

## Article

# Results on Second-Order Hankel Determinants for Convex Functions with Symmetric Points

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**Abstract:** One of the most important problems in the study of geometric function theory is knowing how to obtain the sharp bounds of the coefficients that appear in the Taylor–Maclaurin series of univalent functions. In the present investigation, our aim is to calculate some sharp estimates of problems involving coefficients for the family of convex functions with respect to symmetric points and associated with a hyperbolic tangent function. These problems include the first four initial coefficients, the Fekete–Szegő and Zalcman inequalities, and the second-order Hankel determinant. Additionally, the inverse and logarithmic coefficients of the functions belonging to the defined class are also studied in relation to the current problems.

**Keywords:** convex functions with respect to symmetric points; subordinations; Zalcman functionals; logarithmic and inverse coefficients; Hankel determinant problems



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## 1. Introduction and Definitions

To properly understand the basic terminology used throughout our primary findings, we must first explain some basic concepts. For this, let  $\mathbb{U}_d = \{z \in \mathbb{C} : |z| < 1\}$  represent the open unit disc and the symbol  $\mathcal{A}$  denote the holomorphic (analytic) functions class normalized by  $f(0) = f'(0) - 1 = 0$ . This signifies that  $f \in \mathcal{A}$  has Taylor's series representation

$$f(z) = \sum_{l=1}^{\infty} a_l z^l, \quad (a_1 = 1), \quad (1)$$

and if an analytic function takes no values more than once in  $\mathbb{U}_d$ , it is univalent in region  $\mathbb{U}_d$ . That is,  $f$  being univalent in  $\mathbb{U}_d$  means mathematically that  $f(z_1) = f(z_2)$  implies  $z_1 = z_2$  for  $z_1, z_2 \in \mathbb{U}_d$ . Thus, by the notation  $\mathcal{S}$ , we utilize series expansion (1) to denote the family of univalent functions. Koebe discovered this family in 1907.

The most famous result of function theory, known as the “Bieberbach conjecture”, was stated by Bieberbach [1] in 1916. According to this conjecture, if  $f \in \mathcal{S}$ , then  $|a_n| \leq n$  for all  $n \geq 2$ . He also proved this problem for  $n = 2$ . Many eminent scholars have used a variety of techniques to address this problem. For  $n = 3$ , this conjecture was solved by Löwner [2] and also by Schaeffer and Spencer [3] using the Löwner differential equation and variational method, respectively. Later, Jenkins [4] used quadratic differentials to prove the same coefficient inequality  $|a_3| \leq 3$ . The variational technique was used by Garabedian and Schiffer [5] to determine that  $|a_4| \leq 4$ . The Garabedian-Schiffer inequality [6] (p. 108) was used by Pederson and Schiffer [7] to calculate that  $|a_5| \leq 5$ . Additionally, by using the Grunsky inequality [6] (p. 60), Pederson [8] and Ozawa [9,10] have both proved that  $|a_6| \leq 6$ . This conjecture has been long sought to be resolved by numerous academics, but nobody has been able to prove it for  $n \geq 7$ . Finally, in 1985, de-Branges [11] proved this conjecture for all  $n \geq 2$  by using hypergeometric functions.

The estimates of the  $n$ th coefficient bounds for several subfamilies of the univalent function family, particularly starlike  $\mathcal{S}^*$ , convex  $\mathcal{C}$ , close-to-convex  $\mathcal{K}$ , and so on, were examined in an attempt to solve the mentioned problems between 1916 and 1985. Some of the basic classes are described below:

$$\begin{aligned}\mathcal{S}^* &= \left\{ f \in \mathcal{S} : \Re \left( \frac{zf'(z)}{f(z)} \right) > 0, \quad (z \in \mathbb{U}_d) \right\}, \\ \mathcal{C} &= \left\{ f \in \mathcal{S} : \Re \left( \frac{(zf'(z))'}{f'(z)} \right) > 0, \quad (z \in \mathbb{U}_d) \right\}, \\ \mathcal{K} &= \left\{ f \in \mathcal{S} : \Re \left( \frac{zf'(z)}{h(z)} \right) > 0 \text{ with } h \in \mathcal{S}^* (z \in \mathbb{U}_d) \right\}.\end{aligned}$$

By selecting  $h(z) = z$ , the  $\mathcal{K}$  family is reduced to the family of bounded turning functions, and we denote such a family of functions by the symbol  $\mathcal{BT}$ . In 1992, the authors [12] considered a univalent function  $\phi$  in  $\mathbb{U}_d$  with the properties that  $\phi'(0) > 0$  and  $\Re \phi > 0$ . Additionally, the region  $\phi(\mathbb{U}_d)$  is star-shaped about the point  $\phi(0) = 1$  and is symmetric along the real line axis. Applying the familiar idea of subordination " $\prec$ ", they defined the following unified subfamily of the class  $\mathcal{S}$ .

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \phi(z), \quad (z \in \mathbb{U}_d) \right\}.$$

They focused on certain consequences, such as the covering, growth, and distortion theorems. Over the past few years, a number of collection  $\mathcal{S}$  subfamilies have been considered as specific options for the class  $\mathcal{S}^*(\phi)$ . The following families stand out as being remarkable in the study that has lately been introduced.

- (i).  $\mathcal{S}_{\mathcal{L}}^* \equiv \mathcal{S}^*(\sqrt{1+z})$  [13],  $\mathcal{S}_{car}^* \equiv \mathcal{S}^*\left(1 + \frac{2}{3}z + \frac{1}{3}z^2\right)$  [14],  $\mathcal{S}_{exp}^* \equiv \mathcal{S}^*(\exp(z))$  [15],
- (ii).  $\mathcal{S}_{\cos}^* \equiv \mathcal{S}^*(\cos(z))$  [16],  $\mathcal{S}_{\sin}^* \equiv \mathcal{S}^*(1 + \sin(z))$  [17],  $\mathcal{S}_{pet}^* \equiv \mathcal{S}^*(1 + \sinh^{-1} z)$  [18],
- (iii).  $\mathcal{S}_{\cosh}^* \equiv \mathcal{S}^*(\cosh(z))$  [19],  $\mathcal{S}_{\tanh}^* \equiv \mathcal{S}^*(1 + \tanh(z))$  [20],  $\mathcal{S}_c^* \equiv \mathcal{S}^*(1 + z + \frac{1}{2}z^2)$  [21],
- (iv).  $\mathcal{S}_{(n-1)\mathcal{L}}^* \equiv \mathcal{S}^*(\Psi_{n-1}(z))$  [22] with  $\Psi_{n-1}(z) = 1 + \frac{n}{n+1}z + \frac{1}{n+1}z^n$  for  $n \geq 2$ . Also see

the articles [23–26] for more recently studied generalised classes.

The below described determinant  $\mathcal{H}_{\lambda,n}(f)$  with  $n, \lambda \in \mathbb{N} = \{1, 2, \dots\}$  is known as the Hankel determinant and has entries consisting of coefficients of the function  $f \in \mathcal{S}$

$$\mathcal{H}_{\lambda,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+\lambda-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+\lambda} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+\lambda-1} & a_{n+\lambda} & \cdots & a_{n+2\lambda-2} \end{vmatrix}.$$

This determinant was contributed to by Pommerenke [27,28]. The first- and second-order Hankel determinants, respectively, are known in particular as the following determinants:

$$\begin{aligned}\mathcal{H}_{2,1}(f) &= \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2, \\ \mathcal{H}_{2,2}(f) &= \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.\end{aligned}$$

In the literature, the first two determinants are extensively studied for various subfamilies of univalent functions, but the work carried out by the authors [29–40], in which they determined the sharp bounds of the second-order determinant, is noteworthy. For more about the study of this determinant, see the articles [41–48].

Motivated from the classes established by Sakaguchi [49] and by Das and Singh [50], we now introduce the family  $\mathcal{SK}_{\tanh}$  of the convex function with respect to symmetric points connected with the tan hyperbolic function, and it is given by

$$\mathcal{SK}_{\tanh} = \left\{ f \in \mathcal{S} : \frac{2(zf'(z))'}{(f(z) - f(-z))'} \prec 1 + \tanh(z) \quad (z \in \mathbb{U}_d) \right\}. \quad (2)$$

In this article, we propose a new approach that depends on the connection between the coefficients of functions belonging to a particular family and the coefficients of corresponding Schwarz functions. In many cases, it is simple to determine the exact estimate of the functional and do the required calculations. Our aim is to calculate the sharp estimates of coefficients, Fekete–Szegő, Zalcman inequalities for the family  $\mathcal{SK}_{\tanh}$  of convex function with respect to symmetric points connected with the tan hyperbolic function. We also find the sharp bound of the determinant  $|\mathcal{H}_{2,2}(f)|$  for the same class. Further, we study the logarithmic and inverse coefficients for the same class.

## 2. A Set of Lemmas

An analytic function  $w : \mathbb{U}_d \rightarrow \mathbb{U}_d$  with  $w(0) = 0$  is called a Schwarz function, and let the family of such functions be denoted by the symbol  $\mathfrak{B}_0$ . Thus, we can represent the function  $w \in \mathfrak{B}_0$  by the following power series expansion:

$$w(z) = \sum_{n=1}^{\infty} w_n z^n. \quad (3)$$

**Lemma 1** ([51]). Let  $w(z) = \sum_{n=1}^{\infty} w_n z^n$  be a Schwarz function and let

$$|w_3 + \sigma w_1 w_2 + \varsigma w_1^3| = \gamma(w)$$

with  $\sigma$  and  $\varsigma$  are real numbers. Then the following sharp estimate hold

$$\gamma(w) \leq \Phi(\sigma, \varsigma),$$

where

$$\Phi(\sigma, \varsigma) = \begin{cases} 1 & \text{if } (\sigma, \varsigma) \in \mathbb{D}_1 \cup \mathbb{D}_2 \cup \{(2, 1)\} \\ \frac{2}{3}(|\sigma| + 1) \left( \frac{|\sigma| + 1}{2(|\sigma| + 1 + \varsigma)} \right)^{1/2} & \text{if } (\sigma, \varsigma) \in \mathbb{D}_3 \cup \mathbb{D}_4 \end{cases},$$

with

$$\begin{aligned} \mathbb{D}_1 &= \left\{ |\sigma| \leq \frac{1}{2}, -1 \leq \varsigma \leq 1 \right\}, \\ \mathbb{D}_2 &= \left\{ \frac{1}{2} \leq |\sigma| \leq 2, \frac{4}{27}(1 + |\sigma|)^3 - (1 + |\sigma|) \leq \varsigma \leq 1 \right\}, \\ \mathbb{D}_3 &= \left\{ |\sigma| \geq 2, -\frac{2}{3}(1 + |\sigma|) \leq \varsigma \leq \frac{2(1 + |\sigma|)|\sigma|}{4 + \sigma^2 + 2|\sigma|} \right\}, \\ \mathbb{D}_4 &= \left\{ \frac{1}{2} \leq |\sigma| \leq 2, -\frac{2}{3}(|\sigma| + 1) \leq \varsigma \leq \frac{4}{27}(1 + |\sigma|)^3 - (1 + |\sigma|) \right\}. \end{aligned}$$

**Lemma 2** ([52]). If  $w \in \mathfrak{B}_0$  is in the form (3), then

$$|w_2| \leq 1 - |w_1|^2, \quad (4)$$

$$|w_n| \leq 1, \quad n \geq 1. \quad (5)$$

Furthermore, the inequality of (4) can be improved in the manner

$$|w_2 + \eta w_1^2| \leq \max\{1, |\eta|\}, \quad \eta \in \mathbb{C}. \quad (6)$$

**Lemma 3** ([53]). Let  $\mathfrak{w}(z) = w_1 z + w_2 z^2 + \dots$  be a Schwarz function. Then,

$$|w_3| \leq 1 - |w_1|^2 - \frac{|w_2|^2}{1 + |w_1|}, \quad (7)$$

$$|w_4| \leq 1 - |w_1|^2 - |w_2|^2. \quad (8)$$

**Lemma 4** ([54]). Let  $\mathfrak{w}(z) = w_1 z + w_2 z^2 + \dots$  be a Schwarz function. Then,

$$|w_1 w_3 - w_2^2| \leq 1 - |w_1|^2.$$

### 3. Coefficient Estimates on Function Belonging to the Class $\mathcal{SK}_{\tanh}$

We first discuss the bounds on some initial coefficients for  $f \in \mathcal{SK}_{\tanh}$ .

**Theorem 1.** Let  $f \in \mathcal{SK}_{\tanh}$ . Then,

$$\begin{aligned} |a_2| &\leq \frac{1}{4}, \\ |a_3| &\leq \frac{1}{6}, \\ |a_4| &\leq \frac{1}{16}, \\ |a_5| &\leq \frac{1}{20}. \end{aligned}$$

All of these bounds are sharp.

**Proof.** Assume that  $f \in \mathcal{SK}_{\tanh}$ . It follows from the definition that a Schwarz function  $\mathfrak{w}$  exists such that

$$\frac{2(zf'(z))'}{(f(z) - f(-z))'} = 1 + \tanh \mathfrak{w}(z). \quad (9)$$

Utilizing (1), we obtain

$$\frac{2(zf'(z))'}{(f(z) - f(-z))'} := 1 + 4a_2 z + 6a_3 z^2 + (-12a_2 a_3 + 16a_4)z^3 + (-18a_3^2 + 20a_5)z^4 + \dots \quad (10)$$

Let

$$\mathfrak{w}(z) = w_1 z + w_2 z^2 + w_3 z^3 + w_4 z^4 + \dots \quad (11)$$

By some easy computation and utilizing the series representation of (11), we achieve

$$1 + \tanh(\mathfrak{w}(z)) = 1 + w_1 z + w_2 z^2 + \left(-\frac{1}{3}w_1^3 + w_3\right)z^3 + (-w_1^2 w_2 + w_4)z^4 + \dots \quad (12)$$

Now, by comparing (10) and (12), we obtain

$$a_2 = \frac{1}{4}w_1, \quad (13)$$

$$a_3 = \frac{1}{6}w_2, \quad (14)$$

$$a_4 = \frac{1}{16}w_3 - \frac{1}{48}w_1^3 + \frac{1}{32}w_1w_2, \quad (15)$$

$$a_5 = \frac{1}{40}w_2^2 - \frac{1}{20}w_1^2w_2 + \frac{1}{20}w_4. \quad (16)$$

From the use of (13) and (14) along with Lemma 2, we easily obtain

$$|a_2| \leq \frac{1}{4} \quad \text{and} \quad |a_3| \leq \frac{1}{6}.$$

By rearranging (15), we have

$$|a_4| = \frac{1}{16} \left| w_3 + \frac{1}{2}w_1w_2 - \frac{1}{3}w_1^3 \right|.$$

By using Lemma 1 with  $\sigma = \frac{1}{2}$  and  $\varsigma = -\frac{1}{3}$ , and then by applying the triangle inequality, we obtain

$$|a_4| \leq \frac{1}{16}.$$

Rearranging (16), we have

$$\begin{aligned} |a_5| &= \frac{1}{20} \left| w_4 + \frac{1}{2}w_2^2 - w_1^2w_2 \right|, \\ &\leq \frac{1}{20} \left\{ |w_4| + \frac{1}{2}|w_2|^2 + |w_1|^2|w_2| \right\}. \end{aligned}$$

By using Lemma 3 along with some simple computations, we obtain

$$|a_5| \leq \frac{1}{20} \left\{ 1 - \frac{|w_2|^2}{2} \right\} \leq \frac{1}{20}.$$

The bounds on the estimation of  $|a_2|$ ,  $|a_3|$ ,  $|a_4|$ , and  $|a_5|$  are sharp with the extremal functions given, respectively, by

$$\frac{2(zf'(z))'}{(f(z) - f(-z))'} = 1 + \tanh(z) = 1 + z - \frac{1}{3}z^3 + \dots, \quad (17)$$

$$\frac{2(zf'(z))'}{(f(z) - f(-z))'} = 1 + \tanh(z^2) = 1 + z^2 - \frac{1}{3}z^6 + \dots, \quad (18)$$

$$\frac{2(zf'(z))'}{(f(z) - f(-z))'} = 1 + \tanh(z^3) = 1 + z^3 - \frac{1}{3}z^9 + \dots, \quad (19)$$

$$\frac{2(zf'(z))'}{(f(z) - f(-z))'} = 1 + \tanh(z^4) = 1 + z^4 - \frac{1}{3}z^{12} + \dots. \quad (20)$$

□

**Theorem 2.** Let  $f \in \mathcal{SK}_{\tanh}$ . Then, for  $\eta \in \mathbb{C}$

$$|a_3 - \eta a_2^2| \leq \frac{1}{6} \max \left\{ 1, \frac{3}{8}|\eta| \right\}.$$

This result is sharp.

**Proof.** From (13) and (14), we obtain

$$\begin{aligned} |a_3 - \eta a_2^2| &= \frac{1}{6} \left| w_2 - \frac{3}{8} \eta w_1^2 \right|, \\ &= \frac{1}{6} \left| w_2 + \left( -\frac{3\eta}{8} \right) w_1^2 \right|. \end{aligned}$$

Using Lemma 2 and then applying the triangle inequality, we obtain

$$|a_3 - \eta a_2^2| \leq \max \left\{ \frac{1}{6}, \left| -\frac{3\eta}{48} \right| \right\}.$$

□

By putting  $\eta = 1$ , we obtain the below corollary.

**Corollary 1.** If  $f \in \mathcal{SK}_{\tanh}$  is of the form (1), then

$$|a_3 - a_2^2| \leq \frac{1}{6}.$$

This result is sharp with the extremal function given by (18).

Now, we give estimates on the Zalcman functionals for  $f \in \mathcal{SK}_{\tanh}$ .

**Theorem 3.** Suppose that  $f \in \mathcal{SK}_{\tanh}$  is the form of (1); then,

$$|a_4 - a_2 a_3| \leq \frac{1}{16}, \quad (21)$$

and

$$|a_5 - a_3^2| \leq \frac{1}{20}. \quad (22)$$

The inequalities (21) and (22) are sharp for the extremal function given by (19) and (20).

**Proof.** It is noted that

$$|a_4 - a_2 a_3| = \frac{1}{16} \left| w_3 - \frac{1}{6} w_1 w_2 - \frac{1}{3} w_1^3 \right|,$$

so, taking  $\sigma = -\frac{1}{6}$  and  $\varsigma = -\frac{1}{3}$  in Lemma 1 yields

$$|a_4 - a_2 a_3| \leq \frac{1}{16}.$$

For  $a_5 - a_3^2$ , we have

$$\begin{aligned} |a_5 - a_3^2| &= \frac{1}{20} \left| w_4 - \frac{1}{18} w_2^2 - w_1^2 w_2 \right|, \\ &\leq \frac{1}{20} \left\{ |w_4| + \frac{1}{18} |w_2|^2 + |w_1|^2 |w_2| \right\}. \end{aligned}$$

By using Lemma 3 and some simple calculations, we obtain

$$\begin{aligned} |a_5 - a_3^2| &\leq \frac{1}{20} \left\{ 1 - \frac{17}{18} |w_2|^2 \right\}, \\ &\leq \frac{1}{20}. \end{aligned}$$

Thus, the proof is completed.  $\square$

**Theorem 4.** Let  $f \in \mathcal{SK}_{\tanh}$ . Then,

$$|\mathcal{H}_{2,2}(f)| \leq \frac{1}{36}.$$

This result is sharp with the extremal function given by (18).

**Proof.** From (13), (14), and (15), we have

$$\begin{aligned} |a_2 a_4 - a_3^2| &= \frac{1}{36} \left| w_2^2 + \frac{3}{16} w_1^4 - \frac{9}{16} w_1 w_3 - \frac{9}{32} w_1^2 w_2 \right| \\ &= \frac{1}{36} \left| \frac{1}{2} (w_2^2 - w_1 w_3) + \frac{1}{2} \left( \frac{3}{8} w_1^4 - \frac{1}{8} w_1 w_3 - \frac{9}{16} w_1^2 w_2 + w_2^2 \right) \right| \\ &\leq \frac{1}{72} |w_2^2 - w_1 w_3| + \frac{1}{72} \left| \frac{3}{8} w_1^4 - \frac{1}{8} w_1 w_3 - \frac{9}{16} w_1^2 w_2 + w_2^2 \right| \\ &= \frac{1}{72} L_1 + \frac{1}{72} L_2, \end{aligned}$$

where

$$L_1 = |w_2^2 - w_1 w_3|$$

and

$$L_2 = \left| \frac{3}{8} w_1^4 - \frac{1}{8} w_1 w_3 - \frac{9}{16} w_1^2 w_2 + w_2^2 \right|.$$

Using Lemma 4, we obtain  $L_1 \leq 1$ . For finding the bound of  $L_2$ , we use Lemma 3 and the triangle inequality in the below expression:

$$\begin{aligned} |L_2| &\leq \frac{3}{8} |w_1|^4 + \frac{1}{8} |w_1| \left( 1 - |w_1|^2 - \frac{|w_2|^2}{1 + |w_1|} \right) + \frac{9}{16} |w_1|^2 |w_2| + |w_2|^2, \\ &\leq \frac{3}{8} |w_1|^4 + \frac{1}{8} |w_1| - \frac{1}{8} |w_1|^3 - \frac{|w_1| |w_2|^2}{8(1 + |w_1|)} + \frac{9}{16} |w_1|^2 |w_2| + |w_2|^2, \\ &\leq \frac{3}{8} |w_1|^4 + \frac{1}{8} |w_1| - \frac{1}{8} |w_1|^3 + |w_2|^2 \left( 1 - \frac{|w_1|}{8(1 + |w_1|)} \right) + \frac{9}{16} |w_1|^2 |w_2|. \quad (23) \end{aligned}$$

Since  $\left( 1 - \frac{|w_1|}{8(1 + |w_1|)} \right) > 0$  and  $|w_2| \leq 1 - |w_1|^2$ , we have

$$|L_2| \leq \frac{3}{8} |w_1|^4 + \frac{1}{8} |w_1| - \frac{1}{8} |w_1|^3 + (1 - |w_1|^2)^2 \left( 1 - \frac{|w_1|}{8(1 + |w_1|)} \right) + \frac{9}{16} |w_1|^2 (1 - |w_1|^2).$$

By putting  $|w_1| = x$  and  $x \in (0, 1]$ , we obtain

$$|L_2| \leq 1 - \frac{21}{16} x^2 + \frac{11}{16} x^4 = F(x).$$

As  $F'(x) \leq 0$ ,  $F(x)$  is a decreasing function of  $x$ , it gives the maximum value at  $x = 0$

$$|L_2| \leq 1.$$

Hence,

$$|\mathcal{H}_{2,2}(f)| \leq \frac{1}{72}L_1 + \frac{1}{72}L_2 \leq \frac{1}{36}.$$

The proof is thus completed.  $\square$

#### 4. Logarithmic Coefficient for $\mathcal{SK}_{\tanh}$

The logarithmic coefficients of a given function  $f$ , represented by  $\gamma_n = \gamma_n(f)$ , are defined by

$$\frac{1}{2} \log\left(\frac{f(z)}{z}\right) = \sum_{n=1}^{\infty} \gamma_n z^n. \quad (24)$$

It is natural to consider the Hankel determinant whose entries are the logarithmic coefficients. In [32,33], Kowalczyk et al. first introduced the Hankel determinant containing logarithmic coefficients as the elements, which is given by

$$\mathcal{H}_{q,n}(F_f/2) := \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2q-2} \end{vmatrix}. \quad (25)$$

In particular, it is noted that

$$\mathcal{H}_{2,1}(F_f/2) = \begin{vmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{vmatrix} = |\gamma_1\gamma_3 - \gamma_2^2|.$$

For more about the study of logarithmic coefficients, see articles [38,55,56].

If  $f$  is given by (1), then its logarithmic coefficients are given as follows:

$$\gamma_1 = \frac{1}{2}a_2 \quad (26)$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right) \quad (27)$$

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right). \quad (28)$$

**Theorem 5.** Let  $f \in \mathcal{SK}_{\tanh}$ . Then,

$$\begin{aligned} |\gamma_1| &\leq \frac{1}{8}, \\ |\gamma_2| &\leq \frac{1}{12}, \\ |\gamma_3| &\leq \frac{1}{32}. \end{aligned}$$

All of these bounds are sharp.

**Proof.** Applying (13)–(15) in (26)–(28), we obtain

$$\gamma_1 = \frac{1}{8}w_1, \quad (29)$$

$$\gamma_2 = \frac{1}{12}w_2 - \frac{1}{64}w_1^2, \quad (30)$$

$$\gamma_3 = \frac{1}{32}w_3 - \frac{1}{192}w_1w_2 - \frac{1}{128}w_1^3. \quad (31)$$



The bounds of  $\gamma_1$  and  $\gamma_2$  are directly obtained by using simple computations. For the bound of  $\gamma_3$ , first we rearrange (31) in the form

$$|\gamma_3| = \frac{1}{32} \left| w_3 - \frac{1}{6} w_1 w_2 - \frac{1}{4} w_1^3 \right|,$$

and then by applying Lemma 1 with  $\sigma = -\frac{1}{6}$  and  $\varsigma = -\frac{1}{4}$ , we obtain the required result. Equalities holds for the function given by (17)–(19) and using (26)–(28).  $\square$

**Theorem 6.** If  $f \in \mathcal{SK}_{\tanh}$  is of the form (1), then

$$|\gamma_2 - \eta \gamma_1^2| \leq \frac{1}{12} \max \left\{ 1, \frac{3}{16} |1 + \eta| \right\}.$$

This inequality is sharp.

**Proof.** From (29) and (30), we have

$$\begin{aligned} |\gamma_2 - \eta \gamma_1^2| &= \frac{1}{12} \left| w_2 - \frac{3}{16} w_1^2 - \frac{3\eta}{16} w_1^2 \right|, \\ &= \frac{1}{12} \left| w_2 + \left( \frac{-3(1+\eta)}{16} \right) w_1^2 \right|. \end{aligned}$$

Using Lemma 2 and the triangle inequality, we obtain the required result.  $\square$

Putting  $\eta = 1$ , we obtained the following corollary.

**Corollary 2.** If  $f \in \mathcal{SK}_{\tanh}$  is of the form (1), then

$$|\gamma_2 - \gamma_1^2| \leq \frac{1}{12}.$$

Equality is determined by using (26), (27), and (18).

**Theorem 7.** If  $f \in \mathcal{SK}_{\tanh}$  is of the form (1), then

$$|\gamma_3 - \gamma_1 \gamma_2| \leq \frac{1}{32}.$$

Equality is determined by using (26)–(28), and (19).

**Proof.** From (29)–(31), we obtain

$$|\gamma_3 - \gamma_1 \gamma_2| = \frac{1}{32} \left| w_3 - \frac{1}{2} w_1 w_2 - \frac{3}{16} w_1^3 \right|,$$

so taking  $\sigma = -\frac{1}{2}$  and  $\varsigma = -\frac{3}{16}$  in Lemma 1 yields

$$|\gamma_3 - \gamma_1 \gamma_2| \leq \frac{1}{32},$$

which completes the proof.  $\square$

**Theorem 8.** If  $f \in \mathcal{SK}_{\tanh}$  is of the form (1), then

$$\left| \mathcal{H}_{2,1}(F_f/2) \right| = |\gamma_1 \gamma_3 - \gamma_2^2| \leq \frac{1}{144}.$$

This inequality is sharp, and equality is determined by using (26)–(28), and (18).

**Proof.** From (29)–(31), we have

$$\begin{aligned} |\gamma_1\gamma_3 - \gamma_2^2| &= \frac{1}{144} \left| w_2^2 + \frac{45}{256} w_1^4 - \frac{9}{16} w_1 w_3 - \frac{9}{32} w_1^2 w_2 \right| \\ &= \frac{1}{144} \left| \frac{1}{2} (w_2^2 - w_1 w_3) + \frac{1}{2} \left( \frac{45}{128} w_1^4 - \frac{1}{8} w_1 w_3 - \frac{9}{16} w_1^2 w_2 + w_2^2 \right) \right| \\ &\leq \frac{1}{288} |w_2^2 - w_1 w_3| + \frac{1}{288} \left| \frac{45}{128} w_1^4 - \frac{1}{8} w_1 w_3 - \frac{9}{16} w_1^2 w_2 + w_2^2 \right| \\ &= \frac{1}{288} Q_1 + \frac{1}{288} Q_2, \end{aligned}$$

where

$$Q_1 = |w_2^2 - w_1 w_3|$$

and

$$Q_2 = \left| \frac{45}{128} w_1^4 - \frac{1}{8} w_1 w_3 - \frac{9}{16} w_1^2 w_2 + w_2^2 \right|$$

Using Lemma 4, we obtain  $Q_1 \leq 1$ . For  $Q_2$ , using Lemma 3 and the triangle inequality, we have

$$\begin{aligned} |Q_2| &\leq \frac{45}{128} |w_1|^4 + \frac{1}{8} |w_1| \left( 1 - |w_1|^2 - \frac{|w_2|^2}{1 + |w_1|} \right) + \frac{9}{16} |w_1|^2 |w_2| + |w_2|^2, \\ &\leq \frac{45}{128} |w_1|^4 + \frac{1}{8} |w_1| - \frac{1}{8} |w_1|^3 - \frac{|w_1| |w_2|^2}{8(1 + |w_1|)} + \frac{9}{16} |w_1|^2 |w_2| + |w_2|^2, \\ &\leq \frac{45}{128} |w_1|^4 + \frac{1}{8} |w_1| - \frac{1}{8} |w_1|^3 + |w_2|^2 \left( 1 - \frac{|w_1|}{8(1 + |w_1|)} \right) + \frac{9}{16} |w_1|^2 |w_2|. \quad (32) \end{aligned}$$

Since  $\left( 1 - \frac{|w_1|}{8(1 + |w_1|)} \right) > 0$  and  $|w_2| \leq 1 - |w_1|^2$  in (32), we have

$$|Q_2| \leq \frac{45}{128} |w_1|^4 + \frac{1}{8} |w_1| - \frac{1}{8} |w_1|^3 + (1 - |w_1|^2)^2 \left( 1 - \frac{|w_1|}{8(1 + |w_1|)} \right) + \frac{9}{16} |w_1|^2 (1 - |w_1|^2).$$

After the elementary calculus of maxima and minima, we obtain

$$|Q_2| \leq 1.$$

Hence

$$\left| \mathcal{H}_{2,1} \left( F_f/2 \right) \right| \leq \frac{1}{288} Q_1 + \frac{1}{288} Q_2 \leq \frac{1}{144}.$$

The proof is thus completed.  $\square$

## 5. Inverse Coefficient for $\mathcal{SK}_{\tanh}$

The renowned Koebe 1/4-theorem ensures that, for each univalent function  $f$  defined in  $\mathbb{U}_d$ , its inverse  $f^{-1}$  exists at least on a disc of radius 1/4 with Taylor's series representation form

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n, \quad \left( |w| < \frac{1}{4} \right). \quad (33)$$

Using the representation  $f(f^{-1}(w)) = w$ , we obtain

$$A_2 = -a_2 \quad (34)$$

$$A_3 = -a_3 + 2a_2^2 \quad (35)$$

$$A_4 = -a_4 + 5a_2 a_3 - 5a_2^3. \quad (36)$$

Researchers have demonstrated a significant interest in understanding the geometric behavior of the inverse function in recent years. For example, Krzyz et al. [57] calculated the upper bounds of the initial coefficient contained in the inverse function  $f^{-1}$  when  $f \in \mathcal{S}^*(\beta)$  with  $0 \leq \beta \leq 1$ . In addition, Ali [58] examined the sharp bounds of the first four initial coefficients for the class  $\mathcal{SS}^*(\zeta)$  ( $0 < \zeta \leq 1$ ) of a strongly starlike function as well as the sharp estimate of the Fekete–Szegő coefficient functional of the inverse function. For more about the study of inverse coefficients, see the articles [59,60].

**Theorem 9.** If  $f \in \mathcal{SK}_{\tanh}$  is of the form (1), then

$$\begin{aligned} |A_2| &\leq \frac{1}{4}, \\ |A_3| &\leq \frac{1}{6}, \\ |A_4| &\leq \frac{23\sqrt{874}}{8208}. \end{aligned}$$

The first two bounds are sharp.

**Proof.** Applying (13)–(15) in (34)–(36), we obtain

$$A_2 = -\frac{1}{4}w_1 \quad (37)$$

$$A_3 = \frac{1}{8}w_1^2 - \frac{1}{6}w_2 \quad (38)$$

$$A_4 = -\frac{11}{192}w_1^3 + \frac{17}{96}w_1w_2 - \frac{1}{16}w_3. \quad (39)$$

The bounds of  $A_2$  and  $A_3$  are simple and straightforward. For  $A_4$ , consider the following:

$$A_4 = \frac{1}{16} \left| w_3 - \frac{17}{6}w_1w_2 + \frac{11}{12}w_1^3 \right|.$$

Now, using Lemma 1 with  $\sigma = -\frac{17}{6}$  and  $\varsigma = \frac{11}{12}$  and the triangle inequality, we obtain

$$|A_4| \leq \frac{23\sqrt{874}}{8208}.$$

Equalities holds for the function given (17), (18), and using (34), (35).  $\square$

**Theorem 10.** If  $f \in \mathcal{SK}_{\tanh}$  is of the form (1), then

$$\left| A_3 - \eta A_2^2 \right| \leq \frac{1}{6} \max \left\{ 1, \left| \frac{3(\eta - 2)}{8} \right| \right\}.$$

This inequality is sharp.

**Proof.** From (37) and (38), we have

$$\begin{aligned} \left| A_3 - \eta A_2^2 \right| &= \frac{1}{6} \left| w_2 - \frac{3}{4}w_1^2 + \frac{3\eta}{8}w_1^2 \right| \\ &= \frac{1}{6} \left| w_2 + \left( \frac{3(\eta - 2)}{8} \right) w_1^2 \right|. \end{aligned}$$

Using Lemma 2 and the triangle inequality, we obtain the needed result.  $\square$

Putting  $\eta = 1$ , we obtained the below inequality.

**Corollary 3.** If  $f \in \mathcal{SK}_{\tanh}$  is of the form (1), then

$$|A_3 - A_2^2| \leq \frac{1}{6}.$$

Equality is determined by using (34), (35), and (18).

**Theorem 11.** If  $f \in \mathcal{SK}_{\tanh}$  is of the form (1), then

$$|A_4 - A_2A_3| \leq \frac{19\sqrt{4902}}{18576}.$$

**Proof.** From (37)–(39), we obtain

$$|A_4 - A_2A_3| = \frac{1}{16} \left| w_3 - \frac{13}{6}w_1w_2 + \frac{5}{12}w_1^3 \right|,$$

and so by taking  $\sigma = -\frac{13}{6}$  and  $\varsigma = \frac{5}{12}$  in Lemma 1 yields

$$|A_4 - A_2A_3| \leq \frac{19\sqrt{4902}}{18576}.$$

This completes the proof.  $\square$

**Theorem 12.** If  $f \in \mathcal{SK}_{\tanh}$  is of the form (1), then

$$|\mathcal{H}_{2,2}(f^{-1})| = |A_2A_4 - A_3^2| \leq \frac{1}{36}.$$

Equality is determined by using (34)–(36), and (18).

**Proof.** From (37)–(39), we have

$$\begin{aligned} |A_2A_4 - A_3^2| &= \frac{1}{36} \left| \frac{3}{64}w_1^4 + \frac{3}{32}w_1^2w_2 - \frac{9}{16}w_1w_3 + w_2^2 \right| \\ &= \frac{1}{36} \left| \frac{1}{2}(w_2^2 - w_1w_3) + \frac{1}{2} \left( \frac{3}{32}w_1^4 + \frac{3}{16}w_1^2w_2 - \frac{1}{8}w_1w_3 + w_2^2 \right) \right| \\ &\leq \frac{1}{72} |w_2^2 - w_1w_3| + \frac{1}{72} \left| \frac{3}{32}w_1^4 + \frac{3}{16}w_1^2w_2 - \frac{1}{8}w_1w_3 + w_2^2 \right| \\ &= \frac{1}{72}Q_1 + \frac{1}{72}Q_2, \end{aligned}$$

where

$$Q_1 = |w_2^2 - w_1w_3|$$

and

$$Q_2 = \left| \frac{3}{32}w_1^4 + \frac{3}{16}w_1^2w_2 - \frac{1}{8}w_1w_3 + w_2^2 \right|$$

Using Lemma 4, we obtain  $Q_1 \leq 1$ . For  $Q_2$  using Lemma 3, we have

$$\begin{aligned} |Q_2| &\leq \frac{3}{32}|w_1|^4 + \frac{1}{8}|w_1| \left( 1 - |w_1|^2 - \frac{|w_2|^2}{1 + |w_1|} \right) + \frac{3}{16}|w_1|^2|w_2| + |w_2|^2, \\ &\leq \frac{3}{32}|w_1|^4 + \frac{1}{8}|w_1| - \frac{1}{8}|w_1|^3 - \frac{|w_1||w_2|^2}{8(1 + |w_1|)} + \frac{3}{16}|w_1|^2|w_2| + |w_2|^2, \\ &\leq \frac{3}{32}|w_1|^4 + \frac{1}{8}|w_1| - \frac{1}{8}|w_1|^3 + |w_2|^2 \left( 1 - \frac{|w_1|}{8(1 + |w_1|)} \right) + \frac{3}{16}|w_1|^2|w_2|. \quad (40) \end{aligned}$$

Since  $\left(1 - \frac{|w_1|}{8(1+|w_1|)}\right) > 0$  and  $|w_2| \leq 1 - |w_1|^2$  in (40), we have

$$|Q_2| \leq \frac{3}{32}|w_1|^4 + \frac{1}{8}|w_1| - \frac{1}{8}|w_1|^3 + (1 - |w_1|^2)^2 \left(1 - \frac{|w_1|}{8(1+|w_1|)}\right) + \frac{3}{16}|w_1|^2(1 - |w_1|^2).$$

After elementary calculus of maxima and minima, we obtain

$$|Q_2| \leq 1.$$

Hence,

$$\left|\mathcal{H}_{2,2}(f^{-1})\right| \leq \frac{1}{72}Q_1 + \frac{1}{72}Q_2 \leq \frac{1}{36}.$$

The proof is thus completed.  $\square$

## 6. Conclusions

The basic idea behind investigating coefficient problems in various families of holomorphic functions is to represent the coefficients of the corresponding functions with the well-known class  $\mathcal{P}$ , which includes functions with a positive real part in the open unit disc. Many fascinating results were recently attained using this technique. Most of the bounds, however, were non-sharp for analytic univalent functions linked to symmetric points. In this work, we determine the estimates of the problems containing coefficients for functions belonging to the family  $SK_{\tanh}$  of the function, which are starlike with respect to symmetric points associated with  $\tan$  hyperbolic function, respectively. In proof of the main results, we use the Lemmas derived by Prokhorov and Szynal, Libera, and Zlotkiwicz, and Carlson's inequality and bounds on the Schwarz function obtained by Eframidis. The approach is focused on the relationship between the coefficients of functions in the given family and the coefficients of corresponding Schwarz functions. Most of the bounds are proved to be sharp. This work may inspire more investigations on the sharp bounds of analytic functions connected with symmetric points.

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