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Certain Properties of Δ_h Multi-Variate Hermite Polynomials

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Abstract: The research described in this paper follows the hypothesis that the monomiality principle leads to novel results that are consistent with past knowledge. Thus, in line with prior facts, our aim is to introduce the Δ_h multi-variate Hermite polynomials ${}_{\Delta_h}\mathfrak{H}_m(q_1, q_2, \cdots, q_r; h)$. We obtain their recurrence relations by using difference operators. Furthermore, symmetric identities satisfied by these polynomials are established. The operational rules are helpful in demonstrating the novel characteristics of the polynomial families, and thus the operational principles satisfied by these polynomials are derived and will prove beneficial for future observations.

Keywords: Δ_h Hermite polynomials; symmetric identities; monomiality principle; operational formula

MSC: 26A33; 33E20; 33B10; 33E30; 33C45

1. Introduction

One of the most significant families of polynomial sequences is the Hermite family, the Hermite polynomials are a sequence of polynomials that are widely used in mathematics and physics, particularly in the study of quantum mechanics. They are named after Charles Hermite, a French mathematician who first studied them in the late 19th century. The Hermite polynomials are a family of orthogonal polynomials that satisfy a specific differential equation known as the Hermite differential equation.

Many issues in applied mathematics, theoretical physics, approximation theory, and other branches of mathematics regularly include Hermite polynomial sequences [1], which have many applications in mathematical analysis, numerical analysis, statistics, physics, and other fields of mathematics. Particularly in recent years, several extensions of special functions in mathematical physics have seen significant progress. The majority of precisely resolved problems in mathematical physics and engineering that have a variety of broad applications have an analytical foundation due to this groundbreaking discovery.

Gaussian quadrature, numerical analysis, the quantum harmonic oscillator, and Schrondinger's equation all contain the Hermite polynomials. A notable development in the theory of generalized special functions is the addition of multitudinous-index and variable special functions. Both in the context of pure mathematics and in real-world applications, these functions have been recognized for their importance. It is recognized to address the problems that appear in a variety of mathematical fields, from the theory of partial differential equations to abstract group theory, complex polynomials with many variables and indices are necessary. Hermite polynomials with multiple indexes and variables were initially proposed by Hermite himself.

Gaussian quadrature, numerical analysis, the quantum harmonic oscillator, and Schrondinger's equation all contain Hermite polynomials. A large number of authors have

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taken interest in introducing and finding several characteristics of Δ_h special polynomials; see, for example, refs. [2–6]. Wani and colleagues recently developed a number of doped polynomials of a certain kind and deduced their numerous features and attributes, which are significant from an engineering perspective; see, for example, refs. [7–11]. Summation formulas, determinant forms, approximation qualities, explicit and implicit formulae, generating expressions, etc. are a few examples of these features.

In the event when $g : I \subset \mathbb{R} \to \mathbb{R}$ and h $\in \mathbb{R}_+$, the forward difference operator denoted by Δ_h is provided by [12] (p. 2):

$$x(u+h) - x(u) = \Delta_h[x](u). \tag{1}$$

As a result, it follows for finite differences of degree $i \in \mathbb{N}$.

$$\Delta_h\big(\Delta_h^{i-1}[x](u)\big) = \sum_{l=0}^{i}(-1)^{i-l}\binom{i}{l}x(u+lh) = \Delta_h^i[x](u), \tag{2}$$

where $\Delta_h^1 = \Delta_h$ and $\Delta_h^0 = I$, with I as the identity operator.

Costabile et al. [3,13,14] recently made the first attempt to introduce Δ_h polynomial sequences, namely Δ_h Appell polynomials, and they explored their many features, including generating functions, differential equations, and determinant forms.

Further, in [3], Δ_h Appell sequences $\mathcal{Q}_m(q)$, $m \in \mathbb{N}$ were defined by the power series of the product of two functions $\gamma(t)(1+h\xi)^{\frac{q_1}{h}}$ by

$$\gamma(t)(1+h\xi)^{\frac{q_1}{h}} = \mathcal{Q}_0(q;h) + \mathcal{Q}_1(q;h)\frac{t}{1!} + \mathcal{Q}_2(q;h)\frac{t^2}{2!} + \cdots \mathcal{Q}_m(q;h)\frac{t^m}{m!}\cdots, \tag{3}$$

where

$$\gamma(t) = \gamma_{0,h} + \gamma_{1,h}\frac{t}{1!} + \gamma_{2,h}\frac{t^2}{2!} + \cdots + \gamma_{m,h}\frac{t^m}{m!}\cdots. \tag{4}$$

Δ_h Appell sequences transform into popular sequences and polynomials, such as extended falling factorials $(q)^h m \equiv (q)m$ [12], Bernoulli sequences of the second kind $b_m(q)$ [12], Boole sequences $B_{lm}(q;\lambda)$ [12], and Poisson–Charlier sequences $C_m(q;\gamma)$ [12] (p. 2).

It is possible to trace the beginnings of monomiality back to 1941 when Steffenson created the poweroid concept [15], which was subsequently improved by Dattoli [16]. The multiplicative and derivative operations for a polynomial set $\{b_m(u)\}_{m\in\mathbb{N}}$ are called the $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ operators, and they fulfill the following expressions:

$$b_{m+1}(u) = \hat{\mathcal{M}}\{b_m(u)\} \tag{5}$$

and

$$m\, b_{m-1}(u) = \hat{\mathcal{D}}\{b_m(u)\}. \tag{6}$$

When multiplicative and derivative operators are used to alter a set $\{b_m(q)\}_{m\in\mathbb{N}}$, it is referred to as a quasi-monomial and must adhere to the following formula:

$$[\hat{\mathcal{D}}, \hat{\mathcal{M}}] = \hat{\mathcal{D}}\hat{\mathcal{M}} - \hat{\mathcal{M}}\hat{\mathcal{D}} = \hat{1}, \tag{7}$$

and the result shows a Weyl group structure.

When the underlying set is quasi-monomial, the characteristics of the operators $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ can be utilized to ascertain the characteristics of the set. Hence, the following characteristics apply:

(i) $b_m(q)$ demonstrates the differential equation

$$\hat{\mathcal{M}}\hat{\mathcal{D}}\{b_m(q)\} = m\, b_m(q), \tag{8}$$

if $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ possesses differential realizations.

(ii) The following forms can be used to cast the explicit form of $b_m(u)$:

$$b_m(q) = \hat{\mathcal{M}}^m \{1\}, \tag{9}$$

while taking $b_0(q) = 1$.

(iii) Moreover, the following may be used to cast a relation in exponential form for $b_m(q)$:

$$e^{t\hat{\mathcal{M}}}\{1\} = \sum_{m=0}^{\infty} b_m(q) \frac{t^m}{m!}, \quad |t| < \infty \ , \tag{10}$$

by the usage of identity (9).

Numerous branches of mathematical physics, quantum mechanics, and classical optics still employ these operational methods. Hence, these methods offer powerful and efficient research tools; for instance, see [17–19].

Motivated by the work of Costabile and Longo [3], we introduce Δ_h multi-variate Hermite polynomials, which can be represented by the generating expression:

$$(1+h\xi)^{\frac{q_1}{h}} \ (1+h\xi^2)^{\frac{q_2}{h}} \ (1+h\xi^3)^{\frac{q_3}{h}} \cdots (1+h\xi^r)^{\frac{q_r}{h}} = \sum_{m=0}^{\infty} {}_{\Delta_h}\mathfrak{H}_m^{[r]}(q_1, q_2, \cdots, q_r; h) \frac{\xi^m}{m!} \tag{11}$$

We derive several of their properties, and the rest of the manuscript is written as follows: multi-variate Δ_h Hermite polynomials are introduced in Section 2 along with some of their specific features. In Section 3, symmetric identities for these polynomials are established. Quasi-monomial characteristics for these polynomials are established in Section 4. A conclusion is added in the last section.

2. Δ_h Multi-Variate Hermite Polynomials

In this section, we offer an alternative generic technique for identifying multivariate Δ_h Hermite sequences. In actuality, we have

Theorem 1. *Since, we observe Δ_h multi-variate Hermite are given by (11); therefore, we have*

$$\begin{aligned}
{}_{q_1}\Delta_h[{}_{\Delta_h}\mathfrak{H}_m^{[r]}(q_1, q_2, \cdots, q_r; h)] &= \ mh \ {}_{\Delta_h}\mathfrak{H}_{m-1}^{[r]}(q_1, q_2, \cdots, q_r; h) \\
{}_{q_2}\Delta_h[{}_{\Delta_h}\mathfrak{H}_m^{[r]}(q_1, q_2, \cdots, q_r; h)] &= \ m(m-1)h \ {}_{\Delta_h}\mathfrak{H}_{m-2}^{[r]}(q_1, q_2, \cdots, q_r; h) \\
{}_{q_3}\Delta_h[{}_{\Delta_h}\mathfrak{H}_m^{[r]}(q_1, q_2, \cdots, q_r; h)] &= \ m(m-1)(m-2)h \ {}_{\Delta_h}\mathfrak{H}_{m-3}^{[r]}(q_1, q_2, \cdots, q_r; h) \\
&\vdots \\
&\vdots \\
{}_{q_r}\Delta_h[{}_{\Delta_h}\mathfrak{H}_m^{[r]}(q_1, q_2, \cdots, q_r; h)] &= \ m(m-1)(m-2)\cdots(m-r+1)h \ {}_{\Delta_h}\mathfrak{H}_{m-r}^{[r]}(q_1, q_2, \cdots, q_r; h).
\end{aligned} \tag{12}$$

Proof. Differentiating Equation (11) w.r.t. q_1 via difference operators, we have

$$\begin{aligned}
{}_{q_1}\Delta_h\Big\{ \sum_{m=0}^{\infty} {}_{\Delta_h}\mathfrak{H}_m(q_1, q_2, \cdots, q_r; h) \frac{\xi^m}{m!} \Big\} = {}_{q_1}\Delta_h\Big\{ (1+h\xi)^{\frac{q_1}{h}} \ (1+h\xi^2)^{\frac{q_2}{h}} \ (1+h\xi^3)^{\frac{q_3}{h}} \cdots \\
\times (1+h\xi^r)^{\frac{q_r}{h}} \Big\}
\end{aligned}$$

Thus, in view of difference operators given by Equations (1) and (2), it follows that

$$
{}_{q_1}\Delta_h\Big\{\sum_{m=0}^{\infty}{}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\frac{\xi^m}{m!}\Big\} = (1+h\xi)^{\frac{q_1+h}{h}}
\times (1+h\xi^2)^{\frac{q_2}{h}}\cdots(1+h\xi^r)^{\frac{q_r}{h}} - (1+h\xi)^{\frac{q_1}{h}}(1+h\xi^2)^{\frac{q_2}{h}}\cdots(1+h\xi^r)^{\frac{q_r}{h}}
$$
$$
= \Big(1+h\xi-1\Big)(1+h\xi)^{\frac{q_1}{h}}(1+h\xi^2)^{\frac{q_2}{h}}(1+h\xi^3)^{\frac{q_3}{h}}\cdots(1+h\xi^r)^{\frac{q_r}{h}}
$$
$$
= \Big(h\xi\Big)(1+h\xi)^{\frac{q_1}{h}}(1+h\xi^2)^{\frac{q_2}{h}}(1+h\xi^3)^{\frac{q_3}{h}}\cdots(1+h\xi^r)^{\frac{q_r}{h}}
$$
$$
= h\sum_{m=0}^{\infty}{}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\frac{t^{m+1}}{m!}
$$

Replace m with $m-1$, then equalize the coefficients of the same powers of ξ in the resultant expression, and we have

$$
{}_{q_1}\Delta_h\{\sum_{m=0}^{\infty}{}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\frac{\xi^m}{m!}\} = \sum_{m=0}^{\infty} m\{\sum_{m=0}^{\infty}{}_{\Delta_h}\mathfrak{H}_{m-1}(q_1,q_2,\cdots,q_r;h)\frac{\xi^m}{m!}\}.
$$

The proof of the first equation of the system of Equation (12) is obtained by comparing the coefficients of the same powers of ξ on b/s of the previous equation.

Next, differentiating Equation (11) w.r.t. q_2 via difference operators, we have

$$
{}_{q_2}\Delta_h\Big\{\sum_{m=0}^{\infty}{}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\frac{\xi^m}{m!}\Big\} = {}_{q_2}\Delta_h\Big\{(1+h\xi)^{\frac{q_1}{h}}(1+h\xi^2)^{\frac{q_2}{h}}(1+h\xi^3)^{\frac{q_3}{h}}\cdots
\times(1+h\xi^r)^{\frac{q_r}{h}}\Big\}
$$

Thus, in view of the difference operators given by Equations (1) and (2), it follows that

$$
{}_{q_2}\Delta_h\Big\{\sum_{m=0}^{\infty}{}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\frac{\xi^m}{m!}\Big\} = (1+h\xi)^{\frac{q_1}{h}}
\times (1+h\xi^2)^{\frac{q_2+h}{h}}\cdots(1+h\xi^r)^{\frac{q_r}{h}} - (1+h\xi)^{\frac{q_1}{h}}(1+h\xi^2)^{\frac{q_2}{h}}\cdots(1+h\xi^r)^{\frac{q_r}{h}}
$$
$$
= \Big(1+h\xi^2-1\Big)(1+h\xi)^{\frac{q_1}{h}}(1+h\xi^2)^{\frac{q_2}{h}}(1+h\xi^3)^{\frac{q_3}{h}}\cdots(1+h\xi^r)^{\frac{q_r}{h}}
$$
$$
= \Big(h\xi^2\Big)(1+h\xi)^{\frac{q_1}{h}}(1+h\xi^2)^{\frac{q_2}{h}}(1+h\xi^3)^{\frac{q_3}{h}}\cdots(1+h\xi^r)^{\frac{q_r}{h}}
$$
$$
= h\sum_{m=0}^{\infty}{}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\frac{t^{m+2}}{m!} .
$$

Replace m with $m-2$, then equalize the coefficients of the same powers of ξ in the resultant expression, and we have

$$
{}_{q_1}\Delta_h\{\sum_{m=0}^{\infty}{}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\frac{\xi^m}{m!}\} = \sum_{m=0}^{\infty} m(m-1)\{\sum_{m=0}^{\infty}{}_{\Delta_h}\mathfrak{H}_{m-2}(q_1,q_2,\cdots,q_r;h)\frac{\xi^m}{m!}\},
$$

the proof of the second equation of the system of Equation (12) is obtained by comparing the coefficients of the same powers of ξ on b/s of the previous equation.

In a similar fashion, differentiating Equation (11) w.r.t. q_r via difference operators, we have

$$
{}_{q_r}\Delta_h\Big\{\sum_{m=0}^{\infty} {}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\frac{\xi^m}{m!}\Big\} = {}_{q_r}\Delta_h\Big\{(1+h\xi)^{\frac{q_1}{h}}\,(1+h\xi^2)^{\frac{q_2}{h}}\,(1+h\xi^3)^{\frac{q_3}{h}}\cdots \times (1+h\xi^r)^{\frac{q_r}{h}}\Big\}
$$

Thus, in view of the difference operators given by Equations (1) and (2), it follows that

$$
\begin{aligned}
{}_{q_r}\Delta_h\Big\{\sum_{m=0}^{\infty} {}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\frac{\xi^m}{m!}\Big\} &= (1+h\xi)^{\frac{q_1}{h}} \\
&\times (1+h\xi^2)^{\frac{q_2}{h}}\cdots(1+h\xi^r)^{\frac{q_r}{h}} - (1+h\xi)^{\frac{q_1}{h}}\,(1+h\xi^2)^{\frac{q_2}{h}}\cdots(1+h\xi^r)^{\frac{q_r+h}{h}} \\
&= \Big(1+h\xi^r-1\Big)(1+h\xi)^{\frac{q_1}{h}}\,(1+h\xi^2)^{\frac{q_2}{h}}\,(1+h\xi^3)^{\frac{q_3}{h}}\cdots(1+h\xi^r)^{\frac{q_r}{h}} \\
&= \Big(h\xi^r\Big)(1+h\xi)^{\frac{q_1}{h}}\,(1+h\xi^2)^{\frac{q_2}{h}}\,(1+h\xi^3)^{\frac{q_3}{h}}\cdots(1+h\xi^r)^{\frac{q_r}{h}} \\
&= h\sum_{m=0}^{\infty} {}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\frac{t^{m+r}}{m!} .
\end{aligned}
$$

Replace m with $m-r$, then equalize the coefficients of the same powers of ξ in the resultant expression, and we have

$$
{}_{q_r}\Delta_h\{\sum_{m=0}^{\infty} {}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\frac{\xi^m}{m!}\} = \sum_{m=0}^{\infty} m(m-1)(m-2)\cdots(m-r+1) \times\Bigg\{\sum_{m=0}^{\infty} {}_{\Delta_h}\mathfrak{H}_{m-r}(q_1,q_2,\cdots,q_r;h)\frac{\xi^m}{m!}\Bigg\},
$$

the proof of the rth equation of the system of Equation (12) is obtained by comparing the coefficients of the same powers of ξ on b/s of the previous equation. □

Theorem 2. *Further, the Δ_h multi-variate Hermite polynomials ${}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)$, $m\in\mathbb{N}$ are determined by the power series expansion of the product $(1+h\xi)^{\frac{q_1}{h}}\,(1+h\xi^2)^{\frac{q_2}{h}}$ $(1+h\xi^3)^{\frac{q_3}{h}}\cdots(1+h\xi^r)^{\frac{q_r}{h}}$, that is*

$$
\begin{aligned}
(1+h\xi)^{\frac{q_1}{h}}\,(1+h\xi^2)^{\frac{q_2}{h}}\,(1+h\xi^3)^{\frac{q_3}{h}} &= {}_{\Delta_h}\mathfrak{H}_0(q_1,q_2,\cdots,q_r;h) + {}_{\Delta_h}\mathfrak{H}_1(q_1,q_2,\cdots,q_r;h)\tfrac{\xi}{1!} \\
&+ {}_{\Delta_h}\mathfrak{H}_2(q_1,q_2,\cdots,q_r;h)\tfrac{\xi^2}{2!} + \cdots + {}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\tfrac{\xi^m}{m!} + \cdots .
\end{aligned} \tag{13}
$$

Proof. Expanding $(1+h\xi)^{\frac{q_1}{h}}\,(1+h\xi^2)^{\frac{q_2}{h}}\,(1+h\xi^3)^{\frac{q_3}{h}}\cdots(1+h\xi^r)^{\frac{q_r}{h}}$ by a Newton series for finite differences at $q_1=q_2=\cdots\ q_r=0$ and the order the product of the developments of function $(1+h\xi)^{\frac{q_1}{h}}\,(1+h\xi^2)^{\frac{q_2}{h}}\,(1+h\xi^3)^{\frac{q_3}{h}}\cdots(1+h\xi^r)^{\frac{q_r}{h}}$ w.r.t. the powers of ξ, then, in view of expression (3) with $\gamma(t)=1$, we observe the polynomials ${}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)$ expressed in Equation (13) as coefficients of $\frac{\xi^m}{m!}$ as the generating function of Δ_h multi-variate Hermite polynomials. □

Next, we establish the quasi-monomial properties satisfied by Δ_h multi-variate Hermite polynomials by proving the following results:

Theorem 3. *The Δ_h multi-variate Hermite polynomials satisfy the following multiplicative and derivative operators:*

$$
\begin{gathered}
{}_{\Delta_h}\mathfrak{H}_{m+1}(q_1,q_2,\cdots,q_r;h)=\hat{\mathcal{M}}_{\Delta_h}\{{}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\}= \\
\left(q_1\frac{1}{1+_{q_1}\Delta_h}+2q_2\frac{_{q_1}\Delta_h}{h+_{q_1}\Delta_h{}^2}+3q_3\frac{_{q_1}\Delta_h}{h^2+_{q_1}\Delta_h{}^3}+\cdots+rq_r\frac{_{q_1}\Delta_h{}^{r-1}}{h^{r-1}+_{q_1}\Delta_h{}^r}\right)\{{}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\}
\end{gathered} \tag{14}
$$

and

$$
\begin{gathered}
{}_{\Delta_h}\mathfrak{H}_{m-1}(q_1,q_2,\cdots,q_r;h)=\hat{\mathcal{D}}_{\Delta_h}\left\{{}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\right\}= \\
\frac{log(1+_{q_1}\Delta_h)}{mh}\left\{{}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\right\},
\end{gathered} \tag{15}
$$

respectively.

Proof. In view of finite difference operator Δ_h, we have

$$
{}_{q_1}\Delta_h\left[{}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\right]=h\,\xi\left[{}_{\Delta_h}\mathfrak{H}_{m-1}(q_1,q_2,\cdots,q_r;h)\right], \tag{16}
$$

or

$$
\frac{_{q_1}\Delta_h}{h}\left[{}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\right]=\xi\left[{}_{\Delta_h}\mathfrak{H}_{m-1}(q_1,q_2,\cdots,q_r;h)\right]. \tag{17}
$$

Differentiating (11) w.r.t. ξ and q_1, separately, we find

$$
\begin{gathered}
{}_{\Delta_h}\mathfrak{H}_{m+1}(q_1,q_2,\cdots,q_r;h)=\hat{\mathcal{M}}_{\Delta_h}\left\{{}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\right\}= \\
\left(\frac{q_1}{1+h\xi}+\frac{2\,q_2\xi}{1+h\xi^2}+\frac{3\,q_3\xi^2}{1+h\xi^3}+\cdots+\frac{r\,q_r\xi^{r-1}}{1+h\xi^r}\right)\left\{{}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\right\}
\end{gathered} \tag{18}
$$

and

$$
\begin{gathered}
{}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)=\hat{\mathcal{D}}_{\Delta_h}\left\{{}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\right\}= \\
\frac{log(1+h\xi)}{mh}\left\{{}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)\right\}.
\end{gathered} \tag{19}
$$

Using identity (17) in view of (5) and (6) in above Equations (18) and (19), we are led to assertions (14) and (15). □

Theorem 4. *The Δ_h multi-variate Hermite polynomials satisfy the differential equation listed as:*

$$
\begin{gathered}
\left(q_1\frac{1}{1+_{q_1}\Delta_h}+2q_2\frac{_{q_1}\Delta_h}{h+_{q_1}\Delta_h{}^2}+3q_3\frac{_{q_1}\Delta_h}{h^2+_{q_1}\Delta_h{}^3}+\cdots+rq_r\frac{_{q_1}\Delta_h{}^{r-1}}{h^{r-1}+_{q_1}\Delta_h{}^r}-\frac{m^2h}{log(1+_{q_1}\Delta_h)}\right) \\
\times\,{}_{\Delta_h}\mathfrak{H}_m(q_1,q_2,\cdots,q_r;h)=0.
\end{gathered} \tag{20}
$$

Proof. Making use of expressions (14) and (15) in (8), we are led to assertion (20). □

Since, $(1+h\xi)^{\frac{1}{h}}\rightarrow e^{\xi}$ as $h\rightarrow 0$, it is clear that expression (11) reduces to multi-variable Hermite polynomials.

3. Symmetric Identities

We provide symmetric identities for the multivariate Hermite Kampé de Fériet polynomials in this section. Furthermore, we learn some of the multi-variate Hermite Kampé de Fériet polynomials' formulae and characteristics.

Theorem 5. *For, $\beta\neq\alpha$ and $\beta,\alpha.0$, we have*

$$
\alpha^m{}_{\Delta_h}\mathfrak{H}_m(\beta q_1,\beta q_2,\cdots,\beta q_r;h)=\beta^m{}_{\Delta_h}\mathfrak{H}_m(\alpha q_1,\alpha q_2,\cdots,\alpha q_r;h). \tag{21}
$$

Proof. Since, $\beta\neq\alpha$ and $\beta,\alpha.0$, we start by writing:

$$
\mathfrak{R}(\xi;q_1,q_2,q_3,\cdots,q_r;h)=(1+h\xi)^{\frac{\alpha\beta q_1}{h}}(1+h\xi^2)^{\frac{\alpha^2\beta^2q_2}{h}}\cdots(1+h\xi^r)^{\frac{\alpha^r\beta^rq_r}{h}}. \tag{22}
$$

Therefore, the above expression $\mathfrak{R}(\xi; q_1, q_2, q_3, \cdots, q_r; h)$ is symmetric in α and β.

Further, we can write

$$\mathfrak{R}(\xi; q_1, q_2, q_3, \cdots, q_r; h) = {}_{\Delta_h}\mathfrak{H}_m(\alpha q_1, \alpha^2 q_2, \cdots, \alpha^r q_r; h)\frac{(\beta\xi)^m}{m!} = \beta^m\, {}_{\Delta_h}\mathfrak{H}_m(\alpha q_1, \alpha^2 q_2, \cdots, \alpha^r q_r; h)\frac{\xi^m}{m!}. \tag{23}$$

Thus, it follows that

$$\mathfrak{R}(\xi; q_1, q_2, q_3, \cdots, q_r; h) = {}_{\Delta_h}\mathfrak{H}_m(\beta q_1, \beta^2 q_2, \cdots, \beta^r q_r; h)\frac{(\alpha\xi)^m}{m!} = \alpha^m\, {}_{\Delta_h}\mathfrak{H}_m(\beta q_1, \beta^2 q_2, \cdots, \beta^r q_r; h)\frac{\xi^m}{m!}. \tag{24}$$

The assertion (21) is obtained by equating the coefficients of the like term of xi in the final two equations (23) and (24). □

Theorem 6. *For, $\beta \neq \alpha$ $\beta, \alpha > 0$, and it follows that*

$$\sum_{i=0}^{m}\sum_{n=0}^{i}\binom{m}{i}\binom{i}{n}\alpha^i\beta^{m+1-i}{}_{\Delta_h}\mathfrak{H}_m(\beta q_1, \beta^2 q_2, \cdots, \beta^r q_r; h)\mathcal{P}_{m-i}(\alpha-1; h) =$$

$$\sum_{i=0}^{m}\sum_{n=0}^{i}\binom{m}{i}\binom{i}{n}\beta^i\alpha^{m+1-i}{}_{\Delta_h}\mathfrak{H}_m(\alpha q_1, \alpha^2 q_2, \cdots, \alpha^r q_r; h)\mathcal{P}_{m-i}(\beta-1; h). \tag{25}$$

Proof. Since, $\beta \neq \alpha$ $\beta, \alpha > 0$, we start by writing:

$$\mathfrak{S}(\xi; q_1, q_2, q_3, \cdots, q_r; h) = \frac{\alpha\beta\xi(1+h\xi)^{\frac{\alpha\beta q_1}{h}}(1+h\xi^2)^{\frac{\alpha^2\beta^2 q_2}{h}}\cdots(1+h\xi^r)^{\frac{\alpha^r\beta^r q_r}{h}}\left((1+h\xi)^{\frac{\alpha\beta q_1}{h}-1}\right)}{\left((1+h\xi)^{\frac{\alpha}{h}-1}\right)\left((1+h\xi)^{\frac{\beta}{h}-1}\right)}. \tag{26}$$

In a similar way to the previous theorem, we obtain statement (25). □

Theorem 7. *For, $\beta \neq \alpha$ $\beta, \alpha > 0$, we have*

$$\sum_{i=0}^{m}\sum_{n=0}^{i}\binom{m}{i}\binom{i}{n}\alpha^i\beta^{m+1-i}\,\mathcal{B}_m(h){}_{\Delta_h}\mathfrak{H}_{i-m}(\beta q_1, \beta^2 q_2, \cdots, \beta^r q_r; h)\mathcal{P}_{m-i}(\alpha-1; h)\;\sigma_{m-i}(\alpha-1; h) =$$

$$\sum_{i=0}^{m}\sum_{n=0}^{i}\binom{m}{i}\binom{i}{n}\beta^i\alpha^{m+1-i}\,\mathcal{B}_m(h){}_{\Delta_h}\mathfrak{H}_{i-m}(\alpha q_1, \alpha^2 q_2, \cdots, \alpha^r q_r; h)\;\sigma_{m-i}(\beta-1; h). \tag{27}$$

Proof. Since, $\beta \neq \alpha$ $\beta, \alpha > 0$, we start by writing:

$$\mathfrak{T}(\xi; q_1, q_2, q_3, \cdots, q_r; h) = \frac{\alpha\beta\xi(1+h\xi)^{\frac{\alpha\beta q_1}{h}}(1+h\xi^2)^{\frac{\alpha^2\beta^2 q_2}{h}}\cdots(1+h\xi^r)^{\frac{\alpha^r\beta^r q_r}{h}}\left((1+h\xi)^{\frac{\alpha\beta q_1}{h}-1}\right)}{\left((1+h\xi)^{\frac{\alpha}{h}-1}\right)\left((1+h\xi)^{\frac{\beta}{h}-1}\right)}. \tag{28}$$

The preceding equation may be expressed as

$$\mathfrak{T}(\xi; q_1, q_2, q_3, \cdots, q_r; h) = \frac{\alpha\beta\xi}{\left((1+h\xi)^{\frac{\alpha}{h}-1}\right)}(1+h\xi)^{\frac{\alpha\beta q_1}{h}}(1+h\xi^2)^{\frac{\alpha^2\beta^2 q_2}{h}}\cdots(1+h\xi^r)^{\frac{\alpha^r\beta^r q_r}{h}} \times \frac{\left((1+h\xi)^{\frac{\alpha\beta q_1}{h}-1}\right)}{\left((1+h\xi)^{\frac{\beta}{h}-1}\right)}. \tag{29}$$

Using $\frac{\alpha\beta\xi}{\left((1+h\xi)^{\frac{\alpha}{h}-1}\right)} = \alpha\sum\limits_{m=0}^{\infty}\mathcal{B}_m(h)\frac{(\beta\xi)^m}{m!}$, $\frac{\left((1+h\xi)^{\frac{\alpha\beta q_1}{h}-1}\right)}{\left((1+h\xi)^{\frac{\beta}{h}-1}\right)} = \sigma_{m-i}(\alpha-1;h)\frac{(\beta\xi)^m}{m!}$ and (11), we have

$$\begin{aligned}\mathfrak{T}(\xi;q_1,q_2,q_3,\cdots,q_r;h) &= \alpha\sum_{m=0}^{\infty}\mathcal{B}_m(h)\frac{(\beta\xi)^m}{m!}\sum_{m=0}^{\infty}{}_{\Delta_h}\mathfrak{H}_m(\beta q_1,\beta^2 q_2,\cdots,\beta^r q_r;h)\frac{(\alpha\xi)^m}{m!}\\ &\times\sum_{m=0}^{\infty}\sigma_{m-i}(\alpha-1;h)\frac{(\beta\xi)^m}{m!}\\ = \sum_{m=0}^{\infty}&\left(\sum_{i=0}^{m}\sum_{n=0}^{i}\binom{m}{i}\binom{i}{n}\alpha^i\beta^{m+1-i}\,\mathcal{B}_m(h)_{\Delta_h}\mathfrak{H}_{i-m}(\beta q_1,\beta^2 q_2,\cdots,\beta^r q_r;h)\sigma_{m-i}(\alpha-1;h)\right)\frac{\xi^m}{m!}.\end{aligned} \tag{30}$$

If we proceed in the same way, we have

$$\begin{aligned}\mathfrak{T}(\xi;q_1,q_2,q_3,\cdots,q_r;h) = \sum_{m=0}^{\infty}&\left(\sum_{i=0}^{m}\sum_{n=0}^{i}\binom{m}{i}\binom{i}{n}\beta^i\alpha^{m+1-i}\,\mathcal{B}_m(h)\right.\\ &\left.\times{}_{\Delta_h}\mathfrak{H}_{i-m}(\beta q_1,\beta^2 q_2,\cdots,\beta^r q_r;h)\,\sigma_{m-i}(\beta-1;h)\right)\frac{\xi^m}{m!}.\end{aligned} \tag{31}$$

If we compare the coefficients of expressions (30) and (31), we obtain the result (27). □

4. Operational Formalism and Identities

The creation of new functional families and the facilitation of the derivation of the attributes associated with regular and generalized special functions are both possible through the application of operational approaches. Dattoli and their collaborators [15,16,20–23] are interested in the study of special functions that aim to discover explicit solutions for families of partial differential equations, such those of the Heat and D'Alembert type, and their applications have acknowledged the value of using operational processes.

Differentiating successively (11) w.r.t. q_1 via the forward difference operator concept by taking into consideration expression (2), we find

$$\begin{aligned}{}_{q_1}\Delta_h[{}_{\Delta_h}\mathfrak{H}_m^{[r]}(q_1,q_2,\cdots,q_r;h)] &= mh\,{}_{\Delta_h}\mathfrak{H}_{m-1}^{[r]}(q_1,q_2,\cdots,q_r;h)\\ {}_{q_1}\Delta_h^2[{}_{\Delta_h}\mathfrak{H}_m^{[r]}(q_1,q_2,\cdots,q_r;h)] &= m(m-1)h\,{}_{\Delta_h}\mathfrak{H}_{m-2}^{[r]}(q_1,q_2,\cdots,q_r;h)\\ &\vdots\\ {}_{q_1}\Delta_h^r[{}_{\Delta_h}\mathfrak{H}_m^{[r]}(q_1,q_2,\cdots,q_r;h)] &= m(m-1)(m-2)\cdots(m-r+1)h\\ \times\,{}_{\Delta_h}\mathfrak{H}_{m-r}^{[r]}(q_1,q_2,\cdots,q_r;h).&\end{aligned} \tag{32}$$

Next, differentiating (11) w.r.t. $q_2,q_3,\cdots,q_r$ via the forward difference operator concept by taking into consideration expression (2), we find

$$\begin{aligned}{}_{q_2}\Delta_h[{}_{\Delta_h}\mathfrak{H}_m^{[r]}(q_1,q_2,\cdots,q_r;h)] &= m(m-1)h\,{}_{\Delta_h}\mathfrak{H}_{m-2}^{[r]}(q_1,q_2,\cdots,q_r;h)\\ {}_{q_3}\Delta_h[{}_{\Delta_h}\mathfrak{H}_m^{[r]}(q_1,q_2,\cdots,q_r;h)] &= m(m-1)(m-2)h\,{}_{\Delta_h}\mathfrak{H}_{m-3}^{[r]}(q_1,q_2,\cdots,q_r;h)\\ &\vdots\\ &\vdots\\ {}_{q_r}\Delta_h[{}_{\Delta_h}\mathfrak{H}_m^{[r]}(q_1,q_2,\cdots,q_r;h)] &= m(m-1)(m-2)\cdots(m-r+1)h\\ \times{}_{\Delta_h}\mathfrak{H}_{m-r}^{[r]}(q_1,q_2,\cdots,q_r;h).&\end{aligned} \tag{33}$$

In view of system of expressions (32) and (33), we find that ${}_{\Delta_h}\mathfrak{H}^{[r]}_{m-r}(q_1, q_2, \cdots, q_r; h)$ are solutions of the expressions:

$$\begin{aligned}
{}_{q_2}\Delta_h[{}_{\Delta_h}\mathfrak{H}^{[r]}_m(q_1, q_2, \cdots, q_r; h)] &= {}_{q_1}\Delta_h^2 {}_{\Delta_h}\mathfrak{H}^{[r]}_{m-2}(q_1, q_2, \cdots, q_r; h) \\
{}_{q_3}\Delta_h[{}_{\Delta_h}\mathfrak{H}^{[r]}_m(q_1, q_2, \cdots, q_r; h)] &= {}_{q_1}\Delta_h^3 {}_{\Delta_h}\mathfrak{H}^{[r]}_{m-3}(q_1, q_2, \cdots, q_r; h) \\
&\vdots \\
&\vdots \\
{}_{q_r}\Delta_h[{}_{\Delta_h}\mathfrak{H}^{[r]}_m(q_1, q_2, \cdots, q_r; h)] &= {}_{q_1}\Delta_h^r {}_{\Delta_h}\mathfrak{H}^{[r]}_{m-r}(q_1, q_2, \cdots, q_r; h),
\end{aligned} \tag{34}$$

under the listed initial condition

$${}_{\Delta_h}\mathfrak{H}^{[r]}_m(q_1, 0, 0, \cdots, 0; h) = {}_{\Delta_h}\mathfrak{H}_m(q_1; h). \tag{35}$$

Therefore, from system of expressions (34) and expression (35), we find that

$${}_{\Delta_h}\mathfrak{H}^{[r]}_m(q_1, q_2, \cdots, q_r; h) = \exp\left(q_2 \, {}_{q_1}\Delta_h^2 + q_3 \, {}_{q_1}\Delta_h^3 + \cdots + q_r \, {}_{q_1}\Delta_h^r\right) {}_{\Delta_h}\mathfrak{H}_m(q_1; h). \tag{36}$$

In light of previous mentioned expression, the Δ_h multi-variate Hermite polynomials ${}_{\Delta_h}\mathfrak{H}^{[r]}_m(q_1, q_2, \cdots, q_r; h)$ can be constructed from the the Δ_h polynomial ${}_{\Delta_h}\mathfrak{H}_m(q_1, ; h)$ by applying the operational rule (36).

5. Conclusions

Here, we introduced the Δ_h multi-variate Hermite polynomials, and some of their specific features were presented: Quasi-monomial characteristics for these polynomials were established in Section 2, and forward difference relations were established in Theorem 1. Furthermore, symmetric identities were given in Section 3, and the operational rule was established in Section 4.

Many mathematicians and physicists employ the Hermite polynomials, a set of polynomials that is particularly useful for studying quantum mechanics. They are named after the French mathematician Charles Hermite, who conducted the initial investigation into them in the late 19th century. A collection of orthogonal polynomials known as the Hermite polynomials fulfills the Hermite differential equation—a particular type of differential equation. In addition to Gaussian quadrature, these polynomials may be found in physics, numerical analysis, the quantum harmonic oscillator, and Schrondinger's equation. They also come up often in problems involving theoretical physics, approximation theory, applied mathematics, and other disciplines of mathematics including biological and medical science.

The above-mentioned polynomials may also be proven to have extended generalized forms, integral representations, and other features through further research and observation. Furthermore, the interpolation form, recurrence relations, shift operators, and summation formulae can also be a problem for new observations. Moreover, the hybrid forms of these polynomials can be studied in future investigations, such as when convoluting with Δ_h Bernoulli, Euler, Genocchi, and Tangent polynomials.

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