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Localized Symmetric and Asymmetric Solitary Wave Solutions of Fractional Coupled Nonlinear Schrödinger Equations

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Abstract: The existence of solutions with localized solitary wave structures is one of the significant characteristics of nonlinear integrable systems. Darboux transformation (DT) is a well-known method for constructing multi-soliton solutions, using a type of localized solitary wave, of integrable systems, but there are still no reports on extending DT techniques to construct such solitary wave solutions of fractional integrable models. This article takes the coupled nonlinear Schrödinger (CNLS) equations with conformable fractional derivatives as an example to illustrate the feasibility of extending the DT and generalized DT (GDT) methods to construct symmetric and asymmetric solitary wave solutions for fractional integrable systems. Specifically, the traditional n -fold DT and the first-, second-, and third-step GDTs are extended for the fractional CNLS equations. Based on the extended GDTs, explicit solutions with symmetric/asymmetric soliton and soliton–rogon (solitrogon) spatial structures of the fractional CNLS equations are obtained. This study found that the symmetric solitary wave solutions of the integer-order CNLS equations exhibit asymmetry in the fractional order case.

Keywords: soliton solution; solitrogon solution; coupled nonlinear Schrödinger equations; Darboux transformation; generalized Darboux transformation



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1. Introduction

In mathematical physics, there are some effective methods [1–8] for the solitary wave solutions of nonlinear evolution equations, such as the inverse scattering method [1], DT [3], Hirota bilinear approach [4], and the exp-function method [6]. The DT [3] shows its algebraic operation characteristics in constructing multi-soliton solutions [9–14], which creates favorable conditions for the GDT [15–17] to construct semirational solutions with solitrogon structures [18–21]. The integrable NLS equations [22–25], a class of most typical soliton equations, can be used to describe nonlinear phenomena in various fields, such as nonlinear optics, Bose–Einstein condensation, plasma physics, and optical communication. In addition, the NLS-type equations have clear quantum mechanical significance, and can also describe various weakly dispersive and slowly modulated waves. Due to the wide applications of the NLS-type equations, solving such equations is an eternal hot topic for scholars. At the same time, deriving some important nonlinear integrable systems, especially NLS-type equations, is also meaningful research work. As for the new developments in this area, we are willing to mention the reduced NLS equations [26], and modified Korteweg–de Vries (mKdV) equations [27] generated through group reductions of matrix spectral problems. Moreover, the DT method has been applied to such reduced NLS equations [28] and mKdV equations [29]. The derivation of nonlocal integrable equations by group reductions is also feasible. For concrete details, see [30] where N -soliton solutions of the derived nonlocal integrable equations have also been systematically studied by solving Riemann–Hilbert problems. With the developments in fractional calculus [31], the question of how to extend the existing methods for fractional differential equations is a natural research trend [32–38].

This article aims to extend the DT and GDT for the following conformable fractional CNLS equation:

$$\begin{cases} iD_t^\alpha p + D_x^{2\alpha} p + 2(a^2|p|^2 + a|q|^2)p = 0 \\ iD_t^\alpha q + D_x^{2\alpha} q + 2(a^2|p|^2 + a|q|^2)q = 0 \end{cases} \quad (1)$$

which is our generalization of the CNLS equation [39], not only to the arbitrary nonzero constant-coefficient a^2 and a but also to the fractional order α ($0 < \alpha < 1$). In Equation (1), $|p|$ and $|q|$ are the models of potential functions $p(x, t)$ and $q(x, t)$, i stands for the imaginary unit, and D_t^α and $D_x^{2\alpha} = D_x^\alpha D_x^\alpha$ represent the conformable fractional derivative operators with respect to t and x [40,41]. Replacing D_t^α and $D_x^{2\alpha}$ with ∂_t and ∂_x^2 , respectively, and letting $a = 1$, the fractional CNLS Equation (1) is the integer-order CNLS equation [39]. Further taking $q = 0$, then the fractional CNLS Equation (1) degenerates into the classical focusing NLS equation [24]. Interestingly, in the case where $a = i$, the fractional CNLS Equation (1) can reduce to the classical defocusing NLS equation.

The fractional CNLS Equation (1) has Lax integrability. This is due to the fact that Equation (1) has the fractional Lax pair as below:

$$D_x^\alpha \varphi = M\varphi = (i\lambda J + U)\varphi, \quad (2)$$

$$D_t^\alpha \varphi = N\varphi = (-2i\lambda^2 J - 2\lambda U - iU_1)\varphi. \quad (3)$$

with

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, U = \begin{bmatrix} 0 & 0 & p \\ 0 & 0 & q \\ r_1 & r_2 & 0 \end{bmatrix}, U_1 = \begin{bmatrix} pr_1 & pr_2 & p_x \\ qr_1 & qr_2 & q_x \\ r_{1x} & r_{2x} & pr_1 + qr_2 \end{bmatrix},$$

where φ denotes the vector eigenfunction of the independent variables x and t and the isospectral parameter λ , $r_1 = -a^2 p^*$, $r_2 = -aq^*$, and $*$ stands for the complex conjugation.

The other sections of this article are as follows. In Section 2, the n -fold DT for the fractional CNLS Equation (1) is constructed. In Section 3, the GDT with the first three steps for the fractional CNLS Equation (1) is derived based on the constructed n -fold DT. In Section 4, some explicit solitary wave solutions of the fractional CNLS Equation (1) are obtained, including one-soliton and solitron solutions with symmetric/asymmetric localized spatial structures. In Section 5, this article is summarized.

2. n -Fold DT for the Fractional CNLS Equation (1)

Firstly, we introduce a gauge transformation in matrix form [18]:

$$\varphi^{[1]} = T\varphi, \det T \neq 0, \quad (4)$$

Then, Equation (3) transforms the Lax pair (2) into a new one:

$$D_x^\alpha \varphi^{[1]} = M^{[1]} \varphi^{[1]}, \quad (5)$$

$$D_t^\alpha \varphi^{[1]} = N^{[1]} \varphi^{[1]}. \quad (6)$$

where the iso spectral parameter λ is complex, $\varphi = \varphi(x, t, \lambda)$ is the eigen function.

According to Equations (2), (4) and (5), we arrive at:

$$D_x^\alpha \varphi^{[1]} = (D_x^\alpha T)\varphi + TD_x^\alpha \varphi = (D_x^\alpha T)\varphi + TM\varphi = M^{[1]} \varphi^{[1]} = M^{[1]} T\varphi,$$

namely:

$$D_x^\alpha T + TM = M^{[1]} T. \quad (7)$$

Similarly, by virtue of Equations (3), (4) and (6) we have:

$$D_t^\alpha T + TN = N^{[1]} T, \quad (8)$$

Then, using Equations (5)–(8) yields:

$$D_t^\alpha M^{[1]} - D_x^\alpha N^{[1]} + [M^{[1]}, N^{[1]}] = T(D_t^\alpha M - D_x^\alpha N + [M, N])T^{-1}, \quad (9)$$

which tells us that the zero curvature equation $D_t^\alpha M^{[1]} - D_x^\alpha N^{[1]} + [M^{[1]}, N^{[1]}]$ is equivalent to $D_t^\alpha M - D_x^\alpha N + [M, N]$. In other words, $M^{[1]}$ and $N^{[1]}$ have the same forms as M and N except that p and q are respectively replaced with $p^{[1]}$ and $q^{[1]}$.

We next assume that T is a polynomial matrix of the first degree:

$$T = T(\lambda) = \lambda \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix},$$

where b_{ij} and c_{ij} ($i = 1, 2, 3; j = 1, 2, 3$) are undetermined functions of x and t .

By comparing the coefficients of λ^2 , λ and λ^0 in Equation (5), we sequentially have:

$$b_{13} = b_{23} = b_{31} = b_{32} = 0,$$

$$\begin{cases} D_x^\alpha b_{11} = D_x^\alpha b_{12} = D_x^\alpha b_{21} = D_x^\alpha b_{22} = D_x^\alpha b_{33} = 0 \\ p^{[1]}b_{33} - pb_{11} - qb_{12} + 2ic_{13} = 0 \\ q^{[1]}b_{33} - pb_{21} - qb_{22} + 2ic_{23} = 0 \\ a^2 p^{[1]*}b_{11} + aq^{[1]*}b_{21} - a^2 p^*b_{33} + 2ic_{31} = 0 \\ a^2 p^{[1]*}b_{12} + aq^{[1]*}b_{22} - aq^*b_{33} + 2ic_{32} = 0 \end{cases},$$

$$\begin{cases} p^{[1]}c_{31} + a^2 c_{13}p^* - D_x^\alpha c_{11} = 0 \\ p^{[1]}c_{32} + ac_{13}q^* - D_x^\alpha c_{12} = 0 \\ p^{[1]}c_{33} - pc_{11} - qc_{12} - D_x^\alpha c_{13} = 0 \\ q^{[1]}c_{31} + a^2 c_{23}p^* - D_x^\alpha c_{21} = 0 \\ q^{[1]}c_{32} + ac_{23}q^* - D_x^\alpha c_{22} = 0 \\ q^{[1]}c_{33} - pc_{21} - qc_{22} - D_x^\alpha c_{23} = 0 \\ aq^{[1]*}c_{21} + a^2 c_{11}p^{[1]*} + D_x^\alpha c_{31} - a^2 c_{33}p^* = 0 \\ aq^{[1]*}c_{22} + a^2 c_{12}p^{[1]*} + D_x^\alpha c_{32} - ac_{33}q^* = 0 \\ aq^{[1]*}c_{23} + a^2 c_{13}p^{[1]*} + pc_{31} + qc_{32} + D_x^\alpha c_{33} = 0 \end{cases},$$

which shows that b_{11} , b_{12} , b_{21} , b_{22} and b_{33} are all independent to x . Similarly, substituting above T , N and $N^{[1]}$ into Equation (6) and comparing all the coefficients of the same powers of λ , we also find that b_{11} , b_{12} , b_{21} , b_{22} and b_{33} are all independent to t . In this case, we write T as:

$$T_1 = T_1(\lambda) = \lambda \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}.$$

To calculate the concrete expressions for c_{ij} ($i = 1, 2, 3; j = 1, 2, 3$), we consider the vector eigen functions $\varphi_j = (\varphi_{j1}, \varphi_{j2}, \varphi_{j3})^T$, here $\varphi_{j1} = \varphi_{j1}(x, t, \lambda_j)$, $\varphi_{j2} = \varphi_{j2}(x, t, \lambda_j)$ and $\varphi_{j3} = \varphi_{j3}(x, t, \lambda_j)$ for $j = \{1, 2, 3\}$. Then, it is not difficult to prove that when $\varphi_1 = (\varphi_{11}, \varphi_{12}, \varphi_{13})^T = (\varphi_{11}(x, \lambda_1), \varphi_{12}(x, \lambda_1), \varphi_{13}(x, \lambda_1))^T$ is the column vector solution of Equation (2) in the case of $\lambda = \lambda_1$, then $\varphi_2 = (\varphi_{21}, \varphi_{22}, \varphi_{23})^T = (\varphi_{12}^*(x, \lambda_1^*), -a\varphi_{11}^*(x, \lambda_1^*), 0)^T$ is also the column vector solution of Equation (2). At the same time, we know the fact that the vector $\varphi_3 = (\varphi_{31}, \varphi_{32}, \varphi_{33})^T = (\varphi_{13}^*(x, \lambda_1^*), 0, -a^2\varphi_{11}^*(x, \lambda_1^*))^T$ satisfies Equation (2). For convenience, we set $\lambda_3 = \lambda_1^*$ and $\lambda_2 = \lambda_1^*$.

With the above preparations, we select $b_{11} = b_{22} = b_{33} = 1$ and $b_{12} = b_{21} = 0$, then the first-step DT for the fractional CNLS Equation (1) turns out as:

$$T_1 = T_1(\lambda) = \lambda I_3 + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}, \quad (10)$$

$$p^{[1]} = p - 2ic_{13}, \quad q^{[1]} = q - 2ic_{23}, \quad (11)$$

where the Darboux matrix T_1 satisfies:

$$T_1(\lambda; \lambda_j) \varphi_j = 0, \quad (j = 1, 2, 3). \quad (12)$$

Equation (12) gives the following linear algebraic equations:

$$\begin{cases} c_{11}\varphi_{11} + c_{12}\varphi_{12} + c_{13}\varphi_{13} = -\lambda_1\varphi_{11} \\ c_{11}\varphi_{21} + c_{12}\varphi_{22} + c_{13}\varphi_{23} = -\lambda_2\varphi_{21} \\ c_{11}\varphi_{31} + c_{12}\varphi_{32} + c_{13}\varphi_{33} = -\lambda_3\varphi_{31} \end{cases}, \quad (13)$$

$$\begin{cases} c_{21}\varphi_{11} + c_{22}\varphi_{12} + c_{23}\varphi_{13} = -\lambda_1\varphi_{12} \\ c_{21}\varphi_{21} + c_{22}\varphi_{22} + c_{23}\varphi_{23} = -\lambda_2\varphi_{22} \\ c_{21}\varphi_{31} + c_{22}\varphi_{32} + c_{23}\varphi_{33} = -\lambda_3\varphi_{32} \end{cases}, \quad (14)$$

$$\begin{cases} c_{31}\varphi_{11} + c_{32}\varphi_{12} + c_{33}\varphi_{13} = -\lambda_1\varphi_{13} \\ c_{31}\varphi_{21} + c_{32}\varphi_{22} + c_{33}\varphi_{23} = -\lambda_2\varphi_{23} \\ c_{31}\varphi_{31} + c_{32}\varphi_{32} + c_{33}\varphi_{33} = -\lambda_3\varphi_{33} \end{cases}. \quad (15)$$

Solving the linear algebraic Equations (13)–(15) by Cramer's rule with the help of

$$\omega = \begin{vmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{vmatrix}, \quad (16)$$

one can obtain the following solutions:

$$c_{11} = \frac{\begin{vmatrix} -\lambda_1\varphi_{11} & \varphi_{12} & \varphi_{13} \\ -\lambda_2\varphi_{21} & \varphi_{22} & \varphi_{23} \\ -\lambda_3\varphi_{31} & \varphi_{32} & \varphi_{33} \end{vmatrix}}{\omega}, \quad c_{12} = \frac{\begin{vmatrix} \varphi_{11} & -\lambda_1\varphi_{11} & \varphi_{13} \\ \varphi_{21} & -\lambda_2\varphi_{21} & \varphi_{23} \\ \varphi_{31} & -\lambda_3\varphi_{31} & \varphi_{33} \end{vmatrix}}{\omega}, \quad c_{13} = \frac{\begin{vmatrix} \varphi_{11} & \varphi_{12} & -\lambda_1\varphi_{11} \\ \varphi_{21} & \varphi_{22} & -\lambda_2\varphi_{21} \\ \varphi_{31} & \varphi_{32} & -\lambda_3\varphi_{31} \end{vmatrix}}{\omega},$$

$$c_{21} = \frac{\begin{vmatrix} -\lambda_1\varphi_{12} & \varphi_{12} & \varphi_{13} \\ -\lambda_2\varphi_{22} & \varphi_{22} & \varphi_{23} \\ -\lambda_3\varphi_{32} & \varphi_{32} & \varphi_{33} \end{vmatrix}}{\omega}, \quad c_{22} = \frac{\begin{vmatrix} \varphi_{11} & -\lambda_1\varphi_{12} & \varphi_{13} \\ \varphi_{21} & -\lambda_2\varphi_{22} & \varphi_{23} \\ \varphi_{31} & -\lambda_3\varphi_{32} & \varphi_{33} \end{vmatrix}}{\omega}, \quad c_{23} = \frac{\begin{vmatrix} \varphi_{11} & \varphi_{12} & -\lambda_1\varphi_{12} \\ \varphi_{21} & \varphi_{22} & -\lambda_2\varphi_{22} \\ \varphi_{31} & \varphi_{32} & -\lambda_3\varphi_{32} \end{vmatrix}}{\omega},$$

$$c_{31} = \frac{\begin{vmatrix} -\lambda_1\varphi_{13} & \varphi_{12} & \varphi_{13} \\ -\lambda_2\varphi_{23} & \varphi_{22} & \varphi_{23} \\ -\lambda_3\varphi_{33} & \varphi_{32} & \varphi_{33} \end{vmatrix}}{\omega}, \quad c_{32} = \frac{\begin{vmatrix} \varphi_{11} & -\lambda_1\varphi_{13} & \varphi_{13} \\ \varphi_{21} & -\lambda_2\varphi_{23} & \varphi_{23} \\ \varphi_{31} & -\lambda_3\varphi_{33} & \varphi_{33} \end{vmatrix}}{\omega}, \quad c_{33} = \frac{\begin{vmatrix} \varphi_{11} & \varphi_{12} & -\lambda_1\varphi_{13} \\ \varphi_{21} & \varphi_{22} & -\lambda_2\varphi_{23} \\ \varphi_{31} & \varphi_{32} & -\lambda_3\varphi_{33} \end{vmatrix}}{\omega}.$$

Then, T_1 in Equation (10) is finally determined by the form:

$$T_1 = (\lambda - \lambda_1^*)I_3 - (\lambda_1 - \lambda_1^*) \frac{\varphi_1 \psi_1}{\psi_1 \varphi_1}, \quad (17)$$

where $\varphi_1 = (\varphi_{11}, \varphi_{12}, \varphi_{13})^T$ is, as mentioned earlier, the column vector solution of Equations (2) and (3) with $\lambda = \lambda_1$, while $\psi_1 = (a^3 \varphi_{11}^*, a^2 \varphi_{12}^*, a \varphi_{13}^*)$. Thus, the relations between the new and former potentials can be derived as follows:

$$p^{[1]} = p + 2i(\lambda_1 - \lambda_1^*) \left(\frac{\varphi_1 \psi_1}{\psi_1 \varphi_1} \right)_{13}, \quad q^{[1]} = q + 2i(\lambda_1 - \lambda_1^*) \left(\frac{\varphi_1 \psi_1}{\psi_1 \varphi_1} \right)_{23}. \quad (18)$$

If all the n different basic solutions $\varphi_k = (\varphi_{k1}, \varphi_{k2}, \varphi_{k3})^T$ ($k = 1, 2, \dots, n$) of Equations (2) and (3) with $\lambda = \lambda_k$ ($k = 1, 2, \dots, n$), then the n -fold DT for the fractional CNLS Equation (1) can be iterated smoothly. Continuing the above process and combining all the Darboux matrices, the n -th step DT turns out to be:

$$T_n = (\lambda - \lambda_n^*) I_3 - (\lambda_n - \lambda_n^*) \frac{\varphi_n \psi_n}{\psi_n \varphi_n}, \quad (19)$$

$$p^{[n]} = p + 2i \sum_{m=1}^n (\lambda_m - \lambda_m^*) \left(\frac{\varphi_m \psi_m}{\psi_m \varphi_m} \right)_{13}, \quad q^{[n]} = q + 2i \sum_{m=1}^n (\lambda_m - \lambda_m^*) \left(\frac{\varphi_m \psi_m}{\psi_m \varphi_m} \right)_{23} \quad (20)$$

where $\varphi_m = (\varphi_{m1}, \varphi_{m2}, \varphi_{m3})^T$ is the column vector solution of Equations (2) and (3) with $\lambda = \lambda_m$, and $\psi_m = (a^3 \varphi_{m1}^*, a^2 \varphi_{m2}^*, a \varphi_{m3}^*)$.

3. GDT for the Fractional CNLS Equation (1)

We have constructed in Section 2 the n -th step DT for the fractional CNLS Equation (1), which contains n eigenfunctions φ_i ($i = 1, 2, \dots, n$) combined with the given eigenvalues λ_i ($i = 1, 2, \dots, n$). Based on the constructed DT, we gain $T_1 \varphi_1 = 0$. It is obvious that the DT under this case cannot be applied on φ_1 again. We then suppose that $\gamma^{[1]} = \varphi_1(\lambda_1 + \delta)$ is a special solution of Equations (2) and (3). In this way, we can infer that $\varphi_1(\lambda_1 + \delta)/\delta$ is also a solution of Equations (2) and (3), where δ is a small parameter. Expanding $\gamma^{[1]}$ in Taylor series at λ_1 yields:

$$\gamma^{[1]} = \varphi_1(\lambda_1 + \delta) = \gamma_0^{[1]} + \gamma_1^{[1]} \delta + \gamma_2^{[1]} \delta^2 + \dots + \gamma_n^{[1]} \delta^n + \dots \quad (21)$$

with

$$\gamma_k^{[1]} = \frac{1}{k!} D_{\lambda}^{k\alpha} \varphi_1(\lambda) \Big|_{\lambda=\lambda_1}, \quad (k = 0, 1, 2, \dots).$$

We find that $\gamma_0^{[1]} = \varphi_1(\lambda_1)$ solves Equations (2) and (3) with $\lambda = \lambda_1$ and associated seed solutions p and q . Thus, we have the first-step GDT for the fractional CNLS Equation (1):

$$T_1[1] = (\lambda - \lambda_1^*) I_3 - (\lambda_1 - \lambda_1^*) \frac{\varphi_1 \psi_1}{\psi_1 \varphi_1}, \quad (22)$$

$$p[1] = p + 2i(\lambda_1 - \lambda_1^*) \left(\frac{\varphi_1 \psi_1}{\psi_1 \varphi_1} \right)_{13}, \quad q[1] = q + 2i(\lambda_1 - \lambda_1^*) \left(\frac{\varphi_1 \psi_1}{\psi_1 \varphi_1} \right)_{23}, \quad (23)$$

where $\varphi_1 = (\varphi_{11}, \varphi_{12}, \varphi_{13})^T$ and $\psi_1 = (a^3 \varphi_{11}^*, a^2 \varphi_{12}^*, a \varphi_{13}^*)$.

For the second-step GDT for the fractional CNLS Equation (1), we then obtain the second iterated GDT via the limit process:

$$\lim_{\delta \rightarrow 0} \frac{[T_1[1]]_{\lambda=\lambda_1+\delta} \gamma^{[1]}}{\delta} = \lim_{\delta \rightarrow 0} \frac{[\delta + T_1[1]]_{\lambda=\lambda_1} \gamma^{[1]}}{\delta} = \gamma_0^{[1]} + T_1[1]_{\lambda=\lambda_1} \gamma_1^{[1]} = \varphi_1[1], \quad (24)$$

where $\gamma_0^{[1]}$ and $\gamma_1^{[1]}$ are already given in Equation (21). Thus, $\varphi_1[1]$ associated with λ_1 can be determined by means of Equation (24). Let us assume $\varphi_1[1] = (\phi_{11}, \phi_{12}, \phi_{13})^T$. When

$n = 2$, we can obtain the second-step GDT of the fractional CNLS Equation (1) by replacing related variables in Equation (19) in the limit case of $\lambda_2 \rightarrow \lambda_1$:

$$T_1[2] = (\lambda - \lambda_1^*)I_3 - (\lambda_1 - \lambda_1^*) \frac{\varphi_1[1]\psi_1[1]}{\psi_1[1]\varphi_1[1]}, \quad (25)$$

$$p[2] = p[1] + 2i(\lambda_1 - \lambda_1^*) \left(\frac{\varphi_1[1]\psi_1[1]}{\psi_1[1]\varphi_1[1]} \right)_{13}, \quad q[2] = q[1] + 2i(\lambda_1 - \lambda_1^*) \left(\frac{\varphi_1[1]\psi_1[1]}{\psi_1[1]\varphi_1[1]} \right)_{23}, \quad (26)$$

where $\varphi_1[1] = (\phi_{11}, \phi_{12}, \phi_{13})^T$ and $\psi_1[1] = (a^3\phi_{11}^*, a^2\phi_{12}^*, a\phi_{13}^*)^T$.

For the third-step GDT of the fractional CNLS Equation (1), we calculate:

$$\lim_{\delta \rightarrow 0} \frac{[T_1[2](\lambda_1 + \delta)] \left[\frac{\delta + T_1[1](\lambda_1)}{\delta} \right] \gamma^{[1]}}{\delta} \quad (27)$$

$$= \gamma_0^{[1]} + [T_1[1](\lambda_1) + T_1[2](\lambda_1)]\gamma_1^{[1]} + T_1[2](\lambda_1)T_1[1](\lambda_1)\gamma_2^{[1]} = \varphi_1[2],$$

where $\gamma_0^{[1]}$, $\gamma_1^{[1]}$ and $\gamma_2^{[1]}$ are known via Equation (18). Therefore, $\varphi_1[2]$ is determined by means of Equation (27). We suppose $\varphi_1[2] = (\phi_{21}, \phi_{22}, \phi_{23})^T$, then the third-step GDT for the fractional CNLS Equation (1) can be constructed as follows:

$$T_1[3] = (\lambda - \lambda_1^*)I_3 - (\lambda_1 - \lambda_1^*) \frac{\varphi_1[2]\psi_1[2]}{\psi_1[2]\varphi_1[2]}, \quad (28)$$

$$p[3] = p[2] + 2i(\lambda_1 - \lambda_1^*) \left(\frac{\varphi_1[2]\psi_1[2]}{\psi_1[2]\varphi_1[2]} \right)_{13}, \quad q[3] = q[2] + 2i(\lambda_1 - \lambda_1^*) \left(\frac{\varphi_1[2]\psi_1[2]}{\psi_1[2]\varphi_1[2]} \right)_{23}, \quad (29)$$

where $\varphi_1[2] = (\phi_{21}, \phi_{22}, \phi_{23})^T$ and $\psi_1[2] = (a^3\phi_{21}^*, a^2\phi_{22}^*, a\phi_{23}^*)^T$.

4. Symmetric and Asymmetric Solitary Wave Solutions of the Fractional NLS Equation (1)

In the previous sections, we have constructed the necessary formulae to generate exact solutions of the fractional NLS Equation (1), including soliton solutions and solitron solutions. In this part, we calculate several explicit solutions of the fractional NLS Equation (1) by employing these formulae.

We start with the one-soliton solutions by taking the seed solutions $p = 0$ and $q = 0$. Substituting these two seed solutions into Equations (2) and (3), and solving the resultant equations, we derive the following column vector solution with $\lambda = \lambda_1$:

$$\varphi_1 = \begin{bmatrix} d_1 e^{i \frac{\lambda_1}{\alpha} (x^\alpha - 2\lambda_1 t^\alpha)} \\ d_2 e^{i \frac{\lambda_1}{\alpha} (x^\alpha - 2\lambda_1 t^\alpha)} \\ d_3 e^{-i \frac{\lambda_1}{\alpha} (x^\alpha - 2\lambda_1 t^\alpha)} \end{bmatrix}$$

Substituting above φ_1 into Equation (23), we can obtain the one-soliton solutions of the fractional NLS Equation (1). When we set $\lambda_1 = i$, $d_1 = 1$, $d_2 = 1$ and $d_3 = 2$, the one-soliton solutions are obtained as follows:

$$p[1] = q[1] = -\frac{8e^{\frac{2}{\alpha}(2it^\alpha + x^\alpha)}}{a + a^2 + 4e^{\frac{4}{\alpha}x^\alpha}}. \quad (30)$$

When letting $\lambda_1 = -1 + i$, $d_1 = 1$, $d_2 = 2$ and $d_3 = 1$, the one-soliton solutions of the fractional NLS Equation (1) read:

$$p[1] = q[1] = -\frac{4e^{\frac{2}{\alpha}[4t^\alpha + (1-i)x^\alpha]}}{4a + a^2 + e^{\frac{4}{\alpha}(4t^\alpha + x^\alpha)}}. \quad (31)$$

Using Formulae (24) and (26), we can straight-forwardly obtain the solitron solutions of the fractional CNLS Equation (1). Based on Equation (30), the solutions with $\lambda_1 = i$, $d_1 = 1$, $d_2 = 1$ and $d_3 = 2$ can be expressed as:

$$p[2] = q[2] = \frac{16e^{\frac{2}{\alpha}(2it^\alpha + x^\alpha)}[a(1+a)(-1 - \frac{8i}{\alpha}t^\alpha - \frac{2}{\alpha}x^\alpha) + 4e^{\frac{4}{\alpha}x^\alpha}(-1 - \frac{8i}{\alpha}t^\alpha + \frac{2}{\alpha}x^\alpha)]}{a^2 + 2a^3 + a^4 + 16e^{\frac{8}{\alpha}x^\alpha} + a(8+a)\{8e^{\frac{4}{\alpha}x^\alpha}[1 + 128(\frac{t^\alpha}{\alpha})^2 + 8(\frac{x^\alpha}{\alpha})^2]\}}. \quad (32)$$

When $\lambda_1 = -1 + i$, $d_1 = 1$, $d_2 = 2$ and $d_3 = 1$, Formulas (24) and (26) help us use Solution (31) to obtain the solitron solutions of the fractional NLS Equation (1):

$$p[2] = \frac{1}{2}q[2] = \frac{8e^{\frac{2}{\alpha}[4t^\alpha + (1-i)x^\alpha]}\{e^{\frac{4}{\alpha}(4t^\alpha + x^\alpha)}[-1 + \frac{8(1-i)}{\alpha}t^\alpha + \frac{2}{\alpha}x^\alpha] - a(4-a)[1 + \frac{8(1+i)}{\alpha}t^\alpha + \frac{2}{\alpha}x^\alpha]\}}{16a^2 + 8a^3 + a^4 + e^{\frac{8}{\alpha}(4t^\alpha + x^\alpha)} + 2a(4+a)e^{\frac{4}{\alpha}(4t^\alpha + x^\alpha)}[1 + 256(\frac{t^\alpha}{\alpha})^2 + \frac{64}{\alpha^2}t^\alpha x^\alpha + 8(\frac{x^\alpha}{\alpha})^2]}$$

Two space-time structures of Solution (30) are shown in Figure 1 by setting $a = 0.1$. We can see from Figure 2 that the symmetric one-soliton determined by Solution (30) with $\alpha = 1$ has asymmetry when $\alpha = 5/11$ is selected. With the same fractional order $\alpha = 5/11$ as Figure 2 but different constants $a = 0.01i$ and $a = 0.005i$, we show in Figure 3 another two asymmetric one-solitons determined by Solution (30).

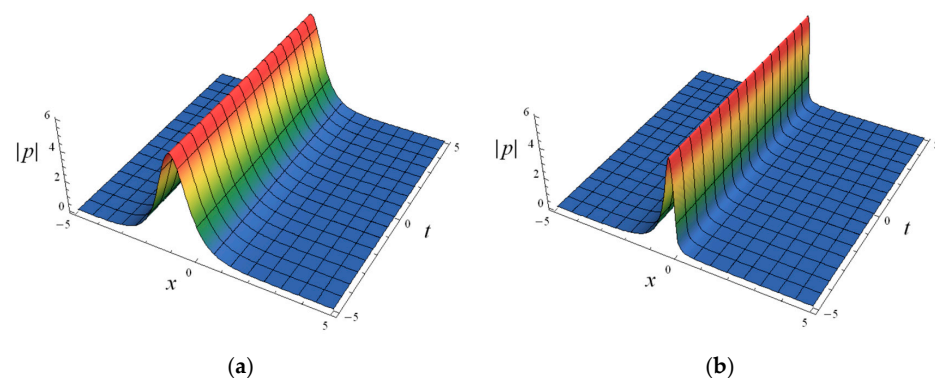


Figure 1. Symmetric and asymmetric space-time structures of Solution (30) with different fractional orders: (a) $\alpha = 1$; (b) $\alpha = 5/11$.

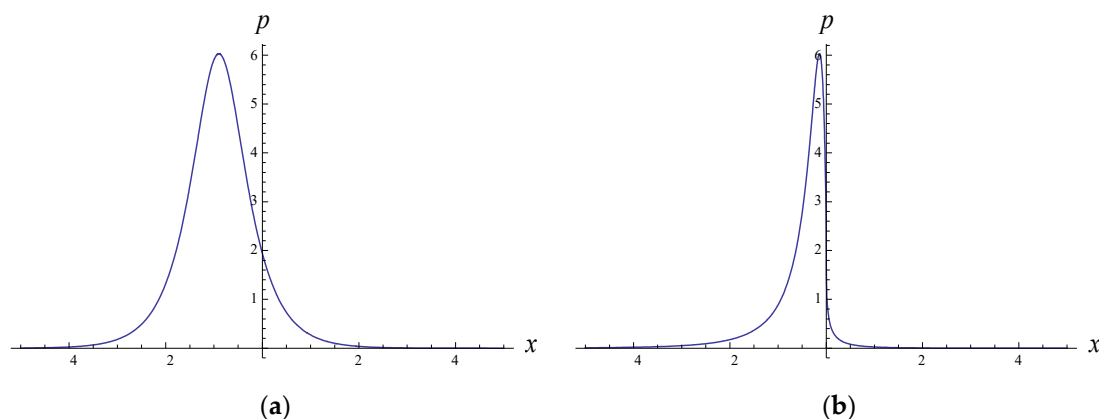


Figure 2. Symmetric and asymmetric bell one-solitons determined by Solution (30) with different fractional orders: (a) $\alpha = 1$; (b) $\alpha = 5/11$ at the time $t = 0$.

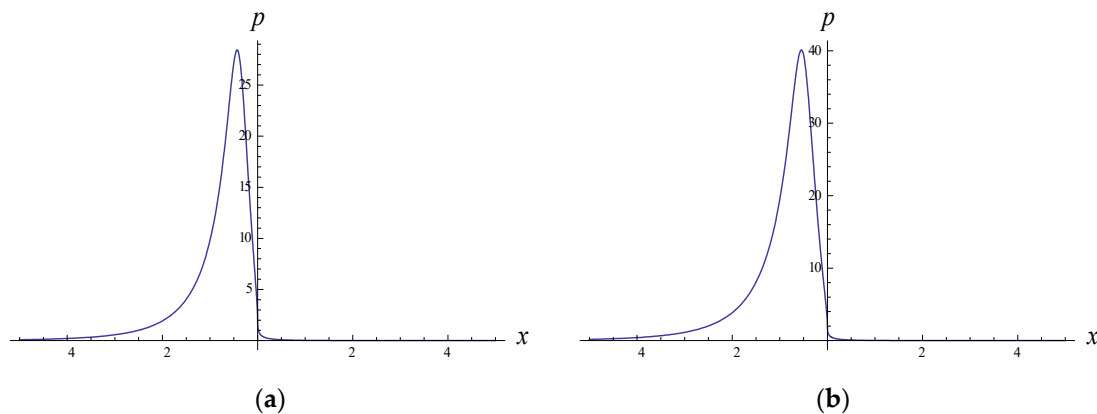


Figure 3. Asymmetric bell one-solitons determined by solution (30) with fractional order $\alpha = 5/11$ and different values of a : (a) $a = 0.01i$; (b) $a = 0.05i$ at the time $t = 0$.

In Figures 4–6, the asymmetric solitron and two-soliton structures determined by Solution (32) are shown, there $a = 0.1$. It is seen that whether the fractional order $\alpha = 1$ or $\alpha = 3/11$, the bell solitrogons are all asymmetric at the time $t = 0$, but the bell two-soliton part of the solitron with $\alpha = 1$ is symmetric while the one corresponding to $\alpha = 3/11$ is asymmetric at $t = 30$. In Figure 7, another pair of asymmetric solitron structures determined by Solution (32) are shown, there the constant a has been selected as $0.1i$ and $0.05i$, respectively. Figures 3 and 7 show that all the amplitudes of the solitary waves increase with the decrease in the imaginary part of a . As the real value of a decreases, a similar phenomenon of increasing the amplitudes of solitary waves also occurs. The law of amplitude variation shown in Figures 3 and 7 can also be understood mathematically by analyzing the expressions of Equations (30) and (32).

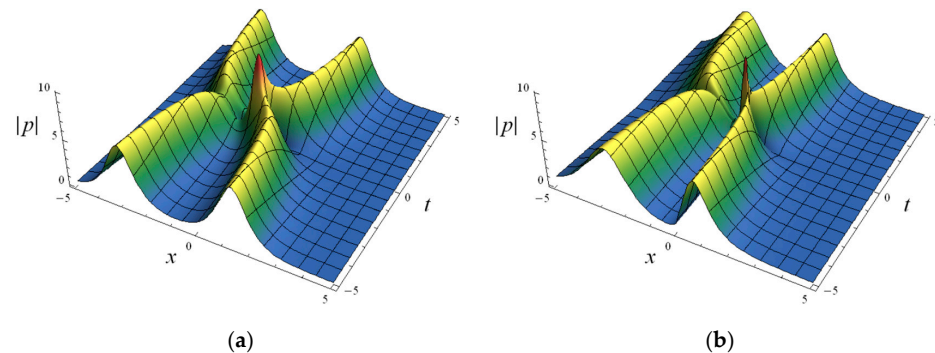


Figure 4. Asymmetric space-time structures of Solution (32) with different fractional orders: (a) $\alpha = 1$; (b) $\alpha = 7/11$.

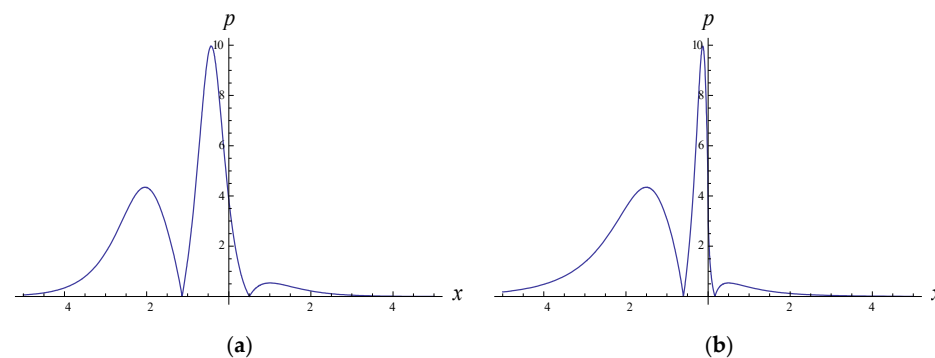


Figure 5. Asymmetric bell solitrogons determined by Solution (32) with different fractional orders: (a) $\alpha = 1$; (b) $\alpha = 7/11$ at the time $t = 0$.

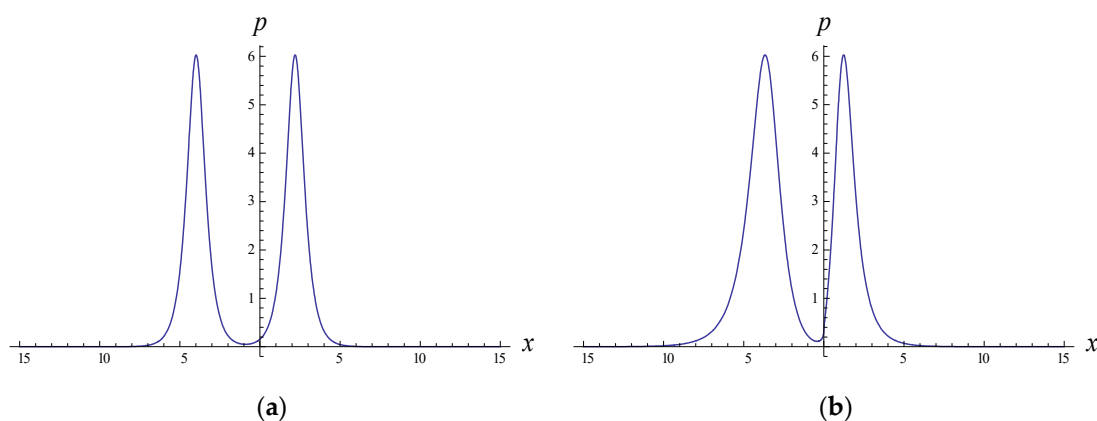


Figure 6. Symmetric and asymmetric bell two-soliton parts of the solitrogons determined by Solution (32) with different fractional orders: (a) $\alpha = 1$; (b) $\alpha = 7/11$ at the time $t = 30$.

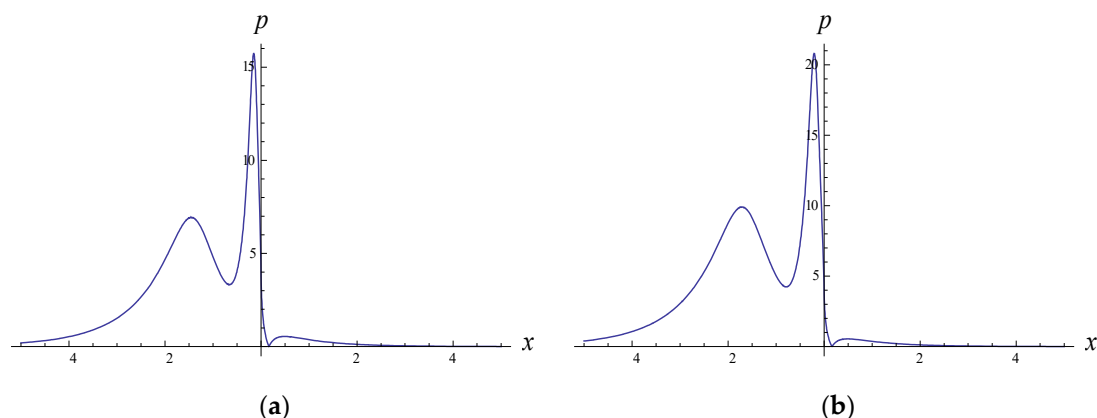


Figure 7. Asymmetric bell solitrogons determined by Solution (32) with fractional order $\alpha = 7/11$ and different values of a : (a) $a = 0.01i$; (b) $a = 0.05i$ at the time $t = 0$.

5. Conclusions and Discussions

In terms of the first iteration Formulae (18) and (23), we can easily recognize that there is essentially no difference between the DT and GDT for the fractional CNLS Equation (1). However, the n -fold GDT with $n > 1$ has advantage over the n -fold DT in constructing other types of solutions, for example, solitron solutions and rogue wave solutions for the fractional CNLS Equation (1). As for the findings of this article, we would like to mention the following aspects: (i) the derivation of the integrable fractional CNLS Equation (1) with fractional Lax pair (2); (ii) the construction of the n -fold DT (20) and the first, second, and third iteration Formulae (23), (26) and (29) of GDT for the fractional NLS Equation (1); (iii) the example illustration of the feasibility simulating asymmetric solitary wave solutions of the fractional NLS Equation (1) via the dominant role of fractional order α ; (iv) the adjustability of the amplitudes of solitary waves localized in Solutions (30) and (32) can be controlled by the value of constant a ; (v) the beginning of extending DT and GDT to construct soliton and solitron solutions of nonlinear fractional integrable systems; (vi) the predictability of constructing rogue wave solutions, especially the asymmetric ones for the fractional NLS Equation (1) using the GDT. Due to the asymmetry being a significant feature of rogue waves, it is worth studying how to fully utilize the dominant role of fractional order α in the construction of rogue waves for fractional models.

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