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Certain Subclasses of Analytic and Bi-Univalent Functions Governed by the Gegenbauer Polynomials Linked with q-Derivative

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Abstract: In this paper, we introduce and investigate two new subclasses of analytic and bi-univalent functions using the q-derivative operator D_q $(0 < q < 1)$ and the Gegenbauer polynomials in a symmetric domain, which is the open unit disc $\Lambda = \{\wp : \wp \in \mathbb{C} \text{ and } |\wp| < 1\}$. For these subclasses of analytic and bi-univalent functions, the coefficient estimates and Fekete–Szegö inequalities are solved. Some special cases of the main results are also linked to those in several previous studies. The symmetric nature of quantum calculus itself motivates our investigation of the applications of such quantum (or q-) extensions in this paper.

Keywords: analytic functions; q-derivative operator; bi-univalent functions; subordination; Fekete–Szegö inequality; Gegenbauer polynomials

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1. Introduction

In the advancement of the Geometric Function Theory as a part of the Complex Analysis research field, the q-derivative is a handy gadget for applications. Jackson was the first one to work on the application of q-calculus (see [1–4]). Recently, incorporating the q-derivative operator into the criterion of differential subordination, many researchers have introduced new subclasses of analytic functions and investigated their geometric properties (see [5–15]). In addition, q-calculus has been widely applied in various realistic systems such as viscoelastic models, neural network models and so on [16–21].

Orthogonal polynomials are main concept in mathematical analysis and researchers have extensively studied since they were discovered in the 19th century. Relating to a particular weight function, on a given interval, they constitute an orthogonal sequence of functions. In mathematics researches have used in various areas, including approximation theory, number theory, and differential equation theory. One of the most important classes of orthogonal polynomials is the class of classical orthogonal polynomials containing the Laguerre, Hermite, Legendre, and Gegenbauer polynomials. Having served as the base for many mathematical applications, researchers have found interest in studying these polynomials thoroughly.

In fact, in the recent past years, many researchers have put vital effort on studying and investigating particular subclasses of analytic and bi-univalent functions related with orthogonal polynomials. They have been interested in obtaining coefficient estimates containing the initial coefficients, general coefficients, Fekete-Szegö functional, and Hankel determinants for these subclasses.

Inspired by their works, in our paper, we define and study two new families $\mathcal{J}_{\mathcal{A}}^{q,s}[U, V, \tau, \xi]$ and $\mathcal{J}_{\Sigma}^{q,s}[\phi, \tau, \xi]$ of the class of analytic and bi-univalent functions related to the q-derivative operator and the Gegenbauer polynomials. For every subclass, we consider the coefficient estimates and Fekete–Szegö inequality. By comparing the results of the present paper with some previous studies in the subject, we show that our results are extending and generalizing theirs. We suggest studies [12,22–25] to be reviewed thoroughly in order for the reader to relate our work with the affiliated recent advancements concerning the coefficient estimates and coefficient inequalities of numerous subclasses of analytic, univalent, and bi-univalent functions containing the Fekete–Szegö functional so that he/she may be motivated for further studies on the subject.

We call $\mathcal{A}$ the class of all analytic functions $\hbar$ defined in the open unit disk

$$\Lambda = \{\wp : \wp \in \mathbb{C} \text{ and } |\wp| < 1\}$$

with the normalization circumstances $\hbar'(0) - 1 = 0$ and $\hbar(0) = 0$. Consequently, a Taylor–Maclaurin series expansion of the type exists for each $\hbar \in \mathcal{A}$

$$\hbar(\wp) = \wp + \sum_{k=2}^{\infty} a_k \wp^k. \tag{1}$$

In addition, we call $\mathcal{S}$ the class of all functions $\hbar \in \mathcal{A}$ that are univalent in Λ. Clearly, it is well known that every function $\hbar \in \mathcal{S}$ has an inverse $\hbar^{-1}$, defined by

$$\hbar^{-1}(\hbar(\wp)) = \wp \;\; (\wp \in \Lambda)$$

and

$$w = \hbar(g(w)) \;\; \left(|w| < r_0(\hbar);\; r_0(\hbar) \geq \frac{1}{4}\right),$$

where

$$g(w) := \hbar^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots . \tag{2}$$

As a well-known definition in the field Complex Analysis, one can recall that we call a function $\hbar(\wp)$ bi-univalent in the case that both $\hbar(\wp)$ and $\hbar^{-1}(\wp)$ are univalent in Λ, and Σ denotes the class of bi-univalent functions. Some examples of functions in the class Σ are

$$\frac{\wp}{1-\wp}, \quad -\log(1-\wp) \quad \text{and} \quad \frac{1}{2}\log\left(\frac{1+\wp}{1-\wp}\right).$$

However, the familiar Koebe function is not a member of the bi-univalent function class Σ.

We note that Srivastava et al. [26] presented groundbreaking and inspiring results on the investigation of the normalized class Σ of analytic and bi-univalent functions in Λ such that their article became a leading work flooding the literature in the field with many sequels to [26].

For a function $\hbar \in \mathcal{A}$, given by (1), and a function $g \in \mathcal{A}$, written as

$$g(\wp) = \wp + \sum_{k=2}^{\infty} b_k \wp^k \;\; (\wp \in \Lambda),$$

the Hadamard product of $\hbar$ and g is written as

$$(\hbar * g)(\wp) := \wp + \sum_{k=2}^{\infty} a_k b_k \wp^k := (g * \hbar)(\wp) \;\; (\wp \in \Lambda).$$

We call $\mathcal{P}$ the class of Carathéodory functions Υ which are analytic in Λ and satisfy

$$\Upsilon(\wp) = 1 + \sum_{k=1}^{\infty} c_k \wp^k$$

and

$$\Re(\Upsilon(\wp)) > 0 \ \ (\wp \in \Lambda).$$

Let $\hbar$ and g be analytic functions in Λ. If there exists a Schwartz function w that is analytic in Λ with

$$w(0) = 0 \text{ and } |w(\wp)| < 1 \ \ (\wp \in \Lambda)$$

such that

$$\hbar(\wp) = g(w(\wp)),$$

then we say that the function $\hbar$ is subordinate to g written as $\hbar \prec g$. Additionally, the following equivalence holds if the function g is univalent in Λ :

$$\hbar(\wp) \prec g(\wp) \iff \hbar(0) = g(0)$$

and

$$\hbar(\Lambda) \subset g(\Lambda).$$

Let $q \in (0,1)$, and define the q-number $[\eta]_q$ as follows:

$$[\eta]_q := \begin{cases} \dfrac{1-q^\eta}{1-q} & (\eta \in \mathbb{C}) \\ 1 + \displaystyle\sum_{k=1}^{n-1} q^k & (\eta = n \in \mathbb{N}). \end{cases}$$

Especially, we note that $[0]_q = 0$.

Now, we recall here the q-difference or the q-derivative operator D_q $(0 < q < 1)$ of a function $\hbar \in \mathcal{A}$ as follows:

$$(D_q\hbar)(\wp) = \begin{cases} \dfrac{\hbar(\wp) - \hbar(q\wp)}{(1-q)\wp} & (\wp \neq 0) \\ \hbar'(0) & (\wp = 0), \end{cases}$$

where $\hbar'(0)$ exists. In addition, we write

$$\left(D_q^{(2)}\hbar\right)(\wp) = \left(D_q\left(D_q\hbar\right)\right)(\wp).$$

Recently, orthogonal polynomials have been broadly investigated from numerous viewpoints due to their importance in probability theory, mathematical physics, engineering and mathematical statistics. From a mathematical perspective, orthogonal polynomials frequently emanate from ordinary differential equation solutions under specific circumstances required by a specific model. The orthogonal polynomials that pop up most ordinarily in utilization are the Gegenbauer, Chebyshev, Legendre, Horadam, (p,q)-Lucas, Jacobi, Bernoulli and Fibonacci polynomials. We recommend the reader to see the recent studies [27–41] in connection with orthogonal polynomials and the geometric function theory.

The definition of the Gegenbauer polynomials [42] is offered in terms of the Jacobi polynomials $P_n^{(u,v)}$, with $u = v = \alpha - \frac{1}{2}$, $\left(\alpha > -\frac{1}{2}, \alpha \neq 0\right)$, which are given by

$$\begin{aligned} C_n^\alpha(x) &= \frac{\Gamma(n+2)\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(2\alpha)\Gamma\left(n+\alpha+\frac{1}{2}\right)} P_n^{\left(\alpha+\frac{1}{2},\, \alpha-\frac{1}{2}\right)}(x) \\ &= \binom{n-1+2\alpha}{n} \sum_{k=0}^{n} \frac{(nk)(2\alpha+n)_k}{\left(\alpha+\frac{1}{2}\right)_k} \left(\frac{x-1}{2}\right)^k. \end{aligned} \tag{3}$$

From (3), it follows that $C_n^\alpha(x)$ is a polynomial of degree n with real coefficients and $C_n^\alpha(1) = \begin{pmatrix} n-1+2\alpha \\ n \end{pmatrix}$, while the leading coefficient of $C_n^\alpha(x)$ is $2^n\begin{pmatrix} n-1+\alpha \\ n \end{pmatrix}$. According to Jacobi polynomial theory, for $u = v = \alpha - \frac{1}{2}$, $\left(\alpha > -\frac{1}{2}, \alpha \neq 0\right)$, we have

$$C_n^\alpha(-x) = (-1)^n C_n^\alpha(x).$$

In [42,43], the Gegenbauer polynomials' generating function is determined by

$$\frac{2^{\alpha-\frac{1}{2}}}{(1-2x\wp+\wp^2)^{\frac{1}{2}}\left(1-x\wp+\sqrt{1-2x\wp+\wp^2}\right)^{\alpha-\frac{1}{2}}} = \frac{\left(\alpha-\frac{1}{2}\right)_n}{(2\alpha)_n} C_n^\alpha(x)\wp^n, \tag{4}$$

and this evenness may be understood from the Jacobi polynomial-generating function.

In 2020, Amourah et al. [44] took into consideration the generating function of Gegenbauer polynomials as follows:

$$\phi_x^\alpha(\wp) = \frac{1}{(1-2x\wp+\wp^2)^\alpha}. \tag{5}$$

The function ϕ_x^α is analytic in Λ for a fixed x, and as a result, its Taylor series expansion is written as:

$$\phi_x^\alpha(\wp) = \sum_{k=0}^{\infty} C_k^\alpha(x)\wp^k, \tag{6}$$

where $|x| \leq 1$, $\alpha \in \left(-\frac{1}{2}, \infty\right)\backslash\{0\}$, $\wp \in \Lambda$ and $C_n^\alpha(x)$ is a Gegenbauer polynomial of degree n.

Obviously, ϕ_x^α generates nothing when $\alpha = 0$. As a result, the generating function of the Gegenbauer polynomial is defined as:

$$\phi_x^0(\wp) = \sum_{k=0}^{\infty} C_k^0(x)\wp^k, \tag{7}$$

for $\alpha = 0$. Furthermore, a normalization of greater than $1/2$ is preferable [45]. The recurrence relation for Gegenbauer polynomials is as follows:

$$C_n^\alpha(x) = \frac{1}{n}\left[2x(n+\alpha-1)C_{n-1}^\alpha(x) - (n+2\alpha-2)C_{n-2}^\alpha(x)\right] \tag{8}$$

with the initial values:

$$C_0^\alpha(x) = 1 \;\; C_1^\alpha(x) = 2\alpha x \;\text{ and }\; C_2^\alpha(x) = \alpha\left[(2+2\alpha)x^2 - 1\right]. \tag{9}$$

Recently, Gegenbauer polynomials and subclasses of the bi-univalent functions were studied by [46–50].

Remark 1. *Particular cases:*

(i) *For $\alpha = 1$, we get the Chebyshev Polynomials.*

(ii) *For $\alpha = \frac{1}{2}$, we get the Legendre Polynomials.*

Moreover, Shah and Noor [51] introduced the q-analogue of the Hurwitz–Lerch zeta function by the following series:

$$\chi_q(s,\tau;\wp) = \sum_{k=0}^{\infty} \frac{\wp^k}{[k+\tau]_q^s}, \tag{10}$$

where $\tau \in \mathbb{C} \backslash \mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|\wp| < 1$ and $\Re(s) > 1$ when $|\wp| = 1$. The normalized form of (10) is as follows:

$$\Psi_q(s,\tau;\wp) = [1+\tau]_q^s \Big\{ \chi_q(s,\tau;\wp) - [\tau]_q^{-s} \Big\} = \wp + \sum_{k=2}^{\infty} \Delta_k \wp^k \tag{11}$$

where $\Delta_k = \left(\frac{[1+\tau]_q}{[k+\tau]_q} \right)^s$, $k \in \mathbb{Z}^+$.

From (11) and (1), Shah and Noor [51] defined the q-Srivastava Attiya operator $J_{q,\tau}^s \hbar(\wp) : \mathcal{A} \to \mathcal{A}$ by

$$J_{q,\tau}^s \hbar(\wp) = \Psi_q(s,\tau;\wp) * \hbar(\wp) = \wp + \sum_{k=2}^{\infty} \Delta_k a_k \wp^k \tag{12}$$

where $*$ denotes the Hadamard product.

We note that:

(i) If $q \to 1^-$, then the function $\chi_q(s,\tau;\wp)$ starts reducing to the Hurwitz–Lerch zeta function, and the operator $J_{q,\tau}^s$ overlaps with the Srivastava–Attiya operator (see [52,53]).

(ii) $J_{q,0}^1 \hbar(\wp) = \int_0^{\wp} \frac{\hbar(t)}{t} d_q t$ (q-Alexander operator).

(iii) $J_{q,\tau}^1 \hbar(\wp) = \frac{[1+\tau]_q}{\wp^{\tau}} \int_0^{\wp} \frac{\hbar(t)}{t^{1-\tau}} d_q t$ (q-Bernardi operator [54]).

(iv) $J_{q,1}^1 \hbar(\wp) = \frac{[2]_q}{\wp} \int_0^{\wp} \frac{\hbar(t)}{t^{1-b}} d_q t$ (q-Libera operator [54]).

Next, we define the analytic function family $\mathcal{J}_{\mathcal{A}}^{q,s}[U,V,\tau,\xi]$ and the bi-univalent function class $\mathcal{J}_{\Sigma}^{q,s}[\phi_x^{\alpha},\tau,\xi]$.

Definition 1. *Let* $-1 \leqq V < U \leqq 1$ *and* $0 \leqq \xi \leqq 1$. *A function* $\hbar \in \mathcal{A}$ *is in the class* $\mathcal{J}_{\mathcal{A}}^{q,s}[U,V,\tau,\xi]$ *if*

$$\frac{\wp\Big(D_q J_{q,\tau}^s \hbar\Big)(\wp) + \xi \wp^2 \Big(D_q^{(2)} J_{q,\tau}^s \hbar\Big)(\wp)}{(1-\xi) J_{q,\tau}^s \hbar(\wp) + \xi \wp \big(D_q J_{q,\tau}^s \hbar\big)(\wp)} \prec \frac{1+U\wp}{1+V\wp} \quad (\wp \in \Lambda), \tag{13}$$

or equivalently,

$$\left| \frac{\frac{\wp\left(D_q J_{q,\tau}^s \hbar\right)(\wp) + \xi \wp^2 \left(D_q^{(2)} J_{q,\tau}^s \hbar\right)(\wp)}{(1-\xi) J_{q,\tau}^s \hbar(\wp) + \xi \wp \left(D_q J_{q,\tau}^s \hbar\right)(\wp)} - 1}{U - V \frac{\wp\left(D_q J_{q,\tau}^s \hbar\right)(\wp) + \xi \wp^2 \left(D_q^{(2)} J_{q,\tau}^s \hbar\right)(\wp)}{(1-\xi) J_{q,\tau}^s \hbar(\wp) + \xi \wp \left(D_q J_{q,\tau}^s \hbar\right)(\wp)}} \right| < 1. \tag{14}$$

Remark 2. (i) *For* $s = 0$ *and* $\xi = 0$, *the class*

$$\mathcal{J}_{\mathcal{A}}^{q,0}[U,V,\tau,0] = \mathcal{S}_q^*[U,V]$$

was introduced by Srivastava et al. [55].

(ii) *For* $s = 0$, $\xi = 0$ *and* $q \to 1^-$, *the class*

$$\lim_{q \to 1^-} \mathcal{J}_{\mathcal{A}}^{q,0}[U,V,\tau,0] = \mathcal{S}^*[U,V]$$

was introduced by Janowski [56].

Definition 2. *Let* $0 \leqq \xi \leqq 1$. *A function* $\hbar \in \Sigma$ *is in the class* $\mathcal{J}_{\Sigma}^{q,s}[\phi_x^{\alpha},\tau,\xi]$ *if*

$$\begin{cases} \dfrac{\wp\left(D_q J^s_{q,\tau}\hbar\right)(\wp) + \xi\wp^2\left(D_q^{(2)} J^s_{q,\tau}\hbar\right)(\wp)}{(1-\xi)J^s_{q,\tau}\hbar(\wp) + \xi\wp\left(D_q J^s_{q,\tau}\hbar\right)(\wp)} \prec \phi^\alpha_x(\wp) \\ \dfrac{w\left(D_q J^s_{q,\tau}g\right)(w) + \xi w^2\left(D_q^{(2)} J^s_{q,\tau}g\right)(w)}{(1-\xi)J^s_{q,\tau}g(w) + \xi w\left(D_q J^s_{q,\tau}g\right)(w)} \prec \phi^\alpha_x(w), \end{cases} \tag{15}$$

where g and ϕ are given by (2) and (5), respectively.

Before proceeding to the main results, the following Lemmas shall be necessary.

Lemma 1 (see [57])**.** *Let $p(\wp) = 1 + c_1\wp + c_2\wp^2 + \cdots$, $\wp \in \Lambda$ is a function that has a positive real part in Λ, and μ is a complex number, then*

$$\left|c_2 - \mu c_1^2\right| \leqq 2\max\{1; |2\mu - 1|\}. \tag{16}$$

Lemma 2 (see [13])**.** *Let*

$$M(\wp) = 1 + \sum_{k=1}^{\infty} C_k\wp^k \prec H(\wp) = 1 + \sum_{k=1}^{\infty} d_k\wp^k.$$

If $H(\wp)$ is univalent and convex in Λ, then

$$|C_k| \leqq |d_1| \quad (k \in \mathbb{N}).$$

Lemma 3 (see [58])**.** *If $p(\wp) = 1 + \sum_{k=1}^{\infty} c_k\wp^k \in \mathcal{P}$, then*

$$|c_k| \leqq 2 \quad (k \in \mathbb{N}).$$

2. Main Results

In this section, for functions in the classes $\mathcal{J}^{q,s}_{\mathcal{A}}[U,V,\tau,\xi]$ and $\mathcal{J}^{q,s}_{\Sigma}[\phi^\alpha_x,\tau,\xi]$, which are defined above (see Definition 1 and 2), the coefficient estimates and the Fekete–Szegö inequality are solved. Many special cases and implications of our main findings are highlighted.

Theorem 1. *If $\hbar$ given by (1) is in the class $\mathcal{J}^{q,s}_{\mathcal{A}}[U,V,\tau,\xi]$, then*

$$\begin{aligned} |a_k| \leq & \frac{q(1-\xi)+\xi}{\left(q(1-\xi)+\xi[k]_q\right)|\Delta_k|} \\ & \times \prod_{j=1}^{k-1} \frac{[j-1]_q\left(q(1-\xi)+\xi[j]_q\right) + \left((1-\xi)+\xi[j]_q\right)(U-V)}{\left(q(1-\xi)+\xi[j]_q\right)[j]_q} \quad (k \geq 2), \end{aligned} \tag{17}$$

where $\Delta_k = \left(\frac{[1+\tau]_q}{[k+\tau]_q}\right)^s$, $k \in \mathbb{Z}^+$.

Proof. For $\hbar \in \mathcal{J}^{q,s}_{\mathcal{A}}[U,V,\tau,\xi]$, we have

$$v(\wp) := \frac{\wp\left(D_q J^s_{q,\tau}\hbar\right)(\wp) + \xi\wp^2\left(D_q^{(2)} J^s_{q,\tau}\hbar\right)(\wp)}{(1-\xi)J^s_{q,\tau}\hbar(\wp) + \xi\wp\left(D_q J^s_{q,\tau}\hbar\right)(\wp)} \prec \frac{1+U\wp}{1+V\wp} \quad (\wp \in \Lambda), \tag{18}$$

where

$$\frac{1+U\wp}{1+V\wp} = 1 + \sum_{k=0}^{\infty}(U-V)(-V)^k\wp^{k+1} = 1 + (U-V)\wp - V(U-V)\wp^2 + \cdots.$$

Since $v(\wp) = 1 + \sum_{k=0}^{\infty} v_k \wp^k$, from Lemma 2, we obtain

$$|v_k| \leqq U - V \quad (k \in \mathbb{N}). \tag{19}$$

From (18), we have

$$\wp\Big(D_q J_{q,\tau}^s \hbar\Big)(\wp) + \xi \wp^2 \Big(D_q^{(2)} J_{q,\tau}^s \hbar\Big)(\wp) = v(\wp)\Big[(1-\xi) J_{q,\tau}^s \hbar(\wp) + \xi \wp \Big(D_q J_{q,\tau}^s \hbar\Big)(\wp)\Big],$$

which shows that

$$\begin{aligned} &\wp + \sum_{k=2}^{\infty} [k]_q \Big[1 + \xi [k-1]_q\Big] a_k \Delta_k \wp^k \\ &\quad = \left(1 + \sum_{k=1}^{\infty} v_k \wp^k\right)\left(\wp + \sum_{k=2}^{\infty} \Big[1 + \xi\Big([k]_q - 1\Big)\Big] a_k \Delta_k \wp^k\right). \end{aligned} \tag{20}$$

In Equation (20), by comparing the coefficients $\wp^k$ for both sides, we obtain

$$[k-1]_q \Big[q(1-\xi) + \xi [k]_q\Big] a_k \Delta_k = \sum_{l=1}^{k-1} \Big[1 + \xi\Big([l]_q - 1\Big)\Big] a_l \Delta_l v_{k-l},$$

where $a_1 = 1,\ v_1 = 1$ and $\Delta_1 = 1$. The above equation gives

$$|a_k| \leq \frac{U - V}{[k-1]_q \Big[q(1-\xi) + \xi [k]_q\Big] |\Delta_k|} \sum_{l=1}^{k-1} \Big[1 + \xi\Big([l]_q - 1\Big)\Big] |a_l| |\Delta_l|.$$

Thus, we obtain

$$\begin{aligned} |a_2| &\leq \frac{U - V}{(q+\xi)|\Delta_2|} \\ |a_3| &\leq \frac{U - V}{\Big(q(1-\xi) + \xi[3]_q\Big)|\Delta_3|}\left(\frac{(q+\xi) + (1+\xi q)(U-V)}{[2]_q (q+\xi)}\right) \\ |a_4| &\leq \frac{U - V}{\Big(q(1-\xi) + \xi[4]_q\Big)|\Delta_4|}\left(\frac{(q+\xi) + (1+\xi q)(U-V)}{[2]_q (q+\xi)}\right) \\ &\quad \times \left(\frac{[2]_q \Big[q(1-\xi) + \xi[3]_q\Big] + \Big((1-\xi) + \xi[3]_q\Big)(U-V)}{\Big(q(1-\xi) + \xi[3]_q\Big)[3]_q}\right) \\ &\cdots \\ |a_k| &\leq \frac{q(1-\xi) + \xi}{\Big(q(1-\xi) + \xi[k]_q\Big)|\Delta_k|} \\ &\quad \times \prod_{j=1}^{k-1} \frac{[j-1]_q \Big(q(1-\xi) + \xi[j]_q\Big) + \Big((1-\xi) + \xi[j]_q\Big)(U-V)}{\Big(q(1-\xi) + \xi[j]_q\Big)[j]_q} \end{aligned}$$

The theorem's proof is now complete. □

For $s = 0$ and $\xi = 0$ in Theorem 1, we obtain a similar consequence of the class $\mathcal{S}_q^*[U, V]$.

Corollary 1. *Let $\hbar \in \mathcal{S}_q^*[U,V]$. Then,*

$$|a_k| \le \prod_{j=1}^{k-1} \frac{q[j-1]_q + (U-V)}{q[j]_q} \quad (k \ge 2).$$

Theorem 2. *If $\hbar$ given by (1) is in the class $\mathcal{J}_\Sigma^{q,s}[\phi_x^\alpha, \tau, \xi]$, then*

$$|a_2| \le \min\left\{ \begin{array}{l} \frac{2|\alpha|}{(q+\xi)|\Delta_2|}, \\ \frac{2|\alpha|}{\sqrt{\left|[2]_q(q+\xi(q^2+1))2\alpha x^2\Delta_3 - (q+\xi)\left[(1+\xi q)2\alpha x^2 + (q+\xi)\left(\alpha x^2 + x^2 - x - \frac{1}{2}\right)\right]\Delta_2^2\right|}} \end{array}\right. \tag{21}$$

and

$$|a_3| \le \min\left\{ \begin{array}{l} \frac{4|\alpha|^2}{(q+\xi)^2|\Delta_2|^2} + \frac{2|\alpha|}{[2]_q(q+\xi(q^2+1))|\Delta_3|}, \\ \frac{4|\alpha|^2}{\left|[2]_q(q+\xi(q^2+1))2\alpha x^2\Delta_3 - (q+\xi)\left[(1+\xi q)2\alpha x^2 + (q+\xi)\left(\alpha x^2 + x^2 - x - \frac{1}{2}\right)\right]\Delta_2^2\right|} + \frac{2|\alpha|}{[2]_q(q+\xi(q^2+1))|\Delta_3|}, \end{array}\right. \tag{22}$$

where $\Delta_k = \left(\frac{[1+\tau]_q}{[k+\tau]_q}\right)^s$, $k \in \mathbb{Z}^+$.

Proof. Let $\hbar \in \mathcal{J}_\Sigma^{q,s}[\phi_x^\alpha, \tau, \xi]$ and $g = \hbar^{-1}$. From (15), we are aware of the existence of two Schwartz functions $u(\wp)$ and $v(w)$, so that

$$\frac{\wp\left(D_q J_{q,\tau}^s \hbar\right)(\wp) + \xi\wp^2\left(D_q^{(2)} J_{q,\tau}^s \hbar\right)(\wp)}{(1-\xi)J_{q,\tau}^s\hbar(\wp) + \xi\wp\left(D_q J_{q,\tau}^s \hbar\right)(\wp)} = \phi_x^\alpha(u(\wp)) \tag{23}$$

and

$$\frac{w\left(D_q J_{q,\tau}^s g\right)(w) + \xi w^2\left(D_q^{(2)} J_{q,\tau}^s g\right)(w)}{(1-\xi)J_{q,\tau}^s g(w) + \xi w\left(D_q J_{q,\tau}^s g\right)(w)} = \phi_x^\alpha(v(w)). \tag{24}$$

We define the functions $s(\wp)$ and $t(w)$ as follows:

$$s(\wp) = \frac{1+u(\wp)}{1-u(\wp)} = 1 + s_1\wp + s_2\wp^2 + \cdots \in \mathcal{P}$$

and

$$t(w) = \frac{1+v(w)}{1-v(w)} = 1 + t_1 w + t_2 w^2 + \cdots \in \mathcal{P}.$$

Since

$$\phi_x^\alpha(\wp) = \frac{1}{(1-2x\wp+\wp^2)^\alpha} = \sum_{k=0}^{\infty} C_k^\alpha(x)\wp^k,$$

we obtain

$$\phi_x^\alpha(u(\wp)) = 1 + \frac{1}{2}C_1^\alpha(x)s_1\wp + \left[\frac{1}{2}C_1^\alpha(x)\left(s_2 - \frac{s_1^2}{2}\right) + \frac{1}{4}C_2^\alpha(x)s_1^2\right]\wp^2 + \cdots \tag{25}$$

and

$$\phi_x^\alpha(v(w)) = 1 + \frac{1}{2}C_1^\alpha(x)t_1 w + \left[\frac{1}{2}C_1^\alpha(x)\left(t_2 - \frac{t_1^2}{2}\right) + \frac{1}{4}C_2^\alpha(x)t_1^2\right]w^2 + \cdots. \tag{26}$$

Using the Taylor series formula, we have

$$\frac{\wp\left(D_q J^s_{q,\tau}\hbar\right)(\wp)+\xi\wp^2\left(D^{(2)}_q J^s_{q,\tau}\hbar\right)(\wp)}{(1-\xi)J^s_{q,\tau}\hbar(\wp)+\xi\wp\left(D_q J^s_{q,\tau}\hbar\right)(\wp)}=1+(q+\xi)\Delta_2 a_2\wp$$
$$+\left\{\left(q(q+1)+\left(q^2+1\right)[2]_q\xi\right)\Delta_3 a_3-(1+\xi q)(q+\xi)\Delta_2^2 a_2^2\right\}\wp^2+\cdots$$

and

$$\frac{w\left(D_q J^s_{q,\tau}g\right)(w)+\xi w^2\left(D^{(2)}_q J^s_{q,\tau}g\right)(w)}{(1-\xi)J^s_{q,\tau}g(w)+\xi w\left(D_q J^s_{q,\tau}g\right)(w)}=1-(q+\xi)\Delta_2 a_2 w$$
$$+\left\{\left(q(q+1)+\left(q^2+1\right)[2]_q\xi\right)\Delta_3\left(2a_2^2-a_3\right)-(1+\xi q)(q+\xi)\Delta_2^2 a_2^2\right\}w^2+\cdots.$$

As a result of comparing the coefficients in (23) and (24), we have

$$(q+\xi)\Delta_2 a_2=\frac{1}{2}C_1^\alpha(x)s_1, \tag{27}$$

$$[2]_q\left(q+\xi\left(q^2+1\right)\right)\Delta_3 a_3-(1+\xi q)(q+\xi)\Delta_2^2 a_2^2=\frac{1}{2}C_1^\alpha(x)\left(s_2-\frac{s_1^2}{2}\right)+\frac{1}{4}C_2^\alpha(x)s_1^2, \tag{28}$$

$$-(q+\xi)\Delta_2 a_2=\frac{1}{2}C_1^\alpha(x)t_1, \tag{29}$$

and

$$[2]_q\left(q+\xi\left(q^2+1\right)\right)\Delta_3\left(2a_2^2-a_3\right)-(1+\xi q)(q+\xi)\Delta_2^2 a_2^2$$
$$=\frac{1}{2}C_1^\alpha(x)\left(t_2-\frac{t_1^2}{2}\right)+\frac{1}{4}C_2^\alpha(x)t_1^2. \tag{30}$$

From (27) and (29), we have

$$a_2=\frac{C_1^\alpha(x)s_1}{2(q+\xi)\Delta_2}=\frac{-C_1^\alpha(x)t_1}{2(q+\xi)\Delta_2}. \tag{31}$$

Thus, we find that

$$s_1=-t_1 \tag{32}$$

and

$$8(q+\xi)^2\Delta_2^2 a_2^2=[C_1^\alpha(x)]^2\left(s_1^2+t_1^2\right). \tag{33}$$

Using Lemma 3, from (31) we find that

$$|a_2|\leq\frac{2|\alpha|}{(q+\xi)|\Delta_2|}. \tag{34}$$

Now from (28), (30)–(32), we have

$$4\left\{[2]_q\left(q+\xi\left(q^2+1\right)\right)[C_1^\alpha(x)]^2\Delta_3-(q+\xi)\left[(1+\xi q)[C_1^\alpha(x)]^2+(q+\xi)(C_2^\alpha(x)-C_1^\alpha(x))\right]\Delta_2^2\right\}a_2^2$$
$$=[C_1^\alpha(x)]^3(s_2+t_2).$$

Since

$$C_1^\alpha(x)=2\alpha x\ \text{ and }\ C_2^\alpha(x)=\alpha\left[2(1+\alpha)x^2-1\right],$$

we obtain

$$
\begin{aligned}
a_2^2 &= \frac{\left[C_1^\alpha(x)\right]^3 (s_2+t_2)}{4\left\{[2]_q(q+\xi(q^2+1))\left[C_1^\alpha(x)\right]^2\Delta_3-(q+\xi)\left[(1+\xi q)\left[C_1^\alpha(x)\right]^2+(q+\xi)\left(C_2^\alpha(x)-C_1^\alpha(x)\right)\right]\Delta_2^2\right\}} \\
&= \frac{\alpha^2 x^3 (s_2+t_2)}{[2]_q(q+\xi(q^2+1))2\alpha x^2\Delta_3-(q+\xi)\left[(1+\xi q)2\alpha x^2+(q+\xi)\left(\alpha x^2+x^2-x-\frac{1}{2}\right)\right]\Delta_2^2}.
\end{aligned} \tag{35}
$$

Applying Lemma 3 to the coefficients s_2 and t_2, we have

$$
|a_2| \le \frac{4|\alpha|}{\sqrt{\left|[2]_q(q+\xi(q^2+1))2\alpha x^2\Delta_3-(q+\xi)\left[(1+\xi q)2\alpha x^2+(q+\xi)\left(\alpha x^2+x^2-x-\frac{1}{2}\right)\right]\Delta_2^2\right|}}.
$$

By subtracting (28) from (30), we have

$$
2[2]_q\left(q+\xi\left(q^2+1\right)\right)\left(a_3-a_2^2\right)\Delta_3 = \frac{1}{2}C_1^\alpha(x)(s_2-t_2)+\frac{1}{4}\left(C_2^\alpha(x)-C_1^\alpha(x)\right)\left(s_1^2-t_1^2\right) \tag{36}
$$

which yields

$$
a_3 = a_2^2 + \frac{C_1^\alpha(x)(s_2-t_2)}{4[2]_q(q+\xi(q^2+1))\Delta_3}. \tag{37}
$$

Now taking the modulus of (37) and using Lemma 3, we obtain

$$
|a_3| \le |a_2|^2 + \frac{\left|C_1^\alpha(x)\right|}{[2]_q(q+\xi(q^2+1))|\Delta_3|} \tag{38}
$$

and thus, by using (34)

$$
|a_3| \le \frac{4|\alpha|^2}{(q+\xi)^2\Delta_2^2} + \frac{2|\alpha|}{[2]_q(q+\xi(q^2+1))|\Delta_3|}.
$$

In addition, using (35) and (38), we obtain

$$
\begin{aligned}
|a_3| \le & \frac{4|\alpha|^2}{\left|[2]_q(q+\xi(q^2+1))2\alpha x^2\Delta_3-(q+\xi)\left[(1+\xi q)2\alpha x^2+(q+\xi)\left(\alpha x^2+x^2-x-\frac{1}{2}\right)\right]\Delta_2^2\right|} \\
&+ \frac{2|\alpha|}{[2]_q(q+\xi(q^2+1))|\Delta_3|}.
\end{aligned}
$$

The theorem's proof is now complete. □

Theorem 3. *If* $\hbar$ *given by (1) is in the class* $\mathcal{J}_{\mathcal{A}}^{q,s}[U,V,\tau,\xi]$, *then*

$$
\left|a_3-\mu a_2^2\right| \le \frac{U-V}{(1+q)(q+(q^2+1)\xi)|\Delta_3|}\max\{1;|2\mu_1(q)-1|\}, \tag{39}
$$

where

$$
\mu_1(q) = \frac{(q+\xi)[(q+\xi)(V+1)-(1+\xi q)(U-V)]\Delta_2^2+\mu(1+q)\left[q+(q^2+1)\xi\right](U-V)\Delta_3}{2(q+\xi)^2\Delta_2^2}
$$

and $\Delta_k = \left(\frac{[1+\tau]_q}{[k+\tau]_q}\right)^s, k \in \mathbb{Z}^+$.

Proof. Let $\hbar \in \mathcal{J}_{\mathcal{A}}^{q,s}[U,V,\tau,\xi]$. Using the Taylor series formula, we have

$$\begin{aligned}\frac{\wp\left(D_q J_{q,\tau}^s \hbar\right)(\wp)+\xi\wp^2\left(D_q^{(2)} J_{q,\tau}^s \hbar\right)(\wp)}{(1-\xi)J_{q,\tau}^s \hbar(\wp)+\xi\wp\left(D_q J_{q,\tau}^s \hbar\right)(\wp)} &= 1+(q+\xi)\Delta_2 a_2\wp \\ &+\left\{\left(q(q+1)+\left(q^2+1\right)[2]_q\xi\right)\Delta_3 a_3-(1+\xi q)(q+\xi)\Delta_2^2 a_2^2\right\}\wp^2+\cdots. \quad (40)\end{aligned}$$

From (13), we are aware of the existence of the Schwarz function h such that

$$\frac{\wp\left(D_q J_{q,\tau}^s \hbar\right)(\wp)+\xi\wp^2\left(D_q^{(2)} J_{q,\tau}^s \hbar\right)(\wp)}{(1-\xi)J_{q,\tau}^s \hbar(\wp)+\xi\wp\left(D_q J_{q,\tau}^s \hbar\right)(\wp)}=\frac{1+Uh(\wp)}{1+Vh(\wp)}.$$

We now define a function $w \in \mathcal{P}$ by

$$w(\wp)=\frac{1+h(\wp)}{1-h(\wp)}=1+w_1\wp+w_2\wp^2+\cdots.$$

This implies that

$$h(\wp)=\frac{w(\wp)-1}{w(\wp)+1}=1+\frac{1}{2}w_1\wp+\left(\frac{1}{2}w_2-\frac{1}{4}w_1^2\right)\wp^2+\cdots.$$

In addition, we have

$$\frac{1+Uh(\wp)}{1+Vh(\wp)}=1+\frac{1}{2}(U-V)w_1\wp+\left[\frac{1}{2}(U-V)w_2-\frac{1}{4}(V+1)(U-V)w_1^2\right]\wp^2+\cdots. \quad (41)$$

Therefore, we obtain

$$a_2 = \frac{U-V}{2(q+\xi)\Delta_2}w_1, \quad (42)$$

$$a_3 = \frac{U-V}{2(1+q)(q+(q^2+1)\xi)\Delta_3}\left\{w_2-\frac{1}{2}\left[(V+1)-\left(\frac{1+\xi q}{q+\xi}\right)(U-V)\right]w_1^2\right\}. \quad (43)$$

Now, we can find that

$$\begin{aligned}\left|a_3-\mu a_2^2\right| &= \left|\frac{U-V}{2(1+q)(q+(q^2+1)\xi)\Delta_3}\left\{w_2-\frac{1}{2}\left[(V+1)-\left(\frac{1+\xi q}{q+\xi}\right)(U-V)\right]w_1^2\right\}\right. \\ &\quad \left.-\frac{(U-V)^2}{4(q+\xi)^2\Delta_2^2}w_1^2\right| \\ &= \frac{U-V}{2(1+q)(q+(q^2+1)\xi)|\Delta_3|}\left|\left\{w_2-\frac{1}{2}\left[(V+1)\right.\right.\right. \\ &\quad \left.\left.\left.-\left(\frac{(1+\xi q)(q+\xi)\Delta_2^2+\mu(1+q)\left[q+(q^2+1)\xi\right]\Delta_3}{(q+\xi)^2\Delta_2^2}\right)(U-V)\right]w_1^2\right\}\right| \\ &= \frac{U-V}{2(1+q)(q+(q^2+1)\xi)|\Delta_3|}\left|w_2-\mu_1(q)w_1^2\right|, \quad (44)\end{aligned}$$

where

$$\mu_1(q)=\frac{(q+\xi)[(q+\xi)(V+1)-(1+\xi q)(U-V)]\Delta_2^2+\mu(1+q)\left[q+(q^2+1)\xi\right](U-V)\Delta_3}{2(q+\xi)^2\Delta_2^2}.$$

□

Using Lemma 1 in (44), we achieve the desired results. The proof of Theorem 3 is now finished.

When $s = 0$, $\xi = 0$ and $q \to 1^-$, we obtain a consequence of the class $\mathcal{S}^*[U, V]$ that was described by Janowski [56].

Corollary 2. *Let $\hbar \in \mathcal{S}^*[U, V]$. Then,*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{U - V}{2} \max\{1; |(U - V)(2\mu - 1) + V|\}.$$

Theorem 4. *If $\hbar$ given by (1) is in the class $\mathcal{J}_\Sigma^{q,s}[\phi_x^\alpha, \tau, \xi]$, then*

$$\left|a_3 - \mu a_2^2\right| \leq |\alpha|(|M + N| + |M - N|), \tag{45}$$

where

$$M = \frac{(1-\mu)\alpha x^2}{[2]_q(q + \xi(q^2+1))2\alpha x^2 \Delta_3 - (q+\xi)\left[(1+\xi q)2\alpha x^2 + (q+\xi)\left(\alpha x^2 + x^2 - x - \frac{1}{2}\right)\right]\Delta_2^2}$$
$$N = \frac{1}{2[2]_q(q + \xi(q^2+1))\Delta_3} \tag{46}$$

and $\Delta_k = \left(\frac{[1+\tau]_q}{[k+\tau]_q}\right)^s$, $k \in \mathbb{Z}^+$.

Proof. From (37), we have

$$a_3 - \mu a_2^2 = (1-\mu)a_2^2 + \frac{C_1^\alpha(x)(s_2 - t_2)}{4[2]_q(q + \xi(q^2+1))\Delta_3} \tag{47}$$

Using (35) and (47), we obtain

$$\begin{aligned} a_3 - \mu a_2^2 = & \frac{\alpha x(s_2 - t_2)}{2[2]_q(q + \xi(q^2+1))\Delta_3} \\ & + \frac{(1-\mu)\alpha^2 x^3 (s_2 + t_2)}{[2]_q(q + \xi(q^2+1))2\alpha x^2 \Delta_3 - (q+\xi)\left[(1+\xi q)2\alpha x^2 + (q+\xi)\left(\alpha x^2 + x^2 - x - \frac{1}{2}\right)\right]\Delta_2^2}. \end{aligned}$$

After making the necessary arrangements, we rewrite the previous equality as

$$a_3 - \mu a_2^2 = \alpha x[(M+N)s_2 + (M-N)t_2], \tag{48}$$

where M and N are given by (46). Taking the absolute value of (48), we derive the desired inequality from Lemma 3.
□

3. Conclusions

We used the q-derivative operator D_q $(0 < q < 1)$ and the Gegenbauer polynomials in this study to ventilate and work on two new subclasses of the class of q-starlike functions and the class of analytic and bi–univalent functions. We obtained several coefficient estimates and Fekete–Szegö–type inequalities for each subclass. We also show that our findings extend and generalize those found in previous works. These results will stimulate various new studies for research in this and related fields. Considering this study, someone can define different general subclasses of analytic and bi-univalent functions by using special polynomials. For these subclasses, some problems essentially subordination, inclusion, coefficient inequalities and coefficient estimates containing the second, third, and fourth Hankel determinants and the Fekete–Szegö functional can be considered. In two recent survey cum expository review published articles (see [59,60]), the triviality of any attempts to translate any known q-results to the corresponding rather inconsequential (p, q)-results by forcing-in a redundant parameter p has already been demonstrated, so any such amateurish-type ventures should be discouraged.

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