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A Symmetry Chaotic Model with Fractional Derivative Order via Two Different Methods

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Abstract: In this article, we have investigated solutions to a symmetry chaotic system with fractional derivative order using two different methods—the numerical scheme for the ABC fractional derivative, and the Laplace decomposition method, with help from the MATLAB and Mathematica platforms. We have explored progressive and efficient solutions to the chaotic model through the successful implementation of two mathematical methods. For the phase portrait of the model, the profiles of chaos are plotted by assigning values to the attached parameters. Hence, the offered techniques are relevant for advanced studies on other models. We believe that the unique techniques that have been proposed in this study will be applied in the future to build and simulate a wide range of fractional models, which can be used to address more challenging physics and engineering problems.

Keywords: numerical solutions; numerical scheme for AB operator; analytical solutions; Laplace decomposition method; chaos

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1. Introduction

The modeling of diffusion, control, and viscoelasticity in fractional calculus has increased the popularity of applied mathematics during the last decades. Research in physics and engineering makes use of fractional differential equations [1–14], and diverse methods have been devised to solve these fractional differential equations [15,16]. Recent years have seen a surge in the literature on modeling chaotic and hyperchaotic systems, with several applications across various fields including electrical circuits, biology, and physics [17–20].

One of the most well-known uses of chaos is electrical circuit modeling, which has been covered in several papers. Given how challenging it is to predict many real-world events, the use of chaotic models is therefore justified. Asymptotic stability, which clarifies how model parameters affect the dynamics of chaotic models, and Lyapunov exponents, which identify the precise nature of the chaos, are just a few of the many novel techniques for evaluating chaotic systems that have arisen in recent years. The mathematical and scientific fields of fractional calculus are remarkably diverse. The use of fractional calculus in science, mathematics, biology, and other fields is expanding rapidly [21–31] for some cutting-edge research and applications in the area of fractional calculus. This discovery is significant since there are numerous meanings for fractional operators. The Caputo derivative and the Riemann-Liouville derivative are two examples of fractional derivatives with singular kernels. Singular-free derivatives include those with exponential and

Mittag–Leffler kernels [32]. Of note, fractional derivatives are particularly useful since they incorporate the influences of long-term memory into account.

In this paper, we used a pair of techniques to examine the solutions to a symmetric chaotic system with fractional derivative order. We introduced the chaotic system with fractional derivative order as follows:

$$\begin{aligned} {}_{0}^{ABC}D_t^{\alpha}x(t) &= a(y-x)+byz^2, \\ {}_{0}^{ABC}D_t^{\alpha}y(t) &= cx+dxz^2, \\ {}_{0}^{ABC}D_t^{\alpha}z(t) &= \beta z+\in|x|. \end{aligned} \tag{1}$$

The physical importance of our system lies in the study of complex dynamics and the display of behaviors in order to fully control the chaos, with this particular system being used in the fields of physics and engineering, especially in electrical circuits and coding.

Through the effective use of two mathematical methodologies, we have investigated advanced and efficient solutions to the symmetry chaotic model. Therefore, the presented approaches hold promise for further study of other models.

Recent studies [33] have shown that there are several solid justifications for employing fractional derivatives in real-world situations. The Lorenz attractor, Chua's electrical circuit, Chen's chaotic system, Lu's chaotic system, and the fundamental chaotic system are only a few examples of chaotic systems that can be found in the literature. It is common knowledge that chaotic systems react violently to both initial circumstances and slight changes in their parameters. For research on the use of fractional derivatives to model these chaotic systems, the fractional calculus papers mentioned below can be referred [34]. The Laplace transform decomposition method [35,36] is essential for solving a wide variety of problems. In certain cases, it has been shown to be successful in overcoming problems.

Several methodologies have been used to address challenges in management, economics, biology, physics, and engineering [37–41].

The significance of this study lies in its ability to solve a chaotic system with fractional derivative order using two distinct approaches. A numerical approach has been displayed for the ABC fractional derivative and the Laplace decomposition technique using the MATLAB and Mathematica platforms.

The numerical results demonstrate that our approach carries out its operations when dealing with fractions in a manner that is satisfactory in terms of its numerical stability. For the phase portrait of the model, chaotic results are produced by assigning specific values to the attached parameters. Thus, the methodologies offered are meritoriously pertinent for future studies on various models. This research was conducted in the hope that it might be useful for future fractional system applications.

2. Preliminaries

2.1. The Mittag–Leffler Function Is Defined as [41]:

$$E_{\alpha}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+1)}. \tag{2}$$

2.2. Atangana–Baleanu–Caputo Fractional Derivative

The Atangana–Baleanu–Caputo derivative (ABC) of a function $y \in H^1(0,\mathrm{c})$, with $\alpha \in (0,1]$ is defined as [42]:

$${}_{0}^{ABC}D_t^{\alpha}y(t)=\frac{B(\alpha)}{1-\alpha}\int_0^t y'(\tau)E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right)d\tau.\ 0<\alpha<1, \tag{3}$$

where $H^1(0,c), c>0$ is a space of square-integrable functions and is itself defined as:

$$H^1(0,c)=\{y(t)\in L^2(0,c)|y'(t)\in L^2(0,c)\}\text{ and }B(\alpha)=1-\alpha+\frac{\alpha}{\sqrt{\alpha}} \tag{4}$$

2.3. The Atangana–Baleanu Fractional Integral

The Atangana–Baleanu fractional integral of the function $y \in H^1(0,c)$, with $c > 0$, is as follows [43]:

$$ {}_0^{AB}I_t^{\alpha} y(t) = \frac{1-\alpha}{M(\alpha)} y(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t y(\tau)(t-\tau)^{\alpha-1} d\tau. \tag{5} $$

2.4. Laplace Transform of an Atangana–Baleanu–Caputo Derivative

The Laplace transform of the fractional derivative [41] given in Equation (3) is defined as:

$$ l\left[{}_0^{ABC}D_t^{\alpha} y(t)\right] = \frac{B(\alpha)}{1-\alpha} \frac{s^{\alpha}\, l[y(t)] - s^{\alpha-1} y(0)}{s^{\alpha} + \frac{\alpha}{1-\alpha}}. \tag{6} $$

3. Numerical Scheme for the *ABC* Fractional Derivative (*ABC*-FD)

This section focuses on chaotic models in which the ABC fractional derivative was present. In this case, the following non-linear fractional ordinary equation was considered:

$$ {}_0^{ABC}D_t^{\alpha} y(t) = \mathcal{F}(t, y(t)), y(0) = y_0. \tag{7} $$

Using the fundamental theorem of fractional calculus, the equation above can be transformed into a fractional integral equation:

$$ y(t) - y_0 = \frac{(1-\alpha)}{ABC(\alpha)} \mathcal{F}(t, y(t)) \frac{(\alpha)}{ABC(\alpha)\Gamma(\alpha+2)} \int_0^t \mathcal{F}(t, y(t))(t-\tau)^{\alpha-1} d\tau. \tag{8} $$

We presented the numerical method of this system using a new approach at t_{n+1}, n = 0, 1,2. . ., and reformulated this method as follows:

$$ \begin{aligned} y(t_{n+1}) - y_0 &= \frac{(1-\alpha)}{ABC(\alpha)} \mathcal{F}(t_n, y(t_n)) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha+2)} \sum_{k=0}^{n} \int_0^{\zeta_{n+1}} \mathcal{F}(\tau, y(t))(t_{n+1}-\tau)^{\alpha-1} d\tau \\ &= \frac{(1-\alpha)}{ABC(\alpha)} \mathcal{F}(t_n, y(t_n)) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \mathcal{F}(\tau, y(t))(t_{n+1}-\tau)^{\alpha-1} d\tau. \end{aligned} \tag{9} $$

With $[t_k, t_{k+1}]$, and the function $\mathcal{F}(\tau, y(\tau))$, the following can therefore be approximated using two-step Lagrange polynomial interpolation:

$$ \begin{aligned} P_k(\tau) &= \frac{\tau - t_{k-1}}{t_k - t_{k-1}} \mathcal{F}(t_k, y(t_k)) - \frac{\tau - \zeta_{k-1}}{\zeta_k - \zeta_{k-1}} \mathcal{F}(t_{k-1}, y(t_{k-1})) \\ &= \frac{\mathcal{F}(\zeta_k, y(\zeta_k))(\tau - \zeta_{k-1})}{\hbar} - \frac{\mathcal{F}(\zeta_{k-1}, y(\zeta_{k-1}))(\tau - \zeta_k)}{\hbar} \approx \frac{\mathcal{F}(t_k, y_k))(\tau - \zeta_{k-1})}{\hbar} - \frac{\mathcal{F}(\zeta_{k-1}, y(\zeta_{k-1}))(\tau - \zeta_k)}{\hbar}. \end{aligned} \tag{10} $$

This approximation may thus be used into Equation (3) to obtain

$$ \begin{aligned} y_{n+1} = y_0 + \frac{(1-\alpha)}{ABC(\alpha)} \mathcal{F}(t_n, y(t_n)) \frac{(\alpha)}{ABC(\alpha)\Gamma(\alpha+2)} \sum_{k=0}^{n} \left(\frac{\mathcal{F}(t_k, y_k)}{h}\right) \int_{t_k}^{t_{k+1}} (\tau - t_{k-1})(t_{n+1}-\tau)^{\alpha-1} d\tau - \\ \left(\frac{\mathcal{F}(t_k, y_k)}{h}\right) \int_{t_k}^{t_{k+1}} (\tau - t_k)(t_{n+1}-\tau)^{\alpha-1} d\tau. \end{aligned} \tag{11} $$

For simplicity, we implemented the following equations:

$$ \begin{aligned} A^{\circ}_{\alpha,k,1} &= \int_{t_k}^{t_{k+1}} (\tau - t_{k-1})(t_{n+1}-\tau)^{\alpha-1} d\tau, \\ A^{\circ}_{\alpha,k,2} &= \int_{t_k}^{t_{k+1}} (\tau - t_k)(t_{n+1}-\tau)^{\alpha-1} d\tau. \end{aligned} \tag{12} $$

$$ \begin{aligned} A^{\circ}_{\alpha,k,1} &= \frac{h^{\alpha+1(n+1-k)(n-k+2+\alpha)-(n-k)^{\alpha}(n-k+2+2\alpha)}}{\alpha(\alpha+1)}, \\ A^{\circ}_{\alpha,k,1} &= \frac{h^{\alpha+1(n+1-k)-(n-k)^{\alpha}(n-k+1+\alpha)}}{\alpha(\alpha+1)}. \end{aligned} \tag{13} $$

From combining Equations (11) and (12) and substituting them into Equation (10), we obtained

$$
\begin{aligned}
y_{n+1} = y_0 + \frac{(1-\alpha)}{ABC(\alpha)}\mathcal{F}(t_n, y(t_n)) \\
+\frac{(\alpha)}{ABC(\alpha)}\sum_{k=0}^{n}\frac{h^{\alpha}\mathcal{F}(t_{\kappa}, y_{\kappa})}{\Gamma(\alpha+2)}\left((n+1-\kappa)^{\alpha}(n-\kappa+2+\alpha)-(n-\kappa)^{\alpha}(n-\kappa+2(1+2\alpha))\right). \\
-\frac{h^{\alpha}\mathcal{F}(t_{\kappa-1}, y_{\kappa-1})}{\Gamma(\alpha+2)}\left((n+1-\kappa)^{\alpha}-(n-\kappa)^{\alpha}(n-\kappa+1+\alpha)\right).
\end{aligned}
\tag{14}
$$

4. Applications of (ABC-FD)

In this section, we investigated the utility of a numerical scheme for the ABC fractional derivative for solving systems and resorted to numerical simulations to examine the solutions to our test issues. In order to describe this fractional chaotic system, we wrote it down as a system of three fractional-order differential equations (FODEs) which are as follows:

$$
\begin{aligned}
{}_{0}^{ABC}D_t^{\alpha}x(t) = a(y-x)+byz^2, \\
{}_{0}^{ABC}D_t^{\alpha}y(t) = cx+dxz^2, \\
{}_{0}^{ABC}D_t^{\alpha}z(t) = \beta z + \in |x|.
\end{aligned}
\tag{15}
$$

When $a = 7.5, b = 1, c = 5, d = -1, \beta = 1,$ and $\epsilon = 1$, the system was deemed chaotic, as shown in Figure 1 with $x(0) = \frac{1}{2}$, $y(0) = 0.4$, and $z(0) = -0.4$, respectively.

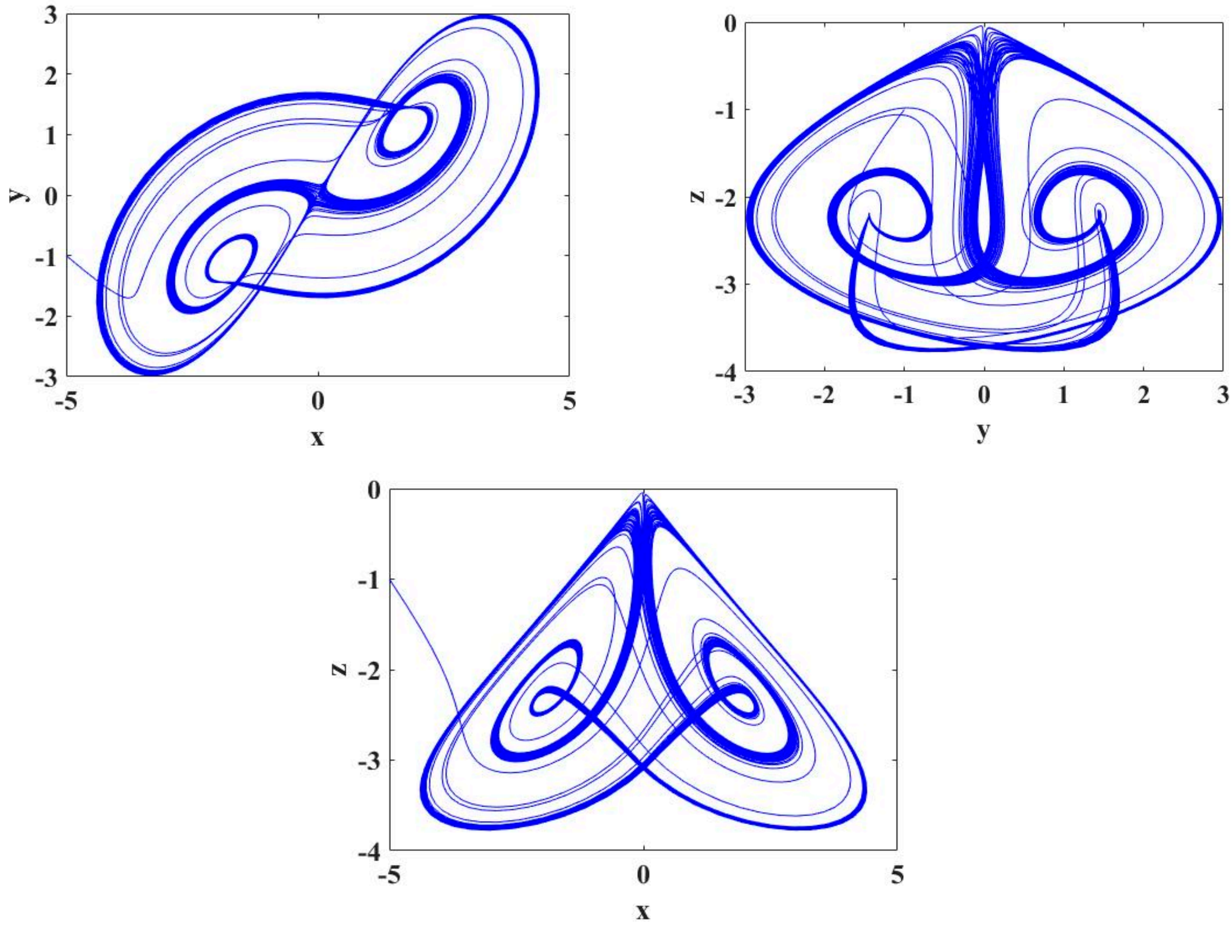

Figure 1. Chaotic attractor of Equation (15), when $\alpha = 0.97$.

Table 1 provides numerical results from the ABC method to the fractional model (15), when $\alpha = 1$, $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \beta, \epsilon) = (7.5,\ 1, 5, -1, 1, 1)$, and $(x_0, y_0, z_0) = (-5, -1, -1)$, respectively, using the numerical scheme for the ABC operator and the RK4 method at $t = 0.2$. Furthermore, our numerical solutions were found to be in great agreement with those obtained with the RK4 technique when the step size h was sufficiently small. We presented accurate and close solutions that were remarkably similar to RK4. In Table 2, the numerical solutions were presented in fractional order, and this provides evidence in that the method is effective and can be used in different fractional systems.

Table 1. Solutions of Equation (20) when $\alpha = 1$, $x(0) = 0.5, y(0) = 0.4, z(0) = -0.4$, and $t = 0.2$, respectively.

h	x	y	z
1/320	0.467020507081402	0.477135507758732	−0.384820957568090
1/640	0.467484966883951	0.478385773744663	−0.384554656198324
1/1280	0.467724698261518	0.479009970990537	−0.384422673415633
1/2560	0.467846465168544	0.479321827468676	−0.384356986890225
1/5120	0.467907826528056	0.479477694202676	−0.384324221394378
1/10240	0.467938626970537	0.479555612075837	−0.384307858278194
1/20480	0.467954057165821	0.479594567124480	−0.384299681651556
1/40960	0.467961779760992	0.479614043675021	−0.384295594574026
R K4	0.467969508867656	0.479633519438750	−0.384291508484103

Table 2. Solutions of Equation (15) when $\alpha = 0.75$, $x(0) = 0.5, y(0) = 0.4, z(0) = -0.4$, and $t = 0.2$, respectively.

h	x	y	z
1/320	0.476834615196041	0.477608450387220	−0.383951033804631
1/640	0.477214259207429	0.477680964115402	−0.383847719119722
1/1280	0.477402328114390	0.477716831041644	−0.383796524583421
1/2560	0.477495834460713	0.477734624943283	−0.383771103225998
1/5120	0.477542430419032	0.477743474458066	−0.383758454328070
1/10240	0.477565681810709	0.477747883627248	−0.383752150593742
1/20480	0.477577293723388	0.477750083206768	−0.383749005467426
1/40960	0.467583095603098	0.477751181416007	−0.383747435053161
R K4	0.467969508867656	0.479633519438750	−0.384291508484103

5. Laplace Decomposition Method (LDM)

In this part, we explain the algorithm of LDM by considering ODE (7) with the Atangana– Baleanu fractional derivative, with Equation (7) written as:

$$ {}_{0}^{ABC}D_{t}^{\alpha}y(t) = \mathcal{G}y(t) + \mathcal{H}y(t) + \mathcal{U}(t), \tag{16} $$

where $\mathcal{G}y(t)$ is a linear term, $\mathcal{H}y(t)$ is a nonlinear term, and $\mathcal{U}(t)$ is the source term.

After applying the Laplace decomposition method displayed in Equation (16), the following equation was devised:

$$ l[y(t)] = \frac{y(0)}{s} + \frac{1-\alpha}{ABC(\alpha)}\left(1 + \frac{\alpha}{s^{\alpha}(1-\alpha)}\right) l[\mathcal{U}(t) + \mathcal{G}y(t) + \mathcal{H}y(t)], \tag{17} $$

From taking the inverse Laplace transform of Equation (17), we obtained:

$$ y(t) = \mathcal{P}(t) + l^{-1}\left[\frac{1-\alpha}{ABC(\alpha)}\left(1 + \frac{\alpha}{s^{\alpha}(1-\alpha)}\right) l[\mathcal{G}y(t) + \mathcal{H}y(t)]\right], \tag{18} $$

where the LDM takes a solution as

$$y(t) = \sum_{n=0}^{\infty} y_n(t), \tag{19}$$

and the nonlinear terms are represented as

$$\mathcal{H}y(t) = \sum_{n=0}^{\infty} A_n, \tag{20}$$

where A_n, a domain polynomial can be computed using

$$A_n = \frac{1}{n!}\frac{d^n}{d\mathcal{J}^n}\sum_{i=0}^{n}\left[\mathcal{J}^i y_i(t)\right]_{\mathcal{J}=0}. \tag{21}$$

By substituting Equations (19) and (20) into Equation (18), we obtain

$$\sum_{n=0}^{\infty} y_n(t) = \mathcal{P}(t) + l^{-1}\left[\frac{1-\alpha}{ABC(\alpha)}\left(1+\frac{\alpha}{s^{\alpha}(1-\alpha)}\right)l\left[\mathcal{G}\sum_{n=0}^{\infty} y_n(t) + A_n\right]\right], \tag{22}$$

which gives the general recursive formula

$$\begin{aligned} y_0(t) &= \mathcal{P}(t), \\ y_{n+1}(t) &= l^{-1}\left[\tfrac{1-\alpha}{ABC(\alpha)}\left(1+\tfrac{\alpha}{s^{\alpha}(1-\alpha)}\right)l[\mathcal{G}y_n(t) + A_n]\right], \ \ n \geq 0. \end{aligned} \tag{23}$$

The final solution can be ultimately written as

$$y(t) = \sum_{n=0}^{\infty} y_n(t). \tag{24}$$

6. Application of (LDM)

In this part, we explain the solution of a fractional chaotic system (15) using LDM. From applying the Laplace decomposition method to Equation (15), we obtain

$$\begin{aligned} l[x(t)] &= \tfrac{x(0)}{s} + \tfrac{1-\alpha}{B(\alpha)}\left(1+\tfrac{\alpha}{s^{\alpha}(1-\alpha)}\right)l\left[a(y-x) + byz^2\right], \\ l[y(t)] &= \tfrac{y(0)}{s} + \tfrac{1-\alpha}{B(\alpha)}\left(1+\tfrac{\alpha}{s^{\alpha}(1-\alpha)}\right)l\left[cx + dxz^2\right], \\ l[z(t)] &= \tfrac{z(0)}{s} + \tfrac{1-\alpha}{B(\alpha)}\left(1+\tfrac{\alpha}{s^{\alpha}(1-\alpha)}\right)l[\beta z + \epsilon|x|]. \end{aligned} \tag{25}$$

Operating the inverse Laplace transform of Equation (25) obtained the following equation.

$$\begin{aligned} x(t) &= x(0) + l^{-1}\left[\tfrac{1-\alpha}{ABC(\alpha)}\left(1+\tfrac{\alpha}{s^{\alpha}(1-\alpha)}\right)l\left[a(y-x) + byz^2\right]\right], \\ y(t) &= y(0) + l^{-1}\left[\tfrac{1-\alpha}{ABC(\alpha)}\left(1+\tfrac{\alpha}{s^{\alpha}(1-\alpha)}\right)l\left[cx + dxz^2\right]\right], \\ z(t) &= z(0) + l^{-1}\left[\tfrac{1-\alpha}{ABC(\alpha)}\left(1+\tfrac{\alpha}{s^{\alpha}(1-\alpha)}\right)l[\beta z + \epsilon|x|]\right]. \end{aligned} \tag{26}$$

The LDM then represents a solution as an infinite series,

$$\begin{aligned} x(t) &= \sum_{n=0}^{\infty} x_n(t), \\ y(t) &= \sum_{n=0}^{\infty} y_n(t), \\ z(t) &= \sum_{n=0}^{\infty} z_n(t). \end{aligned} \tag{27}$$

and the nonlinear terms are decomposed as:

$$yz^2 = \sum_{n=0}^{\infty} A_n\,, \ \text{ and } \ \ xz^2 = \sum_{n=0}^{\infty} B_n, \tag{28}$$

where A_n and B_n are Adomain polynomials which can be calculated by Equation (21). Substituting Equations (27) and (28) into Equation (26), yields

$$\begin{aligned}
&\sum_{n=0}^{\infty} x_n(t) = x(0) + l^{-1}\left[\tfrac{1-\alpha}{B(\alpha)}\left(1+\tfrac{\alpha}{s^{\alpha}(1-\alpha)}\right)l[a(\sum_{n=0}^{\infty} x_n(t) - \sum_{n=0}^{\infty} y_n(t)) + b\sum_{n=0}^{\infty} A_n\ ]\right],\\
&\sum_{n=0}^{\infty} y_n(t) = y(0) + l^{-1}\left[\tfrac{1-\alpha}{B(\alpha)}\left(1+\tfrac{\alpha}{s^{\alpha}(1-\alpha)}\right)l[c\sum_{n=0}^{\infty} x_n(t) + d\sum_{n=0}^{\infty} B_n\ ]\right],\\
&\sum_{n=0}^{\infty} z_n(t) = z(0) + l^{-1}\left[\tfrac{1-\alpha}{B(\alpha)}\left(1+\tfrac{\alpha}{s^{\alpha}(1-\alpha)}\right)l[\beta\sum_{n=0}^{\infty} z_n(t) + \epsilon|\sum_{n=0}^{\infty} x_n(t)|]\right].
\end{aligned} \tag{29}$$

With the following recursive formula,

$$\begin{aligned}
&x_0(t) = x(0)\\
&x_{n+1}(t) = l^{-1}\left[\tfrac{1-\alpha}{B(\alpha)}\left(1+\tfrac{\alpha}{s^{\alpha}(1-\alpha)}\right)l[a(x_n(t) - y_n(t)) + bA_n],\right], n\geq 0,\\
&y_0(t) = y(0)\\
&y_{n+1}(t) = l^{-1}\left[\tfrac{1-\alpha}{B(\alpha)}\left(1+\tfrac{\alpha}{s^{\alpha}(1-\alpha)}\right)l[cx_n(t) + dB_n]\right], n\geq 0,\\
&z_0(t) = z(0)\\
&z_{n+1}(t) = l^{-1}\left[\tfrac{1-\alpha}{B(\alpha)}\left(1+\tfrac{\alpha}{s^{\alpha}(1-\alpha)}\right)l[\beta z_n(t) + \epsilon|x_n(t)|]\right], n\geq 0.
\end{aligned} \tag{30}$$

this gave the following:

$$x_0(t) = 0.5,$$

$$x_1(t) = -\frac{0.1(a-0.64b)(t^{\alpha}\alpha + (1-\alpha)\Gamma[1+\alpha])}{B(\alpha)\Gamma[1+\alpha]},$$

$$\begin{aligned}
x_2 = -\tfrac{1}{(B(\alpha))^2\Gamma[1+\alpha]\Gamma[1+2\alpha]}&(t^{\alpha}\alpha(a^2(-0.2+0.2\alpha) + a(-c-0.16d\\
&+b(0.128-0.128\alpha) + c\alpha + 0.16d\alpha) + b(-0.0256d\\
&+c(-0.16+0.16\alpha) + 0.0256d\alpha - 0.256\beta + 0.256\alpha\beta\\
&+0.32\epsilon - 0.326\alpha\epsilon))\Gamma[1+2\alpha] + \Gamma[1+\alpha](t^{2\alpha}\alpha^2(-0.1a^2\\
&+a(0.064b - 0.5c - 0.08d) + b(-0.08c - 0.0128d\\
&-0.128\beta + 0.16\epsilon)) + (-0.1a^2(1-1.\alpha)^2 + a(0.064b\\
&-0.5c - 0.08d)(1.-1.\alpha)^2 + b\Big(-0.08(1-\alpha)^2\\
&-0.0128d(1.-1.\alpha)^2 - 0.128\beta + 0.256\alpha\beta - 0.128\alpha^2\beta\\
&+0.16\epsilon - 0.32\alpha\epsilon + 0.16\alpha^2\epsilon))\Gamma[1+2\alpha])),
\end{aligned}$$

$$y_0(t) = 0.4,$$

$$y_1(t) = \frac{0.5(1.c+0.16d)(t^{\alpha}\alpha + (1-1.\alpha)\Gamma[1+\alpha])}{B(\alpha)\Gamma[1+\alpha]},$$

$$\begin{aligned}
y_2(t) = -\tfrac{1}{(B(\alpha))^2\Gamma[1+\alpha]\Gamma[1+2\alpha]}&(t^{\alpha}\alpha(ac(0.2-0.2\alpha)\ ,\\
&+ad(0.032-0.032\alpha) + bd(-0.02048 + 0.02048\alpha)\\
&+bc(-0.128+0.128\alpha) + d(-0.32+0.32\alpha)\beta\\
&+d(0.4-0.4\alpha)\epsilon)\Gamma[1+2\alpha]\\
&+\Gamma[1\\
&+\alpha]\big(t^{2\alpha}\alpha^2(0.2ac - 0.064bc + 0.016ad - 0.01024bd\\
&-0.16d\beta + 0.2d\epsilon)\\
&+0.2(1.-1.\alpha)^2(1.ac - 0.64bc + 0.16ad - 0.1024bd\\
&-1.6d\beta + 2d\epsilon)\Gamma[1+2\alpha])),
\end{aligned}$$

$$z_0(t) = -0.4,$$

$$z_1(t) = -\frac{0.4(1.\beta - 1.25\epsilon)(t^{\alpha}\alpha + (1-1.\alpha)\Gamma[1+\alpha])}{B(\alpha)\Gamma[1+\alpha]},$$

$$\begin{aligned} z_2(t) = & -\frac{1}{(B(\alpha))^2\Gamma[1+\alpha]\Gamma[1+2\alpha]}\left(t^{\alpha}\alpha\Big((0.8-0.8\alpha)\beta^2+a(0.2\right. \\ & -0.2\alpha)\epsilon+b(-0.128+0.1283\alpha)\epsilon+(-1+\alpha)\beta\epsilon)\Gamma[1 \\ & +2\alpha]+\Big(0.4\beta^2+0.1a\epsilon-0.064b\epsilon-0.5\beta\epsilon\Big)\Gamma[1 \\ & \left.+\alpha](t^{2\alpha}\alpha^2+(1-\alpha)^2\Gamma[1+2\alpha])\right). \end{aligned}$$

Ultimately, this series solution can be expressed as

$$\begin{aligned} x(t) &= \textstyle\sum_{n=0}^{\infty} x_n(t), \\ y(t) &= \textstyle\sum_{n=0}^{\infty} y_n(t), \\ z(t) &= \textstyle\sum_{n=0}^{\infty} z_n(t). \end{aligned} \tag{31}$$

Table 3 provides numerical results from the ABC method to the fractional model (15), when $\alpha = 1$, $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \beta, \epsilon) = (7.5,\ 1, 5, -1, 1, 1)$, and $(x_0, y_0, z_0) = (-5, -1, -1)$, respectively, using (LDM), In Table 4, the numerical solutions when α = 0.75 and this provides evidence that the method is effective and can be used in different fractional systems.

Table 3. A solution of fractional chaotic system (15) using LDM, when a = 7.5, b = 1, c = 5, d = −1, β = 1, ϵ = 1, α = 1, x(0) = 0.5, y(0) = 0.4, and z(0) = −0.4, respectively.

t	x	y	z
0	0.5	0.4	−0.4
0.1	0.549651	0.6255988	−0.39293
0.2	0.835804	0.8183952	−0.39172
0.3	1.3584590000000003	0.9783892000000001	−0.39637
0.4	2.1176160000000004	1.1055807999999998	−0.40687999999999996
0.5	3.113275	1.19997	−0.42325
0.6	4.345436000000001	1.2615568000000004	−0.44548000000000004
0.7	5.814099000000001	1.2903412000000003	−0.47357000000000005
0.8	7.519264000000001	1.2863232	−0.50752
0.9	9.46093	1.2495028000000004	−0.54733

Table 4. A solution of fractional chaotic system (15) using LDM, when a = 7.5, b = 1, c = 5, d = −1, β = 1, ϵ = 1, α = 0.75, x(0) = 0.5, y(0) = 0.4, and z(0) = −0.4, respectively.

t	x	y	z
0	1.5330999766570965	0.7774994575127766	−0.4070046374413804
0.1	3.075738916272227	0.865837863878042	−0.4344350715747163
0.2	4.418837606033031	0.8856582296785278	−0.4603556734189791
0.3	5.790621851656716	0.8760830290256982	−0.48789448134169433
0.4	7.21034260681244	0.8451086645601766	−0.5171476747850935
0.5	8.682368702935259	0.7967085706251742	−0.5480600096176654
0.6	10.207209101765235	0.7333298178257313	−0.5805542106330043
0.7	11.784083641366404	0.6566582074503424	−0.614554623507307
0.8	13.411754094460488	0.5679391266589542	−0.6499922360919338
0.9	15.088840563199248	0.46813773976555995	−0.686805232061098

In Figures 1–3, we plotted numerical solutions to Equation (15) when $(a, b, c) = (0.4,\ 0.4,\ 4.5)$, and $(x_0, y_0, z_0) = (1.5,\ 1,\ 1)$, respectively. In these figures, we displayed the chaotic attractors of Equation (15) that were obtained using the numerical scheme for the ABC operator for certain parameter values. It is observable from Figures 1–3, and from Equation (15) where $(a,\ b,\ c) = (0.4,\ 0.4,\ 4.5)$, how they can show the same kind of chaotic attractor as its integer order [44] when $\alpha = 0.97,\ 0.98$, and 0.99, respectively. The advantage of this method lies in the accuracy of its graphics and the display of chaos in a clear and effective manner, which is remarkably similar to the integer order.

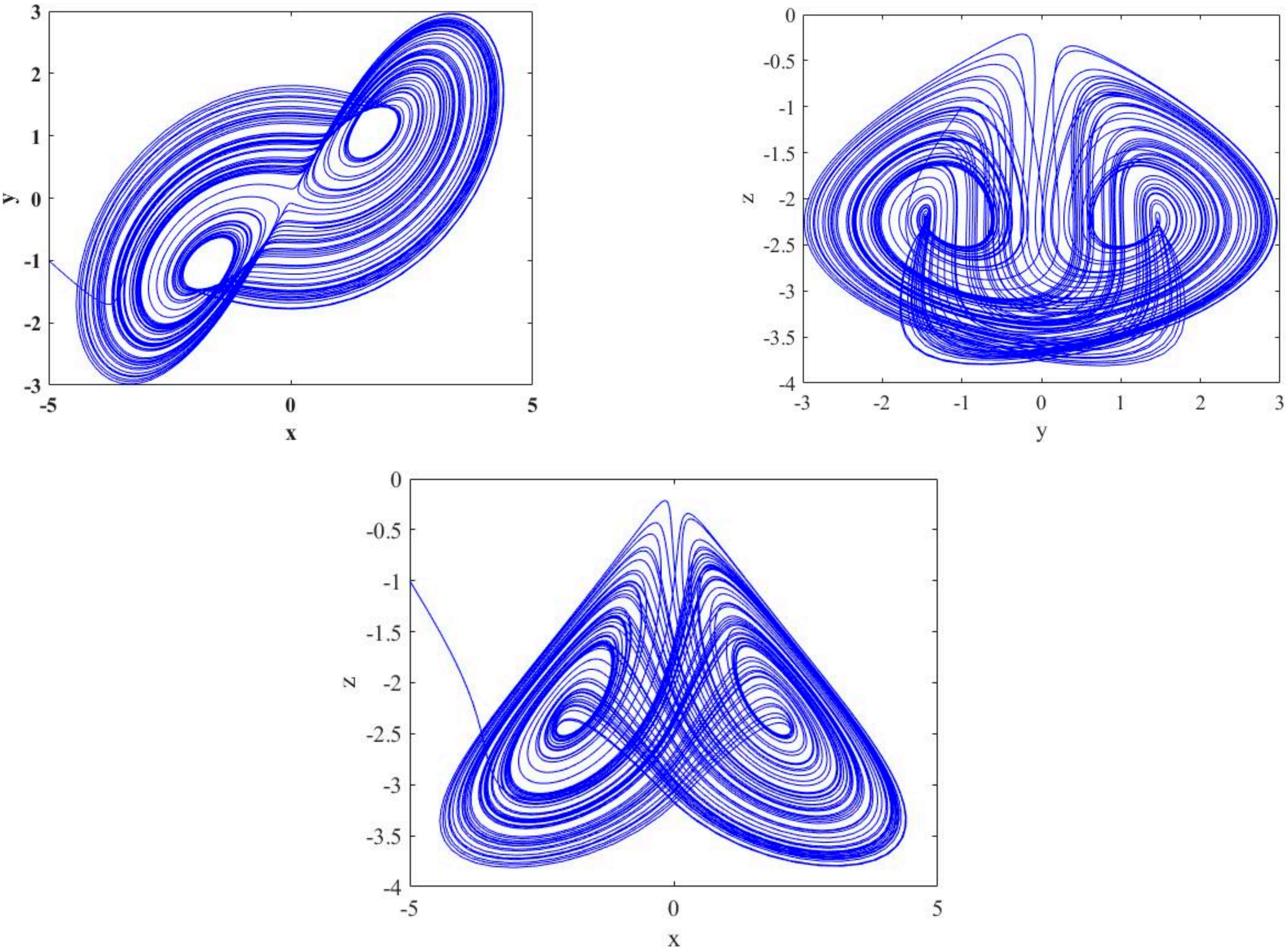

Figure 2. Chaotic attractor of Equation (15), when $\alpha = 0.98$.

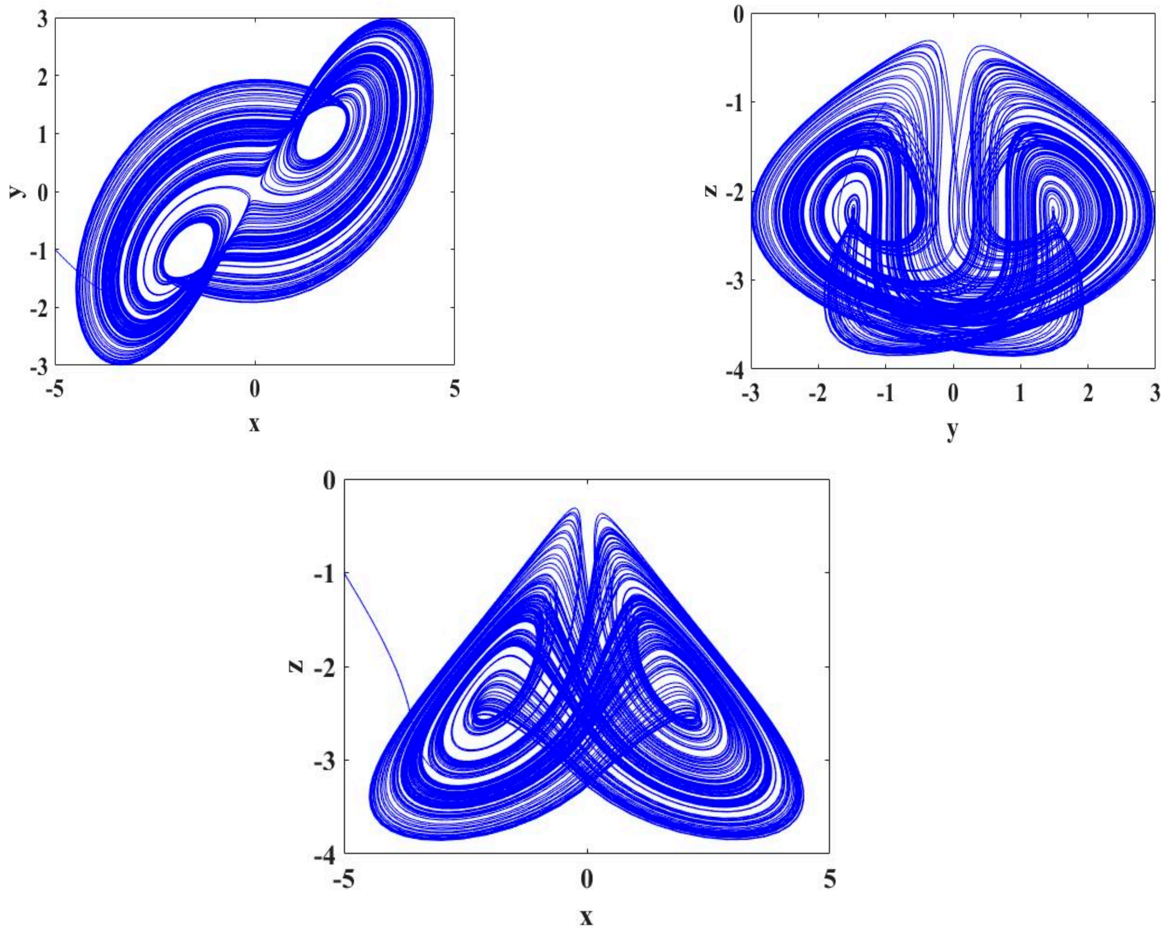

Figure 3. Chaotic attractor of Equation (15), when $\alpha = 0.99$.

7. Conclusions

This work successfully used two different approaches to solve a chaotic system with fractional derivative order. Utilizing the MATLAB and Mathematica platforms, a numerical strategy was provided for the ABC fractional derivative and Laplace decomposition method. Numerical simulations demonstrated that the numerical approach for the ABC fractional derivative produces numerical results that were remarkably close to the exact solutions, or RK4 solutions, in the integer order situation as the step size h was decreased. The acquired numerical results show that our procedure executes its operations in the case of fractions in a manner that is satisfactory in terms of its numerical stability. By giving specific values to the attached parameters, chaotic results were produced for the phase portrait of the model. The methodologies provided are thus meritoriously relevant for further research on various other models. This study was conducted in the anticipation that it would be a valuable tool for the upcoming fractional system applications.

We recommend a wider use of this method to address both physics and engineering challenges that are becoming ever more complicated. In the future, we intend to solve several of the newer fractional models and make comparisons with other numerical methods.

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References

1. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1999.
2. Oldham, K.B.; Spanier, J. *The Fractional Calculus*; Academic Press: New York, NY, USA, 1974.
3. Nguyen, A.T.; Tuan, N.H.; Yang, C. On Cauchy Problem for Fractional Parabolic-Elliptic Keller-Segel Model. *Adv. Nonlinear Anal.* **2022**, *12*, 97–116. [CrossRef]
4. Thach, T.N.; Tuan, N.H. Stochastic Pseudo-Parabolic Equations with Fractional Derivative and Fractional Brownian Motion. *Stoch. Anal. Appl.* **2022**, *40*, 328–351.
5. Tuan, N.H.; Phuong, N.D.; Thach, T.N. New Well-Posedness Results for Stochastic Delay Rayleigh-Stokes Equations. *Discret. Contin. Dyn. Systems. Ser. B* **2023**, *28*, 347. [CrossRef]
6. Ali, K.K.; Abd El Salam, M.A.; Mohamed, M.S. Chebyshev Fifth-Kind Series Approximation for Generalized Space Fractional Partial Differential Equations. *AIMS Math.* **2022**, *7*, 7759–7780. [CrossRef]
7. Ali Khalid, K.; Mukheimer, A.; Younis Jihad, A.; Abd El Salam, M.A.; Aydi, H. Spectral Collocation Approach with Shifted Chebyshev Sixth-Kind Series Approximation for Generalized Space Fractional Partial Differential Equations. *AIMS Math.* **2022**, *7*, 8622–8644. [CrossRef]
8. Asjad, M.I.; Ullah, N.; Rehman, H.U.; Baleanu, D. Optical Solitons for Conformable Space-Time Fractional Nonlinear Model. *J. Math. Comput. Sci.* **2022**, *27*, 28–41. [CrossRef]
9. Asjad, M.I.; Ullah, N.; Rehman, H.U.; Gia, T.N. Novel Soliton Solutions to the Atangana–Baleanu Fractional System of Equations for the ISALWs. *Open Phys.* **2021**, *19*, 770–779. [CrossRef]
10. Chawla, R.; Deswal, K.; Kumar, D.; Baleanu, D. A Novel Finite Difference Based Numerical Approach for Modified Atangana-Baleanu Caputo Derivative. *AIMS Math.* **2022**, *7*, 17252–17268. [CrossRef]
11. Nadeem, S.; Akber, R.; Almutairi, S.; Ghazwani, H.A.; Mahmoud, O. Numerical Analysis of Hydrothermal Flow and Heat Transfer inside a Cavity Formed Due to Faults Causing Earthquakes. *Front. Phys.* **2022**, *10*, 674. [CrossRef]
12. Rezazadeh, H.; Adel, W.; Tala-Tebue, E.; Yao, S.-W.; Mustafa Inc. Bright and Singular Soliton Solutions to the Atangana-Baleanu Fractional System of Equations for the ISALWs. *J. King Saud Univ. Sci.* **2021**, *33*, 101420. [CrossRef]
13. Ur Rehman, H.; Mustafa Inc.; Asjad, M.I.; Habib, A.; Munir, Q. New Soliton Solutions for the Space-Time Fractional Modified Third Order Korteweg–de Vries Equation. *J. Ocean. Eng. Sci.* **2022**, *2022*, 2174806. [CrossRef]
14. Younis, M.; Rehman, H.U.; Iftikhar, M. Computational Examples of a Class of Fractional Order Nonlinear Evolution Equations Using Modified Extended Direct Algebraic Method. *J. Comput. Methods Sci. Eng.* **2015**, *15*, 359–365. [CrossRef]
15. Carpinteri, A.; Mainardi, F. *Fractals and Fractional Calculus in Continuum Mechanics*; Springer: Berlin/Heidelberg, Germany, 1997; pp. 277–290.
16. Samko, G.; Kilbas, A.; Marichev, O. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach: Amsterdam, The Netherlands, 1993.
17. Dudkowski, D.; Jafari, S.; Kapitaniak, T.; Kuznetsov, N.V.; Leonov, G.A.; Prasad, A. Hidden Attractors in Dynamical Systems. *Phys. Rep.* **2016**, *637*, 1–50. [CrossRef]
18. Xu, G.; Shekofteh, Y.; Akgül, A.; Li, C.; Panahi, S. A New Chaotic System with a Self-Excited Attractor: Entropy Measurement, Signal Encryption, and Parameter Estimation. *Entropy* **2018**, *20*, 86. [CrossRef]
19. Lai, Q.; Akgul, A.; Li, C.; Xu, G.; Çavuşoğlu, Ü. A New Chaotic System with Multiple Attractors: Dynamic Analysis, Circuit Realization and S-Box Design. *Entropy* **2017**, *20*, 12. [CrossRef]
20. Petras, I. A note on the fractional-order chua's system. *Chaos Solitons Fractals* **2008**, *38*, 140–147. [CrossRef]
21. Abdoon, M.A.; Hasan, F.L.; Taha, N.E. Computational Technique to Study Analytical Solutions to the Fractional Modified KDV-Zakharov-Kuznetsov Equation. *Abstr. Appl. Anal.* **2022**, *2022*, 2162356. [CrossRef]
22. El-Maksoud, A.J.A.; El-Kader, A.A.A.; Hassan, B.G.; Rihan, N.G.; Tolba, M.; Said, L.A.; Radwan, A.G.; Abu-Elyazeed, M.F. FPGA implementation of sound encryption system based on fractional-order chaotic systems. *Microelectron. J.* **2019**, *90*, 323–335. [CrossRef]
23. Fahd, J.; Abdeljawad, T. A modified Laplace transform for certain generalized fractional operators. *Res. Nonl. Anal.* **2018**, *1*, 88–98.
24. Sene, N. Global Asymptotic Stability of the Fractional Differential Equations. *J. Nonlinear Sci. Appl.* **2019**, *13*, 171–175. [CrossRef]
25. Singh, J.; Kumar, D.; Hammouch, Z.; Atangana, A. A Fractional Epidemiological Model for Computer Viruses Pertaining to a New Fractional Derivative. *Appl. Math. Comput.* **2018**, *316*, 504–515. [CrossRef]
26. Kumar, S.; Kumar, R.; Cattani, C.; Samet, B. Chaotic Behaviour of Fractional Predator-Prey Dynamical System. *Chaos Solitons Fractals* **2020**, *135*, 109811. [CrossRef]

27. Hammouch, Z.; Mekkaoui, T.; Belgacem, F.B.M. Numerical Simulations for a Variable Order Fractional Schnakenberg Model. *AIP Publ. LLC* **2014**, *1637*, 1450–1455.
28. Abdoon, M.A.; Saadeh, R.; Berir, M.; Guma, F.E. Analysis, Modeling and Simulation of a Fractional-Order Influenza Model. *Alex. Eng. J.* **2023**, *74*, 231–240.
29. Hammouch, Z.; Mekkaoui, T. Traveling-wave solutions of the generalized Zakharov equation with time-space fractional derivatives. *J. MESA* **2014**, *5*, 489–498.
30. Belgacem, F.B.M.; Silambarasan, R.; Zakia, H.; Mekkaoui, T. New and Extended Applications of the Natural and Sumudu Transforms: Fractional Diffusion and Stokes Fluid Flow Realms. In *Trends in Mathematics*; Springer: Singapore, 2017; pp. 107–120.
31. Toufik, M.; Atangana, A. New Numerical Approximation of Fractional Derivative with Non-Local and Non-Singular Kernel: Application to Chaotic Models. *Eur. Phys. J. Plus* **2017**, *132*, 444. [CrossRef]
32. Ghanbari, B.; Atangana, A. Some New Edge Detecting Techniques Based on Fractional Derivatives with Nonlocal and Non-Singular Kernels. *Adv. Differ. Equ.* **2020**, *2020*, 435. [CrossRef]
33. Sene, N. Second-Grade Fluid Model with Caputo–Liouville Generalized Fractional Derivative. *Chaos Solitons Fractals* **2020**, *133*, 109631. [CrossRef]
34. Petras, I. Control of fractional-order Chua's system. *J. Electr. Eng.* **2002**, *53*, 219–222.
35. Elbadri, M. Initial Value Problems with Generalized Fractional Derivatives and Their Solutions via Generalized Laplace Decomposition Method. *Adv. Math. Phys.* **2022**, *2022*, 3586802. [CrossRef]
36. Jafari, H.; Khalique, C.; Nazari, M. Application of the Laplace Decomposition Method for Solving Linear and Nonlinear Fractional Diffusion–Wave Equations. *Appl. Math. Lett.* **2011**, *24*, 1799–1805. [CrossRef]
37. Elbadri, M.; Ahmed, S.A.; Abdalla, Y.T.; Hdidi, W. A New Solution of Time-Fractional Coupled KdV Equation by Using Natural Decomposition Method. *Abstr. Appl. Anal.* **2020**, *2020*, 3950816. [CrossRef]
38. Hosseini, K.; Samadani, F.; Kumar, D.; Faridi, M. New Optical Solitons of Cubic-Quartic Nonlinear Schrödinger Equation. *Optik* **2017**, *157*, 1101–1105. [CrossRef]
39. Faeza, L.; Hasan Mohamed, A. The Generalized (2 + 1) and (3 + 1)-Dimensional with Advanced Analytical Wave Solutions via Computational Applications. *Int. J. Nonlinear Anal. Appl.* **2021**, *12*, 1213–1241.
40. Hosseini, K.; Mayeli, P.; Ansari, R. Bright and Singular Soliton Solutions of the Conformable Time-Fractional Klein–Gordon Equations with Different Nonlinearities. *Waves Random Complex Media* **2017**, *28*, 426–434. [CrossRef]
41. Abdoon, M.A.; Hasan, F.L. Advantages of the Differential Equations for Solving Problems in Mathematical Physics with Symbolic Computation. *Math. Model. Eng. Probl.* **2022**, *9*, 268–276. [CrossRef]
42. Atangana, A.; Baleanu, D. New Fractional Derivatives with Nonlocal and Non-Singular Kernel: Theory and Application to Heat Transfer Model. *Therm. Sci.* **2016**, *20*, 763–769. [CrossRef]
43. Saad, K.M.; Khader, M.M.; Gómez-Aguilar, J.F.; Baleanu, D. Numerical Solutions of the Fractional Fisher's Type Equations with Atangana-Baleanu Fractional Derivative by Using Spectral Collocation Methods. *Chaos Interdiscip. J. Nonlinear Sci.* **2019**, *29*, 023116. [CrossRef]
44. Qureshi, S.; Atangana, A.; Shaikh, A.A. Strange Chaotic Attractors under Fractal-Fractional Operators Using Newly Proposed Numerical Methods. *Eur. Phys. J. Plus* **2019**, *134*, 523. [CrossRef]