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Fixed-Point Results for (α-ψ)-Fuzzy Contractive Mappings on Fuzzy Double-Controlled Metric Spaces

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Abstract: We introduce the novel concept of (α-ψ)-fuzzy contractive mappings on fuzzy double-controlled metric spaces and demonstrate some fixed-point results. The theorems presented generalize some intriguing findings in the literature. Thus, we prove the fixed-point theorem in the settings of fuzzy double-controlled metric spaces. Furthermore, we provide several examples and an application of our result on the existence of the solution to an integral equation.

Keywords: fixed point; controlled fuzzy metric space; (α-ψ)-contraction; fuzzy metric space; fuzzy double-controlled metric space; α-admissible mapping; integral equations

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1. Introduction

The fixed-point theory is a fast-growing and exciting field of mathematics with various applications in diverse areas of mathematics, including nonlinear analysis. It stems from the Banach contraction principle, which was proved in 1922 in a metric space setting [1]. Numerous expansions of the Banach Theorem appeared, which led to the generalization of the notion of classical metric spaces, such as the concept of b-metric spaces by Bakhtin [2], expanded b-metric spaces by Kamran et al. [3], controlled metric-type spaces by Mlaiki et al. [4], which was later developed into double-controlled metric-type spaces by Abdeljawad et al. [5], and double-controlled quasi metric-like spaces by Haque et al. [6]. Recently, Azmi [7] produced some fixed-point results on double-controlled metric-type spaces by utilizing the (α-ψ)-contractive mappings. Lately, a new geometric generalization of the fixed-point theory appeared as the fixed-circle problem [8–11].

In 1965, Zadeh proposed the fuzzy set theory [12] as a natural extension of the concept of a set and established the groundwork for fuzzy mathematics, and the interest in the fuzzy set has grown since then. Combining the probabilistic metric space with the fuzzy set, a new concept of fuzzy space was introduced in [13], with applications in applied sciences, such as signal processing and medical imaging, and in a variety of mathematical disciplines, such as topology, logic, analysis, algebra, artificial intelligence, and fixed-point theory. Many authors have used fuzzy sets extensively in many branches of mathematics. For instance, Puri and Ralescu [14] introduced the differentials of fuzzy functions, whereas Buckley and Feuring [15] established the theory of fuzzy partial differential equations, and Kaleva [16] pioneered fuzzy differential equations. Fuzzy metric spaces are one of the most studied topics in fuzzy set theory, introduced by Kramosil and Michalek [17]. Afterward, many authors extended the fuzzy metric space notion and developed it in various directions. For example, George and Veeramani [18] modified the notion of fuzzy metric space and illustrated that every fuzzy metric produces a Hausdorff topology. Nadaban introduced the concept of fuzzy b-metric space [19], and some fixed-point results in fuzzy b-metric space were carried out by Kim et al. [20]. Then, Mehmood et al. [21] defined the concept of extended fuzzy b-metric space and established the contraction principle. Afterward, Saleem et al. presented the concept of fuzzy double-controlled metric space and illustrated the Banach contraction principle [22], while in [23] they examined the notion of extended b-rectangular and controlled rectangular fuzzy metric-like spaces. The fixed-point results on fuzzy triple-controlled metric spaces were explored by Furqan et al. [24]. The α-admissible mappings and the notion of (α-ψ)-contractive mappings on complete metric spaces with various fixed-point theorems were developed by Samet et al. [25]. Motivated by Samet's work, Gopal and Vetro [26] discussed the concept of (α-ψ)-fuzzy contractive mappings and established some fixed-point results.

Inspired by the work of Gopal and Vetro [26], we introduce the notion of (α-ψ)-fuzzy contractive mappings on fuzzy double-controlled metric spaces and establish some fixed-point results.

2. Preliminaries

We recall some notions and definitions which will be needed in the sequel.

Definition 1 ([22])**.** *A binary operation $* : [0,1]^2 \to [0,1]$ is a continuous t-norm if it satisfies the following conditions:*

1. *$*$ is commutative and associative.*
2. *$*$ operation is continuous.*
3. *$\zeta * 1 = \zeta$ for all $\zeta \in [0,1]$.*
4. *$\zeta_1 * \xi_1 \leq \zeta_2 * \xi_2$, if $\zeta_1 \leq \zeta_2$ and $\xi_1 \leq \xi_2$, for all $\zeta_1, \zeta_2, \xi_1, \xi_2 \in [0,1]$.*

Next, we recall the definition of fuzzy metric space as stated by George and Veeramani [18].

Definition 2. *Let $\mathcal{X}$ be a nonempty set. A fuzzy metric space is a triple $(\mathcal{X}, \mathcal{M}, *)$, where $*$ is a continuous t-norm and $\mathcal{M}$ is a fuzzy set on $\mathcal{X}^2 \times (0, +\infty)$, satisfying the following, for all $\zeta, \xi \in \mathcal{X}$:*

(F1) $\mathcal{M}(\zeta, \xi, t) > 0$ for all $t > 0$;
(F2) $\mathcal{M}(\zeta, \xi, t) = 1$ for all $t > 0$, if and only if $\zeta = \xi$;
(F3) $\mathcal{M}(\zeta, \xi, t) = \mathcal{M}(\xi, \zeta, t)$, symmetric in ζ and ξ, and for all $t > 0$;
(F4) $\mathcal{M}(\zeta, \xi, .) : (0, +\infty) \to [0,1]$ is continuous;
(F5) $\mathcal{M}(\zeta, \varpi, t+s) \geq \mathcal{M}(\zeta, \xi, t) * \mathcal{M}(\xi, \varpi, s)$ for all $\varpi \in \mathcal{X}$ and for all $t, s > 0$.

A more general concept of a fuzzy metric space is the fuzzy b-metric space [19].

Definition 3. *Let $\mathcal{X}$ be a nonempty set, given any real number $b \geq 1$, let $*$ be a continuous t-norm. A fuzzy set $\mathcal{M}$ on $\mathcal{X}^2 \times (0, +\infty)$ is called a fuzzy b-metric on $\mathcal{X}$, if for all $\zeta, \xi, \varpi \in \mathcal{X}$, and $t, s > 0$, the following conditions hold:*

(F1) $\mathcal{M}(\zeta, \xi, t) > 0$;
(F2) $\mathcal{M}(\zeta, \xi, t) = 1$, if and only if $\zeta = \xi$;
(F3) $\mathcal{M}(\zeta, \xi, t) = \mathcal{M}(\xi, \zeta, t)$, symmetric in ζ and ξ for all $t > 0$;
(F4) $\mathcal{M}(\zeta, \varpi, t+s) \geq \mathcal{M}(\zeta, \xi, t/b) * \mathcal{M}(\xi, \varpi, s/b)$;
(F5) $\mathcal{M}(\zeta, \xi, .) : (0, +\infty) \to [0,1]$ is continuous.
The quadruple $(\mathcal{X}, \mathcal{M}, *, b)$ is called a fuzzy b-metric space.

Next, we define the notion of fuzzy double-controlled metric space [22].

Definition 4 ([22])**.** *Consider two non-comparable functions $\beta, \mu : \mathcal{X}^2 \to [1, +\infty)$, defined on a nonempty set $\mathcal{X}$, and let $*$ be a continuous t-norm operation. A fuzzy set $\mathcal{M}_{\beta,\mu}$ on $\mathcal{X}^2 \times (0, +\infty)$ is called a fuzzy double-controlled metric on $\mathcal{X}$, if for all $\zeta, \xi, \varpi \in \mathcal{X}$ the following conditions hold:*

(FD1) $\mathcal{M}_{\beta,\mu}(\zeta, \xi, t) > 0$ for all $t > 0$;
(FD2) $\mathcal{M}_{\beta,\mu}(\zeta, \xi, t) = 1$ for all $t > 0$, if and only if $\zeta = \xi$;
(FD3) $\mathcal{M}_{\beta,\mu}(\zeta, \xi, t) = \mathcal{M}_{\beta,\mu}(\xi, \zeta, t)$, symmetric in ζ and ξ, and for all $t > 0$;
(FD4) $\mathcal{M}_{\beta,\mu}(\zeta, \varpi, t+s) \geq \mathcal{M}_{\beta,\mu}(\zeta, \xi, \frac{t}{\beta(\zeta,\xi)}) * \mathcal{M}_{\beta,\mu}(\xi, \varpi, \frac{s}{\mu(\xi,\varpi)})$, for all $s, t > 0$;
(FD5) $\mathcal{M}_{\beta,\mu}(\zeta, \xi, .) : (0, +\infty) \to [0,1]$ is continuous.

Then, $(\mathcal{X}, \mathcal{M}_{\beta,\mu}, *)$ is called a fuzzy double-controlled metric space. From now on, the class of fuzzy double-controlled metric space will be denoted as $FDCM$.

Remark 1. *The class of fuzzy double-controlled metric space is larger than the class of fuzzy b-metric spaces, as one can see by taking $\beta(\zeta,\xi)=\mu(\zeta,\xi)=b$. Moreover, the class of fuzzy b-metric spaces is effectively larger than that of fuzzy metric spaces, taking $b=1$.*

Example 1 ([22]). *Let $\mathcal{X}=[0,1]$, and define $\beta,\mu:\mathcal{X}^2\to[1,+\infty)$ by $\beta(\zeta,\xi)=2(\zeta+\xi)$ and $\mu(\zeta,\xi)=2(\zeta^2+\xi^2+1)$. Let*

$$\mathcal{M}_{\beta,\mu}(\zeta,\xi,t)=\exp^{-\frac{(\zeta-\xi)^2}{t}},\ \zeta,\xi\in\mathcal{X},t>0.$$

Then, $(\mathcal{X},\mathcal{M}_{\beta,\mu},*)$ is an FDCM with product t-norm.
Axioms (FD1) to (FD3) and (FD5) are straightforward; we only prove (FD4). Note

$$\begin{aligned}\mathcal{M}_{\beta,\mu}(\zeta,\xi,\frac{t}{\beta(\zeta,\xi)}) &= \exp^{-\frac{(\zeta-\xi)^2}{\frac{t}{\beta(\zeta,\xi)}}}=\exp^{-\frac{(\zeta-\xi)^2}{t/2(\zeta+\xi)}}.\\ &\leq \exp^{-\frac{(\zeta-\xi)^2}{t}}\leq\exp^{-\frac{(\zeta-\xi)^2}{t+s}}.\end{aligned}$$

Moreover,

$$\begin{aligned}\mathcal{M}_{\beta,\mu}(\xi,\varpi,\frac{s}{\mu(\xi,\varpi)}) &= \exp^{-\frac{(\xi-\varpi)^2}{\frac{s}{\mu(\xi,\varpi)}}}=\exp^{-\frac{(\xi-\varpi)^2}{s/2(\xi^2+\varpi^2+1)}}.\\ &\leq \exp^{-\frac{(\xi-\varpi)^2}{s}}\leq\exp^{-\frac{(\xi-\varpi)^2}{t+s}}.\end{aligned}$$

Observe

$$\begin{aligned}\mathcal{M}_{\beta,\mu}(\zeta,\varpi,t+s) &= \exp^{-\frac{(\zeta-\varpi)^2}{t+s}}=\exp^{-\frac{(\zeta-\xi+\xi-\varpi)^2}{t+s}}.\\ &\geq \exp^{-\frac{(\zeta-\xi)^2}{t}}\exp^{-\frac{(\xi-\varpi)^2}{s}}.\\ &\geq \mathcal{M}_{\beta,\mu}(\zeta,\xi,\frac{t}{\beta(\zeta,\xi)})*\mathcal{M}_{\beta,\mu}(\xi,\varpi,\frac{s}{\mu(\xi,z)}).\end{aligned}$$

Thus, $(\mathcal{X},\mathcal{M}_{\beta,\mu},*)$ is an FDCM, which is not a fuzzy b-metric space, because $\beta(\zeta,\xi)=2(\zeta+\xi)\neq\mu(\zeta,\xi)=2(\zeta^2+\xi^2+1)\neq b$.

The next example illustrates a fuzzy b-metric space which is not a fuzzy metric space.

Example 2. *Let $\mathcal{X}=[0,1]$ and let $b=2$. Define*

$$\mathcal{M}(\zeta,\xi,t)=\exp^{-\frac{(\zeta-\xi)^2}{t}},\textit{for all}\ \zeta,\xi\in\mathcal{X},t>0.$$

Then, $(\mathcal{X},\mathcal{M},*)$ is a fuzzy b-metric space which is not a fuzzy metric space.
Axioms (F1) to (F3) and (F5) are straightforward; we only prove (F4). Note that

$$\frac{(\zeta-\xi)^2}{t+s}\leq 2\frac{(\zeta-\varpi)^2}{t+s}+2\frac{(\varpi-\xi)^2}{t+s}\leq\frac{(\zeta-\varpi)^2}{\frac{t}{2}}+\frac{(\varpi-\xi)^2}{\frac{s}{2}}.$$

Hence,

$$\begin{aligned}\mathcal{M}(\zeta,\xi,t+s) &= \exp^{-\frac{(\zeta-\xi)^2}{t+s}}\geq\exp^{-\frac{(\zeta-\varpi)^2}{t}}\exp^{-\frac{(\varpi-\xi)^2}{s}}.\\ &\geq \mathcal{M}(\zeta,\varpi,\frac{t}{2})*\mathcal{M}(\varpi,\xi\frac{s}{2}).\end{aligned}$$

This shows that $(\mathcal{X},\mathcal{M},*)$ is a fuzzy b-metric space which is not a fuzzy metric space.

Next, we define the concept of sequence convergence in $FDCM$ and the notion of the open ball in this topology.

Definition 5. *Let $(\mathcal{X}, \mathcal{M}_{\beta,\mu}, *)$ be an FDCM. Then, the following:*

(1) A sequence $\{\zeta_n\}$ converges to $\zeta \in \mathcal{X}$, if $\lim_{n\to+\infty} \mathcal{M}_{\beta,\mu}(\zeta_n, \zeta, t) = 1$ for all $t > 0$.
(2) A sequence $\{\zeta_n\}$ is called Cauchy, if $\mathcal{M}_{\beta,\mu}(\zeta_n, \zeta_{n+m}, t) = 1$, for each $m \in \mathbb{N}$ and $t > 0$.
(3) $(\mathcal{X}, \mathcal{M}_{\beta,\mu}, *)$ is called complete FDCM, if every Cauchy sequence is convergent.
(4) The open ball $\mathcal{B}(\zeta_0, r, t)$ with center ζ_0, radius $r \in (0, 1)$, and $t > 0$ is defined as follows:

$$\mathcal{B}(\zeta_0, r, t) = \{\xi \in \mathcal{X} : \mathcal{M}_{\beta,\mu}(\zeta_0, \xi, t) > 1 - r\}.$$

Remark 2. *The topology in the fuzzy metric space is different from the topology in the metric space because the definition of the open balls is different in both spaces. For instance, if $(\mathcal{Y}, \mathcal{D})$ is any metric space, then a circle with center y_0 is defined as*

$$C_{y_0,r} = \{y \in \mathcal{Y} : \mathcal{D}(y, y_0) = r\},$$

while the open ball with center y_0 is defined as

$$B_{y_0,r} = \{y \in \mathcal{Y} : \mathcal{D}(y, y_0) < r\},$$

which is different from the way the open ball is defined in Definition 5.

Definition 6 ([22])**.** *Let $(\mathcal{X}, \mathcal{M}_{\beta,\mu}, *)$ be an FDCM. The fuzzy double-controlled metric $\mathcal{M}_{\beta,\mu}$ is said to be triangular if the following condition holds:*

$$\left(\frac{1}{\mathcal{M}_{\beta,\mu}(\zeta, \xi, t)} - 1\right) \leq \left(\frac{1}{\mathcal{M}_{\beta,\mu}(\zeta, \varpi, t)} - 1\right) + \left(\frac{1}{\mathcal{M}_{\beta,\mu}(\xi, \varpi, t)} - 1\right), \tag{1}$$

for all $\zeta, \xi, \varpi \in \mathcal{X}$ and for all $t > 0$.

Next, we state a lemma which is useful in proving our results, for details consult [22].

Lemma 1. *Let $\{\zeta_n\}$ be a Cauchy sequence in an FDCM $(\mathcal{X}, \mathcal{M}_{\beta,\mu}, *)$ such that $\zeta_n \neq \zeta_m$ whenever $m, n \in \mathbb{N}$ with $n \neq m$. Then, the sequence $\{\zeta_n\}$ can converge to at most one limit point.*

3. The Main Results

Inspired by Gopal and Vetro [26], who introduced the concept of (α-ψ)-fuzzy contractive mapping on fuzzy metric spaces, we introduce two concepts: α-admissible mappings and (α-ψ)-fuzzy contractive mappings on fuzzy double-controlled metric space $(\mathcal{X}, \mathcal{M}_{\beta,\mu}, *)$ as follows:

Definition 7. *Let $(\mathcal{X}, \mathcal{M}_{\beta,\mu}, *)$ be an FDCM. We say $T : \mathcal{X} \longrightarrow \mathcal{X}$ is α-admissible if there exists $\alpha : \mathcal{X}^2 \times (0, +\infty) \to [0, +\infty)$, such that for all $t > 0$,*

$$\zeta, \xi \in \mathcal{X},\ \alpha(\zeta, \xi, t) \geq 1 \Longrightarrow \alpha(T\zeta, T\xi, t) \geq 1. \tag{2}$$

Let Ψ denote the family of all right continuous functions $\psi : [0, +\infty) \to [0, +\infty)$ with $\psi(r) < r$ for all $r > 0$.

Remark 3. *Note that for any $\psi \in \Psi$, then $\lim_{n\to+\infty} \psi^n(r) = 0$, for all $r > 0$, where ψ^n is the n-th iterate of ψ.*

Definition 8. *Let $(\mathcal{X}, \mathcal{M}_{\beta,\mu}, *)$ be an FDCM, and we say $T : \mathcal{X} \longrightarrow \mathcal{X}$ is an (α-ψ)-fuzzy contractive mapping if there exists two functions $\alpha : \mathcal{X}^2 \times (0, +\infty) \to [0, +\infty)$ and $\psi \in \Psi$ such that*

$$\alpha(\zeta,\xi,t)\left(\frac{1}{\mathcal{M}_{\beta,\mu}(T\zeta,T\xi,t)} - 1\right) \leq \psi\left(\frac{1}{\mathcal{M}_{\beta,\mu}(\zeta,\xi,t)} - 1\right), \tag{3}$$

for all $\zeta, \xi \in \mathcal{X}$ and for all $t > 0$.

We now state and prove our first main finding.

Theorem 1. *Let $(\mathcal{X}, \mathcal{M}_{\beta,\mu}, *)$ be a complete FDCM, where $\beta, \mu : \mathcal{X}^2 \to [1, 1/\tau)$ are two non-comparable functions (for some $\tau \in (0,1)$). Let $T : \mathcal{X} \to \mathcal{X}$ be an (α-ψ)-fuzzy contractive mapping, for some $\psi \in \Psi$, satisfying the following conditions:*

(1) T is α-admissible;
(2) There exists $\zeta_0 \in \mathcal{X}$ such that $\alpha(\zeta_0, T\zeta_0, t) \geq 1$ for all $t > 0$;
(3) T is continuous;
(4) For any $\zeta \in \mathcal{X}$, both

$$\lim_{n\to+\infty} \beta(\zeta_n, \zeta), \ \text{and} \ \lim_{n\to+\infty} \mu(\zeta, \zeta_n) \ \text{exist and are finite}, \tag{4}$$

where the sequence $\{\zeta_n\}$ is defined as $\zeta_n = T^n\zeta_0$, for some $\zeta_0 \in \mathcal{X}$.

Then, T admits a fixed point, i.e., there exists some $\zeta^* \in \mathcal{X}$ such that $T(\zeta^*) = \zeta^*$.

Proof. Let $\zeta_0 \in \mathcal{X}$ so that $\alpha(\zeta_0, T\zeta_0, t) \geq 1$ for all $t > 0$, and we have a sequence $\{\zeta_n\}$ in $\mathcal{X}$ with $T^n\zeta_0 = \zeta_n$, for all $n \in \mathbb{N}$.

Note that if $\zeta_m = \zeta_{m+1}$ for some $m \in \mathbb{N}$, then this implies that $T^m\zeta_0$ is a fixed point of the mapping T. Thus, without loss of generality, we may assume that $\zeta_n \neq \zeta_{n+1}$ for all $n \in \mathbb{N}$.

From the hypotheses we have that $\alpha(\zeta_0, \zeta_1, t) = \alpha(\zeta_0, T\zeta_0, t) \geq 1$, as T is α-admissible, this implies that $\alpha(T\zeta_0, T\zeta_1, t) = \alpha(\zeta_1, \zeta_2, t) \geq 1$. By induction, we can easily deduce

$$\alpha(\zeta_n, \zeta_{n+1}, t) \geq 1, \ \text{for all } n \in \mathbb{N}, \ \text{and for all } t > 0. \tag{5}$$

Thus, utilizing equation (5), and equation (3), we obtain

$$\begin{aligned}
\left(\frac{1}{\mathcal{M}_{\beta,\mu}(\zeta_n,\zeta_{n+1},t)} - 1\right) &= \left(\frac{1}{\mathcal{M}_{\beta,\mu}(T\zeta_{n-1},T\zeta_n,t)} - 1\right). \\
&\leq \alpha(\zeta_{n-1},\zeta_n,t)\left(\frac{1}{\mathcal{M}_{\beta,\mu}(T\zeta_{n-1},T\zeta_n,t)} - 1\right). \\
&\leq \psi\left(\frac{1}{\mathcal{M}_{\beta,\mu}(\zeta_{n-1},\zeta_n,t)} - 1\right), \ \text{ repeating the process.} \\
&\leq \psi\left(\psi\left(\frac{1}{\mathcal{M}_{\beta,\mu}(\zeta_{n-2},\zeta_{n-1},t)} - 1\right)\right) = \psi^2\left(\frac{1}{\mathcal{M}_{\beta,\mu}(\zeta_{n-2},\zeta_{n-1},t)} - 1\right). \\
&\leq \cdots \leq \psi^n\left(\frac{1}{\mathcal{M}_{\beta,\mu}(\zeta_0,\zeta_1,t)} - 1\right).
\end{aligned} \tag{6}$$

Considering the limit as n goes to infinity in equation (6) and using the fact that $\lim_{n\to+\infty} \psi^n(r) = 0$ with $r = \dfrac{1}{\mathcal{M}_{\beta,\mu}(\zeta_0,\zeta_1,t)} - 1$, we obtain

$$\lim_{n\to+\infty} \mathcal{M}_{\beta,\mu}(\zeta_{n-1},\zeta_n,t) = 1 \ \text{ for all } t > 0. \tag{7}$$

For any $n, m \in \mathbb{N}$, with $n < m$, then

$$\begin{aligned}
\mathcal{M}_{\beta,\mu}(\zeta_n,\zeta_m,t) &\geq \mathcal{M}_{\beta,\mu}(\zeta_n,\zeta_{n+1},\frac{\frac{t}{2}}{\beta(\zeta_n,\zeta_{n+1})}) * \mathcal{M}_{\beta,\mu}(\zeta_{n+1},\zeta_m,\frac{\frac{t}{2}}{\mu(\zeta_{n+1},\zeta_m)}).\\
&\geq \mathcal{M}_{\beta,\mu}(\zeta_n,\zeta_{n+1},\frac{\frac{t}{2}}{\beta(\zeta_n,\zeta_{n+1})}) * \mathcal{M}_{\beta,\mu}(\zeta_{n+1},\zeta_{n+2},\frac{\frac{t}{2^2}}{\beta(\zeta_{n+1},\zeta_{n+2})\mu(\zeta_{n+1},\zeta_m)})\\
&* \mathcal{M}_{\beta,\mu}(\zeta_{n+2},\zeta_m,\frac{\frac{t}{2^2}}{\mu(\zeta_{n+1},\zeta_m)\mu(\zeta_{n+2},\zeta_m)}).\\
&\geq \mathcal{M}_{\beta,\mu}(\zeta_n,\zeta_{n+1},\frac{\frac{t}{2}}{\beta(\zeta_n,\zeta_{n+1})}) * \mathcal{M}_{\beta,\mu}(\zeta_{n+1},\zeta_{n+2},\frac{\frac{t}{2^2}}{\beta(\zeta_{n+1},\zeta_{n+2})\mu(\zeta_{n+1},\zeta_m)})\\
&* \mathcal{M}_{\beta,\mu}(\zeta_{n+2},\zeta_{n+3},\frac{\frac{t}{2^3}}{\beta(\zeta_{n+2},\zeta_{n+3})\mu(\zeta_{n+1},\zeta_m)})\\
&* \mathcal{M}_{\beta,\mu}(\zeta_{n+3},\zeta_m,\frac{\frac{t}{2^3}}{\mu(\zeta_{n+1},\zeta_m)\mu(\zeta_{n+2},\zeta_m)\mu(\zeta_{n+3},\zeta_m)}).
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_{\beta,\mu}(\zeta_n,\zeta_m,t) &\geq \mathcal{M}_{\beta,\mu}(\zeta_n,\zeta_{n+1},\frac{t}{2\beta(\zeta_n,\zeta_{n+1})}) * \mathcal{M}_{\beta,\mu}(\zeta_{n+1},\zeta_{n+2},\frac{t}{2^2\beta(\zeta_{n+1},\zeta_{n+2})\mu(\zeta_{n+1},\zeta_m)})\\
&* \mathcal{M}_{\beta,\mu}(\zeta_{n+2},\zeta_{n+3},\frac{t}{2^3\beta(\zeta_{n+2},\zeta_{n+3})\mu(\zeta_{n+1},\zeta_m)})\\
&\vdots\\
&* \mathcal{M}_{\beta,\mu}(\zeta_{m-2},\zeta_{m-1},\frac{t}{2^{m-1}\beta(\zeta_{m-2},\zeta_{m-1})\mu(\zeta_{m-2},\zeta_m)\mu(\zeta_{m-3},\zeta_m)\cdots\mu(\zeta_{n+1},\zeta_m)})\\
&* \mathcal{M}_{\beta,\mu}(\zeta_{m-1},\zeta_m,\frac{t}{2^m\mu(\zeta_{m-1},\zeta_m)\mu(\zeta_{m-2},\zeta_m)\mu(\zeta_{m-3},\zeta_m)\cdots\mu(\zeta_{n+1},\zeta_m)}).
\end{aligned}$$

Taking the limit as $n \to +\infty$ in the above inequality and using equation (7) and equation (4), we obtain

$$\lim_{n\to+\infty} \mathcal{M}_{\beta,\mu}(\zeta_n,\zeta_m,t) = 1 * 1 * 1 * \cdots * 1 = 1. \tag{8}$$

This implies that the sequence $\{\zeta_n\}$ is a Cauchy sequence in $\mathcal{X}$, as $\mathcal{X}$ is a complete $FDCM$, so there exists some $\zeta^* \in \mathcal{X}$ such that $\zeta_n \to \zeta^*$, i.e.,

$$\lim_{n\to+\infty} \mathcal{M}_{\beta,\mu}(\zeta_n,\zeta^*,t) = 1. \tag{9}$$

The continuity of T implies that $T(\zeta_n) \to T(\zeta^*)$, i.e., $\lim_{n\to+\infty} \mathcal{M}_{\beta,\mu}(T(\zeta_n),T(\zeta^*),t) = 1$, for all $t > 0$. Thus, we have

$$\lim_{n\to+\infty} \mathcal{M}_{\beta,\mu}(\zeta_{n+1},T(\zeta^*),t) = \lim_{n\to+\infty} \mathcal{M}_{\beta,\mu}(T(\zeta_n),T(\zeta^*),t) = 1, \text{ for all } t > 0. \tag{10}$$

This yields $\zeta_n \to T(\zeta^*)$, and by Lemma 1 we obtain $T(\zeta^*) = \zeta^*$, so ζ^* is a fixed point of T. □

As a special case, if we let $\beta(\zeta,\xi) = \mu(\zeta,\xi) = b$, then Theorem 1 provides a proof for the case of complete fuzzy b-metric space as shown in the next corollary.

Corollary 1. *Let $(\mathcal{X},\mathcal{M},*)$ be a complete fuzzy b-metric space, let $T : \mathcal{X} \to \mathcal{X}$ be an (α-ψ)-fuzzy contractive mapping, for some $\psi \in \Psi$, satisfying the following conditions:*

(1) T is α-admissible;

(2) There exists $\zeta_0 \in \mathcal{X}$ such that $\alpha(\zeta_0,T\zeta_0,t) \geq 1$ for all $t > 0$;

(3) *T is continuous.*

Then, T has a fixed point, i.e., there exists some $\zeta^* \in \mathcal{X}$ such that $T(\zeta^*) = \zeta^*$.

Proof. By taking $\beta(\zeta, \xi) = \mu(\zeta, \xi) = b$ in Theorem 1 and repeating the same steps of the proof. Thus, T has a fixed point because it meets all the requirements of Theorem 1. □

It should be observed that Theorem 1 is an extension of Theorem 3.5 in [26], because taking $\beta(\zeta, \xi) = \mu(\zeta, \xi) = 1$, the fuzzy double-controlled metric space becomes fuzzy metric space. In addition, Corollary 2 provides an alternative proof for Theorem 3.5 in [26].

Corollary 2. *Let $(\mathcal{X}, \mathcal{M}, *)$ be a complete fuzzy metric space, let $T : \mathcal{X} \to \mathcal{X}$ be an (α-ψ)-fuzzy contractive mapping, for some $\psi \in \Psi$, satisfying the following conditions:*

(1) *T is α-admissible;*
(2) *There exists $\zeta_0 \in \mathcal{X}$ so that $\alpha(\zeta_0, T\zeta_0, t) \geq 1\, \forall t > 0$;*
(3) *T is continuous.*

Then, T admits a fixed point, i.e., some $\zeta^* \in \mathcal{X}$ can be found so that $T(\zeta^*) = \zeta^*$.

Proof. By taking $\beta(\zeta, \xi) = \mu(\zeta, \xi) = 1$ in Theorem 1 and repeating the proof. Thus, T has a fixed point, because it fulfills every requirement of Theorem 1. □

The following is a supporting example for the main Theorem 1.

Example 3. *Let $\mathcal{X} = [0, 1]$, and the control functions $\beta, \mu : \mathcal{X}^2 \to [1, +\infty)$ are defined as $\beta(\zeta, \xi) = \zeta + \xi + 1$ and $\mu(\zeta, \xi) = \zeta^2 + \xi^2 + 1$. Define the fuzzy set $\mathcal{M}_{\beta,\mu}$ by*

$$\mathcal{M}_{\beta,\mu}(\zeta, \xi, t) = exp^{\frac{-|\zeta - \xi|}{t}}. \tag{11}$$

Then, one can easily show that $(\mathcal{X}, \mathcal{M}_{\beta,\mu}, *)$ is a complete fuzzy double-controlled metric space, and we will verify condition (FD4) only. Note that

$$M(\zeta, \xi, \frac{t}{\beta(\zeta, \xi)}) = exp^{\frac{\frac{-|\zeta-\xi|}{t}}{\beta(\zeta,\xi)}} = exp^{\frac{-\beta(\zeta,\xi)|\zeta-\xi|}{t}} \leq exp^{\frac{-|\zeta-\xi|}{t}} \leq exp^{\frac{-|\zeta-\xi|}{t+s}},\ s > 0.$$

Similarly,

$$M(\xi, \varpi, \frac{s}{\mu(\xi, \varpi)}) = exp^{\frac{\frac{-|\xi-\varpi|}{s}}{\mu(\xi,\varpi)}} = exp^{\frac{-\mu(\xi,\varpi)|\xi-\varpi|}{s}} \leq exp^{\frac{-|\xi-\varpi|}{s}} \leq exp^{\frac{-|\xi-\varpi|}{t+s}},\ t > 0.$$

Hence, for $t, s > 0$,

$$M(\zeta, \xi, \frac{t}{\beta(\zeta, \xi)}) * M(\xi, \varpi, \frac{s}{\mu(\xi, \varpi)}) \leq exp^{\frac{-|\zeta-\xi|}{t+s}}.exp^{\frac{-|\xi-\varpi|}{t+s}} \leq exp^{\frac{-|\zeta-\varpi|}{t+s}} = M(\zeta, \varpi, t + s).$$

Let $T : \mathcal{X} \longrightarrow \mathcal{X}$, $\alpha : \mathcal{X} \times \mathcal{X} \times (0, +\infty) \to [0, +\infty)$, and $\psi : [0, +\infty) \to [0, +\infty)$ be defined as $T(\zeta) = 1 - \frac{\zeta}{4}$, $\psi(r) = r/2$ and

$$\alpha(\zeta, \xi, t) = \begin{cases} 1 & \text{if } \zeta, \xi \in (0, 1], \\ 0 & \text{otherwise,} \end{cases}$$

It is easy to see that T is α-admissible and continuous, because for $\zeta, \xi \in \mathcal{X}$, with $\alpha(\zeta, \xi, t) \geq 1$, then $\alpha(T\zeta, T\xi, t) = \alpha(1 - \frac{\zeta}{4}, 1 - \frac{\xi}{4}, t) \geq 1$.

To illustrate that T is (α-ψ)-fuzzy contractive mapping, we have to show (Equation (3)) holds:

$$\alpha(\zeta,\xi,t)(\frac{1}{\mathcal{M}_{\beta,\mu}(T\zeta,T\xi,t)}-1) \leq (\frac{1}{\mathcal{M}_{\beta,\mu}(T\zeta,T\xi,t)}-1).$$
$$= (\frac{1}{exp^{\frac{-|T\zeta-Ty|}{t}}}-1) = (\frac{1}{exp^{\frac{-|\zeta-\xi|}{4t}}}-1).$$
$$\leq \frac{exp^{\frac{|\zeta-\xi|}{t}}-1}{2} = \psi(\frac{1}{\mathcal{M}_{\beta,\mu}(\zeta,\xi,t)}-1).$$

Thus, T satisfies the hypothesis of Theorem 1; hence, there exists a fixed point $\zeta^* = 4/5$, *such that* $T(4/5) = 4/5$.

In the next theorem, we replace the continuity hypotheses of T in Theorem 1 with another regularity hypothesis.

Theorem 2. *Let* $(\mathcal{X}, \mathcal{M}_{\beta,\mu}, *)$ *be a triangular complete FDCM, where* $\beta, \mu : \mathcal{X}^2 \to [1, 1/\tau)$ *are two non-comparable functions* ($\tau \in (0,1)$). *Let* $T : \mathcal{X} \to \mathcal{X}$ *be an (α-ψ)-fuzzy contractive mapping, for some* $\psi \in \Psi$, *satisfying the following conditions:*

(1) *T is α-admissible;*
(2) *There exists* $\zeta_0 \in \mathcal{X}$ *such that* $\alpha(\zeta_0, T\zeta_0, t) \geq 1$ *for all* $t > 0$;
(3) *If* $\{\zeta_n\}$ *is a sequence in* $\mathcal{X}$ *such that* $\alpha(\zeta_n, \zeta_{n+1}, t) \geq 1$ *for all* $n \in \mathbb{N}$ *and* $\zeta_n \to \zeta$ *as* $n \to +\infty$, *then* $\alpha(\zeta_n, \zeta, t) \geq 1$ *for all* $n \in \mathbb{N}$.
(4) *For any* $\zeta \in \mathcal{X}$, *both*

$$\lim_{n\to+\infty} \beta(\zeta_n, \zeta),\ \textit{and} \lim_{n\to+\infty} \mu(\zeta, \zeta_n)\ \textit{exist and are finite}, \tag{12}$$

where the sequence $\{\zeta_n\}$ *is defined as* $\zeta_n = T^n\zeta_0$, *for some* $\zeta_0 \in \mathcal{X}$.

Then, T has a fixed point, i.e., there exists some $\zeta^* \in \mathcal{X}$ *such that* $T(\zeta^*) = \zeta^*$.

Proof. Following the proof of Theorem 1, we get that $\{\zeta_n\}$ is a Cauchy sequence in a complete *FDCM* $(\mathcal{X}, \mathcal{M}_{\beta,\mu}, *)$. This implies there exists some $\zeta^* \in \mathcal{X}$ such that $\zeta_n \to \zeta^*$ as $n \to +\infty$. Thus, by hypothesis (3), we obtain

$$\alpha(\zeta_n, \zeta^*, t) \geq 1 \text{ for all } n \in \mathbb{N} \text{ and for all } t > 0. \tag{13}$$

Using the fact that $\mathcal{M}_{\beta,\mu}$ is triangular and by Equation (12) and Equation (3) we have

$$(\frac{1}{\mathcal{M}_{\beta,\mu}(T\zeta^*,\zeta^*,t)}-1) \leq (\frac{1}{\mathcal{M}_{\beta,\mu}(T\zeta^*,T\zeta_n,t)}-1) + (\frac{1}{\mathcal{M}_{\beta,\mu}(\zeta_{n+1},\zeta^*,t)}-1).$$
$$\leq \alpha(\zeta_n,\zeta^*,t)(\frac{1}{\mathcal{M}_{\beta,\mu}(T\zeta_n,T\zeta^*,t)}-1) + (\frac{1}{\mathcal{M}_{\beta,\mu}(\zeta_{n+1},\zeta^*,t)}-1).$$
$$\leq \psi(\frac{1}{\mathcal{M}_{\beta,\mu}(\zeta_n,\zeta^*,t)}-1) + (\frac{1}{\mathcal{M}_{\beta,\mu}(\zeta_{n+1},\zeta^*,t)}-1),\ \text{because } \psi(r) < r.$$
$$< (\frac{1}{\mathcal{M}_{\beta,\mu}(\zeta_n,\zeta^*,t)}-1) + (\frac{1}{\mathcal{M}_{\beta,\mu}(\zeta_{n+1},\zeta^*,t)}-1). \tag{14}$$

Letting $n \to +\infty$ in (Equation (14)), we obtain

$$\lim_{n\to+\infty} \mathcal{M}_{\beta,\mu}(T\zeta^*, \zeta^*, t) = 1 \text{ for all } t > 0, \tag{15}$$

that is, $T(\zeta^*) = \zeta^*$, so T has a fixed point. □

Next, we present an example for Theorem 2.

Example 4. *Consider $\mathcal{X} = [0, +\infty)$ and let the control functions $\beta, \mu : \mathcal{X}^2 \to [1, +\infty)$ be defined as*

$$\beta(\zeta, \xi) = \begin{cases} max\{\zeta, \xi\} + 1 & if\ \zeta, \xi \in [0,1], \\ 1 & otherwise, \end{cases}$$

and

$$\mu(\zeta, \xi) = \begin{cases} max\{\zeta, \xi\} + 2 & if\ \zeta, \xi \in [1,2], \\ 1 & otherwise. \end{cases}$$

The fuzzy set $\mathcal{M}_{\beta,\mu}$ is defined by

$$\mathcal{M}_{\beta,\mu}(\zeta, \xi, t) = \frac{t}{t + |\zeta - \xi|}. \tag{16}$$

Then, one can easily show that $(\mathcal{X}, \mathcal{M}_{\beta,\mu}, *)$ is a complete fuzzy double-controlled metric space which is triangular, and we will verify condition (FD4) only. Note that

$$\mathcal{M}_{\beta,\mu}(\zeta, \xi, \frac{t}{\beta(\zeta, \xi)}) = \frac{t}{t + \beta(\zeta, \xi)|\zeta - \xi|} \leq \frac{t}{t + |\zeta - \xi|} \leq \frac{t + s}{t + s + |\zeta - \xi|}.$$

Similarly,

$$\mathcal{M}_{\beta,\mu}(\xi, \varpi, \frac{s}{\mu(\xi, \varpi)}) = \frac{s}{s + \mu(\xi, \varpi)|\xi - \varpi|} \leq \frac{s}{s + |\xi - \varpi|} \leq \frac{t + s}{t + s + |\xi - \varpi|}.$$

Thus,

$$\begin{aligned} \mathcal{M}_{\beta,\mu}(\zeta, \xi, \frac{t}{\beta(\zeta, \xi)}) * \mathcal{M}_{\beta,\mu}(\xi, \varpi, \frac{s}{\mu(\xi, \varpi)}) &\leq (\frac{t + s}{t + s + |\zeta - \xi|})(\frac{t + s}{t + s + |\xi - \varpi|}). \\ &\leq \frac{t + s}{t + s + |\zeta - \varpi|} = M(\zeta, \varpi, t + s). \end{aligned} \tag{17}$$

Note that Equation (17) follows from the below inequality:

$$\begin{aligned} (\frac{1}{1 + |\zeta - \xi|})(\frac{1}{1 + |\xi - \varpi|}) &= \frac{1}{1 + |\zeta - \xi| + |\xi - \varpi| + |\zeta - \xi||\xi - \varpi|}. \\ &\leq \frac{1}{1 + |\zeta - \xi| + |\xi - \varpi|} \leq \frac{1}{1 + |\zeta - \varpi|}, \textit{then replace } 1 \textit{ by } s + t \end{aligned}$$

Let $T : \mathcal{X} \longrightarrow \mathcal{X}$ and $\alpha : \mathcal{X}^2 \times (0, +\infty) \to [0, +\infty)$, be defined as

$$T(\zeta) = \begin{cases} \frac{\zeta^2}{4} & if\ \zeta \in [0,1], \\ 2 & otherwise, \end{cases}$$

and

$$\alpha(\zeta, \xi, t) = \begin{cases} 1 & if\ \zeta, \xi \in [0,1],\ t > 0 \\ 0 & otherwise. \end{cases}$$

Let $\psi(r) = r/2$ for $r \geq 0$. To show T is α-admissible, for any $\zeta, \xi \in \mathcal{X}$, if $\alpha(\zeta, \xi, t) \geq 1$, then $\zeta, \xi \in [0,1]$; hence, both $T(\zeta), T(\xi) \in [0,1]$ which implies that $\alpha(T\zeta, T\xi, t) \geq 1$, for all $t > 0$.

To show that T is (α-ψ)-fuzzy contractive mapping, we need to show Equation (3) holds. For $\zeta, \xi > 1$, the case is trivial; thus, we consider the case $\zeta, \xi \in [0,1]$.

$$\begin{aligned}\alpha(\zeta,\xi,t)(\frac{1}{\mathcal{M}_{\beta,\mu}(T\zeta,T\xi,t)}-1) &\leq (\frac{1}{\mathcal{M}_{\beta,\mu}(T\zeta,T\xi,t)}-1).\\ &= (\frac{1}{\mathcal{M}_{\beta,\mu}(\frac{\zeta^2}{4},\frac{\xi^2}{4},t)}-1) = \frac{1}{4t}|\zeta^2-\xi^2|.\\ &\leq \frac{1}{2}(\frac{|\zeta-\xi|}{t}) = \psi(\frac{1}{\mathcal{M}_{\beta,\mu}(\zeta,\varpi,t)}-1).\end{aligned}$$

If $\{\zeta_n\}$ is any sequence in $\mathcal{X}$ such that $\alpha(\zeta_n,\zeta_{n+1},t) \geq 1$ for all $n \in \mathbb{N}$, and $\zeta_n \to \zeta$ as $n \to +\infty$, then $\{\zeta_n\} \subset [0,1]$. Hence, we have $\zeta \in [0,1]$ which implies that $\alpha(\zeta_n,\zeta,t) \geq 1$ for all $n \in \mathbb{N}$. Therefore, all the hypotheses of theorem 2 are satisfied; consequently, T has fixed points, which are $\zeta = 0$ and $\zeta = 2$.

4. Application

Let $\mathcal{X} = C([0,I],\mathbb{R})$ be the space of all continuous real-valued functions defined on the interval $[0,I]$, for some $I > 0$. Define the control functions $\beta,\mu : \mathcal{X}^2 \to [1,+\infty)$ by $\beta(\zeta,\xi) = \zeta+\xi+1$ and $\mu(\zeta,\xi) = \zeta^2+\xi^2+1$. The fuzzy metric $\mathcal{M}_{\beta,\mu}$ is defined on $\mathcal{X}$ by

$$\mathcal{M}_{\beta,\mu}(\zeta,\xi,t) = e^{-sup_{s\in[0,I]}\frac{|\zeta(s)-\xi(s)|}{t}}, \text{ where } \zeta,\xi \in \mathcal{X},\ t > 0. \tag{18}$$

Then, $(\mathcal{X},\mathcal{M}_{\beta,\mu},*)$ is a complete fuzzy double-controlled metric space.

Theorem 3. *Consider $(\mathcal{X},\mathcal{M}_{\beta,\mu},*)$ a complete FDCM, as defined above. Let $T : \mathcal{X} \longrightarrow \mathcal{X}$ be an integral operator defined by*

$$T\zeta(s) = p(s) + \int_0^s \mathcal{K}(s,x,\zeta(x))dx, \tag{19}$$

where $p \in \mathcal{X}$, and $\mathcal{K}(s,x,\zeta(x)) : [0,I]^2 \longrightarrow \mathbb{R}$ is a continuous function. If there exists a function $g : [0,I]\times[0,I] \to [0,+\infty)$ such that for all $s,x \in [0,I]$, we have $g \in L^1([0,I],\mathbb{R})$ satisfying the following:

- $|\mathcal{K}(s,x,\zeta(x)) - \mathcal{K}(s,x,\xi(x))| \leq g(s,x)|\zeta(x)-\xi(x)|$;
- *The integral $\int_0^s g(s,x)dx$ is bounded, i.e., there exists some $k \in (0,1)$ such that $0 < sup_{s\in[0,I]}\int_0^s g(s,x)dx \leq k < 1$. Furthermore, this holds*

$$e^{-sup_{x\in[0,I]}\frac{k|\zeta(x)-\xi(x)|}{t}} \geq 2\, e^{-sup_{x\in[0,I]}\frac{|\zeta(x)-\xi(x)|}{t}}.$$

Then, the integral Equation (19) has a solution.

Proof. First, we define $\alpha : \mathcal{X}^2 \times (0,+\infty) \to [0,+\infty)$ by

$$\alpha(\zeta,\xi,t) = \begin{cases} 1/2 & \text{if } \zeta = \xi. \\ 1/4 & \text{if } \zeta = 0, \text{or } \xi = 0. \\ 1 & \text{otherwise }, \end{cases} \tag{20}$$

and let $\psi(r) = r/2$. For $\zeta,\xi \in \mathcal{X}$, consider the fuzzy metric:

$$
\begin{aligned}
\mathcal{M}_{\beta,\mu}(T\zeta, T\xi, t) &= e^{-sup_{s\in[0,I]}\frac{|T\zeta(s)-T\xi(s)|}{t}}. \\
&\geq e^{-sup_{s\in[0,I]}\frac{\int_0^s|\mathcal{K}(s,x,\zeta(x))-\mathcal{K}(s,x,\xi(x))|dx}{t}}. \\
&\geq e^{-sup_{s\in[0,I]}\frac{\int_0^s g(s,x)|\zeta(x)-\xi(x)|dx}{t}}. \\
&\geq e^{-sup_{x\in[0,I]}\frac{|\zeta(x)-\xi(x)|sup_s\int_0^s g(s,x)dx}{t}}. \\
&\geq e^{-sup_{x\in[0,I]}\frac{k|\zeta(x)-\xi(x)|}{t}}. \\
&\geq 2e^{-sup_{x\in[0,I]}\frac{|\zeta(x)-\xi(x)|}{t}} = 2(\mathcal{M}_{\beta,\mu}(\zeta,\xi,t)).
\end{aligned} \tag{21}
$$

Hence, to show T is $(\alpha$-$\psi)$ fuzzy contractive mapping, we need to show (Equation (3)) holds.
For any $\zeta, \xi \in \mathcal{X}$, then by (20) and (21) we have

$$
\begin{aligned}
\alpha(\zeta,\xi,t)\left(\frac{1}{\mathcal{M}_{\beta,\mu}(T\zeta,T\xi,t)}-1\right) &\leq \frac{1}{\mathcal{M}_{\beta,\mu}(T\zeta,T\xi,t)}-1. \\
&\leq \frac{1}{2\mathcal{M}_{\beta,\mu}(\zeta,\xi,t)}-1 \leq \frac{1}{2}\left(\frac{1}{\mathcal{M}_{\beta,\mu}(\zeta,\xi,t)}-1\right). \\
&= \psi\left(\frac{1}{\mathcal{M}_{\beta,\mu}(\zeta,\xi,t)}-1\right).
\end{aligned}
$$

We conclude that the operator T has a fixed point $\zeta^* \in C([0,1],\mathbb{R})$ which is a solution of the integral Equation (19), because all the conditions of Theorem 1 are met. □

5. Conclusions

The concept of fuzzy double-controlled metric spaces was the topic of this article. On such spaces, we proposed the notion of $(\alpha$-$\psi)$-fuzzy contractive mappings and established some fixed-point results. Moreover, we provided applications of our finding on the existence of a solution for an integral equation as well as some examples. Recently, some researchers interested in the geometric generalization of the fixed-point theory have studied the fixed-circle problem on metric spaces [8] and on s-metric spaces [9] by utilizing various contractive mappings. For instance, in [10], a new fixed-circle theorem for self-mappings on an s-metric space was presented using Wardowski-type contractions, and more recently, Mlaiki et al. [29] investigated conditions to make any circle as a common fixed circle for two or more self-mappings. We propose some suggestions for future research directions, such as using fuzzy double-controlled metric spaces and $(\alpha$-$\psi)$-fuzzy contractive mappings to examine the fixed-circle problem.

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