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Bi-Starlike Function of Complex Order Involving Mathieu-Type Series Associated with Telephone Numbers

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Abstract: For the first time, we attempted to define two new sub-classes of bi-univalent functions in the open unit disc of the complex order involving Mathieu-type series, associated with generalized telephone numbers. The initial coefficients of functions in these classes were obtained. Moreover, we also determined the Fekete–Szegö inequalities for function in these and several related corollaries.

Keywords: analytic functions; bi-univalent functions; bi-starlike and bi-convex functions; coefficient bounds; Mathieu-type series; Hadamard (convolution) product; Fekete–Szegö problem; telephone numbers

MSC: 30C45; 30C50; 30C80

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1. Introduction and Definitions

Geometric Function Theory has been characterized by the use of a wide range of special functions, like q calculus and special polynomials, such as the following: the Fibonacci polynomials, the Faber polynomials, the Horadam polynomials, the Lucas polynomials, the Pell polynomials, the Pell–Lucas polynomials and the Chebyshev polynomials of the second kind. These functions are potentially applied to a variety of mathematical, physical, statistical, and engineering disciplines. This article briefly describes telephone numbers and the Emilie Leonard Mathieu series that were used to define new sub-classes of bi-univalent functions.

1.1. Analytic Functions

Let $\mathcal{H}$ represent the class of holomorphic (analytic or regular) functions in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

and normalized by the conditions $f(0) = 0$ and $f'(0) = 1.$. Further, let $\mathcal{S}$ denote all functions in which $\mathcal{H}$ might be univalent in $\mathbb{D}$. Some of the great and properly-investigated sub-classes of the univalent function class $\mathcal{S}$ functions are the class $\mathcal{S}^*(\nu)$ of star-like functions, of order ν in $\mathbb{D}$, and the $\mathcal{C}(\nu)$ class of convex features, of order ν $(0 \leq \nu < 1)$ in $\mathbb{D}$.

Let $f_1, f_2 \in \mathcal{H}$ and f_1 be subordinate to f_2,, written as $f_1 \prec f_2$,, provided that in $\mathbb{D}$ there is a function $\varpi \in \mathcal{H}$ with $\varpi(0) = 0$ and $|\varpi(z)| < 1$, sustaining $f_1(z) = f_2(\varpi(z))$. The convolution or Hadamard product of two functions $f, h \in \mathcal{H}$ is denoted by $f * h$, given by:

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \tag{2}$$

where $f(z)$ is given by (1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

Based on the Koebe's one-quarter theorem [1], every $f \in \mathcal{S}$ has the compositional inverse f^{-1} satisfying:

$$f^{-1}(f(z)) = z, \ (z \in \mathbb{D}) \text{ and } f(f^{-1}(w)) = w, \ \ (w \in \mathbb{D}_\rho),$$

where $\rho \geq \frac{1}{4}$ is the radius of the image $f(\mathbb{D})$.It is well-known that $f^{-1}(w)$ has the normalized Taylor–Maclaurin's series:

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{3}$$

A function $f \in \mathcal{H}$ given by (1) is said to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$, and such a class is signified by Σ. For example, we can observe that Σ is not empty. For instance,

$$f_1(z) = \frac{z}{1-z}, \quad f_2(z) = \frac{1}{2} \log \frac{1+z}{1-z} \ \text{ and } \ f_3(z) = -\log(1-z)$$

and, in turn, they have inverses,

$$f_1^{-1}(w) = \frac{w}{1+w}, \quad f_2^{-1}(w) = \frac{e^{2w}-1}{e^{2w}+1} \ \text{ and } \ f_3^{-1}(w) = \frac{e^w - 1}{e^w},$$

are elements of Σ. However, $z - \frac{z^2}{2}; \frac{z}{1-z^2}$ and the Koebe function are not a member of Σ. Formerly, Brannan and Taha [2] proposed certain sub-classes of Σ, explicitly bi-starlike functions of order $\nu(0 < \nu \leq 1)$, symbolized by $\mathcal{S}_\Sigma^*(\nu)$, and bi-convex functions of order ν, represented by $\mathcal{C}_\Sigma(\nu)$. For $f \in \mathcal{S}_\Sigma^*(\nu)$ and $f \in \mathcal{C}_\Sigma(\nu)$, non-sharp estimates on the first two Taylor–Maclaurin coefficients, $|a_2|$ and $|a_3|$, were established in [2,3]. However, the coefficient problem for each of the succeeding Taylor–Maclaurin coefficients,

$$|a_n| \qquad (n \in \mathbb{N} \setminus \{1,2\}; \ \ \mathbb{N} := \{1,2,3,\cdots\})$$

is still an open problem (see [2–6]). Lately, Srivastava et al. [7] fundamentally revived the study of Σ. Followed by such works as [7], several authors [7–21] have familiarized and inspected several interesting sub-classes of Σ, and obtained non-sharp bounds of $|a_2|$ and $|a_3|$ for the initial coefficients . The study of functions in Σ, and associated specific special polynomials, is a current research interest.

1.2. Generalized Telephone Numbers (GTNs)

The usual involution numbers, also known as telephone numbers, are assumed by the recurrence relation

$$\mathcal{V}(n) = \mathcal{V}(n-1) + (n-1)\mathcal{V}(n-2) \quad for \quad n \geq 2$$

with initial conditions

$$\mathcal{V}(0) = \mathcal{V}(1) = 1.$$

In 1800, Heinrich August Rothe noted that $\mathcal{V}(n)$ is the number of involutions (self-inverse permutations) in a symmetric group (see, for example, [22,23]). The relation between involution numbers and symmetric groups were observed for the first time in the year 1800. Since involutions correspond to standard Young tableaux, it is clear that the nth involution number is also the number of Young tableaux on the set $1, 2, ..., n$ (for details, see [24]). According to John Riordan, the above recurrence relation, in fact, produces the number of connection patterns in a telephone system with n subscribers (see [25]). In

2017, Wlochand Wolowiec-Musial [26] introduced generalized telephone numbers $\mathcal{V}(\tau, n)$ defined for integers $n \geq 0$ and $\tau \geq 1$ by the following recursion,

$$\mathcal{V}(\tau, n) = \tau\mathcal{V}(\tau, n-1) + (n-1)\mathcal{V}(\tau, n-2)$$

with initial conditions

$$\mathcal{V}(\tau, 0) = 1, \mathcal{V}(\tau, 1) = \tau,$$

and studied some properties. In 2019, Bednarz and Wolowiec-Musial [27] introduced a new GTN by

$$\mathcal{V}_\tau(n) = \mathcal{V}_\tau(n-1) + \tau(n-1)\mathcal{V}_\tau(n-2)$$

with initial conditions

$$\mathcal{V}_\tau(0) = \mathcal{V}_\tau(1) = 1$$

for integers $n \geq 2$ and $\tau \geq 1$. They gave the generating function, direct formula and matrix generators for these numbers. Moreover,they obtained interpretations and proved some properties of these numbers connected with congruences. Lately, they derived the exponential-generating function and the summation formula for GTNs $\mathcal{V}_\tau(n)$ as follows:

$$e^{x+\tau\frac{x^2}{2}} = \sum_{n=0}^{\infty} \mathcal{V}_\tau(n)\frac{x^n}{n!} \quad (\tau \geq 1)$$

As we can observe, if $\tau = 1$, then we obtain classical telephone numbers $\mathcal{V}(n)$. Clearly, $\mathcal{V}_\tau(n)$ is for some values of n as:

1. $\mathcal{V}_\tau(0) = \mathcal{V}_\tau = 1$,
2. $\mathcal{V}_\tau(2) = 1 + \tau$,
3. $\mathcal{V}_\tau(3) = 1 + 3\tau$
4. $\mathcal{V}_\tau(4) = 1 + 6\tau + 3\tau^2$
5. $\mathcal{V}_\tau(5) = 1 + 10\tau + 15\tau^2$
6. $\mathcal{V}_\tau(6) = 1 + 15\tau + 45\tau^2 + 15\tau^3$.

Lately Deniz [28], (also see [29]) consider the function

$$\begin{aligned} \Xi(z) &:= e^{(z+\tau\frac{z^2}{2})} \\ &= 1 + z + \frac{1+\tau}{2}z^2 + \frac{1+3\tau}{6}z^3 + \frac{3\tau^2+6\tau+1}{24}z^4 + \frac{1+10\tau+15\tau^2}{120}z^5 + \cdots, \end{aligned} \tag{4}$$

for $z \in \mathbb{D}$ and studied $f \in \mathcal{H}$.

1.3. Mathieu-Series

The subsequent collection is named after Leonard Mathieu (1835–1890) who investigated it in his monograph [30] on the elasticity of solid bodies

$$\mathcal{X}(\ell) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+\ell^2)^2} \qquad (\ell > 0). \tag{5}$$

A Closed integral illustration of the series $\mathcal{X}(\ell)$ is given by (see [31])

$$\mathcal{X}(\ell) = \frac{1}{\ell}\int_0^\infty \frac{t\sin(\ell t)}{e^t - 1}dt.$$

The Mathieu-type series is defined by (see [31])

$$\mathcal{X}(\ell; z) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+\ell^2)^2}z^n \quad (\ell > 0, \quad |z| < 1).$$

Initially it was further defined for the function of real variables. However, for complex variables it was defined by Bansal et al. [32]. Since $\mathcal{X}(\ell;z) \notin \mathcal{H}$, using the following normalization, we have:

$$\begin{aligned}\mathcal{X}(\ell;z) &= \frac{\left(\ell^2+1\right)^2}{2}\sum_{n=1}^{\infty}\frac{2n}{\left(n^2+\ell^2\right)^2}z^n \\ &= z+\sum_{n=2}^{\infty}\frac{n\left(\ell^2+1\right)^2}{\left(n^2+\ell^2\right)^2}z^n, \qquad (6) \\ &= z+\sum_{n=2}^{\infty}\mathcal{Y}_n(\ell)z^n, \qquad (7)\end{aligned}$$

where

$$\mathcal{Y}_n(\ell)=\frac{n\left(\ell^2+1\right)^2}{\left(n^2+\ell^2\right)^2}. \qquad (8)$$

for some related work we refer the reader to see [32,33].

We now define a new linear operator $\mathcal{W}_\ell : \mathcal{H} \to \mathcal{H}$ given by:

$$\mathcal{W}_\ell f(z) := \mathcal{X}(\ell;z) * f(z),\ z \in \Delta,$$

where the symbol "$*$" stands for the Hadamard product. Thus, if $f \in \mathcal{H}$ has the form (1), then:

$$\mathcal{W}_\ell f(z) = z+\sum_{n=2}^{\infty}\mathcal{Y}_n(\ell)a_n z^n,\ z\in\Delta. \qquad (9)$$

Stimulated by the work of Silverman and Silvia [34] (also [35]), Srivastava et al. [36], the earlier work in [37,38], and the latest work of Murugusundaramoorthy and Vijaya [29], we introduce, in the present paper, new sub-classes of Σ, of complex order $\vartheta \in \mathbb{C}\backslash\{0\}$, regarding the linear operator $\mathcal{W}_\ell$. We further discover estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new sub-classes of the function class Σ. In addition, numerous associated classes are considered and their relationship to earlier recognized results are explained.

Definition 1. *A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{S}^{\ell,\tau}_{\Sigma,\Xi}(\vartheta,\varphi)$ if it satisfies the following:*

$$1+\frac{1}{\vartheta}\left(\frac{z(\mathcal{W}_\ell f(z))'}{\mathcal{W}_\ell f(z)}+\left(\frac{1+e^{i\varphi}}{2}\right)\frac{z^2(\mathcal{W}_\ell f(z))''}{\mathcal{W}_\ell f(z)}-1\right)\prec \Xi(z) \qquad (10)$$

and

$$1+\frac{1}{\vartheta}\left(\frac{w(\mathcal{W}_\ell g(w))'}{\mathcal{W}_\ell g(w)}+\left(\frac{1+e^{i\varphi}}{2}\right)\frac{w^2(\mathcal{W}_\ell g(w))''}{\mathcal{W}_\ell g(w)}-1\right)\prec \Xi(w) \qquad (11)$$

where $\vartheta \in \mathbb{C}\backslash\{0\}$ $\varphi \in (-\pi,\pi], z, w \in \mathbb{D}$ and g as assumed in (3).

Definition 2. *A function $f \in \Sigma$, given by (1), is said to be in the class $\mathcal{C}^{\ell,\tau}_{\Sigma,\Xi}(\vartheta,\varphi)$ if:*

$$1+\frac{1}{\vartheta}\left(\frac{[z(\mathcal{W}_\ell f(z))'+\left(\frac{1+e^{i\varphi}}{2}\right)z^2(\mathcal{W}_\ell f(z))'']'}{(\mathcal{W}_\ell f(z))'}-1\right)\prec \Xi(z) \qquad (12)$$

and

$$1+\frac{1}{\vartheta}\left(\frac{[w(\mathcal{W}_\ell g(w))'+\left(\frac{1+e^{i\varphi}}{2}\right)w^2(\mathcal{W}_\ell g(w))'']'}{(\mathcal{W}_\ell g(w))'}-1\right)\prec \Xi(w), \qquad (13)$$

where $\vartheta \in \mathbb{C}\backslash\{0\}, \varphi \in (-\pi, \pi], z, w \in \mathbb{D}$ *and* g*, as assumed in* (3)*.*

Remark 1. *Let* $f \in \Sigma$ *be given by* (1) *and for* $\varphi = \pi$*, we note that* $\mathcal{S}_{\Sigma,\Xi}^{\ell,\tau}(\vartheta, \pi) \equiv \mathcal{S}_{\Sigma,\Xi}^{\ell,\tau}(\vartheta)$ *and* $\mathcal{C}_{\Sigma,\Xi}^{\ell,\tau}(\vartheta, \pi) \equiv \mathcal{C}_{\Sigma,\Xi}^{\ell,\tau}(\vartheta)$ *if*

$$\left[1 + \frac{1}{\vartheta}\left(\frac{z(\mathcal{W}_\ell f(z))'}{\mathcal{W}_\ell f(z)} - 1\right)\right] \prec \Xi(z) \text{ and } \left[1 + \frac{1}{\vartheta}\left(\frac{w(\mathcal{W}_\ell g(w))'}{\mathcal{W}_\ell g(w)} - 1\right)\right] \prec \Xi(w)$$

and

$$\left[1 + \frac{1}{\vartheta}\left(\frac{z(\mathcal{W}_\ell f(z))''}{(\mathcal{W}_\ell f(z))'}\right)\right] \prec \Xi(z) \text{ and } \left[1 + \frac{1}{\vartheta}\left(\frac{w(\mathcal{W}_\ell g(w))''}{(\mathcal{W}_\ell g(w))'}\right)\right] \prec \Xi(w),$$

respectively, for $\vartheta \in \mathbb{C}\backslash\{0\}$ $z, w \in \mathbb{D}$ *and* g *as assumed in* (3)*.*

Remark 2. *A function* $f \in \Sigma$ *given by* (1) *and for* $\vartheta = 1$*, we let* $\mathcal{S}_{\Sigma,\Xi}^{\ell,\tau}(\vartheta, \varphi) \equiv \mathcal{S}_{\Sigma,\Xi}^{\ell,\tau}(\varphi)$ *if it satisfies:*

$$\left(\frac{z(\mathcal{W}_\ell f(z))'}{\mathcal{W}_\ell f(z)} + \left(\frac{1+e^{i\varphi}}{2}\right)\frac{z^2(\mathcal{W}_\ell f(z))''}{\mathcal{W}_\ell f(z)}\right) \prec \Xi(z)$$

and

$$\left(\frac{w(\mathcal{W}_\ell g(w))'}{\mathcal{W}_\ell g(w)} + \left(\frac{1+e^{i\varphi}}{2}\right)\frac{w^2(\mathcal{W}_\ell g(w))''}{\mathcal{W}_\ell g(w)}\right) \prec \Xi(w).$$

Furthermore, $\mathcal{C}_{\Sigma,\Xi}^{\ell,\tau}(\vartheta, \varphi) \equiv \mathcal{C}_{\Sigma,\Xi}^{\ell,\tau}(\varphi)$ *if it satisfy:*

$$\left(\frac{[z(\mathcal{W}_\ell f(z))' + \left(\frac{1+e^{i\varphi}}{2}\right)z^2(\mathcal{W}_\ell f(z))'']'}{(\mathcal{W}_\ell f(z))'}\right) \prec \Xi(z)$$

and

$$\left(\frac{[w(\mathcal{W}_\ell g(w))' + \left(\frac{1+e^{i\varphi}}{2}\right)w^2(\mathcal{W}_\ell g(w))'']'}{(\mathcal{W}_\ell g(w))'}\right) \prec \Xi(w),$$

where $\varphi \in (-\pi, \pi], z, w \in \mathbb{D}$ *and the function* g *is given by* (3)*:*

By fixing $\varphi = 0$, we derive the following :

Remark 3. *The function of* $f \in \Sigma$ *given by* (1) *is in class* $\mathcal{M}_{\Sigma,\,Xi}^{\ell,\tau}(\vartheta)$ *if:*

$$1 + \frac{1}{\vartheta}\left(\frac{z(\mathcal{W}_\ell f(z))'}{\mathcal{W}_\ell f(z)} + \frac{z^2(\mathcal{W}_\ell f(z))''}{\mathcal{W}_\ell f(z)} - 1\right) \prec \Xi(z) \tag{14}$$

and

$$1 + \frac{1}{\vartheta}\left(\frac{w(\mathcal{W}_\ell g(w))'}{\mathcal{W}_\ell g(w)} + \frac{w^2(\mathcal{W}_\ell g(w))''}{\mathcal{W}_\ell g(w)} - 1\right) \prec \Xi(w) \tag{15}$$

where $\vartheta \in \mathbb{C}\backslash\{0\} z, w \in \mathbb{D}$ *and* g *as assumed in* (3)*.*

Remark 4. *The function of* $f \in \Sigma$ *given by* (1) *is in class* $\mathcal{F}_{\Sigma,\,Xi}^{\ell,\tau}(\vartheta)$ *if*

$$1 + \frac{1}{\vartheta}\left(\frac{[z(\mathcal{W}_\ell f(z))' + z^2(\mathcal{W}_\ell f(z))'']'}{(\mathcal{W}_\ell f(z))'} - 1\right) \prec \Xi(z) \tag{16}$$

and

$$1 + \frac{1}{\vartheta}\left(\frac{[w(\mathcal{W}_\ell g(w))' + w^2(\mathcal{W}_\ell g(w))'']'}{(\mathcal{W}_\ell g(w))'} - 1\right) \prec \Xi(w), \tag{17}$$

where $\vartheta \in \mathbb{C}\backslash\{0\}, z, w \in \mathbb{D}$ *and* g*, as assumed in* (3)*.*

The unique cases remarked on above yield new sub-classes of Σ-based Mathieu series and these classes have not, so far, been studied in association with telephone numbers. In the following section we investigate coefficient estimates for the function class $\mathcal{S}_{\Sigma,\Xi}^{\ell,\tau}(\vartheta,\varphi)$ and $\mathcal{C}_{\Sigma,\Xi}^{\ell,\tau}(\vartheta,\varphi)$.

2. Coefficient Estimates for f in $\mathcal{S}_{\Sigma,\Xi}^{\ell,\tau}(\vartheta,\varphi)$ and $\mathcal{C}_{\Sigma,\Xi}^{\ell,\tau}(\vartheta,\varphi)$

For notational simplicity, in the sequel we let:

$$\mathcal{Y}_2(\ell) = \frac{2\left(\ell^2+1\right)^2}{\left(4+\ell^2\right)^2}, \tag{18}$$

$$\mathcal{Y}_3(\ell) = \frac{3\left(\ell^2+1\right)^2}{\left(9+\ell^2\right)^2} \tag{19}$$

$$\Xi(z) \quad := \quad e^{\left(z+\tau\frac{z^2}{2}\right)} = 1+z+\frac{1+\tau}{2}z^2+\frac{1+3\tau}{6}z^3+\frac{3\tau^2+6\tau+1}{24}z^4+\cdots. \tag{20}$$

We also let $\vartheta \in \mathbb{C}\backslash\{0\}$ $\varphi \in (-\pi,\pi]$, $z,w \in \mathbb{D}$ and g as in (3), unless otherwise stated. To derive our main results, we need the following lemma.

Lemma 1 ([39]). *If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions h analytic in $\mathbb{D}$, for which $\Re(h(z)) > 0$ and*

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad \textit{for } z \in \mathbb{D}.$$

Define the functions $p(z)$ and $q(z)$ by:

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \cdots$$

and

$$q(z) := \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + \cdots.$$

It follows that:

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \cdots\right]$$

and

$$v(z) := \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1 z + \left(q_2 - \frac{q_1^2}{2}\right)z^2 + \cdots\right].$$

Then, p, q are analytic in $\mathbb{D}$ with $p(0) = 1 = q(0)$.
Since $u, v : \mathbb{D} \to \mathbb{D}$, the functions $p, q \in \mathcal{P}$ and $|p_i| \leq 2$ and $|q_i| \leq 2$ for each i.

Theorem 1. *Let $f \in \mathcal{S}_{\Sigma,\Xi}^{\ell,\tau}(\vartheta,\varphi)$, be given by (1), $\vartheta \in \mathbb{C}\backslash\{0\}$ and $\varphi \in (-\pi,\pi]$. Then:*

$$|a_2| \leq \frac{\sqrt{2}\,|\vartheta|}{\sqrt{2\left|\vartheta[(5+3e^{i\varphi})\mathcal{Y}_3(\ell) - (2+e^{i\varphi})[\mathcal{Y}_2(\ell)]^2] - (\tau-1)(2+e^{i\varphi})^2[\mathcal{Y}_2(\ell)]^2\right|}} \tag{21}$$

and

$$|a_3| \leq \frac{|\vartheta|^2}{|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2} + \frac{|\vartheta|}{|5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}. \tag{22}$$

Proof. From (10) and (11) it follows that:

$$1+\frac{1}{\vartheta}\left(\frac{z(\mathcal{W}_{\ell}f(z))'}{\mathcal{W}_{\ell}f(z)}+\left(\frac{1+e^{i\varphi}}{2}\right)\frac{z^2(\mathcal{W}_{\ell}f(z))''}{\mathcal{W}_{\ell}f(z)}-1\right)=\Xi(u(z)) \tag{23}$$

and

$$1+\frac{1}{\vartheta}\left(\frac{w(\mathcal{W}_{\ell}g(w))'}{\mathcal{W}_{\ell}g(w)}+\left(\frac{1+e^{i\varphi}}{2}\right)\frac{w^2(\mathcal{W}_{\ell}g(w))''}{\mathcal{W}_{\ell}g(w)}-1\right)=\Xi(v(w)), \tag{24}$$

where

$$\begin{aligned}\Xi(u(z)) &= e^{\left[\frac{p(z)-1}{p(z)+1}+\ell\frac{\left(\frac{p(z)-1}{p(z)+1}\right)^2}{2}\right]}\\ &= 1+\frac{p_1}{2}z+\left(\frac{p_2}{2}+\frac{(\tau-1)p_1^2}{8}\right)z^2+\left(\frac{p_3}{2}+(\tau-1)\frac{p_1p_2}{4}+\frac{(1-3\tau)}{48}p_1^3\right)z^3+\cdots.\end{aligned} \tag{25}$$

Similarly we get:

$$\Xi(v(w))=1+\frac{q_1}{2}z+\left(\frac{q_2}{2}+\frac{(\tau-1)q_1^2}{8}\right)z^2+\left(\frac{q_3}{2}+(\tau-1)\frac{q_1q_2}{4}+\frac{(1-3\tau)}{48}q_1^3\right)z^3+\cdots. \tag{26}$$

Now, equating the coefficients in (23) and (24), we obtain:

$$\frac{1}{\vartheta}(2+e^{i\varphi})[\mathcal{Y}_2(\ell)]a_2=\frac{1}{2}p_1, \tag{27}$$

$$\frac{1}{\vartheta}\left[(5+3e^{i\varphi})\mathcal{Y}_3(\ell)a_3-(2+e^{i\varphi})[\mathcal{Y}_2(\ell)]^2a_2^2\right]=\frac{p_2}{2}+\frac{(\tau-1)p_1^2}{8}, \tag{28}$$

$$-\frac{1}{\vartheta}(2+e^{i\varphi})[\mathcal{Y}_2(\ell)]a_2=\frac{1}{2}q_1, \tag{29}$$

and

$$\frac{1}{\vartheta}\left([2(5+3e^{i\varphi})\mathcal{Y}_3(\ell)-(2+e^{i\varphi})[\mathcal{Y}_2(\ell)]^2]a_2^2-(5+3e^{i\varphi})\mathcal{Y}_3(\ell)a_3\right)=\frac{q_2}{2}+\frac{(\tau-1)q_1^2}{8}. \tag{30}$$

From (27) and (29), we obtain:

$$p_1=-q_1 \tag{31}$$

and

$$8(2+e^{i\varphi})^2[\mathcal{Y}_2(\ell)]^2a_2^2=\vartheta^2(p_1^2+q_1^2). \tag{32}$$

Thus, we have:

$$\frac{8(2+e^{i\varphi})^2[\mathcal{Y}_2(\ell)]^2a_2^2}{\vartheta^2}=p_1^2+q_1^2. \tag{33}$$

$$|a_2|\leq\frac{|\vartheta|}{|2+e^{i\varphi}|\mathcal{Y}_2(\ell)}. \tag{34}$$

Now, from (28), (30) and (32), we obtain:

$$\left(2\{2\vartheta[(5+3e^{i\varphi})\mathcal{Y}_3(\ell)-(2+e^{i\varphi})[\mathcal{Y}_2(\ell)]^2]-(\tau-1)(2+e^{i\varphi})^2[\mathcal{Y}_2(\ell)]^2\}\right)a_2^2=\vartheta^2(p_2+q_2). \tag{35}$$

Lemma 1 applied to the coefficients p_2 and q_2, yields

$$|a_2|\leq\frac{\sqrt{2}\,|\vartheta|}{\sqrt{2|\vartheta[(5+3e^{i\varphi})\mathcal{Y}_3(\ell)-(2+e^{i\varphi})[\mathcal{Y}_2(\ell)]^2]-(\tau-1)(2+e^{i\varphi})^2[\mathcal{Y}_2(\ell)]^2|}}.$$

By subtracting (28) from (30) and using (31), we obtain $|a_3|$:

$$\frac{2}{\vartheta}(5+3e^{i\varphi})\mathcal{Y}_3(\ell)(a_3-a_2^2)=\frac{1}{2}(p_2-q_2).$$

When substituting a_2^2 from (32), we get:

$$a_3=\frac{\vartheta^2(p_1^2+q_1^2)}{8(2+e^{i\varphi})^2[\mathcal{Y}_2(\ell)]^2}+\frac{\vartheta(p_2-q_2)}{4(5+3e^{i\varphi})\mathcal{Y}_3(\ell)}.$$

Applying Lemma 1 once again to the coefficients p_1, p_2, q_1 and q_2, we get

$$|a_3|\leq\frac{|\vartheta|^2}{|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2}+\frac{|\vartheta|}{|5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}.$$

□

Theorem 2. *Let $f\in\mathcal{C}_{\Sigma,\Xi}^{\ell,\tau}(\vartheta,\varphi)$ be given by (1),$\vartheta\in\mathbb{C}\backslash\{0\}$ and $\varphi\in(-\pi,\pi]$,then*

$$|a_2|\leq\frac{|\vartheta|}{\sqrt{|\vartheta\{3(5+3e^{i\varphi})\mathcal{Y}_3(\ell)-4(2+e^{i\varphi})[\mathcal{Y}_2(\ell)]^2\}-2(\tau-1)(2+e^{i\varphi})^2[\mathcal{Y}_2(\ell)]^2|}}\tag{36}$$

and

$$|a_3|\leq\frac{|\vartheta|^2}{4|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2}+\frac{|\vartheta|}{3|5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}.\tag{37}$$

Proof. We write the argument inequalities in (refeq5) and (13) equivalently as follows:

$$1+\frac{1}{\vartheta}\left(\frac{[z(\mathcal{W}_\ell f(z))'+\left(\frac{1+e^{i\varphi}}{2}\right)z^2(\mathcal{W}_\ell f(z))'']'}{(\mathcal{W}_\ell f(z))'}-1\right)=\Xi(u(z))\tag{38}$$

and

$$1+\frac{1}{\vartheta}\left(\frac{[w(\mathcal{W}_\ell g(w))'+\left(\frac{1+e^{i\varphi}}{2}\right)w^2(\mathcal{W}_\ell g(w))'']'}{(\mathcal{W}_\ell g(w))'}-1\right)=\Xi(v(w)).\tag{39}$$

Now intending to find the evidence of Theorem 1, from (38) and (39), we reap the subsequent relations:

$$\frac{2}{\vartheta}(2+e^{i\varphi})[\mathcal{Y}_2(\ell)]a_2=\frac{1}{2}p_1,\tag{40}$$

$$\frac{1}{\vartheta}[3(5+3e^{i\varphi})\mathcal{Y}_3(\ell)a_3-4(2+e^{i\varphi})[\mathcal{Y}_2(\ell)]^2a_2^2]=\frac{p_2}{2}+\frac{(\tau-1)p_1^2}{8},\tag{41}$$

and

$$-\frac{2}{\vartheta}(2+e^{i\varphi})[\mathcal{Y}_2(\ell)]a_2=\frac{1}{2}q_1,\tag{42}$$

$$\frac{1}{\vartheta}[3(5+3e^{i\varphi})(2a_2^2-a_3)\mathcal{Y}_3(\ell)-4(2+e^{i\varphi})[\mathcal{Y}_2(\ell)]^2a_2^2]=\frac{q_2}{2}+\frac{(\tau-1)q_1^2}{8}\tag{43}$$

From (40) and (42), we get:

$$p_1=-q_1\tag{44}$$

and

$$\begin{aligned} 32(2+e^{i\varphi})^2[\mathcal{Y}_2(\ell)]^2a_2^2 &= \vartheta^2(p_1^2+q_1^2), &(45)\\ a_2^2 &= \frac{\vartheta^2 p_1^2}{16(2+e^{i\varphi})^2[\mathcal{Y}_2(\ell)]^2}, &(46)\\ |a_2| &\leq \frac{|\vartheta|}{4|2+e^{i\varphi}|\mathcal{Y}_2(\ell)} \end{aligned}$$

Now, from (41), (43) and (45), we obtain:

$$a_2^2 = \frac{\vartheta^2(p_2+q_2)}{4[\vartheta\{3(5+3e^{i\varphi})\mathcal{Y}_3(\ell)-4(2+e^{i\varphi})[\mathcal{Y}_2(\ell)]^2\}-2(\tau-1)(2+e^{i\varphi})^2[\mathcal{Y}_2(\ell)]^2]}. \tag{47}$$

Making use of Lemma (1) to the coefficients p_2 and q_2, we have the preferred inequality, given in (36). Subsequently, this allows us to find the bound on $|a_3|$, by subtracting (41) from (43), and using (44), we get:

$$\frac{6}{\vartheta}(5+3e^{i\varphi})(a_3-a_2^2)\mathcal{Y}_3(\ell) = \frac{1}{2}(p_2-q_2). \tag{48}$$

Upon substituting the value of a_2^2 given by (45), the above equation leads to:

$$a_3 = \frac{\vartheta(p_2-q_2)}{12(5+3e^{i\varphi})\mathcal{Y}_3(\ell)} + \frac{\vartheta^2(p_1^2+q_1^2)}{32(2+e^{i\varphi})^2[\mathcal{Y}_2(\ell)]^2}. \tag{49}$$

Applying the Lemma1 once again to the coefficients p_1, p_2, q_1 and q_2, we get the desired coefficient given in (37). □

Fixing $\varphi = \pi$ in Theorems 1 and 2 , we can state the coefficient estimates for the functions in the sub-classes $\mathcal{S}_{\Sigma,\Xi}^{\ell,\tau}(\vartheta)$ and $\mathcal{C}_{\Sigma,\Xi}^{\ell,\tau}(\vartheta)$ defined in Remark 1.

Corollary 1. *Let $f \in \mathcal{S}_{\Sigma,\Xi}^{\ell,\tau}(\vartheta)$ be given by (1). Then*

$$|a_2| \leq \frac{\sqrt{2}\,|\vartheta|}{\sqrt{2|\vartheta(2\mathcal{Y}_3(\ell)-[\mathcal{Y}_2(\ell)]^2)-(\tau-1)[\mathcal{Y}_2(\ell)]^2|}} \quad \text{and} \quad |a_3| \leq \frac{|\vartheta|^2}{[\mathcal{Y}_2(\ell)]^2} + \frac{|\vartheta|}{2\mathcal{Y}_3(\ell)}.$$

Corollary 2. *Let $f \in \mathcal{C}_{\Sigma,\Xi}^{\ell,\tau}(\vartheta)$ be given by (1) . Then,*

$$|a_2| \leq \frac{|\vartheta|}{\sqrt{2|\vartheta(3\mathcal{Y}_3(\ell)-2[\mathcal{Y}_2(\ell)]^2)-2(\tau-1)[\mathcal{Y}_2(\ell)]^2|}} \quad \text{and} \quad |a_3| \leq \frac{|\vartheta|^2}{4[\mathcal{Y}_2(\ell)]^2} + \frac{|\vartheta|}{6\mathcal{Y}_3(\ell)}.$$

Taking $\vartheta = 1$ in Theorems 1 and 2 , we can state the coefficient estimates for the functions in the sub-classes $\mathcal{S}_{\Sigma,\Xi}^{\ell,\tau}(\varphi)$ and $\mathcal{C}_{\Sigma,\Xi}^{\ell,\tau}(\varphi)$ defined in Remark 2.

Corollary 3. *Let $f \in \mathcal{S}_{\Sigma,\Xi}^{\ell,\tau}(\varphi)$ be given by (1).Then,*

$$|a_2| \leq \frac{\sqrt{2}}{\sqrt{2|(5+3e^{i\varphi})\mathcal{Y}_3(\ell)-(2+e^{i\varphi})[\mathcal{Y}_2(\ell)]^2|-(\tau-1)|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2}}$$

and

$$|a_3| \leq \frac{1}{|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2} + \frac{1}{|5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}.$$

Corollary 4. *Let $f(z)$ given by (1) in the class $\mathcal{C}^{\ell,\tau}_{\Sigma,\Xi}(\varphi)$.. Then,*

$$|a_2| \le \frac{1}{\sqrt{\{|3(5+3e^{i\varphi})\mathcal{Y}_3(\ell) - 4(2+e^{i\varphi})[\mathcal{Y}_2(\ell)]^2 - 2(\tau-1)(2+e^{i\varphi})^2[\mathcal{Y}_2(\ell)]^2|\}}}$$

and

$$|a_3| \le \frac{1}{4|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2} + \frac{1}{3|5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}.$$

3. Fekete–Szegö Inequality for $f \in \mathcal{S}^{\ell,\tau}_{\Sigma,\Xi}(\vartheta,\varphi)$

Fekete–Szegö inequality is one of the famous problems related to coefficients of univalent analytic functions. It was first given by [40], who stated:

$$|a_3 - va_2^2| \le \begin{cases} 3-4v, & if \quad v \le 0, \\ 1+2e^{\frac{-2v}{1-v}}, & if \quad 0 \le v \le 1, \\ 4v-3, & if \quad v \ge 1. \end{cases}$$

In this section, we prove Fekete–Szegö inequalities for functions in the class $\mathcal{S}^{\ell,\tau}_{\Sigma,\Xi}(\vartheta,\varphi)$. We used the following lemmas, which were introduced by Zaprawa in [13,14], and by the technique given in [37,38].

Lemma 2 ([13])**.** *Let $k \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$. If $|z_1| < R$ and $|z_2| < R$, then*

$$|(k+1)z_1 + (k-1)z_2| \le \begin{cases} 2|k|R, & |k| \ge 1, \\ 2R, & |k| \le 1. \end{cases}$$

Lemma 3 ([13])**.** *Let $k, l \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$. If $|z_1| < R$ and $|z_2| < R$, then*

$$|(k+l)z_1 + (k-l)z_2| \le \begin{cases} 2|k|R, & |k| \ge |l|, \\ 2|l|R, & |k| \le |l|. \end{cases}$$

Lemma 4 ([41])**.** *If $p \in \mathcal{P}$, then there exist some x, ζ with $|x| \le 1$, $|\zeta| \le 1$, such that*

$$2p_2 = p_1^2 + x(4-p_1^2),$$

$$4p_3 = p_1^3 + 2p_1x(4-p_1^2) - (4-p_1^2)p_1x^2 + 2(4-p_1^2)(1-|x|^2)\zeta.$$

Theorem 3. *Let f. given by (1), be in the class $\mathcal{S}^{\ell,\tau}_{\Sigma,\Xi}(\vartheta,\varphi)$ and $\aleph \in \mathbb{R}$. Then:*

$$|\ a_3 - \aleph a_2^2\ | \le \begin{cases} \frac{2|\vartheta|}{3|5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}, & |\ 1-\aleph\ | \in [0, \frac{16|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2}{3|\vartheta||5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}) \\ |1-\aleph|\frac{|\vartheta|^2}{4|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2}, & |1-\aleph| \in [\frac{16|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2}{3|\vartheta||5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}, \infty). \end{cases}$$

Proof. From (47) and (49) it follows that:

$$a_3 - \aleph a_2^2 \quad = \quad (1-\aleph)\frac{\vartheta^2 p_1^2}{16(2+e^{i\varphi})^2[\mathcal{Y}_2(\ell)]^2} + \frac{\vartheta(p_2-q_2)}{12(5+3e^{i\varphi})\mathcal{Y}_3(\ell)}$$

From Lemma 4, we have $2p_2 = p_1^2 + x(4-p_1^2)$ and $2q_2 = q_1^2 + y(4-q_1^2)$, and, hence, we get

$$p_2 - q_2 = (\frac{4-p_1^2}{2})(x-y).$$

Using triangle inequality, and taking $|x| = \theta, |y| = \kappa$, we obtain, without difficulty, that:

$$|a_3 - \aleph a_2^2| \leq |1-\aleph|\frac{|\vartheta|^2 t^2}{16|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2} + \frac{|\vartheta|}{24|5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}(4-t_1^2)(\theta+\kappa).$$

Let $\mathcal{M}(t) = |1-\aleph|\frac{|\vartheta|^2 t^2}{16|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2} \geq 0$ and $\mathcal{N}(t) = \frac{|\vartheta|}{24|5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}(4-t_1^2) \geq 0$. Thus,

$$|a_3 - \aleph a_2^2| \leq \mathcal{M}(t) + \mathcal{N}(t)(\theta+\kappa) = \mathcal{W}(\theta,\kappa)$$

It is evident that the maximum of the function $\mathcal{W}(\theta,\kappa)$ occurs at $(\theta,\kappa) = (1,1)$. Thus,

$$\max \mathcal{W}(\theta,\kappa) : \theta,\kappa \in [0,1] = \mathcal{W}(1,1) = \mathcal{M}(t) + 2\mathcal{N}(t).$$

Let $H : [0,2] \to \mathbb{R}$, as follows:

$$H(t) = \mathcal{M}(t) + 2\mathcal{N}(t) \tag{50}$$

for fixed $\vartheta \in \mathbb{C} - \{0\}$. Substituting the value $\mathcal{M}(t), \mathcal{N}(t)$ in (50), we obtain:

$$\begin{aligned} H(t) &= |1-\aleph|\frac{|\vartheta|^2 t^2}{16|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2} + \frac{|\vartheta|}{12|5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}(4-t_1^2) \\ &= \left[|1-\aleph|\frac{|\vartheta|^2}{16|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2} - \frac{|\vartheta|}{3|5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}\right]t^2 + \frac{|\vartheta|}{12|5+3e^{i\varphi}|\mathcal{Y}_3(\ell)} \\ &= \frac{|\vartheta|^2}{16|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2}\left[|1-\aleph| - \frac{16|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2}{3|\vartheta||5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}\right]t^2 + \frac{|\vartheta|}{12|5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}. \end{aligned}$$

Now, we should investigate the maximum of $H(t)$ in $[0,2]$. By simple computation, we have

$$H'(t) = \frac{|\vartheta|^2}{8|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2}\left[|1-\aleph| - \frac{16|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2}{3|\vartheta||5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}\right]t.$$

It is clear that $H'(t) \leq 0$ if $\frac{|\vartheta|^2}{8|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2}\left[|1-\aleph| - \frac{16|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2}{3|\vartheta||5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}\right] \leq 0$; that is if $|1-\aleph| \in \left(0, \frac{16|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2}{3|\vartheta||5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}\right)$. So, the function $H(t)$ is a strictly descending function if $|1-\aleph| \in \left(0, \frac{16|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2}{3|\vartheta||5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}\right)$.

Therefore,

$$\max\{H(t) : t \in [0,2]\} = H(0) = \frac{|\vartheta|}{3|5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}.$$

Also, $H'(t) \geq 0$; that is $H(t)$ is an increasing function for $|1-\aleph| \geq \frac{16|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2}{3|\vartheta||5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}$. Therefore,

$$\max\{H(t) : t \in [0,2]\} = H(2) = |1-\aleph|\frac{|\vartheta|^2}{4|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2}.$$

Thus, we get:

$$| a_3 - \aleph a_2^2 | \leq \begin{cases} \frac{|\vartheta|}{3|5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}, & | 1-\aleph | \in [0, \frac{16|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2}{3|\vartheta||5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}) \\ |1-\aleph|\frac{|\vartheta|^2}{4|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2}, & |1-\aleph| \in [\frac{16|2+e^{i\varphi}|^2[\mathcal{Y}_2(\ell)]^2}{3|\vartheta||5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}, \infty). \end{cases}$$

In particular, by taking $\aleph = 1$, we get:

$$| a_3 - a_2^2 | \leq \frac{|\vartheta|}{3|5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}.$$

$\square$

By taking $\vartheta \in \mathbb{R}$ and $\varphi = n\pi, (n \in \mathbb{Z})$ in the following theorem we prove the following Fekete–Szegö inequalities.

Theorem 4. *Let f, given by (1), be in the class $\mathcal{S}^{\ell,\tau}_{\Sigma,\Xi}(\vartheta,\varphi)$ and $\aleph \in \mathbb{R}$.. Then:*

$$| a_3 - \aleph a_2^2 |\leq \begin{cases} \frac{|\vartheta|}{3|5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}, & 0 \leq| \Psi(\aleph,\varphi) |\leq \frac{|\vartheta|}{3|5+3e^{i\varphi}|\mathcal{Y}_3(\ell)} \\ 2|\vartheta||\Psi(\aleph,\varphi)|, & |\Psi(\aleph,\varphi)| \geq \frac{|\vartheta|}{3|5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}, \end{cases} \tag{51}$$

where

$$\Psi(\aleph,\varphi) = \frac{\vartheta^2(1-\aleph)}{4[\vartheta[3(5+3e^{i\varphi})\mathcal{Y}_3(\ell) - 4(2+e^{i\varphi})[\mathcal{Y}_2(\ell)]^2] - 2(\tau-1)(2+e^{i\varphi})^2[\mathcal{Y}_2(\ell)]^2]}.$$

Proof. From (46) and (48) it follows that:

$$\begin{aligned} a_3 - \aleph a_2^2 &= \frac{(1-\aleph)\vartheta^2(p_2+q_2)}{4[\vartheta[3(5+3e^{i\varphi})\mathcal{Y}_3(\ell) - 4(2+e^{i\varphi})[\mathcal{Y}_2(\ell)]^2] - 2(\tau-1)(2+e^{i\varphi})^2[\mathcal{Y}_2(\ell)]^2]} \\ &+ \frac{\vartheta(p_2-q_2)}{12(5+3e^{i\varphi})\mathcal{Y}_3(\ell)} \\ &= \left[\Psi(\aleph,\varphi) + \frac{\vartheta}{12(5+3e^{i\varphi})\mathcal{Y}_3(\ell)}\right]p_2 + \left[\Psi(\aleph,\varphi) - \frac{\vartheta}{12(5+3e^{i\varphi})\mathcal{Y}_3(\ell)}\right]q_2, \end{aligned}$$

where

$$\Psi(\aleph,\varphi) = \frac{\vartheta^2(1-\aleph)}{4[\vartheta[3(5+3e^{i\varphi})\mathcal{Y}_3(\ell) - 4(2+e^{i\varphi})[\mathcal{Y}_2(\ell)]^2] - 2(\tau-1)(2+e^{i\varphi})^2[\mathcal{Y}_2(\ell)]^2]}.$$

Thus by applying Lemmas 1, 3, we get the desired result given in (51).
In particular, by taking $\aleph = 1$, we obtain:

$$| a_3 - a_2^2 |\leq \frac{|\vartheta|}{3|5+3e^{i\varphi}|\mathcal{Y}_3(\ell)}.$$

□

4. Bi-Univalent Function Class $\mathcal{G}^{\ell}_{\Sigma}(\vartheta,\mu,\Xi)$

In this section we define another new subclass of bi-univalent functions, based on Mathieu–type power series, and associated with telephone numbers, and obtain the initial Taylor estimates $|a_2|; |a_3|$. Making use of this, we derive the Fekete–Szegö inequality for $f \in \mathcal{G}^{\ell}_{\Sigma}(\vartheta,\mu,\Xi)$:

Definition 3. *A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{G}^{\ell}_{\Sigma}(\vartheta,\mu,\Xi)$ if the following conditions are satisfied:*

$$1 + \frac{1}{\vartheta}\left(\frac{z(\mathcal{W}_\ell f(z))'}{(1-\mu)\mathcal{W}_\ell f(z) + \mu z(\mathcal{W}_\ell f(z))'} - 1\right) \prec \Xi(z) \tag{52}$$

and

$$1 + \frac{1}{\vartheta}\left(\frac{w(\mathcal{W}_\ell g(w))'}{(1-\mu)\mathcal{W}_\ell g(w) + \mu z(\mathcal{W}_\ell g(w))'} - 1\right) \prec \Xi(w) \tag{53}$$

where $\vartheta \in \mathbb{C}\backslash\{0\};\ 0 \leq \mu < 1;\ z,w \in \mathbb{D}$ and g is as in (3).

Example 1. *For $\mu = 0$ and $\vartheta \in \mathbb{C}\backslash\{0\}$, the function given by (1) is in class $f \in \Sigma$, is said to be in $\mathcal{S}^{\ell}_{\Sigma}(\vartheta, \Xi)$, if the following conditions are met:*

$$1 + \frac{1}{\vartheta}\left(\frac{z(\mathcal{W}_\ell f(z))'}{\mathcal{W}_\ell f(z)} - 1\right) \prec \Xi(z), \tag{54}$$

and

$$1 + \frac{1}{\vartheta}\left(\frac{w(\mathcal{W}_\ell g(w))'}{\mathcal{W}_\ell g(w)} - 1\right) \prec \Xi(w), \tag{55}$$

where $z, w \in \mathbb{D}$ and the function g is as in (3).

Note that $\mathcal{S}^{\ell}_{\Sigma}(\vartheta, \Xi) = \mathcal{S}^{\ell,\tau}_{\Sigma,\Xi}(\vartheta, \pi) \equiv \mathcal{S}^{\ell,\tau}_{\Sigma,\Xi}(\vartheta)$, as given in Remark 1.

Theorem 5. *Let the function $f(z)$ given by (1) be in the class $\mathcal{G}^{\ell}_{\Sigma}(\vartheta, \mu, \Xi)$. Then,*

$$|a_2| \leq \frac{|\vartheta|\sqrt{2}}{\sqrt{|[\vartheta(\mu^2 - 1) - (\tau - 1)(1 - \mu)^2][\mathcal{Y}_2(\ell)]^2 + 4\vartheta(1 - \mu)\mathcal{Y}_3(\ell)|}}, \tag{56}$$

and

$$|a_3| \leq \frac{|\vartheta|^2}{(1 - \mu)^2[\mathcal{Y}_2(\ell)]^2} + \frac{|\vartheta|}{2(1 - \mu)\mathcal{Y}_3(\ell)}. \tag{57}$$

For $\hbar \in \mathbb{R}$, we have:

$$| a_3 - \hbar a_2^2 | \leq \begin{cases} \frac{|\vartheta|}{2(1-\mu)\mathcal{Y}_3(\ell)}, & 0 \leq | \phi(\hbar) | \leq \frac{|\vartheta|}{8(1-\mu)\mathcal{Y}_3(\ell)} \\ 2|\vartheta||\phi(\hbar)|, & |\phi(\hbar)| \geq \frac{|\vartheta|}{8(1-\mu)\mathcal{Y}_3(\ell)}, \end{cases} \tag{58}$$

where

$$\phi(\hbar) = \frac{\vartheta^2(1 - \hbar)}{2([\vartheta(\mu^2 - 1) + (1 - \mu)^2][\mathcal{Y}_2(\ell)]^2 + 4\vartheta(1 - \mu)\mathcal{Y}_3(\ell))}.$$

Proof. It follows from (52) and (53) that:

$$1 + \frac{1}{\vartheta}\left(\frac{z(\mathcal{W}_\ell f(z))'}{(1 - \mu)\mathcal{W}_\ell f(z) + \mu z(\mathcal{W}_\ell f(z))'} - 1\right) = \Xi(u(z)) \tag{59}$$

and

$$1 + \frac{1}{\vartheta}\left(\frac{w(\mathcal{W}_\ell g(w))'}{(1 - \mu)\mathcal{W}_\ell g(w) + \mu z(\mathcal{W}_\ell g(w))'} - 1\right) = \Xi(v(w)). \tag{60}$$

Now, equating the coefficients in (59) and (60), we get:

$$\frac{(1 - \mu)}{\vartheta}[\mathcal{Y}_2(\ell)]a_2 = \frac{1}{2}p_1, \tag{61}$$

$$\frac{(\mu^2 - 1)}{\vartheta}[\mathcal{Y}_2(\ell)]^2 a_2^2 + \frac{2(1 - \mu)}{\vartheta}\mathcal{Y}_3(\ell)a_3 = \frac{p_2}{2} + \frac{(\tau - 1)p_1^2}{8}, \tag{62}$$

$$-\frac{(1 - \mu)}{\vartheta}[\mathcal{Y}_2(\ell)]a_2 = \frac{1}{2}q_1 \tag{63}$$

and

$$\frac{(\mu^2 - 1)}{\vartheta}[\mathcal{Y}_2(\ell)]^2 a_2^2 + \frac{2(1 - \mu)}{\vartheta}\mathcal{Y}_3(\ell)(2a_2^2 - a_3) = \frac{q_2}{2} + \frac{(\tau - 1)q_1^2}{8}. \tag{64}$$

From (61) and (63), we find that:

$$a_2 = \frac{\vartheta p_1}{2(1 - \mu)[\mathcal{Y}_2(\ell)]} = \frac{-\vartheta q_1}{2(1 - \mu)[\mathcal{Y}_2(\ell)]}, \tag{65}$$

which implies:

$$p_1 = -q_1 \tag{66}$$

and

$$8(1-\mu)^2[\mathcal{Y}_2(\ell)]^2 a_2^2 = \vartheta^2(p_1^2 + q_1^2). \tag{67}$$

$$a_2^2 = \frac{\vartheta^2 p_1^2}{4(1-\mu)^2[\mathcal{Y}_2(\ell)]^2}. \tag{68}$$

Adding (62) and (64), by using (65) and (66), we obtain:

$$2\{[2\vartheta(\mu^2-1) - (\tau-1)(1-\mu)^2][\mathcal{Y}_2(\ell)]^2 + 4\vartheta(1-\mu)\mathcal{Y}_3(\ell)\}a_2^2 = \vartheta^2(p_2+q_2). \tag{69}$$

Thus:

$$a_2^2 = \frac{\vartheta^2(p_2+q_2)}{2\{[2\vartheta(\mu^2-1) - (\tau-1)(1-\mu)^2][\mathcal{Y}_2(\ell)]^2 + 4\vartheta(1-\mu)\mathcal{Y}_3(\ell)\}}. \tag{70}$$

Applying Lemma 1 to the coefficients p_2 and q_2 immediately gives:

$$|a_2|^2 \leq \frac{2|\vartheta|^2}{|[2\vartheta(\mu^2-1) - (\tau-1)(1-\mu)^2][\mathcal{Y}_2(\ell)]^2 + 4\vartheta(1-\mu)\mathcal{Y}_3(\ell)|}. \tag{71}$$

The final inequality gives the desired estimate of a_2.
Then, to find the bounds of a_3, subtract (64) from to get (62):

$$\begin{aligned} \frac{4(1-\mu)}{\vartheta}\mathcal{Y}_3(\ell)a_3 - \frac{4(1-\mu)}{\vartheta}\mathcal{Y}_3(\ell)a_2^2 &= \frac{1}{2}(p_2-q_2) + \frac{(\tau-1)}{8}(p_1^2-q_1^2) \\ a_3 &= a_2^2 + \frac{\vartheta(p_2-q_2)}{8(1-\mu)\mathcal{Y}_3(\ell)}. \end{aligned} \tag{72}$$

It follows from (65), (66) and (72) that:

$$a_3 = \frac{\vartheta^2(p_1^2+q_1^2)}{8(1-\mu)^2[\mathcal{Y}_2(\ell)]^2} + \frac{\vartheta(p_2-q_2)}{8(1-\mu)\mathcal{Y}_3(\ell)}. \tag{73}$$

Applying Lemma 1 again to the coefficients p_2 and q_2, we easily get:

$$|a_3| \leq \frac{|\vartheta|^2}{(1-\mu)^2[\mathcal{Y}_2(\ell)]^2} + \frac{|\vartheta|}{2(1-\mu)\mathcal{Y}_3(\ell)}.$$

Now, by fixing $\vartheta, \hbar \in \mathbb{R}$ from (70) and (72) it follows that:

$$\begin{aligned} a_3 - \hbar a_2^2 &= \frac{(1-\hbar)\vartheta^2(p_2+q_2)}{2\{[2\vartheta(\mu^2-1) - (\tau-1)(1-\mu)^2][\mathcal{Y}_2(\ell)]^2 + 4\vartheta(1-\mu)\mathcal{Y}_3(\ell)\}} + \frac{\vartheta(p_2-q_2)}{8(1-\mu)\mathcal{Y}_3(\ell)} \\ &= \left[\phi(\hbar) + \frac{\vartheta}{8(1-\mu)\mathcal{Y}_3(\ell)}\right]p_2 + \left[\phi(\hbar) - \frac{\vartheta}{8(1-\mu)\mathcal{Y}_3(\ell)}\right]q_2 \end{aligned}$$

where

$$\phi(\hbar) = \frac{\vartheta^2(1-\hbar)}{2\{[2\vartheta(\mu^2-1) - (\tau-1)(1-\mu)^2][\mathcal{Y}_2(\ell)]^2 + 4\vartheta(1-\mu)\mathcal{Y}_3(\ell)\}}.$$

Thus, by applying Lemma 1, we get the desired result in (58)
In particular, by taking $\hbar = 1$, we get

$$| a_3 - a_2^2 | \leq \frac{|\vartheta|}{2(1-\mu)\mathcal{Y}_3(\ell)}.$$

This completes the proof of Theorem 5. $\square$

Theorem 6. *Let the function $f(z)$ given by (1) be in the class $\mathcal{G}^{\ell}_{\Sigma}(\vartheta,\mu,\Xi)$. Then:*

$$| a_3 - \hbar a_2^2 | \leq \begin{cases} \frac{|\vartheta|}{2(1-\mu)\mathcal{Y}_3(\ell)}, & | 1-\hbar | \in [0, \frac{(1-\mu)^2[\mathcal{Y}_2(\ell)]^2}{2|\vartheta|(1-\mu)\mathcal{Y}_3(\ell)}) \\ |1-\hbar|\frac{|\vartheta|^2}{(1-\mu)^2[\mathcal{Y}_2(\ell)]^2}, & |1-\hbar| \in [\frac{(1-\mu)^2[\mathcal{Y}_2(\ell)]^2}{2|\vartheta|(1-\mu)\mathcal{Y}_3(\ell)}, \infty). \end{cases}$$

where $\hbar \in \mathbb{R}$

Proof. From (70) and (72) it follows that:

$$a_3 - \hbar a_2^2 = (1-\hbar)\frac{\vartheta^2 p_1^2}{4(1-\mu)^2[\mathcal{Y}_2(\ell)]^2} + \frac{\vartheta(p_2 - q_2)}{8(1-\mu)\mathcal{Y}_3(\ell)}$$

From Lemma 4, we have $2p_2 = p_1^2 + x(4-p_1^2)$ and $2q_2 = q_1^2 + y(4-q_1^2)$, and, hence, we get

$$p_2 - q_2 = (\frac{4-p_1^2}{2})(x-y).$$

Using the triangle inequality and taking $x = \theta, y = \kappa$, we can easily get:

$$|a_3 - \hbar a_2^2| \leq |1-\hbar|\frac{|\vartheta|^2 t^2}{4(1-\mu)^2[\mathcal{Y}_2(\ell)]^2} + \frac{|\vartheta|}{8(1-\mu)\mathcal{Y}_3(\ell)}(4-t_1^2)(\theta+\kappa).$$

Let $\mathcal{M}_1(t) = |1-\aleph|\frac{|\vartheta|^2 t^2}{4(1-\mu)^2[\mathcal{Y}_2(\ell)]^2} \geq 0$ and $\mathcal{N}_1(t) = \frac{|\vartheta|}{8(1-\mu)\mathcal{Y}_3(\ell)}(4-t_1^2)(\theta+\kappa) \geq 0$. Thus:

$$|a_3 - \aleph a_2^2| \leq \mathcal{M}_1(t) + \mathcal{N}_1(t)(\theta+\kappa) = \mathcal{W}_1(\theta,\kappa)$$

It is clear that the maximum value of the function $\mathcal{W}_1(\theta,\kappa)$ occurs at $(\theta,\kappa) = (1,1)$. So

$$\max \mathcal{W}_1(\theta,\kappa) : \theta,\kappa \in [0,1] = \mathcal{W}_1(1,1) = \mathcal{M}_1(t) + 2\mathcal{N}_1(t).$$

Define $H_1 : [0,2] \to \mathbb{R}$ as

$$H_1(t) = \mathcal{M}_1(t) + 2\mathcal{N}_1(t) \tag{74}$$

for fixed $\vartheta \in \mathbb{C} - \{0\}$, substituting the value $\mathcal{M}_1(t), \mathcal{N}_1(t)$ in (74), we obtain:

$$\begin{aligned} H_1(t) &= |1-\aleph|\frac{|\vartheta|^2 t^2}{4(1-\mu)^2[\mathcal{Y}_2(\ell)]^2} + \frac{|\vartheta|}{8(1-\mu)\mathcal{Y}_3(\ell)}(4-t_1^2) \\ &= \left[|1-\aleph|\frac{|\vartheta|^2 t^2}{4(1-\mu)^2[\mathcal{Y}_2(\ell)]^2} - \frac{|\vartheta|}{8(1-\mu)\mathcal{Y}_3(\ell)}\right]t^2 + \frac{|\vartheta|}{2(1-\mu)\mathcal{Y}_3(\ell)} \\ &= \frac{|\vartheta|^2}{4(1-\mu)^2[\mathcal{Y}_2(\ell)]^2}\left[|1-\aleph| - \frac{(1-\mu)^2[\mathcal{Y}_2(\ell)]^2}{2|\vartheta|(1-\mu)\mathcal{Y}_3(\ell)}\right]t^2 + \frac{|\vartheta|}{2(1-\mu)\mathcal{Y}_3(\ell)} \end{aligned}$$

Now, we need to find the maximum value of $H_1(t)$ on the interval $[0,2]$. with a simple calculation,

$$H_1'(t) = \frac{|\vartheta|^2}{2(1-\mu)^2[\mathcal{Y}_2(\ell)]^2}\left[|1-\hbar| - \frac{(1-\mu)^2[\mathcal{Y}_2(\ell)]^2}{2|\vartheta|(1-\mu)\mathcal{Y}_3(\ell)}\right]t.$$

It is clear that $H_1'(t) \leq 0$ if $\frac{|\vartheta|^2}{2(1-\mu)^2[\mathcal{Y}_2(\ell)]^2}\left[|1-\aleph| - \frac{(1-\mu)^2[\mathcal{Y}_2(\ell)]^2}{2|\vartheta|(1-\mu)\mathcal{Y}_3(\ell)}\right] \leq 0$; that is if $|1-\hbar| \in \left(0, \frac{(1-\mu)^2[\mathcal{Y}_2(\ell)]^2}{2|\vartheta|(1-\mu)\mathcal{Y}_3(\ell)}\right)$. Thus, $H(t)$ is a strictly descending function if $|1-\hbar| \in \left(0, \frac{(1-\mu)^2[\mathcal{Y}_2(\ell)]^2}{2|\vartheta|(1-\mu)\mathcal{Y}_3(\ell)}\right)$.

Therefore,

$$\max\{H_1(t) : t \in [0,2]\} = H_1(0) = \frac{|\vartheta|}{2(1-\mu)\mathcal{Y}_3(\ell)}.$$

So, $H'(t) \geq 0$; that is, $H(t)$ is an increasing function for $|1-\hbar| \geq \frac{(1-\mu)^2[\mathcal{Y}_2(\ell)]^2}{2|\vartheta|(1-\mu)\mathcal{Y}_3(\ell)}$. Thus,

$$\max\{H_1(t) : t \in [0,2]\} = H_1(2) = |1-\hbar|\frac{|\vartheta|^2}{(1-\varphi)^2[\mathcal{Y}_2(\ell)]^2}.$$

Hence:

$$| a_3 - \hbar a_2^2 | \leq \begin{cases} \frac{|\vartheta|}{2(1-\mu)\mathcal{Y}_3(\ell)}, & | 1-\aleph | \in [0, \frac{(1-\mu)^2[\mathcal{Y}_2(\ell)]^2}{2|\vartheta|(1-\mu)\mathcal{Y}_3(\ell)}) \\ |1-\hbar|\frac{|\vartheta|^2}{(1-\mu)^2[\mathcal{Y}_2(\ell)]^2}, & |1-\hbar| \in [\frac{(1-\mu)^2[\mathcal{Y}_2(\ell)]^2}{2|\vartheta|(1-\mu)\mathcal{Y}_3(\ell)}, \infty). \end{cases}$$

In particular, by taking $\hbar = 1$, we get

$$| a_3 - a_2^2 | \leq \frac{|\vartheta|}{2(1-\mu)\mathcal{Y}_3(\ell)}.$$

□

5. Concluding Remarks

The work presented in this article followed the pioneering work of Srivastava et al. [7], and related it to Generalized telephone phone numbers (GTNs). We, then, presented the initial Taylor coefficient and Fekete–Szegö inequality results for this newly defined function of classes $\mathcal{S}_{\Sigma,\Xi}^{\ell,\tau}(\vartheta,\varphi)$ and $\mathcal{C}_{\Sigma,\Xi}^{\ell,\tau}(\vartheta,\varphi)$. We specialized the parameters of the new sub-class, with Remarks 1 and 4 not yet examined for GTNs. Furthermore, this work motivates researchers to extend this idea to meromorphic bi-univalent functions, and gives rise to a particular Erd ély–Kober operator [42], quantum computation operator [43,44] and q–Bernstein–Kantorovich operators [45] for $f \in \Sigma$ (see also references cited there). I believe we can derive a new class relating to GTNs.

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