A.H. The Unit Alpha-Power Kum-Modified Size-Biased Lehmann Type II Distribution: Theory, Simulation, and Applications. *Symmetry* **2023**, *15*, 1283. https://doi.org/10.3390/sym15061283

Academic Editor: José Carlos R. Alcantud

Received: 26 April 2023
Revised: 5 June 2023
Accepted: 14 June 2023
Published: 19 June 2023

1. Introduction

In distribution theory, there will always be attempts to generalize distributions. The objective of generalizing distributions is to produce stronger, more flexible models with a wide range of applications. This is achieved through a variety of methods, as evidenced by the great spectrum of work in the literature. The degree to which the chosen distribution fits the input data greatly influences both the analysis and the empirical results; see [1–3]. Bounded data with random variables for rates and proportions are widely used in many fields of knowledge, including economics and medicine. Given their asymmetry and/or kurtosis, some authors have recently concentrated on generating distributions defined on the bounded interval using any of the parent distribution modification strategies. Moreover, popular flexible distributions known as "skew-symmetric distributions" are useful for modeling non-normal characteristics such as skewness and kurtosis [3]. We refer to recent studies in [4–8] for a better understanding.

Additionally, this is frequently performed using generators [9,10], combining [11] two distributions, or increasing the baseline distribution by one parameter. These changes are intended to simplify the classical distributions so that they can be applied to the analysis of complicated data structures such as in [12–17]. Ref. [18] developed a novel probability distribution by applying the inverse Weibull to the alpha power family of distributions. Ref. [9] also introduced the idea of the T-X family of continuous distributions, which substitutes any continuous random variable's probability density function (PDF) for the beta distribution's PDF. A technique for creating continuous single-variable distributions was developed by [10].

The alpha power transformation (APT), a new technique for adding an extra parameter to a continuous distribution, was recently presented by [19]. The major objective of this family was to make use of the non-symmetric behaviour of the parent distribution. Let $F(x)$ be the CDF of the continuous random variable X to define the alpha power transformation. The CDF of the APT family is expressed as

$$F_{APT}(x) = \begin{cases} \frac{\alpha^{F(x)}-1}{\alpha-1} & if\ \alpha > 0, \alpha \neq 1 \\ \\ F(x) & if\ \alpha = 1. \end{cases} \tag{1}$$

The corresponding PDF is

$$f_{APT}(x) = \begin{cases} \frac{\log \alpha}{\alpha-1} \alpha^{F(x)} f(x) & if\ \alpha > 0, \alpha \neq 1 \\ \\ f(x) & if\ \alpha = 1. \end{cases} \tag{2}$$

The alpha power transformation has been used by many researchers. Dey et al. [20], for example, applied the exponential and Rayleigh distributions to the alpha power family of distributions to study the new probability distributions. When modelling reliability, researchers such as Mahmood et al. [21] explored the traits and applications of the extended cosine generalized family of distributions. A novel flexible adaptation of the log-logistic distribution was also addressed by Muse et al. [22] to model the COVID-19 death rate. However, it is relevant because it might be preferable to the one-parameter exponential distribution when modeling data with a poor developing hazard rate function (HRF), such as the transmuted M-generated class of distributions that was proposed in [23], and the power M distribution that was investigated in [24].

A few examples of recent developments based on the T distribution include the new M-generated class of distributions elaborated in [25], the exponentiated power M distribution discussed in [26], the inverse power M distribution investigated in [27], and the truncated M-generated family of distributions stressed in [28]. Additionally, refs. [29,30] introduced the unit Teissier distribution and its applications, as well as the use of order statistics to identify extreme values in samples from continuous distributions. Additionally, refs. [18,31,32] employed the same transformation to obtain the alpha power Weibull distribution, alpha power inverse Weibull distribution, alpha power extended exponential distribution, etc.

One of the simplest and most useful lifetime models to be introduced to the scientific community [33] was the Lehmann type I (L-I) and type II (L-II) lifetime model. The Lehmann type I (L-I) model is most typically contested using the power function (PF) distribution. We used the Kumaraswamy modified size-biased Lehmann type II (Kum-MSBL-II) distribution from the study of [34] in this research. The Kumaraswamy modified size-biased Lehmann type II distribution is an enhanced version of the modified size-biased Lehmann type II (MSBL-II) distribution that was first described by [34].

The random variable X is said to be modified size-biased Lehmann type II (MSBL-II) if it possesses the following CDF:

$$G(x) = 1 - \left(\frac{1-x}{1+\alpha x}\right)^{\alpha}, \ 0 < x < 1, \tag{3}$$

where $\alpha > 0$ is a shape parameter.

The CDF and PDF of the Kum-MSBL-II distribution with three shape parameters take the form

$$F(x) = 1 - \left(1 - \left(1 - \left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right)^{\beta}\right)^{\gamma} \tag{4}$$

and

$$f(x) = \alpha\beta\gamma\frac{(1+\alpha)(1-x)^{\alpha-1}}{(1+\alpha x)^{\alpha+1}}\left(1 - \left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right)^{\beta-1}\left(1 - \left(1 - \left(\frac{1-x}{1+\alpha x}\right)^{\alpha}\right)^{\beta}\right)^{\gamma-1}, \tag{5}$$

where $\alpha,\ \beta,\ \gamma > 0$ are three shape parameters that control the shape and tail behavior of the Kum-MSBL-II distribution.

As a result, significant additions to the body of literature already in existence have been created. Their expanding significance is evidenced by a sharp increase in research publications that suggest novel distributions in the unit interval. There is still insufficient agreement on which distribution is preferred despite the fact that many other distributions have been given and examined.

In this paper, we examine a new distribution called the alpha-power Kum-modified size-biased Lehmann type II (AP-Kum-MSBL-II) distribution, which we extend to the case of bounded random variables on the $(0, 1)$ range. We also study the process of estimating the distribution's parameters.

The remainder of the essay is structured as follows. The new bounded distribution, survival function, and hazard rate are introduced in Section 2, along with the AP-Kum-MSBL-II distribution. We derive some of its mathematical features in Section 3. The maximum likelihood estimation is shown in Section 4. Section 5 presents the Bayesian method. A simulation exercise is conducted in Section 6 to assess how well the model parameter estimates performed. Section 7 presents three examples that use actual data to show how the suggested approaches can be used. Finally, the conclusions are presented in Section 8.

2. Alpha-Power Kum-Modified Size-Biased Lehmann Type II (AP-Kum-MSBL-II) Distribution

The random variable X is said to have an AP-Kum-MSBL-II distribution if its PDF is of the form

$$f(x) = \begin{cases} \frac{\beta\,\gamma\,\log\alpha}{\alpha-1}\,\alpha^{1-(1-G^{\beta}(x))^{\gamma}}\,g(x)\,G^{\beta-1}(x)(1-G^{\beta}(x))^{\gamma-1} & \alpha \neq 1 \\ \beta\,\gamma\,g(x)\,G^{\beta-1}(x)(1-G^{\beta}(x))^{\gamma-1} & \alpha = 1 \end{cases} \tag{6}$$

where

$$G(x) = 1 - (\frac{1-x}{1+\delta\,x})^{\delta}, \quad 0 < x < 1.$$

$$g(x) = \frac{\delta(1+\delta)(1-x)^{\delta-1}}{(1+\delta\,x)^{\delta+1}}.$$

and 0 otherwise. By setting $1 - G^{\beta}(x) = Z$ in (6), it can be easily verified that

$$\int_0^1 f(x)dx = 1$$

The PDF graphs in Figure 1 for various parameter combinations display a variety of shapes, including reversed J-shaped, U-shaped, left-skewed, and right-skewed. In addition, the UPBXD's HF shapes Figure 2 include increasing, J-shaped, and bathtub-shaped.

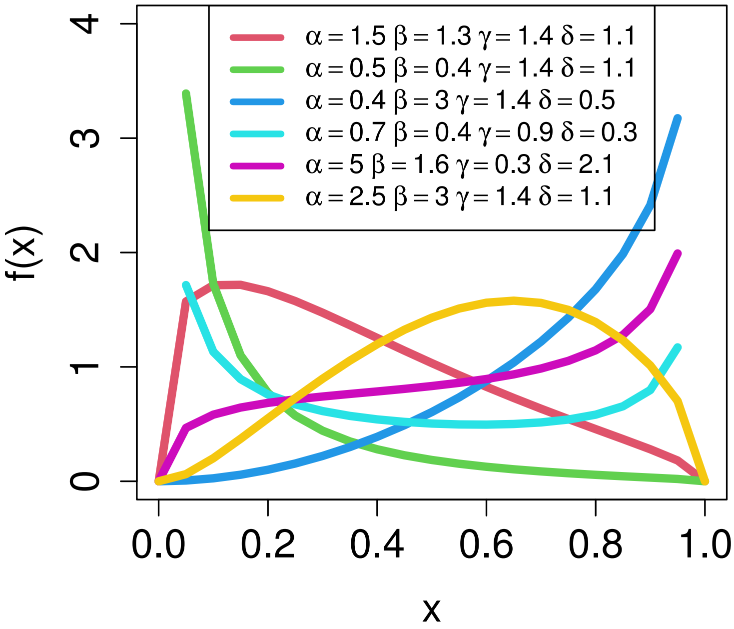

Figure 1. Probability distribution function of AP-Kum-MSBL-II distribution for different values of parameters.

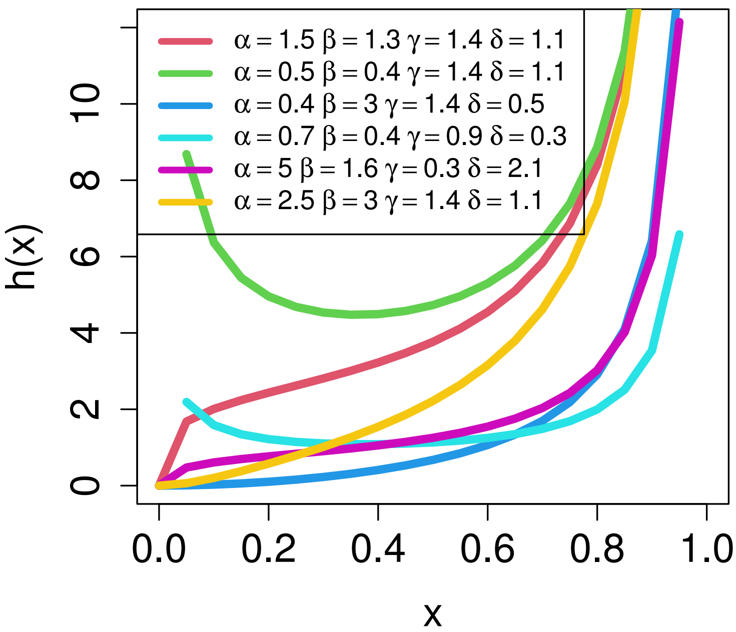

Figure 2. Hazard rate function of AP-Kum-MSBL-II distribution for different values of parameters.

The corresponding CDF of the AP-Kum-MSBL-II distribution is

$$F(x) = \begin{cases} \frac{\alpha^{1-(1-G^{\beta}(x))^{\gamma}}-1}{\alpha-1} & \alpha \neq 1 \\ 1-(1-G^{\beta}(x))^{\gamma} & \alpha = 1. \end{cases} \tag{7}$$

The following, respectively, is how the survival function and hazard rate function are obtained:

$$S(x) = \begin{cases} \frac{\alpha}{\alpha-1}(1-\alpha^{-(1-G^{\beta}(x))^{\gamma}}) & \alpha \neq 1 \\ \\ (1-G^{\beta}(x))^{\gamma} & \alpha = 1. \end{cases} \tag{8}$$

$$h(x) = \begin{cases} \frac{\beta\,\gamma\,\log(\alpha)\,\alpha^{-(1-G^{\beta}(x))^{\gamma}}}{1-\alpha^{-(1-G^{\beta}(x))^{\gamma}}}\; g(x)\,G^{\beta-1}(x)(1-G^{\beta}(x))^{\gamma-1} & \alpha \neq 1 \\ \\ \frac{\beta\,\gamma\,g(x)\,G^{\beta-1}}{(1-G^{\beta}(x))} & \alpha = 1. \end{cases} \tag{9}$$

3. Mathematical Properties

When calculating the mathematical properties, the linear combination provides a far more easy method of describing the CDF and PDF than the usual integral computation. The following binomial expansions are taken into consideration for this:

$$(1-z)^{\beta} = \sum_{i=0}^{\infty}(-1)^{i}\binom{\beta}{i}z^{i}, \quad |z| < 1.$$

From Equations (6) and (7), the PDF and CDF are given as follows

$$f(x) = \frac{\delta\,\beta\,\gamma(1+\delta)\log\alpha}{(\alpha-1)}\,\alpha^{\sum_{i,j,k,l=0}^{\infty}\phi_{ijkl}\,\delta^{l}\,x^{l+k}} \sum_{i,j,k,l=0}^{\infty}\eta_{i,j,k,l}\,\delta^{l}x^{l+k} \tag{10}$$

$$F(x) = \frac{(\alpha^{\sum_{i,j,k,l=0}^{\infty}\phi_{ijkl}\,\delta^{l}\,x^{l+k}} - 1)}{\alpha-1} \tag{11}$$

where $\phi_{ijkl} = (-1)^{i+j+k}\binom{\gamma}{i}\binom{\beta i}{j}\binom{\delta j}{k}\binom{-\delta j}{l}$ and $\eta_{i,j,k,l} = (-1)^{i+j+k}\binom{\gamma-1}{i}\binom{\beta i+\beta-1}{j}\binom{\delta j+j-1}{k}\binom{-\delta j-j-1}{l}$, $\delta,\ \beta,\ \gamma > 0$.

3.1. Quantile Function

An inverse of the distribution function is what the quantile function is described as. Focus on the identity

$$F(X) = U \Rightarrow X = F^{-1}(U)$$

where U follows the standard uniform distribution. The pth quantile of the AP-Kum-MSBL-II distribution is given by

$$x_p = \frac{1-[1-[1-[1-\frac{\log(p(\alpha-1)+1)}{\log\alpha}]^{1/\gamma}]^{1/\beta}]^{1/\delta}}{1+\delta[1-[1-[1-\frac{\log(p(\alpha-1)+1)}{\log\alpha}]^{1/\gamma}]^{1/\beta}]^{1/\delta}}. \tag{12}$$

You can calculate the median of the AP-Kum-MSBL-II distribution by setting $p = 1/2$, that is,

$$x_{\frac{1}{2}} = \frac{1-[1-[1-[1-\frac{\log(\frac{1}{2}(\alpha-1)+1)}{\log\alpha}]^{1/\gamma}]^{1/\beta}]^{1/\delta}}{1+\delta[1-[1-[1-\frac{\log(\frac{1}{2}(\alpha-1)+1)}{\log\alpha}]^{1/\gamma}]^{1/\beta}]^{1/\delta}}. \tag{13}$$

3.2. *rth* Moments

Let $X\sim$AP-Kum-MSBL-II. Then the expression for its rth moments is given as follows

$$\acute{\mu}_r = \sum_{k=0}^{\infty} \frac{(\log\alpha)^{k+1}}{(\alpha-1)(k+1)!}\left(1-\sum_{\ell_3=0}^{\infty}\sum_{m=0}^{k+1}\sum_{\ell_1=0}^{m\gamma}\sum_{\ell_2=0}^{\beta\ell_1} r\,\delta^{\ell_3}B(\ell_3+r,\delta\ell_2+1)\xi_{m,\ell_1,\ell_2,\ell_3}\right), \tag{14}$$

where $\xi_{m,\ell_1,\ell_2,\ell_3} = (-1)^{m+\ell_1+\ell_2+\ell_3}\binom{k+1}{m}\binom{m\gamma}{\ell_1}\binom{\beta\ell_1}{\ell_2}\binom{\delta\ell_2+\ell_3-1}{\ell_3}$

Proof. From the definition of the rth moments, we have

$$\begin{aligned}\acute{\mu}_r = E(X^r) &= \int_0^1 x^r f(x)\,dx\\ &= \int_0^1 x^r \frac{\beta\,\gamma\,\log(\alpha)}{(\alpha-1)}\alpha^{(1-(1-G^{\beta}(x))^{\gamma})}g(x)G^{\beta-1}(x)(1-G^{\beta}(x))^{\gamma-1}dx\\ &= \int_0^1 x^r \frac{\beta\,\gamma\,\log(\alpha)}{(\alpha-1)}\sum_{k=0}^{\infty}\frac{(\log\alpha)^k}{k!}g(x)G^{\beta-1}(x)(1-G^{\beta}(x))^{\gamma-1}dx.\end{aligned}$$

Using integration by parts, we obtain

$$\acute{\mu}_r = \sum_{k=0}^{\infty}\frac{(\log\alpha)^{k+1}}{(\alpha-1)(k+1)!}\left[1-r\int_0^1 x^{r-1}(1-(1-G^{\beta}(x))^{\gamma})^{k+1}dx\right].$$

Using the series representation $\alpha^z = \sum_{k=0}^{\infty}\frac{(\log\alpha)^k}{k!}z^k$ and beta function $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt$ in the above equation, we obtain (14). □

3.3. Moment-Generating Function (MGF)

Let $X\sim$AP-Kum-MSBL-II. Then the MGF of the AP-Kum-MSBL-II distribution has the following form

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{k=0}^{\infty}\frac{(\log\alpha)^{k+1}}{(\alpha-1)(k+1)!}\left(1-\sum_{\ell_3=0}^{\infty}\sum_{m=0}^{k+1}\sum_{\ell_1=0}^{m\gamma}\sum_{\ell_2=0}^{\beta\ell_1} r\,\delta^{\ell_3}B(\ell_3+r,\delta\ell_2+1)\xi_{m,\ell_1,\ell_2,\ell_3}\right) \tag{15}$$

where $\xi_{m,\ell_1,\ell_2,\ell_3} = (-1)^{m+\ell_1+\ell_2+\ell_3}\binom{k+1}{m}\binom{m\gamma}{\ell_1}\binom{\beta\ell_1}{\ell_2}\binom{\delta\ell_2+\ell_3-1}{\ell_3}$ and $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt$ is a beta function.

Proof. From the definition of the moment-generating function, we have

$$M_X(t) = E(e^{tx}) = \int_0^1 e^{xt}\,\frac{\beta\,\gamma\,\log\alpha}{(\alpha-1)}\,\alpha^{1-(1-G^{\beta}(x))^{\gamma}}\,g(x)\,G^{\beta-1}(x)(1-G^{\beta}(x))^{\gamma-1}dx. \tag{16}$$

Using $e^{xt} = \sum_{k=0}^{\infty}\frac{t^r x^r}{r!}$ and the series representation $\alpha^z = \sum_{k=0}^{\infty}\frac{(\log\alpha)^k}{k!}z^k$ in Equation (16), we obtain (15). □

3.4. Mean and Variance

The mean and variance are given, respectively, by

$$E(X) = \sum_{k=0}^{\infty}\frac{(\log\alpha)^{k+1}}{(\alpha-1)(k+1)!}\left(1-\sum_{m=0}^{k+1}\sum_{\ell_1=0}^{m\gamma}\sum_{\ell_2=0}^{\beta\ell_1}\sum_{\ell_3=0}^{\infty}\delta^{\ell_3}B(\ell_3+1,\delta\ell_2+1)\xi_{m,\ell_1,\ell_2,\ell_3}\right) \tag{17}$$

$$
\begin{aligned}
Var(X) &= \sum_{k=0}^{\infty}\frac{(\log\alpha)^{k+1}}{(\alpha-1)(k+1)!}\left(1-\sum_{m=0}^{k+1}\sum_{\ell_1=0}^{m\gamma}\sum_{\ell_2=0}^{\beta\ell_1}\sum_{\ell_3=0}^{\infty}2\,\delta^{\ell_3}B(\ell_3+2,\delta\ell_2+1)\xi_{m,\ell_1,\ell_2,\ell_3}\right)\\
&- \left(\sum_{k=0}^{\infty}\frac{(\log\alpha)^{k+1}}{(\alpha-1)(k+1)!}\left(1-\sum_{m=0}^{k+1}\sum_{\ell_1=0}^{m\gamma}\sum_{\ell_2=0}^{\beta\ell_1}\sum_{\ell_3=0}^{\infty}\delta^{\ell_3}B(\ell_3+1,\delta\ell_2+1)\xi_{m,\ell_1,\ell_2,\ell_3}\right)\right)^2
\end{aligned}
\tag{18}
$$

3.5. Stress–Strength Reliability

Suppose X_1 and X_2 are two continuous and independent random variables, where $x_1\sim$AP-Kum-MSBL-II$(x;\alpha,\beta,\delta,\gamma_1)$ and $x_2\sim$AP-Kum-MSBL-II$(x;\alpha,\beta,\delta,\gamma_2)$. Then the stress–strength reliability is defined as

$$
S=\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{r}\sum_{l=0}^{k}\frac{(-1)^{m+l}\gamma_1(\log\alpha)^{r+k+1}}{(\alpha-1)^2\,(\gamma_1 m+\gamma_2 l+\gamma_1)\,r!\,k!}\binom{r}{m}\binom{k}{l}-\frac{1}{\alpha-1}. \tag{19}
$$

Proof.

$$
\begin{aligned}
S &= \int_{-\infty}^{\infty} f_1(x)F_2(x)dx\\
&= \int_0^1 \frac{\beta\,\gamma_1\,\log\alpha}{\alpha-1}\,\alpha^{1-(1-G^{\beta}(x))^{\gamma_1}}\,g(x)\,G^{\beta-1}(x)(1-G^{\beta}(x))^{\gamma_1-1}\left(\frac{\alpha^{1-(1-G^{\beta}(x))^{\gamma_2}}-1}{\alpha-1}\right)dx\\
&= \int_0^1 \frac{\beta\,\gamma_1\,\log\alpha}{(\alpha-1)^2}\sum_{r=0}^{\infty}\frac{(\log\alpha)^r}{r!}(1-(1-G^{\beta}(x))^{\gamma_1})\,g(x)G^{\beta-1}(x)(1-G^{\beta}(x))^{\gamma_1-1}\\
&\quad(\alpha^{1-(1-G^{\beta}(x))^{\gamma_2}})dx-\frac{1}{\alpha-1}\\
&= \int_0^1\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{\beta\,\gamma_1(\log\alpha)^{r+k+1}}{(\alpha-1)^2 r!\,k!}(1-(1-G^{\beta}(x))^{\gamma_1})^r\,g(x)G^{\beta-1}(x)(1-G^{\beta}(x))^{\gamma_1-1}\\
&\quad(1-(1-G^{\beta}(x))^{\gamma_2})^k dx-\frac{1}{\alpha-1}\\
&= \sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{\gamma_1(\log\alpha)^{r+k+1}}{(\alpha-1)^2 r!\,k!}\sum_{m=0}^{r}\sum_{l=0}^{k}(-1)^{m+l}\binom{r}{m}\binom{k}{l}\int_0^1\beta\,(1-G^{\beta}(x))^{\gamma_1 m+\gamma_2 l+\gamma_1-1}\\
&\quad g(x)G^{\beta-1}(x)dx-\frac{1}{\alpha-1}\\
&= \sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{r}\sum_{l=0}^{k}\frac{(-1)^{m+l}\gamma_1(\log\alpha)^{r+k+1}}{(\alpha-1)^2\,(\gamma_1 m+\gamma_2 l+\gamma_1)\,r!\,k!}\binom{r}{m}\binom{k}{l}-\frac{1}{\alpha-1}.
\end{aligned}
$$

□

Figure 3 shows the stress–strength reliability plots with different parameters using Equation (19).

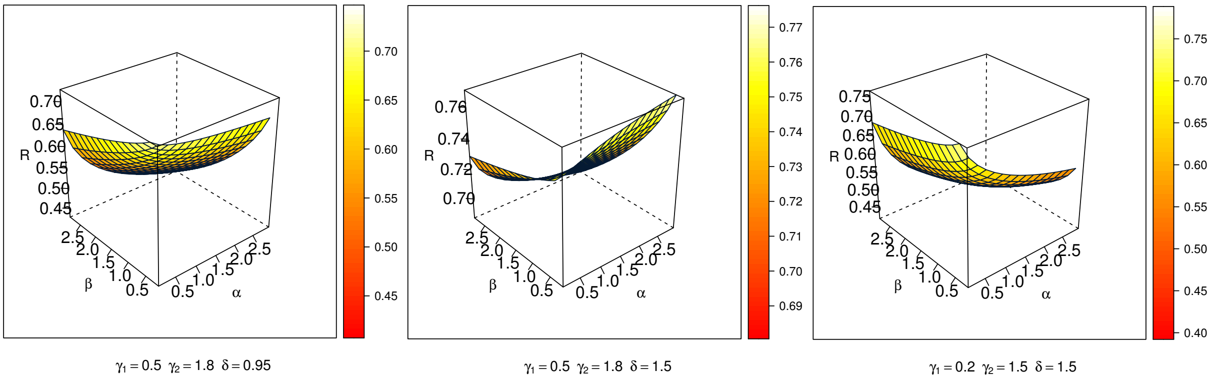

Figure 3. Stress–strength reliability plots with different parameters.

4. Maximum Likelihood Estimation

Let $X_1, X_2, ..., X_n$ be a random sample of size n from the PDF of the AP-Kum-MSBL-II model. Then the likelihood function is given by

$$L(\alpha,\beta,\gamma,\delta) = \left(\alpha\beta\gamma\delta\log(\alpha)\frac{(1+\delta)}{(\alpha-1)}\right)^n \prod_{i=1}^{n} \alpha^{-(1-S_i^{\beta}(\delta))^{\gamma}}(1-S_i(\delta))^{\frac{\delta^2-1}{\delta}} S_i^{\beta-1}(\delta)(1-S_i^{\beta}(\delta)^{\gamma-1}), \tag{20}$$

where $S_i(\delta) = (1-(\frac{1-x_i}{1+\delta x_i})^{\delta})$.

The log-likelihood function, $l(\alpha,\beta,\gamma,\delta)$, reduces to

$$\begin{aligned} l(\alpha,\beta,\gamma,\delta) = {} & n\log(\alpha\beta\gamma\delta) + n\log(\log\alpha) + n\log(1+\delta) - n\log(1-\alpha) - \log\alpha\sum_{i=1}^{n}\left(1-S_i^{\beta}(\delta)\right)^{\gamma} \\ & + \frac{\delta^2-1}{\delta} \times \sum_{i=1}^{n}\log(1-S_i(\delta)) + (\beta-1)\sum_{i=1}^{n}\log(S_i(\delta)) + \gamma - 1\sum_{i=1}^{n}\log\left(1-S_i^{\beta}(\delta)\right). \end{aligned} \tag{21}$$

Taking the derivative of (21) with respect to α, β, γ, and δ, we have the following equations:

$$\frac{\partial l}{\partial\alpha} = \frac{n}{\alpha} + \frac{n}{\alpha\log\alpha} - \frac{n}{\alpha-1} - \frac{1}{\alpha}\sum_{i=1}^{n}(1-S_i^{\beta}(\delta))^{\gamma} \tag{22}$$

$$\frac{\partial l}{\partial\beta} = \frac{n}{\beta} + \gamma\log(\alpha)\sum_{i=1}^{n}(1-S_i^{\beta}(\delta))^{\gamma-1} S_i^{\beta}(\delta)\log S_i(\delta) + \sum_{i=1}^{n}\log(S_i(\delta)) - (\gamma-1)\sum_{i=1}^{n}\frac{S_i^{\beta}(\delta)\log S_i(\delta)}{1-S_i^{\beta}(\delta)} \tag{23}$$

$$\frac{\partial l}{\partial\gamma} = \frac{n}{\gamma} - \log(\alpha)\sum_{i=1}^{n}(1-S_i^{\beta}(\delta))^{\gamma}\ \log(1-S_i^{\beta}(\delta)) + \sum_{i=1}^{n}\log 1 - S_i^{\beta}(\delta). \tag{24}$$

$$\begin{aligned} \frac{\partial l}{\partial\delta} = {} & \frac{n}{\delta} + \frac{n}{1+\delta} - \gamma\beta\log(\alpha)\sum_{i=1}^{n}(1-S_i^{\beta}(\delta))^{\gamma-1}S_i^{\beta-1}(\delta)\ Z_i^{\delta}\left(\log Z_i - \frac{\delta x_i}{1+\delta x_i}\right) + \\ & \sum_{i=1}^{n}(1-x_i) - \sum_{i=1}^{n}\log(1+\delta x_i) - (\delta+1)\sum_{i=1}^{n}\frac{x_i}{1+\delta x_i} - (\beta-1)\sum_{i=1}^{n}\frac{Z_i^{\delta}}{S_i(\delta)} \\ & \left(\log Z_i - \frac{\delta x_i}{1+\delta x_i}\right) + \beta(\gamma-1)\sum_{i=1}^{n}\frac{1}{1-S_i^{\beta}}(\delta)S_i^{\beta-1}(\delta)Z_i^{\delta}\left(\log Z_i - \frac{\delta x_i}{1+\delta x_i}\right) \end{aligned} \tag{25}$$

where $S_i(\delta) = (1-(\frac{1-x_i}{1+\delta x_i})^{\delta})$ and $Z_i = \frac{1-x_i}{1+\delta x_i}$.

By maximizing (21) or simultaneously resolving the aforementioned non-linear equations, one can derive the maximum likelihood estimates (MLEs) (alpha, beta, gamma, and delta) of X. However, these non-linear equations cannot analytically resolve the MLEs or determine the optimal value of $\alpha, \beta, \gamma, \delta$.

5. Bayesian Estimators

In recent years, the Bayesian approach has developed into the most widely used technique in a variety of fields, including but not limited to various applications; for further details see [35–38]. Its ability to use prior information in the analysis makes it especially useful in reliability and quality studies, where one of the significant variables is reliability; see [39,40]. The Bayes estimates and related credible interval of the model parameters α, β, γ, and δ are covered in this section.

5.1. Prior Distribution and Loss Function

The Bayes estimators of the shape parameters $\alpha,\ \beta$, and γ are obtained under the assumption that they are independent random variables with prior distribution $Gamma(\ell_1, m_1)$, $Gamma(\ell_2, m_2)$, and $Gamma(\ell_3, m_3)$, respectively, with PDFs

$$\begin{aligned}
\pi_1(\alpha) &= \frac{m_1^{\ell_1}\ \alpha^{\ell_1-1}\ e^{-m_1\alpha}}{\Gamma\ell_1}, \\
\pi_2(\beta) &= \frac{m_2^{\ell_2}\ \beta^{\ell_2-1}\ e^{-m_2\beta}}{\Gamma\ell_2}, \\
\pi_3(\gamma) &= \frac{m_3^{\ell_3}\ \gamma^{\ell_3-1}\ e^{-m_3\gamma}}{\Gamma\ell_3}, \\
\pi_4(\delta) &= \frac{m_4^{\ell_4}\ \delta^{\ell_4-1}\ e^{-m_4\delta}}{\delta\ell_3}
\end{aligned}$$

where $\alpha,\ \beta,\ \gamma, \delta > 0$ and the hyper-parameters $\ell_1,\ \ell_2,\ \ell_3, \ell_4 > 0$ and $m_1,\ m_2,\ m_3,\ m_4 > 0$ are assumed to be known. The non-normalized joint prior density of α, β, γ, and δ becomes, under the previous assumption,

$$\pi_I(\alpha,\ \beta,\ \gamma,\ \delta) \propto \alpha^{\ell_1-1}\ \beta^{\ell_2-1}\ \gamma^{\ell_3-1}\ \delta^{\ell_4-1}\ e^{-(m_1\alpha+m_2\beta+m_3\gamma+m_4\delta)}. \tag{26}$$

In Bayesian analysis, the selection of symmetric and asymmetric loss functions is essential. The SEL function, $L(.)$, which is defined as follows, is the most frequently employed symmetric loss function in this study for estimating the considered unknown values.

$$L(\eta, \hat{\eta}) = (\hat{\eta} - \eta)^2, \tag{27}$$

where $\hat{\eta}$ is a projection of η. The posterior mean of η provides the objective estimate $\hat{\eta}$ under the condition of (27). However, it is simple to implement any additional loss function.

According to Calabria and Pulcini [41], the entropy loss function (ELF) is an acceptable asymmetric loss function. The form's entropy loss function is thought of as

$$L_E(\eta, \hat{\eta}) \propto \left(\frac{\hat{\eta}}{\eta}\right)^b - b\ \ln\left(\frac{\hat{\eta}}{\eta}\right) - 1, \tag{28}$$

whose minimum occurs at $\hat{\eta} = \eta$. Then the Bayes estimator of η under the entropy loss function is

$$\hat{\eta} = \left[E_\eta(\eta)^{-b}\right]^{\frac{-1}{b}}. \tag{29}$$

5.2. Posterior Analysis

The joint posterior density function is given by

$$\pi_p(\alpha,\ \beta,\ \gamma,\ \delta|\underline{\mathbf{x}}) = \frac{1}{R}\pi_I(\alpha,\ \beta,\ \gamma,\ \delta)L(\alpha,\ \beta,\ \gamma,\ \delta|\underline{\mathbf{x}}), \tag{30}$$

where $R = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \pi_p(\alpha,\ \beta,\ \gamma,\ \delta|\underline{\mathbf{x}})d\alpha\ d\beta\ d\gamma\ d\delta$ is a normalizing constant.

Substituting (26) and (20) into (30), the joint posterior of $\alpha,\ \beta,$ and γ is given by

$$\pi_p(\alpha,\ \beta,\ \gamma|\underline{\mathbf{x}})\frac{\mathcal{Z}}{R}\prod_{i=1}^{n}\alpha^{-(1-S_i^\beta(\delta))^\gamma}(1 - S_i(\delta))^{\frac{\delta^2-1}{\delta}}S_i^{\beta-1}(\delta)(1 - S_i^\beta(\delta)^{\gamma-1}e^{-(m_1\alpha+m_2\beta+m_3\gamma+m_4\delta)}, \tag{31}$$

where $\mathcal{Z} = \alpha^{n+\ell_1-1}\beta^{n+\ell_2-1}\gamma^{n+\ell_3-1}\delta^{n+\ell_4-1}\left(\log(\alpha)\frac{(1+\delta)}{(\alpha-1)}\right)^n$ and $S_i(\delta)) = (1 - (\frac{1-x_i}{1+\delta x_i})^\delta)$.

Since it is obvious that sampling from this joint posterior distribution is difficult, this study used MCMC techniques, namely the Metropolis Hasting methodology. The computation of the conditional posterior was obtained as follows in order to execute this algorithm:

$$\pi_p(\alpha|\ \beta,\ \gamma,\ \delta,\underline{\mathbf{x}}) \propto \prod_{i=1}^{n} \alpha^{-(1-S_i^{\beta}(\delta))^{\gamma}} e^{-m_1\alpha}, \tag{32}$$

$$\pi_p(\beta|\ \alpha,\ \gamma,\ \delta,\underline{\mathbf{x}}) \propto \prod_{i=1}^{n} \alpha^{-(1-S_i^{\beta}(\delta))^{\gamma}} S_i^{\beta-1}(\delta)(1-S_i^{\beta}(\delta)^{\gamma-1} e^{-m_2\beta}, \tag{33}$$

$$\pi_p(\gamma|\ \alpha,\ \beta,\ \delta,\underline{\mathbf{x}}) \propto \prod_{i=1}^{n} \alpha^{-(1-S_i^{\beta}(\delta))^{\gamma}} (1-S_i^{\beta}(\delta)^{\gamma-1} e^{-m_3\gamma}, \tag{34}$$

$$\pi_p(\delta|\ \alpha,\ \beta,\ \gamma,\underline{\mathbf{x}}) \propto \prod_{i=1}^{n} \alpha^{-(1-S_i^{\beta}(\delta))^{\gamma}} (1-S_i(\delta))^{\frac{\delta^2-1}{\delta}} S_i^{\beta-1}(\delta)(1-S_i^{\beta}(\delta)^{\gamma-1} e^{-m_4\delta}. \tag{35}$$

From (32)–(35), that no known distribution can be analytically reduced to conditional posterior distributions of parameters is obvious. As a result, the Metropolis Hasting algorithm employs normal proposal distributions.

6. Simulation

A Monte Carlo simulation was run using different sample sizes $(n = 30, 75, 150)$ and different choices for the true values of the model parameters $(\alpha, \beta, \gamma, \delta)$, as follows.

For Table 1, Case I: $(0.5, 0.6, 0.7, 0.6)$ and $(0.5, 0.6, 0.7, 2)$ and Case II: $(0.5, 0.6, 2, 0.6)$ and $(0.5, 0.6, 2, 2)$ to assess the performance of the MLE and Bayesian estimates of the AP-Kum-MSBL-II distribution. The steps that make up the simulation technique are as follows:

- Establish the sample size and the beginning parameter values.
- Create an n-sized random sample from the AP-Kum-MSBL-II distribution.
- For MLE, the Newton–Raphson iterative approach is used to solve non-linear equations using the "maxLik" package in the R program. For Bayesian estimation, MCMC is used to solve the complex integration using the "coda" package in the R program.
- Calculate the average estimates, along with their mean squared errors (MSE), relative biases (RB), length of asymptotic confidence intervals (LACI), and coverage probability (CP) with 95%.
- Perform the above two steps 5000 times.

The average RB estimates of $\alpha, \beta, \gamma,$ and δ (say Ω) are provided by

$$RB(\Omega_j) = \frac{1}{L} \sum_{i=1}^{L} \frac{\hat{\Omega}_j^{(i)} - \Omega_j}{\Omega_j}.$$

The average MSE estimates of Ω are provided by

$$MSE(\Omega_j) = \frac{1}{L} \sum_{i=1}^{L} \left(\hat{\Omega}_j^{(i)} - \Omega_j\right)^2.$$

On the other hand, the length of confidence intervals (LCI) and coverage percentages (CP) of the interval estimations of Ω were used to compare them:

$$LCI(\Omega_j) = Upper(\Omega_j) - Lower(\Omega_j),$$

$$CP(\Omega_j) = \frac{1}{L} \sum_{i=1}^{L} \begin{cases} 1 & Lower(\Omega_j) < \hat{\Omega}_j^{(i)} < Upper(\Omega_j) \\ 0 & otherwise \end{cases}.$$

Using Tables 1 and 2, it was possible to make the following useful findings regarding the lowest RB, MSE, and LCL values as well as the greatest CP values:

- The provided estimates of $\alpha, \beta, \gamma,$ and δ offer good performance, which is the key general point.
- All estimates perform satisfactorily as n increases.
- Based on the gamma information, the Bayes estimates of $\alpha, \beta, \gamma,$ and δ behaved more predictably than the MLE estimate. Regarding the HPD credible intervals, the same statement might be made.
- When δ increases, the measures of α, γ, and δ increase, and the measure of β decreases.
- In some cases, when γ increases, the measures of all parameters decrease.

Table 1. MLE and Bayesian estimations with different loss functions: Case I.

$\alpha = 0.5, \beta = 0.6$				MLE				SELF			ELF b = 0.5			ELF b = 1.5		
γ	δ	n		RB	MSE	LACI	CP	RB	MSE	LCCI	RB	MSE	LCCI	RB	MSE	LCCI
1	0.6	30	α	0.7811	0.6756	2.8365	95.60%	0.2437	0.1681	1.2929	0.1757	0.1387	1.2184	0.0536	0.0975	1.0990
			β	0.2140	0.0776	0.9696	95.30%	0.1570	0.0326	0.4828	0.1422	0.0284	0.4680	0.1142	0.0219	0.4271
			γ	0.0056	0.1398	1.4665	96.10%	0.2367	0.1305	1.3576	0.1734	0.1828	1.4589	0.0657	0.1222	1.2758
			δ	0.5052	0.3079	1.8228	94.40%	0.2934	0.2047	1.4234	0.2325	0.1668	1.3273	0.1235	0.1129	1.1667
		75	α	0.7087	0.4559	2.2544	94.40%	0.2648	0.1086	1.0929	0.2136	0.0908	1.0358	0.1204	0.0654	0.9570
			β	0.1123	0.0209	0.5019	95.60%	0.0679	0.0048	0.1873	0.0642	0.0044	0.1820	0.0570	0.0039	0.1755
			γ	−0.0153	0.0714	1.0473	96.50%	0.0197	0.0716	1.0481	0.0494	0.0714	0.9367	0.0167	0.0610	0.9168
			δ	0.3854	0.1639	1.3036	95.70%	0.2468	0.1487	1.2876	0.2005	0.1245	1.1896	0.1173	0.0888	1.0586
		150	α	0.4254	0.1970	1.5279	94.80%	0.2593	0.0753	0.8429	0.2263	0.0643	0.7983	0.1641	0.0466	0.7087
			β	0.0982	0.0103	0.3247	96.00%	0.0366	0.0013	0.0875	0.0354	0.0012	0.0873	0.0330	0.0011	0.0869
			γ	−0.0512	0.0371	0.7420	95.60%	0.0152	0.0351	0.6914	0.0412	0.0698	0.6808	0.0106	0.0305	0.6091
			δ	0.3195	0.0753	0.7697	94.50%	0.2250	0.0611	0.6110	0.1903	0.0698	0.7059	0.1265	0.0726	0.5947
	2	30	α	0.9482	1.0988	3.6666	94.20%	0.2485	0.1859	1.4234	0.1812	0.1528	1.3329	0.0613	0.1070	1.1677
			β	0.1647	0.0474	0.7605	95.70%	0.1284	0.0235	0.4659	0.1154	0.0208	0.4487	0.0905	0.0165	0.4218
			γ	0.6276	0.4325	1.9195	95.80%	0.2243	0.2721	1.7664	0.1549	0.2170	1.6261	0.0382	0.1478	1.3700
			δ	−0.1502	0.3776	2.1025	97.10%	0.1602	0.2826	2.0041	0.1104	0.2072	1.8683	−0.0802	0.2609	2.0080
		75	α	0.7268	0.5556	2.5524	94.30%	0.2590	0.1186	1.1939	0.2074	0.0997	1.1327	0.1128	0.0723	1.0004
			β	0.1047	0.0148	0.4081	95.60%	0.0580	0.0040	0.1815	0.0546	0.0037	0.1784	0.0480	0.0033	0.1761
			γ	0.5589	0.2474	1.2044	95.20%	0.2378	0.2096	1.1495	0.1837	0.1705	1.0405	0.0883	0.1162	1.0449
			δ	−0.1713	0.2459	1.4058	95.30%	0.1505	0.2016	1.2947	0.0524	0.1654	1.1692	−0.0820	0.2042	0.9142
		150	α	0.3970	0.2328	1.7247	94.40%	0.2444	0.0657	0.8430	0.2123	0.0567	0.8074	0.1517	0.0428	0.7439
			β	0.1037	0.0090	0.2814	95.20%	0.0356	0.0011	0.0928	0.0343	0.0011	0.0914	0.0317	0.0010	0.0899
			γ	0.5051	0.1875	0.9805	96.00%	0.2207	0.1367	0.9290	0.1810	0.1151	0.8214	0.1097	0.0839	0.8077
			δ	−0.1752	0.1961	1.0620	95.70%	0.1053	0.1624	1.0083	0.0357	0.1425	0.9488	−0.0654	0.1833	0.9106

Table 1. *Cont.*

$\alpha = 0.5, \beta = 0.6$				MLE				SELF			ELF b = 0.5			ELF b = 1.5		
γ	δ	n		RB	MSE	LACI	CP	RB	MSE	LCCI	RB	MSE	LCCI	RB	MSE	LCCI
2	0.6	30	α	0.6886	0.6367	2.8232	95.60%	0.1480	0.1222	1.2245	0.0899	0.1012	1.1488	−0.0147	0.0740	1.0182
			β	0.1218	0.0244	0.5416	95.80%	0.0858	0.0092	0.2979	0.0789	0.0085	0.2894	0.0654	0.0072	0.2766
			γ	0.0719	0.0824	0.9747	94.30%	0.0685	0.0800	0.8717	0.0811	0.0722	0.8715	−0.0986	0.0672	0.8664
			δ	0.1780	0.0563	0.8314	96.40%	0.1586	0.0421	0.8005	0.1200	0.0418	0.7468	0.0946	0.0313	0.7235
		75	α	0.4857	0.3369	2.0676	94.90%	0.2359	0.1114	1.1361	0.1903	0.0950	1.0823	0.1050	0.0695	0.9629
			β	0.0857	0.0105	0.3475	95.30%	0.0421	0.0020	0.1326	0.0400	0.0019	0.1309	0.0360	0.0017	0.1299
			γ	0.0582	0.0365	0.5937	94.70%	0.0623	0.0275	0.4791	0.0981	0.0269	0.4173	−0.0529	0.0251	0.4076
			δ	0.1425	0.0224	0.4813	94.80%	0.1422	0.0204	0.4342	0.1173	0.0401	0.2006	0.0945	0.0209	0.1961
		150	α	0.3035	0.1412	1.3480	95.70%	0.2096	0.0473	0.6753	0.1825	0.0409	0.6393	0.1311	0.0307	0.5983
			β	0.0746	0.0056	0.2338	94.60%	0.0255	0.0045	0.0654	0.0247	0.0035	0.0655	0.0232	0.0025	0.0650
			γ	0.0510	0.0168	0.3131	94.90%	0.1685	0.0120	0.2631	0.0907	0.0120	0.2591	−0.0213	0.0108	0.2438
			δ	0.1272	0.0131	0.3344	95.80%	0.1256	0.0129	0.3200	0.1285	0.0118	0.3096	0.0776	0.0107	0.2905
	2	30	α	0.9757	1.3183	4.0764	93.50%	0.2025	0.1367	1.2301	0.1439	0.1127	1.1566	0.0390	0.0810	1.0251
			β	0.1016	0.0327	0.6679	93.30%	0.0678	0.0078	0.3170	0.0610	0.0073	0.3124	0.0477	0.0063	0.3035
			γ	0.1015	0.2139	1.6298	94.30%	0.1265	0.2059	1.2348	0.0782	0.1939	1.4715	−0.1036	0.1802	0.9110
			δ	0.1139	0.3362	2.0915	96.00%	0.2801	0.2004	2.0056	0.1160	1.1156	2.0065	−0.0853	0.2631	1.7952
		75	α	0.8219	0.9628	3.4946	94.90%	0.1884	0.0901	1.0616	0.1468	0.0779	1.0300	0.0702	0.0598	0.9405
			β	0.0529	0.0134	0.4368	94.80%	0.0319	0.0018	0.1504	0.0299	0.0017	0.1496	0.0260	0.0016	0.1464
			γ	0.0836	0.1048	1.0871	94.60%	0.0819	0.0917	1.0201	0.0662	0.1042	0.9643	−0.1005	0.1046	0.9501
			δ	0.0818	0.1576	1.4187	96.90%	0.0720	0.1208	1.0075	0.0980	0.7726	0.9215	−0.0460	0.1469	0.8250
		150	α	0.7369	0.6090	2.6980	94.60%	0.2138	0.0601	0.8036	0.1871	0.0528	0.7723	0.1364	0.0408	0.7198
			β	0.0410	0.0084	0.3454	94.50%	0.0196	0.0007	0.0888	0.0188	0.0007	0.0889	0.0172	0.0006	0.0881
			γ	0.0987	0.0897	0.8835	93.00%	0.1251	0.0815	0.7178	0.0424	0.0955	0.6826	−0.0732	0.0791	0.5384
			δ	0.0671	0.0742	0.9294	95.80%	0.1588	0.0367	0.8464	0.0901	0.0245	0.8500	−0.0107	0.0278	0.7054

Table 2. MLE and Bayesian estimates with different loss functions: Case II.

$\alpha = 2, \beta = 0.6$				MLE				SELF			ELF b = 0.5			ELF b = 1.5		
γ	δ	n		RB	MSE	LACI	CP	RB	MSE	LCCI	RB	MSE	LCCI	RB	MSE	LCCI
0.7	0.6	30	α	−0.0493	0.8012	3.4910	0.9640	−0.0049	0.0749	1.0592	−0.0093	0.0752	1.0647	−0.0181	0.0764	1.0739
			β	0.4573	0.2465	1.6238	0.9440	0.0955	0.0254	0.5418	0.0872	0.0242	0.5363	0.0708	0.0219	0.5190
			γ	−0.0615	0.1496	1.5081	0.9500	0.0260	0.0333	0.6724	0.0185	0.0324	0.6630	0.0038	0.0309	0.6520
			δ	0.6279	0.3092	1.8008	0.9460	0.1160	0.0302	0.5875	0.1063	0.0287	0.5782	0.0871	0.0259	0.5736
		75	α	−0.0733	0.4472	2.5603	0.9300	−0.0025	0.0260	0.6098	−0.0040	0.0261	0.6147	−0.0071	0.0264	0.6176
			β	0.2635	0.0601	0.7351	0.9540	0.0577	0.0075	0.2965	0.0552	0.0073	0.2957	0.0501	0.0069	0.2909
			γ	−0.1438	0.0791	1.0308	0.9500	0.0284	0.0138	0.4370	0.0255	0.0136	0.4329	0.0199	0.0131	0.4315
			δ	0.5138	0.1652	1.2357	0.9520	0.0480	0.0093	0.3599	0.0448	0.0091	0.3583	0.0385	0.0088	0.3562
		150	α	−0.0451	0.2047	1.7397	0.9380	−0.0020	0.0118	0.4178	−0.0027	0.0118	0.4174	−0.0040	0.0118	0.4164
			β	0.2045	0.0277	0.4406	0.9560	0.0407	0.0032	0.1948	0.0396	0.0032	0.1943	0.0373	0.0031	0.1932
			γ	−0.1196	0.0426	0.7403	0.9540	0.0183	0.0067	0.3098	0.0170	0.0067	0.3092	0.0144	0.0065	0.3064
			δ	0.3720	0.0805	0.8409	0.9340	0.0409	0.0052	0.2698	0.0394	0.0052	0.2698	0.0363	0.0050	0.2704
	2	30	α	−0.0765	1.3555	4.5289	0.9720	−0.0031	0.0703	1.0217	−0.0074	0.0712	1.0309	−0.0160	0.0734	1.0239
			β	0.3770	0.1462	1.2097	0.9560	0.0687	0.0179	0.4800	0.0613	0.0171	0.4743	0.0467	0.0157	0.4597
			γ	0.7982	0.7835	2.6939	0.9560	0.0601	0.0263	0.5891	0.0530	0.0253	0.5823	0.0390	0.0234	0.5747
			δ	−0.1810	0.5246	3.2677	0.9620	−0.0026	0.0590	0.9371	−0.0064	0.0589	0.9341	−0.0140	0.0591	0.9300
		75	α	−0.1347	0.8157	3.3827	0.9800	0.0026	0.0239	0.6057	0.0010	0.0238	0.6089	−0.0022	0.0239	0.6052
			β	0.2439	0.0393	0.5240	0.9420	0.0367	0.0052	0.2744	0.0345	0.0051	0.2735	0.0301	0.0049	0.2736
			γ	0.5773	0.3456	1.6753	0.9640	0.0390	0.0102	0.3746	0.0364	0.0100	0.3717	0.0310	0.0096	0.3650
			δ	−0.1974	0.4480	2.1210	0.9500	0.0055	0.0224	0.5991	0.0041	0.0222	0.6010	0.0012	0.0219	0.5941
		150	α	−0.1265	0.5541	2.7469	0.9740	−0.0016	0.0119	0.4369	−0.0024	0.0119	0.4366	−0.0038	0.0119	0.4317
			β	0.2078	0.0249	0.3792	0.9440	0.0337	0.0026	0.1764	0.0326	0.0025	0.1763	0.0304	0.0024	0.1742
			γ	0.6517	0.3496	1.4761	0.9760	0.0299	0.0043	0.2320	0.0288	0.0042	0.2317	0.0266	0.0041	0.2310
			δ	−0.2462	0.4458	1.7692	0.9480	0.0056	0.0092	0.3626	0.0050	0.0091	0.3611	0.0039	0.0091	0.3596

Table 2. *Cont.*

$\alpha = 2, \beta = 0.6$				MLE				SELF			ELF b = 0.5			ELF b = 1.5		
γ	δ	n		RB	MSE	LACI	CP	RB	MSE	LCCI	RB	MSE	LCCI	RB	MSE	LCCI
2	0.6	30	α	−0.1616	0.9356	3.5774	0.9740	0.0014	0.0704	1.0229	−0.0029	0.0704	1.0368	−0.0114	0.0711	1.0419
			β	0.2311	0.0577	0.7696	0.9580	0.0450	0.0127	0.4090	0.0399	0.0122	0.4064	0.0300	0.0113	0.4018
			γ	0.0443	0.2187	1.8019	0.9320	−0.0052	0.0619	0.9362	−0.0094	0.0623	0.9411	−0.0175	0.0636	0.9446
			δ	0.1588	0.0899	1.1152	0.9600	0.0748	0.0234	0.5405	0.0668	0.0225	0.5376	0.0507	0.0209	0.5275
		75	α	−0.1404	0.6557	2.9802	0.9700	0.0009	0.0242	0.6022	−0.0007	0.0241	0.6030	−0.0039	0.0241	0.6004
			β	0.1849	0.0265	0.4679	0.9720	0.0349	0.0042	0.2408	0.0333	0.0041	0.2392	0.0299	0.0040	0.2361
			γ	0.0478	0.0934	1.1388	0.9400	0.0054	0.0219	0.5693	0.0041	0.0219	0.5692	0.0014	0.0217	0.5656
			δ	0.0950	0.0224	0.5431	0.9680	0.0469	0.0102	0.3762	0.0442	0.0100	0.3746	0.0388	0.0097	0.3690
		150	α	−0.0966	0.2981	2.0037	0.9380	0.0009	0.0115	0.4054	0.0001	0.0115	0.4060	−0.0013	0.0116	0.4066
			β	0.1432	0.0121	0.2707	0.9520	0.0261	0.0018	0.1504	0.0253	0.0018	0.1502	0.0237	0.0018	0.1495
			γ	0.0457	0.0350	0.6401	0.9380	0.0019	0.0109	0.4158	0.0012	0.0109	0.4154	−0.0002	0.0109	0.4139
			δ	0.0956	0.0096	0.3112	0.9460	0.0437	0.0052	0.2543	0.0424	0.0051	0.2530	0.0397	0.0050	0.2504
	2	30	α	−0.1680	1.5486	4.7016	0.9700	−0.0147	0.0639	0.9718	−0.0187	0.0651	0.9721	−0.0264	0.0679	0.9810
			β	0.2751	0.0588	0.6975	0.9600	0.0447	0.0097	0.3519	0.0404	0.0094	0.3506	0.0320	0.0088	0.3469
			γ	0.1527	0.2576	1.5910	0.9380	0.0096	0.0615	0.9515	0.0054	0.0608	0.9486	−0.0029	0.0601	0.9341
			δ	0.0265	0.1698	1.6035	0.9660	0.0170	0.0503	0.8356	0.0133	0.0494	0.8246	0.0060	0.0482	0.8133
		75	α	−0.1679	0.8605	3.3932	0.9620	−0.0082	0.0232	0.5743	−0.0097	0.0234	0.5729	−0.0127	0.0237	0.5790
			β	0.1923	0.0247	0.4193	0.9560	0.0252	0.0029	0.1980	0.0237	0.0028	0.1978	0.0207	0.0027	0.1961
			γ	0.1314	0.1225	0.9072	0.9480	0.0063	0.0234	0.5788	0.0048	0.0233	0.5804	0.0018	0.0230	0.5860
			δ	0.0026	0.0531	0.9043	0.9420	0.0079	0.0207	0.5543	0.0066	0.0205	0.5597	0.0039	0.0203	0.5597
		150	α	−0.1903	0.6417	2.7660	0.9660	0.0028	0.0113	0.4095	0.0021	0.0113	0.4075	0.0007	0.0113	0.4067
			β	0.1708	0.0159	0.2890	0.9500	0.0197	0.0013	0.1287	0.0190	0.0013	0.1283	0.0177	0.0012	0.1275
			γ	0.1843	0.2278	1.1902	0.9840	0.0041	0.0089	0.3600	0.0034	0.0088	0.3601	0.0022	0.0088	0.3596
			δ	−0.0425	0.0795	1.0547	0.9920	0.0072	0.0088	0.3629	0.0065	0.0088	0.3622	0.0053	0.0087	0.3613

7. Applications

In this section, three real data sets are discussed to check the performance of the AP-Kum-MSBL-II distribution.

7.1. The COVID-19 Application

The COVID-19 data in question are from France and span 108 days from 1 March to 16 June 2021. These data have been discussed by [42], who use a transmuted generalization of the Lomax distribution.

The data are as follows:

"0.0023, 0.0023, 0.0023, 0.0046, 0.0065, 0.0067, 0.0069, 0.0069, 0.0091, 0.0093, 0.0093, 0.0093, 0.0111, 0.0115, 0.0116, 0.0116, 0.0119, 0.0133, 0.0136, 0.0138, 0.0138, 0.0159, 0.0161, 0.0162, 0.0162, 0.0162, 0.0163, 0.0180, 0.0187, 0.0202, 0.0207, 0.0208, 0.0225, 0.0230, 0.0230, 0.0239, 0.0245, 0.0251, 0.0255, 0.0255, 0.0271, 0.0275, 0.0295, 0.0297, 0.0300, 0.0302, 0.0312, 0.0314, 0.0326, 0.0346, 0.0349, 0.0350, 0.0355, 0.0379, 0.0384, 0.0394, 0.0394, 0.0412, 0.0419, 0.0425, 0.0461, 0.0464, 0.0468, 0.0471, 0.0495, 0.0501, 0.0521, 0.0571, 0.0588, 0.0597, 0.0628, 0.0679, 0.0685, 0.0715, 0.0766, 0.0780, 0.0942, 0.0960, 0.0988, 0.1223, 0.1343, 0.1781"

Jäntschi [29] discussed nine measures to select the best distribution model. In this paper, we discussed four important measures: Cramér von Mises, Anderson–Darling, Kolmogorov–Smirnov, and P-value for the AP-Kum-MSBL-II distribution. The different measures discussed include Akaike information, Bayesian information, consistent Akaike information, Hannan–Quinn information, Cramér von Mises, and Anderson–Darling, which are denoted M1, M2, M3, M4, M5, and M6, respectively.

$$\begin{cases} M1 = 2k - 2l(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}) \\ M2 = k\ \ln(n) - 2l(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}) \\ M3 = M1 + \dfrac{2k^2 + 2k}{n - k - 1} \\ m4 = 2k\ \ln(\ln(n)) - 2l(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}) \end{cases}$$

For more information about criterion measures, see [43,44].

Table 3 demonstrates that the AP-Kum-MSBL-II distribution for COVID-19 data has the lowest values of M1, M2, M3, M4, M5, M6, and Kolmogorov–Smirnov distance (KSD), as well as the highest p-value for the Kolmogorov–Smirnov test (PVKS), among all fitted competitive models for the COVID-19 data. As a result, for the provided data sets, the AP-Kum-MSBL-II distribution offers a better fit. Table 4 discusses points and intervals estimated with MLE and Bayesian methods for the COVID-19 data. From the result in Table 4, we note that the Bayesian estimation is better than MLE to estimate the parameters of the AP-Kum-MSBL-II distribution for the COVID-19 data.

Table 3. MLE for each model with different measures.

		Estimates	SE	KSD	PVKS	M1	M2	M3	M4	M5	M6
New	α	0.122445	0.045124	0.046364	0.994551	−385.087	−375.46	−384.567	−381.222	0.012091	0.105359
	β	1.764043	0.561574								
	γ	2.679865	1.184638								
	δ	3.152508	1.170303								
TIIPTLIE	α	3718.053	2750.314	0.093446	0.471122	−377.456	−370.235	−377.148	−374.557	0.124102	0.899006
	θ	0.006539	0.001101								
	β	0.009745	0.003613								
K	a	1.492104	0.254537	0.093648	0.468333	−378.694	−373.881	−378.542	−376.762	0.093612	0.632138
	b	124.8496	97.50975								
Beta	a	1.510428	0.214871	0.051362	0.982024	−384.662	−374.849	−383.51	−380.729	0.039691	0.293027
	b	40.65778	6.793717								
UW	α	0.002432	0.000319	0.073747	0.763933	−381.604	−376.79	−381.452	−379.671	0.098829	0.707997
	β	4.315844	0.109935								

Table 3. *Cont.*

		Estimates	SE	KSD	PVKS	M1	M2	M3	M4	M5	M6
MOK	α β θ	0.005264 1.950737 6.387009	0.002879 0.171974 5.103846	0.045966	0.995133	−384.599	−374.379	−384.292	−380.701	0.037555	0.25511
TLWL	β α θ b	27.42895 0.286029 13.14184 2.826471	38.93698 0.288718 19.38104 1.784417	0.049505	0.98794	−384.751	−375.124	−384.231	−380.886	0.017215	0.149338
KMK	a b	1.368003 62.0471	0.112325 22.1059	0.053089	0.974947	−386.466	−381.652	−386.314	−384.533	0.038451	0.287544
UG	α β	0.018102 0.975855	0.007092 0.080349	0.108193	0.29239	−363.699	−358.885	−363.547	−361.766	0.294083	1.982711
UEHL	δ ϕ	1.251462 29.38375	0.102967 9.319743	0.057433	0.94963	−385.054	−380.241	−384.902	−383.122	0.05551	0.393134

Table 4. MLE and Bayesian estimates for COVID-19 data.

	MLE				Bayesian			
	Estimates	**SE**	**Lower**	**Upper**	**Estimates**	**SE**	**Lower**	**Upper**
α	0.1224	0.0451	0.0340	1.0069	0.0943	0.0310	0.0001	0.3460
β	1.7640	0.5616	0.6634	2.8647	1.7985	0.3331	1.2124	2.4505
δ	2.6799	1.1846	0.3580	16.7618	2.5377	0.9327	0.5103	5.3969
γ	3.1525	1.1703	0.8587	13.2863	3.1771	0.9196	1.6023	5.1120

7.2. *Determinants of Economic Development Application*

The trade share data set, the second data set, considers the values of the trade share variable used in the renowned "Determinants of Economic Development Data" together with factors that may be associated with growth. The growth rates of up to 61 different countries are considered. The information is publicly accessible online according to [45].

The trade share data set consists of the following numbers:

"0.140501976, 0.156622976, 0.157703221, 0.160405084, 0.160815045, 0.22145839, 0.299405932, 0.31307286, 0.324612707, 0.324745566, 0.329479247, 0.330021679, 0.337879002, 0.339706242, 0.352317631, 0.358856708, 0.393250912, 0.41760394, 0.425837249, 0.43557933, 0.442142904, 0.444374621, 0.450546652, 0.4557693, 0.46834656, 0.473254889, 0.484600782, 0.488949597, 0.509590268, 0.517664552, 0.527773321, 0.534684658, 0.543337107, 0.544243515, 0.550812602, 0.552722335, 0.56064254, 0.56074965, 0.567130983, 0.575274825, 0.582814276, 0.603035331, 0.605031252, 0.613616884, 0.626079738, 0.639484167, 0.646913528, 0.651203632, 0.681555152, 0.699432909, 0.704819918, 0.729232311, 0.742971599, 0.745497823, 0.779847085, 0.798375845, 0.814710021, 0.822956383, 0.830238342, 0.834204197, 0.979355395".

Table 5 demonstrates that the AP-Kum-MSBL-II distribution for the trade share data has the lowest values of M1, M2, M3, M4, M5, M6, and KSD, as well as the highest PVKS, among all fitted competitive models. Table 6 discusses the points and intervals found using MLE and Bayesian methods for the trade share data. From the result in Table 6, we note that the Bayesian estimation is better than MLE to estimate the parameters of the AP-Kum-MSBL-II distribution for the trade share data.

Table 5. MLE for each model with different measures for trade share data.

		Estimates	SE	KSD	PVKS	M1	M2	M3	M4	M5	M6
New	α	0.0052	0.0026	0.0545	0.9891	−22.2267	−13.7832	−21.5124	−18.9176	0.0321	0.2880
	β	2.7677	0.9986								
	γ	2.4753	1.0920								
	δ	0.4636	0.1510								
TIIPTLIE	α	4429.7791	617.3337	0.1124	0.3949	−11.8840	−5.5514	−11.4630	−9.4022	0.1994	1.4468
	θ	0.5184	0.0783								
	β	0.0309	0.0240								
K	a	2.3308	0.3056	0.0690	0.9137	−21.2503	−13.0285	−21.0434	−18.5957	0.0527	0.4005
	b	2.7646	0.5554								
Beta	a	2.7944	0.4881	0.0618	0.9629	−21.9121	−13.6903	−20.7052	−18.2576	0.0491	0.3864
	b	2.6041	0.4519								
UW	α	1.3395	0.1725	0.0682	0.9208	−21.4872	−13.2654	−21.2803	−18.8326	0.0630	0.5097
	β	1.7346	0.1695								
MOK	α	0.3008	0.3023	0.0582	0.9783	−21.6367	−13.3040	−21.2156	−18.1549	0.0490	0.4139
	β	3.0590	0.6447								
	θ	1.9501	0.9516								
TLWL	β	1.2675	1.7763	0.0552	0.9837	−21.7845	−13.3411	−21.0703	−18.4755	0.0333	0.2896
	α	3.7473	2.2384								
	θ	0.6739	0.4449								
	b	1.0166	1.0642								
KMK	a	2.6118	0.3256	0.0598	0.9721	−21.3240	−13.1022	−21.1171	−18.6694	0.0434	0.3606
	b	2.4206	0.5343								
UG	α	0.6160	0.2657	0.1098	0.4237	−17.7518	−13.5300	−17.5449	−16.0972	0.1585	1.1540
	β	1.0923	0.2469								

Table 6. MLE and Bayesian estimates for trade share data.

	MLE				Bayesian			
	Estimates	**SE**	**Lower**	**Upper**	**Estimates**	**SE**	**Lower**	**Upper**
α	0.0051	0.0026	0.0001	0.0102	0.0051	0.0025	0.0000	0.0119
β	2.7682	0.9986	0.8108	4.7255	3.0349	0.7301	1.6871	4.5326
δ	2.4719	1.0920	0.3315	4.6123	2.1213	1.0418	0.3298	3.9896
γ	0.4640	0.1510	0.1680	0.7599	0.5565	0.1465	0.2359	0.7289

To obtain MLE, we should check that the estimates have maximum log-likelihood. Figures 4–6 confirm that the estimates have maximum log-likelihood with blue points for each data set, respectively. Moreover, one of the main issues with MLEs is that it is frequently impossible to demonstrate their existence and uniqueness. To solve this issue, we suggest using the data sets shown in Figures 7–9 to produce the contour plot of the log-likelihood function concerning the four parameters of the AP-Kum-MSBL-II distribution. In addition, the density and trace plots of α, β, γ, and δ were plotted and are shown in Figures 10–12 using the results produced by both data sets. They provided evidence that the MCMC approach converged well. It is also evident that while the MCMC iterations of β, γ, and δ were symmetric and negatively skewed, α was associated with positive symmetry. Figures 13–15 show graphically the estimated CDF with the empirical CDF, the estimated PDF with a histogram of these data as a symmetry ship, and probability–probability (P-P) plots for the AP-Kum-MSBL-II distributions for both data sets. These figures confirm that the data sets fit the new proposed model.

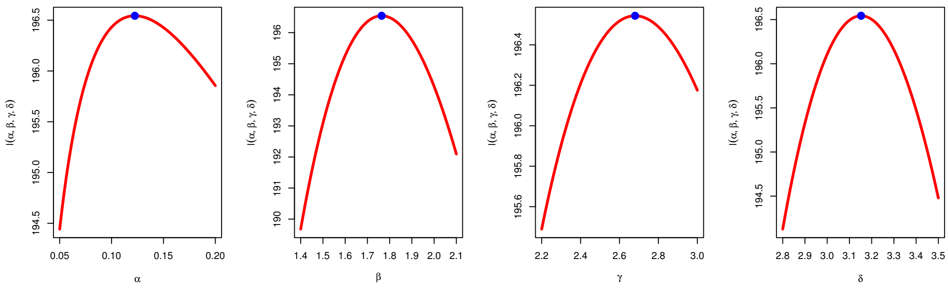

Figure 4. Profile likelihood of α, β, γ, and δ.

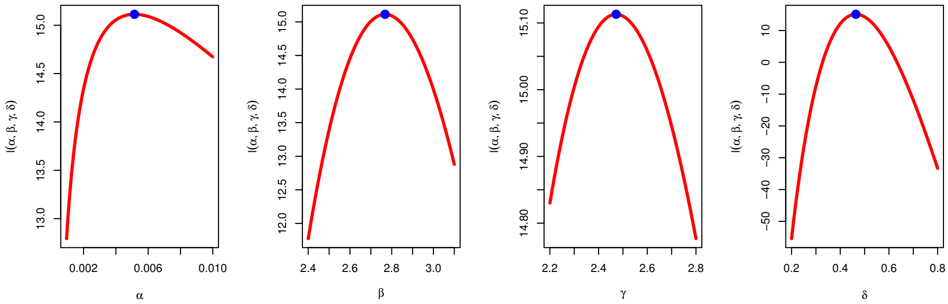

Figure 5. Profile likelihood of α, β, γ, and δ for trade share data.

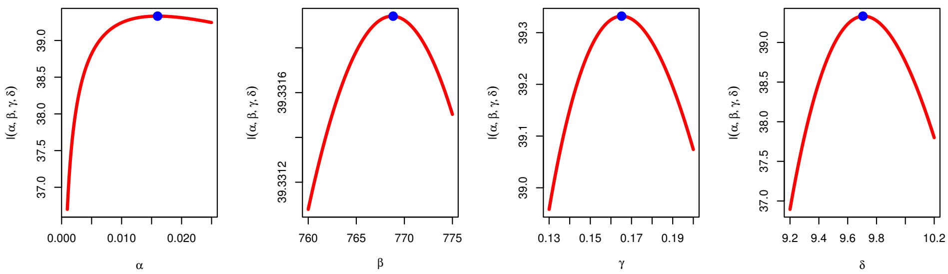

Figure 6. Profile likelihood of α, β, γ, and δ for failure times data.

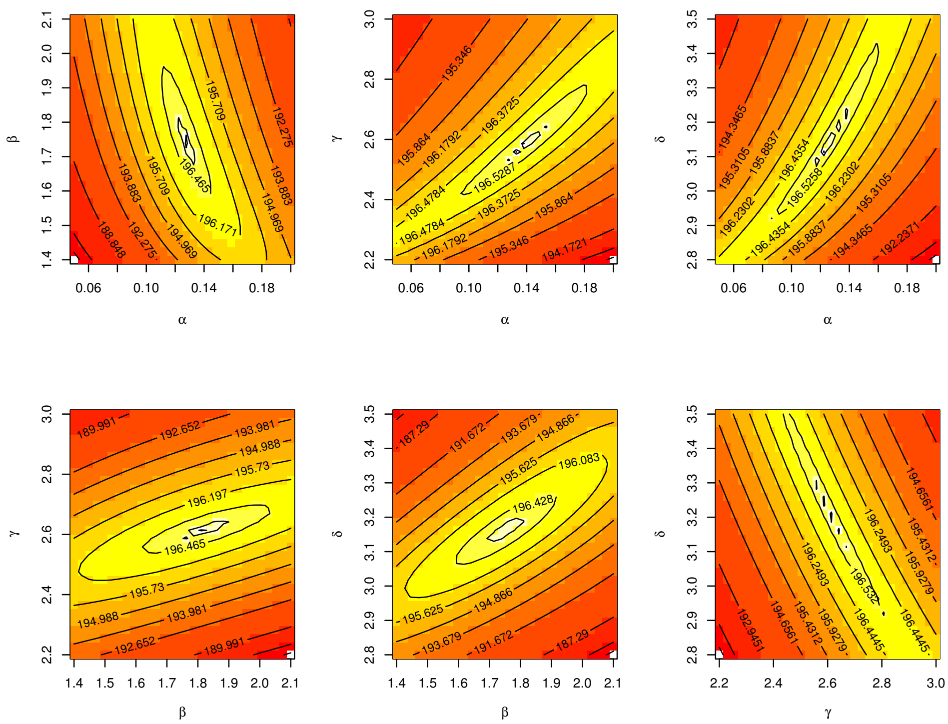

Figure 7. Contour plot of α, β, γ, and δ.

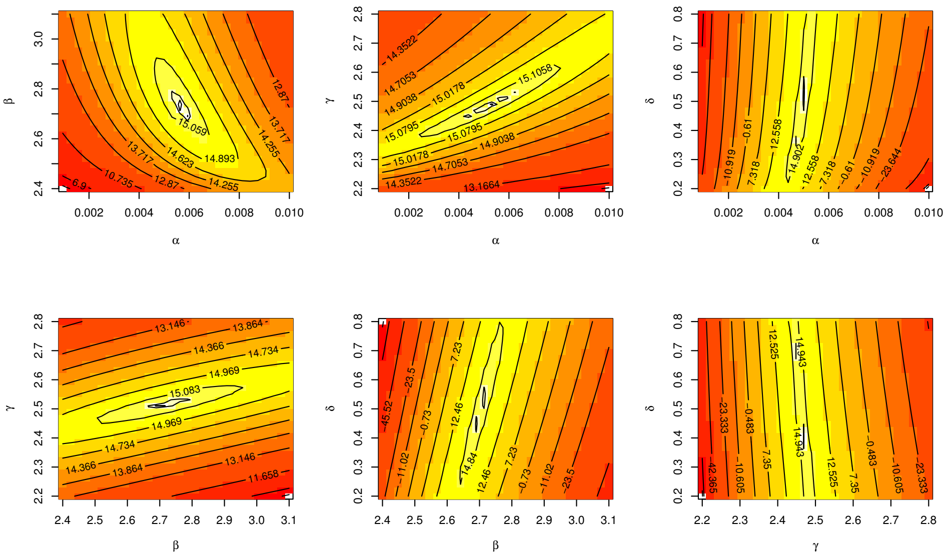

Figure 8. Contour plots of α, β, γ, and δ for trade share data.

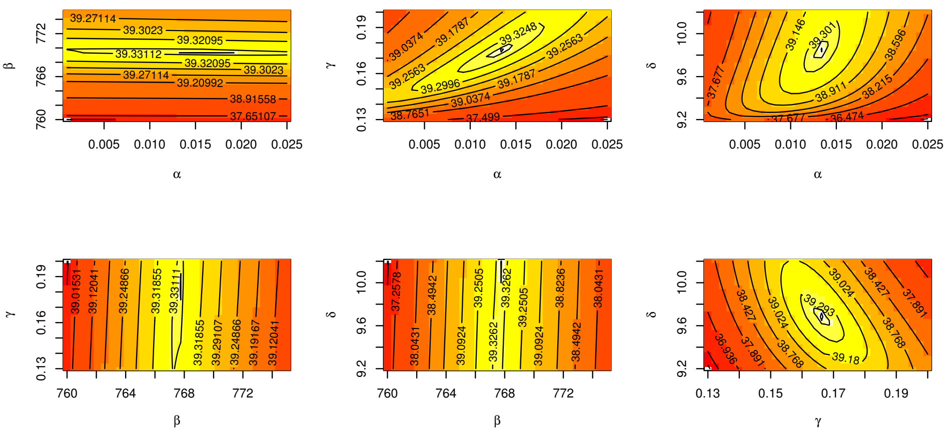

Figure 9. Contour plots of α, β, γ, and δ for failure times data.

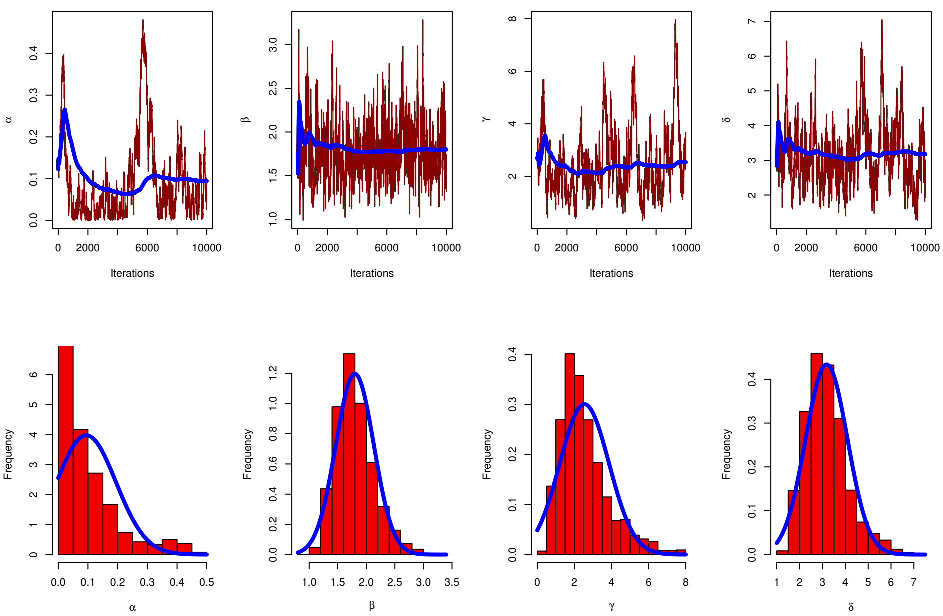

Figure 10. Symmetry and asymmetry MCMC plots of α, β, γ, and δ.

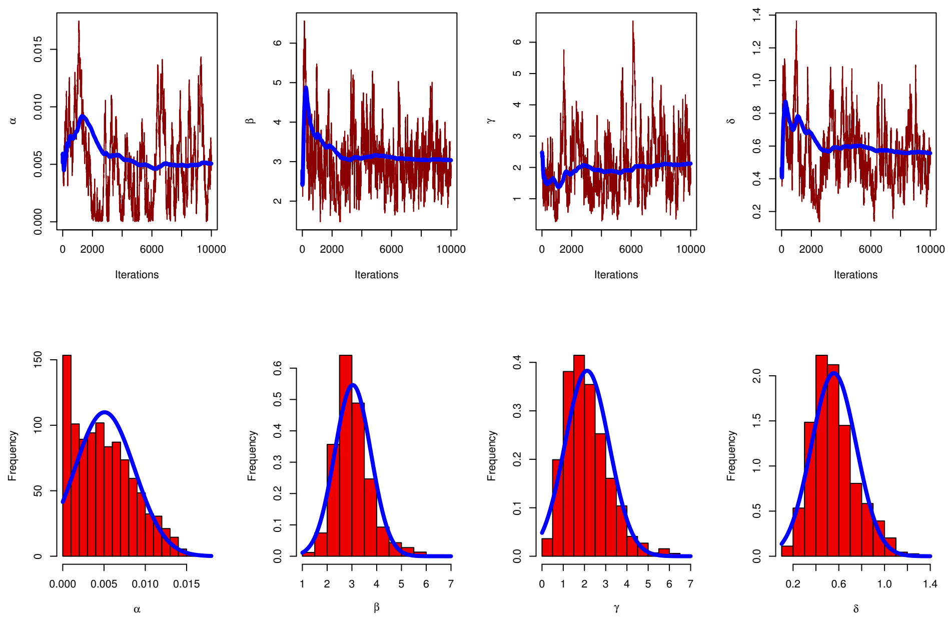

Figure 11. MCMC plots of α, β, γ, and δ for trade share data.

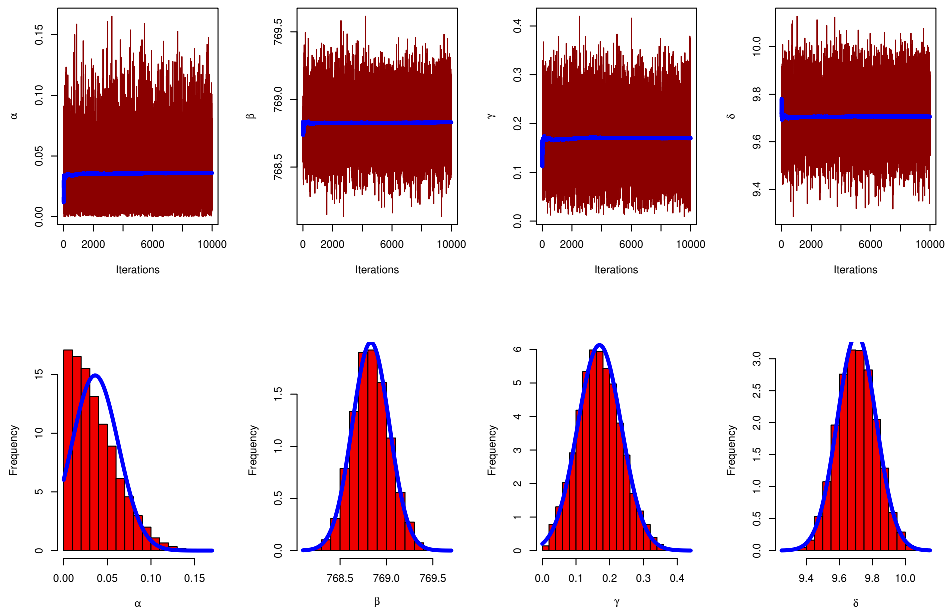

Figure 12. Symmetry and asymmetry MCMC plots of α, β, γ, and δ for failure times data.

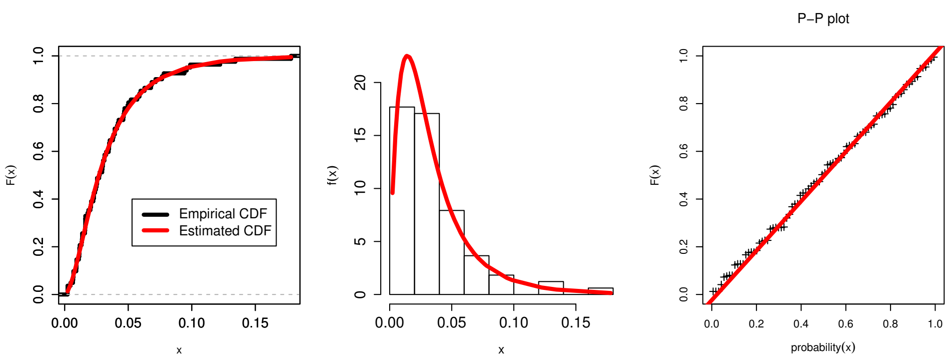

Figure 13. Fitting plots of α, β, γ, and δ.

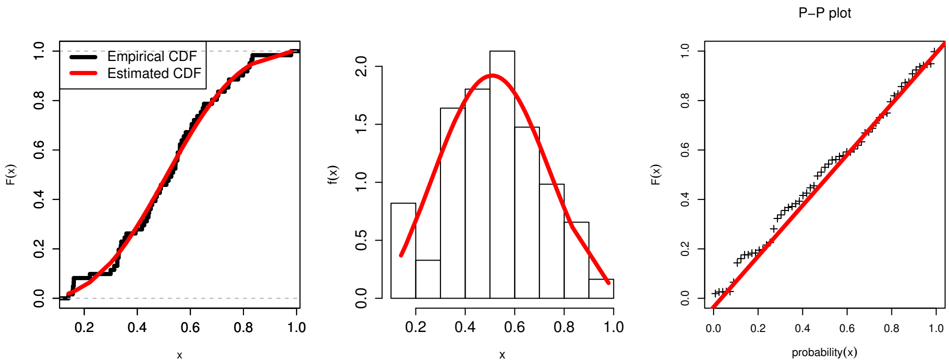

Figure 14. Fitting plots of α, β, γ, and δ for trade share data.

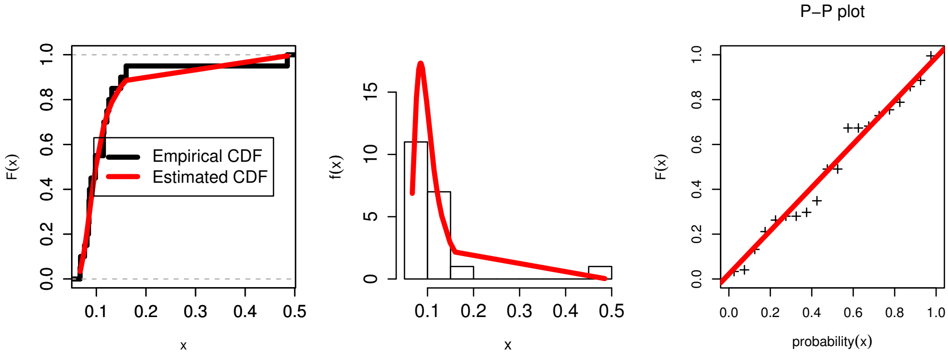

Figure 15. Fitting plots of α, β, γ, and δ for failure times data.

7.3. The Failure Rates Application

In this subsection, we present the analysis of the third real data set from an engineering area as an example of how the suggested estimators can be applied in a real-world setting. Murthy et al. [46] reported this information about the failure rates of 20 mechanical components stated as 0.067, 0.068, 0.076, 0.081, 0.084, 0.085, 0.085, 0.086, 0.089, 0.098, 0.098, 0.114, 0.114, 0.115, 0.121, 0.125, 0.131, 0.149, 0.160, 0.485.

Figure 15 shows graphically the estimated CDF with the empirical CDF, the estimated PDF with a histogram of these failure times data, and probability–probability (P-P) plots for the AP-Kum-MSBL-II distribution for the failure times data. These figures confirm that the failure times data fit the AP-Kum-MSBL-II distribution. Figures 6 and 9 confirm that the estimates have maximum log-likelihood with blue points and unique values of estimates for the failure times data.

Table 7 demonstrates that the AP-Kum-MSBL-II distribution for the failure times data has the lowest values of M1, M2, M3, M4, M5, M6, and KSD, as well as the highest PVKS, among all fitted competitive models.

Table 7. MLE for each model with different measures for failure times data.

		Estimates	SE	KSD	PVKS	M1	M2	M3	M4	M5	M6
New	α	0.0163	0.0456	0.1230	0.9227	−70.6590	−66.6761	−67.9924	−69.8815	0.0407	0.2860
	β	746.8900	702.4795								
	γ	0.1657	0.1257								
	δ	9.6866	0.9798								
TIIPTLIE	α	0.4304	0.5865	0.1465	0.7838	−65.7777	−62.7905	−64.2777	−65.1945	0.1020	0.7769
	θ	0.2217	0.0572								
	β	17.2211	25.4129								
K	a	1.5877	0.2444	0.2627	0.1265	−47.2969	−45.3054	−46.5910	−46.9081	0.4370	2.6508
	b	21.8652	10.2066								

Table 7. *Cont.*

		Estimates	SE	KSD	PVKS	M1	M2	M3	M4	M5	M6
Beta	a b	3.1127 21.8249	0.9368 7.0423	0.2538	0.1521	−51.7626	−49.7711	−51.0567	−51.3738	0.3700	2.3155
UW	α β	3.1127 21.8249	0.9368 7.0423	0.2538	0.1521	−51.7626	−49.7711	−51.0567	−51.3738	0.3700	2.3155
MOK	α β θ	0.0041 3.2326 6.2702	0.0029 0.6036 9.4489	0.1988	0.4080	−61.6321	−58.6449	−60.1321	−61.0490	0.1213	0.9015
TLWL	β α θ b	2029.7292 1.4664 719.4030 0.2248	2150.4535 0.4363 1020.9217 0.0648	0.1340	0.8651	−66.1092	−62.1263	−63.4426	−65.3317	0.0767	0.5985
KMK	a b	1.7994 23.3409	0.2598 11.5510	0.2700	0.1083	−49.2014	−47.2100	−48.4955	−48.8127	0.4015	2.4750
UEHL	δ ϕ	1.6408 12.7893	0.2338 5.6526	0.2615	0.1296	−48.5439	−46.5524	−47.8380	−48.1551	0.4073	2.5031

We provide three real-world applications of COVID-19, trade share, and failure times data that show the AP-Kum-MSBL-II model is superior to all of its rivals in terms of fitting this kind of data set. All estimators obtained have the properties of the only unique estimator and have maximum points for MLE, while the Bayesian estimators had convergence properties.

8. Conclusions

As a novel expansion of the Kum-modified size-biased Lehmann type II distribution, the four-parameter AP-Kum-MSBL-II distribution has been developed. It is based on the alpha power family. The AP-Kum-MSBL-II model has more flexibility than the rival models due to the inclusion of an additional form parameter. We have covered a number of the proposed model's statistical characteristics. The function for stress–strength reliability has also been derived. We provide three real-world applications that show the AP-Kum-MSBL-II model is better than all of its rivals in terms of fitting this kind of data set. Utilizing MLE, the AP-Kum-MSBL-II model's parameters are estimated. Additionally, under the square error loss function, Bayesian estimation is used to estimate the parameters. The performance of the underlying model under various estimation methods is assessed using goodness-of-fit statistics. The three real data sets were applied to compare TIIPTLIE, K, Beta, UW, MOK, TLWL, KMK, and UG distributions.

Author Contributions: Conceptualization, R.S.G., A.M.M., N.A. and A.H.T.; methodology, R.S.G., A.M.M., A.H.T. and E.M.A.; software, E.M.A. and A.H.T.; validation, N.A., R.S.G., A.M.M. and A.H.T.; formal analysis, N.A., R.S.G. and E.M.A.; investigation, R.S.G., A.M.M. and A.H.T.; resources, R.S.G., N.A. and E.M.A.; data curation, R.S.G., A.M.M., N.A., E.M.A. and A.H.T.; writing—original draft preparation, R.S.G., A.M.M., A.H.T., E.M.A. and N.A.; writing—review and editing, E.M.A., R.S.G., A.M.M., N.A. and A.H.T.; funding acquisition N.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data sets are available in this paper.

Acknowledgments: This research is supported by researcher support project number (RSPD2023R548), King Saud University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Alsadat, N.; Hassan, A.S.; Elgarhy, M.; Chesneau, C.; Mohamed, R.E. An Efficient Stress–Strength Reliability Estimate of the Unit Gompertz Distribution Using Ranked Set Sampling. *Symmetry* **2023**, *15*, 1121. [CrossRef]
2. Alghamdi, S.M.; Shrahili, M.; Hassan, A.S.; Gemeay, A.M.; Elbatal, I.; Elgarhy, M. Statistical Inference of the Half Logistic Modified Kies Exponential Model with Modeling to Engineering Data. *Symmetry* **2023**, *15*, 586. [CrossRef]
3. Ghaderinezhad, F.; Ley, C.; Loperfido, N. Bayesian inference for skew-symmetric distributions. *Symmetry* **2020**, *12*, 491. [CrossRef]
4. Ramadan Ahmed, T.; Tolba Ahlam, H.; El-Desouky Beih, S.A. Unit half-logistic geometric distribution and its application in insurance. *Axioms* **2022**, *11*, 676. [CrossRef]
5. Dey, S.; Nassar, M.; Kumar, D. Alpha power transformed inverse Lindley distribution: A distribution with an upside-down bathtub-shaped hazard function. *J. Comput. Appl. Math.* **2019**, *348*, 130–145. [CrossRef]
6. Mazucheli, J.; Menezes, A.F.B.; Ghitany, M.E. The unit-Weibull distribution and associated inference. *J. Appl. Probab. Stat.* **2018**, *13*, 1–22.
7. Mazucheli, J.; Menezes, A.F.B.; Fernandes, L.B.; De Oliveira, R.P.; Ghitany, M.E. The unit-Weibull distribution as an alternative to the Kumaraswamy distribution for the modeling of quantiles conditional on covariates. *J. Appl. Stat.* **2020**, *47*, 954–974. [CrossRef]
8. Mazucheli, J.; Menezes, A.F.B.; Chakraborty, S. On the one parameter unit-Lindley distribution and its associated regression model for proportion data. *J. Appl. Stat.* **2019**, *46*, 700–714. [CrossRef]
9. Alzaatreh, A.; Lee, C.; Famoye, F. A new method for generating families of continuous distributions. *Metron* **2013**, *71*, 63–79. [CrossRef]
10. Lee, C.; Famoye, F.; Alzaatreh, A.Y. Methods for generating families of univariate continuous distributions in the recent decades. *Wiley Interdiscip. Rev. Comput. Stat.* **2013**, *5*, 219–238. [CrossRef]
11. Lemonte, A.J.; Cordeiro, G.M.; Ortega, E.M. On the additive Weibull distribution. *Commun. Stat. Theory Methods* **2014**, *43*, 2066–2080. [CrossRef]
12. Sarhan, A.M.; Apaloo, J. Exponentiated modified Weibull extension distribution. *Reliab. Eng. Syst. Saf.* **2013**, *112*, 137–144. [CrossRef]
13. Sarhan, A.M. A two-parameter discrete distribution with a bathtub hazard shape. *Commun. Stat. Appl. Methods* **2017**, *24*, 15–27. [CrossRef]
14. Sarhan, A.M.; Hamilton, D.C.; Smith, B. Parameter estimation for a two-parameter bathtub-shaped lifetime distribution. *Appl. Math. Model.* **2012**, *36*, 5380–5392. [CrossRef]
15. Tolba, A.H. Bayesian and Non-Bayesian Estimation Methods for Simulating the Parameter of the Akshaya Distribution. *Comput. J. Math. Stat. Sci.* **2022**, *1*, 13–25. [CrossRef]
16. Tolba, A.H.; Almetwally, E.M. Bayesian and Non-Bayesian Inference for The Generalized Power Akshaya Distribution with Application in Medical. *Comput. J. Math. Stat. Sci.* **2023**, *2*, 31–51.
17. Sarhan, A.M.; Smith, B.; Hamilton, D.C. Estimation of P (Y < X) for a two-parameter bathtub shaped failure rate distribution. *Int. J. Stat. Probab.* **2015**, *4*, 33–45.
18. Ramadan, D.A.; Magdy, A.W. On the Alpha-Power Inverse Weibull Distribution. *Int. J. Comput. Appl.* **2018**, *181*, 6–12.
19. Mahdavi, A.; Kundu, D. A new method for generating distributions with an application to an exponential distribution. *Commun. Stat. Theory Methods* **2017**, *46*, 6543–6557. [CrossRef]
20. Dey, S.; Alzaatreh, A.; Zhang, C.; Kumar, D. A new extension of generalized exponential distribution with application to Ozone data. *Ozone Sci. Eng.* **2017**, *39*, 273–285. [CrossRef]
21. Mahmood, Z.; Jawa, T.M.; Sayed-Ahmed, N.; Khalil, E.M.; Muse, A.H.; Tolba, A.H. An extended cosine generalized family of distributions for reliability modeling: Characteristics and applications with simulation study. *Math. Probl. Eng.* **2022**, *2022*, 3634698. [CrossRef]
22. Muse, A.H.; Tolba, A.H.; Fayad, E.; Abu Ali, O.A.; Nagy, M.; Yusuf, M. Modelling the COVID-19 mortality rate with a new versatile modification of the log-logistic distribution. *Comput. Intell. Neurosci.* **2021**, *2021*, 8640794. [CrossRef] [PubMed]
23. Al-Babtain, A.A.; Elbatal, I.; Chesneau, C.; Jamal, F. The transmuted Muth generated class of distributions with applications. *Symmetry* **2020**, *12*, 1677. [CrossRef]
24. Jodra, P.; Gomez, H.W.; Jimenez-Gamero, M.D.; Alba-Fernandez, M.V. The power Muth distribution. *Math. Model. Anal.* **2017**, *22*, 186–201. [CrossRef]
25. Alanzi, A.R.A.; Rafique, M.Q.; Tahir, M.H.; Jamal, F.; Hussain, M.A.; Sami, W. A novel Muth generalized family of distributions: Properties and applications to quality control. *AIMS Math.* **2023**, *8*, 6559–6580. [CrossRef]
26. Irshad, M.R.; Maya, R.; Krishna, A. Exponentiated power Muth distribution and associated inference. *J. Indian Soc. Probab. Stat.* **2021**, *22*, 265–302. [CrossRef]
27. Chesneau, C.; Agiwal, V. Statistical theory and practice of the inverse power Muth distribution. *J. Comput. Math. Data Sci.* **2021**, *1*, 100004. [CrossRef]
28. Almarashi, A.M.; Jamal, F.; Chesneau, C.; Elgarhy, M. A new truncated muth generated family of distributions with applications. *Complexity* **2021**, *2021*, 1211526. [CrossRef]
29. Jäntschi, L. Detecting extreme values with order statistics in samples from continuous distributions. *Mathematics* **2020**, *8*, 216. [CrossRef]

30. Krishna, A.; Maya, R.; Chesneau, C.; Irshad, M.R. The Unit Teissier Distribution and Its Applications. *Math. Comput. Appl.* **2022**, *27*, 12. [CrossRef]
31. Nassar, M.; Alzaatreh, A.; Mead, M.; Abo-Kasem, O. Alpha power Weibull distribution: Properties and applications. *Commun. Stat. Theory Methods* **2017**, *46*, 10236–10252. [CrossRef]
32. Hassan, A.S.; Mohamed, R.E.; Elgarhy, M.; Fayomi, A. Alpha power transformed extended exponential distribution: Properties and applications. *J. Nonlinear Sci. Appl.* **2018**, *12*, 62–67. [CrossRef]
33. Lehmann, E.L. The Power of Rank Tests. *Ann. Math. Stat.* **1953**, *24*, 23–43. [CrossRef]
34. Balogun, O.S.; Iqbal, M.Z.; Arshad, M.Z.; Afify, A.Z.; Oguntunde, P.E. A new generalization of Lehmann type-II distribution: Theory, simulation, and applications to survival and failure rate data. *Sci. Afr.* **2021**, *12*, e00790. [CrossRef]
35. Zhuang, L.; Xu, A.; Wang, X.-L. A prognostic driven predictive maintenance framework based on Bayesian deep learning. *Reliab. Eng. Syst. Saf.* **2023**, *234*, 109181. [CrossRef]
36. Luo, C.; Shen, L.; Xu, A. Modelling and estimation of system reliability under dynamic operating environments and lifetime ordering constraints. *Reliab. Eng. Syst. Saf.* **2022**, *218*, 108136. [CrossRef]
37. Xu, A.; Zhou, S.; Tang, Y. A unified model for system reliability evaluation under dynamic operating conditions. *IEEE Trans. Reliab.* **2019**, *70*, 65–72. [CrossRef]
38. Bernardo, J.M.; Smith, A.F.M. *Bayesian Theory*; Wiley: New York, NY, USA, 1994; Volume 49.
39. Al-Babtain, A.A.; Elbatal, I.; Almetwally, E.M. Bayesian and Non-Bayesian Reliability Estimation of Stress-Strength Model for Power-Modified Lindley Distribution. *J. Comput. Intell. Neurosci.* **2022**, *2022*, 1154705. [CrossRef]
40. Yousef, M.M.; Almetwally, E.M. Bayesian Inference for the Parameters of Exponentiated Chen Distribution Based on Hybrid Censoring. *Pak. J. Statist.* **2022**, *38*, 145–164.
41. Calabria, R.; Pulcini, G.S. An engineering approach to Bayes estimation for the Weibull distribution. *Microelectron. Reliab.* **1994**, *34*, 789–802. [CrossRef]
42. Abu El Azm, W.S.; Almetwally, E.M.; Naji AL-Aziz, S.; El-Bagoury, A.A.A.H.; Alharbi, R.; Abo-Kasem, O.E. A new transmuted generalized Lomax distribution: Properties and applications to COVID-19 data. *Comput. Intell. Neurosci.* **2021**, *2021*, 5918511. [CrossRef] [PubMed]
43. Anderson, D.R.; Burnham, K.P.; White, G.C. Comparison of Akaike information criterion and consistent Akaike information criterion for model selection and statistical inference from capture-recapture studies. *J. Appl. Stat.* **1998**, *25*, 263–282. [CrossRef]
44. Bierens, H.J. *Information Criteria and Model Selection*; Penn State University: State College, PA, USA, 2004.
45. Stock, J.H.; Watson, M.W. *Introduction to Econometrics*; Addison Wesley: Boston, MA, USA, 2003; Volume 104.
46. Murthy, D.P.; Xie, M.; Jiang, R. *Weibull Models*; John Wiley & Sons: Hoboken, NJ, USA, 2004.