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Statistical Inference for the Kavya–Manoharan Kumaraswamy Model under Ranked Set Sampling with Applications

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Abstract: In this article, we introduce a new extension of the Kumaraswamy (Ku) model, which is called the Kavya Manoharan Kumaraswamy (KMKu) model. The shape forms of the pdf for the KMKu model for various values of parameters are similar to the Ku model. It can be asymmetric, such as bathtub, unimodal, increasing and decreasing. In addition, the shape forms of the hrf for the KMKu model can be bathtub, U-shaped, J-shaped and increasing. Several statistical and computational properties were computed. Four different measures of entropy were studied. The maximum likelihood approach was employed to estimate the parameters for the KMKu model under simple and ranked set sampling. A simulation experiment was conducted in order to calculate the model parameters of the KMKu model utilizing simple and ranked set sampling and show the efficiency of the ranked set sampling more than the simple random sampling. The KMKu has more flexibility than the Ku model and other well-known models, and we proved this using three real-world data sets.

Keywords: Kumaraswamy model; asymmetric; ranked set sampling; KM transformation family; simulation; maximum likelihood estimation

MSC: 60E05; 62E15; 62F10

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1. Introduction

There are numerous statistical models in the literature, but it is always possible to construct more flexible models that are better suited to actual real data in various fields such as engineering, environmental science, biomedical science, economics, reliability, biology, energy, and physics. Statisticians have previously proposed a variety of methods for dealing with these problems. When the processes acquire values in the range (0, 1), a main statistical analysis using the usual beta (B) and Kumaraswamy (Ku) models may be used. Before proceeding, a review of these models is required. To begin, they are generated from the B and Ku models. The B model is a continuous model, and it has the range (0, 1) and two positive shape parameters, θ and μ. The cumulative distribution function (cdf) of the B model is provided via:

$$G_B(y;\theta,\mu) = \frac{1}{B(\theta,\mu)} \int_0^y t^{\theta-1}(1-t)^{\mu-1}dt; \qquad 0 < y < 1,\ \theta,\mu > 0,$$

where $B(\theta,\, \mu) = \int_0^1 t^{\theta-1}(1-t)^{\mu-1}dt$.

The appropriate probability density function (pdf) of the B model may take the form of a U, a monotonic (decreasing or increasing) curve. Conversely, the equivalent hazard rate function (hrf) might increase with convex or U-shaped forms [1,2]. The Ku model was created in [3] to supplement the B distribution. The Ku model theoretically depends on uniform-order statistics, and its functions are exceedingly basic, requiring no special functions. The cdf and pdf of the Ku model are provided via

$$G_{Ku}(y;\theta,\mu) = 1 - \left(1 - y^{\theta}\right)^{\mu}; \qquad 0 < y < 1,\ \theta,\mu > 0, \tag{1}$$

and

$$g_{Ku}(y;\theta,\mu) = \theta\mu y^{\theta-1}\left(1 - y^{\theta}\right)^{\mu-1}; \qquad 0 < y < 1,\ \theta,\mu > 0. \tag{2}$$

According to the values of the parameter, the pdf of the Ku model is asymmetric, and it has one of these forms: (i) bathtub when $\theta,\, \mu < 1$, (ii) decreasing when $\theta \leqslant 1$, $\mu > 1$, (iii) increasing when $\theta > 1$, $\mu \leqslant 1$, (iv) unimodal when $\theta, \mu > 1$, (v) steady when $\theta = \mu = 1$.

The behavior of the Ku model is similar to that of the B model, but it is simpler since both its pdf and cdf are closed-form. Both the B and Ku models have the same boundary behavior and significant special models. This distribution might be a viable alternative in instances when the limits are, in fact, finite, i.e., (0, 1).

The Ku was primarily designed as a lifetime model. Several researchers explored and created generalizations of the Ku model, such as an exponentiated Ku model proposed in [4], the Ku Ku model studied in [5], transmuted Ku model discussed in [6], the size-biased Ku model proposed in [7], the Marshall–Olkin Ku model introduced in [8], the exponentiated generalized Ku model suggested in [9], the modified Ku model discussed in [10], the type II half logistic Ku model proposed in [11], the alpha power Ku model discussed in [12], and the bivariate and multivariate weighted Ku models proposed in [13]. All of the previous models are a generalization of the Ku model and have the range (0,1).

Additional parameters provide extra flexibility; however, they may increase the estimation's complexity. Many authors used the approach of adding parameters, such as type II half-logistic odd Fréchet class of distributions by [14], odd Perks class of distributions by [15], type II power Topp-Leone class of distributions by [16], generalized power Akshaya distribution by [17], sec class of distributions by [18]. Recently, the authors of [19] have developed a novel transformation, known as the KM transformation family of distributions, to obtain new parsimonious families of distributions. The cdf and pdf of the KM transformation family of distributions are provided via

$$F_{KM}(y) = \frac{e}{e-1}\left(1 - e^{-G(y)}\right), \quad y \in R, \tag{3}$$

and

$$f_{KM}(y) = \frac{e}{e-1} g(y) e^{-G(y)}, \quad y \in R. \tag{4}$$

where $G(y)$ and $g(y)$ are the cdf and pdf of the parent distribution. The benefit of employing this transformation is that the produced distribution is parameter-parsimonious since no additional parameters are introduced.

The major goal of this article is to provide a novel extension of the Ku model named the KMKu model, which has two shape parameters μ and θ. The below considerations provide sufficient motivation and reason to investigate the KMKu model. We describe them as described below:

1. The new KMKu is more flexible than the Ku model, and they have the same number of parameters.
2. The curves of the pdf for the KMKu model are similar to the Ku model, and it can be asymmetric, such as (i) bathtub when $\theta, \mu < 1$, (ii) decreasing when $\theta \leqslant 1$, $\mu > 1$, (iii) increasing when $\theta > 1$, $\mu < 1$, (iv) unimodal when $\theta > 1, \mu \geqslant 1$.

3. The KMKu model has a closed form for the quantile function, making it simple to generate random numbers from the KMKu proposed model.
4. Several general statistical features of the KMku model were investigated.
5. The maximum likelihood estimation technique was employed to calculate the parameters of the KMKu model, employing simple and ranked set sampling.
6. The KMKu model gives more fit than the Ku model and numerous other well-known models for modeling real-world data sets in different fields, and we recommended that in the application section.

This paper is organized as follows. In Section 2, the construction of the KMKu model is provided by combining the Ku model and the KM transformation. In Section 3, we derive and investigate some important expansions, which we use to calculate the statistical properties of the KMKu model. In Section 4, some general statistical features of the KMKu model are derived. In Section 5, some different measures of entropy are computed. In Section 6, ranked set sampling is discussed. In Section 7, estimations of the unknown parameters using the maximum likelihood methodology under simple and ranked set sampling are studied. In Section 8, simulation outcomes are discussed. In Section 9, the importance and flexibility of the KMKu model are proved by employing three real data sets. Finally, the paper ended with concluding remarks.

2. The Construction of The Suggested Model

In this section, we create a new flexible statistical model defined as the Kavya–Manoharan transformation Kumaraswamy (KMKu) model by introducing (1) into (3) to obtain

$$FCDFF(y;\theta,\mu) = \frac{e}{e-1}\left(1 - e^{-1}e^{\left(1-y^{\theta}\right)^{\mu}}\right); \qquad 0 < y < 1,\ \theta,\mu > 0, \tag{5}$$

and the corresponding pdf is

$$f(y;\theta,\mu) = \frac{1}{e-1}\theta\mu y^{\theta-1}\left(1-y^{\theta}\right)^{\mu-1}e^{\left(1-y^{\theta}\right)^{\mu}}; \qquad 0 < y < 1,\ \theta,\mu > 0. \tag{6}$$

The survival function, hrf and reversed hrf for the KMKu model are

$$R(y;\theta,\mu) = 1 - F(y;\theta,\mu) = 1 - \frac{e}{e-1}\left(1 - e^{-1}e^{\left(1-y^{\theta}\right)^{\mu}}\right); \qquad 0 < y < 1,\ \theta,\mu > 0,$$

$$h(y;\theta,\mu) = \frac{f(y;\theta,\mu)}{R(y;\theta,\mu)} = \frac{\theta\mu y^{\theta-1}\left(1-y^{\theta}\right)^{\mu-1}}{1 - e^{-\left(1-y^{\theta}\right)^{\mu}}},$$

and

$$\tau(y;\theta,\mu) = \frac{f(y;\theta,\mu)}{F(y;\theta,\mu)} = \frac{\theta\mu y^{\theta-1}\left(1-y^{\theta}\right)^{\mu-1}}{e^{1-\left(1-y^{\theta}\right)^{\mu}} - 1}.$$

The cumulative hrf is provided via

$$H(y;\theta,\mu) = -\ln\left[1 - \frac{e}{e-1}\left(1 - e^{-1}e^{\left(1-y^{\theta}\right)^{\mu}}\right)\right].$$

The inverse of the cdf was used to generate random samples from the newly suggested model, and it has the next equation

$$y_u = \left(1 - \left(1 + \ln\left(1 - u\left(1 - e^{-1}\right)\right)\right)^{\frac{1}{\mu}}\right)^{\frac{1}{\theta}}. \tag{7}$$

Figure 1 shows the forms of the KMKu pdf (6) utilizing distinct parameter combinations. According to the values of the parameter, the pdf of the KMKu model is asymmetric, and it has one of these forms: (i) bathtub when $\theta, \mu < 1$, (ii) decreasing when $\theta \leqslant 1, \mu > 1$, (iii) increasing when $\theta > 1, \mu < 1$, (iv) unimodal when $\theta > 1, \mu \geqslant 1$.

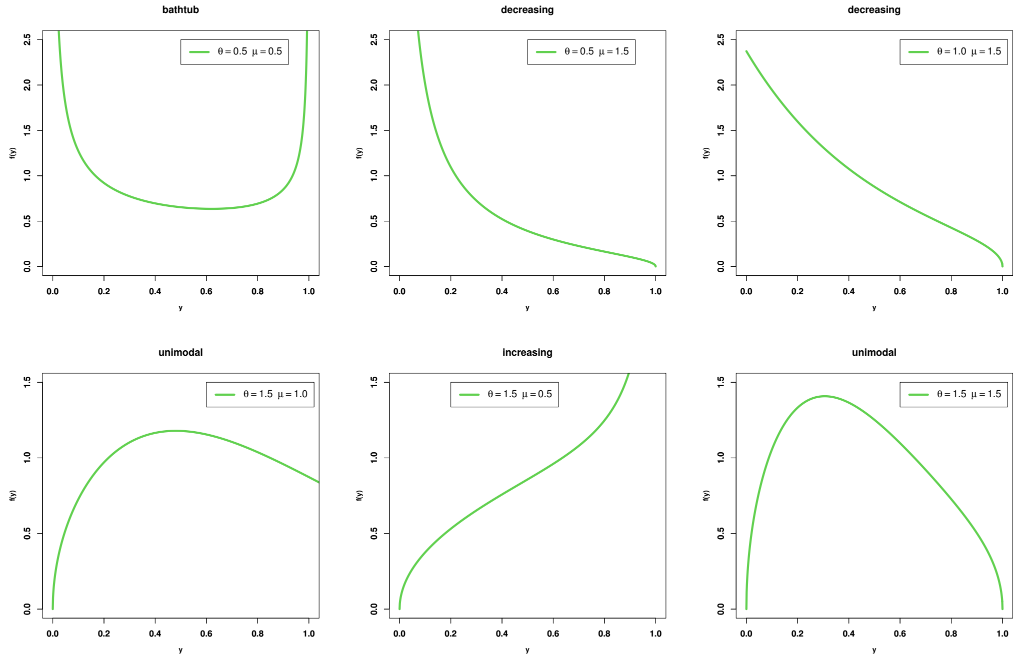

Figure 1. Plots of pdf for the KMKu model.

Figure 2 demonstrates a graphical representation of the hrf of the KMKu model utilizing distinct parameter combinations. According to the values of the parameter, the hrf of the KMKu model can be bathtub, U-shaped and increasing.

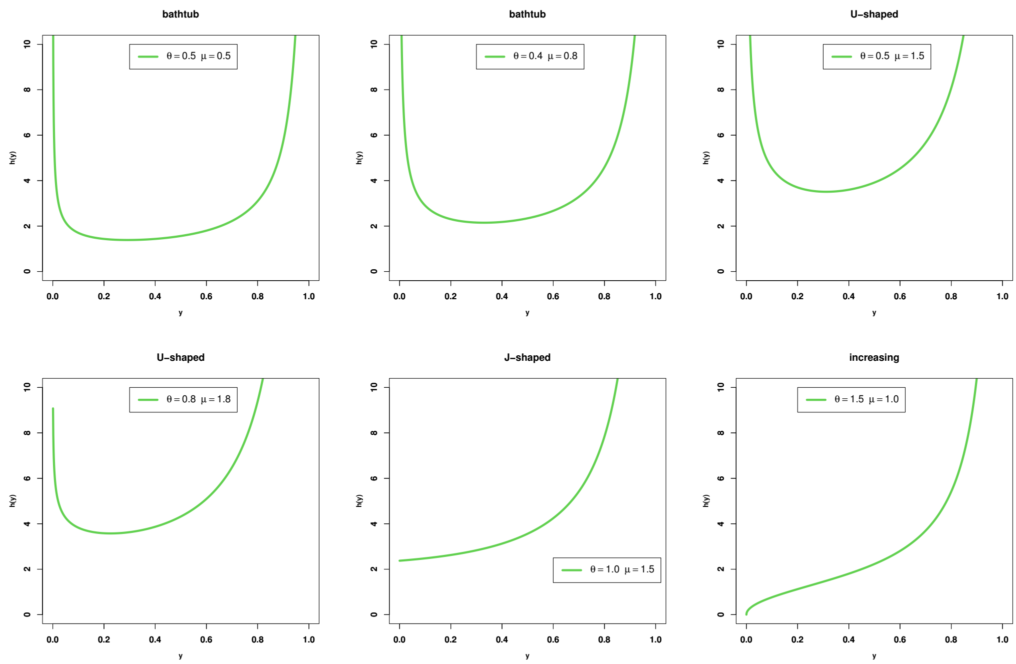

Figure 2. *Cont.*

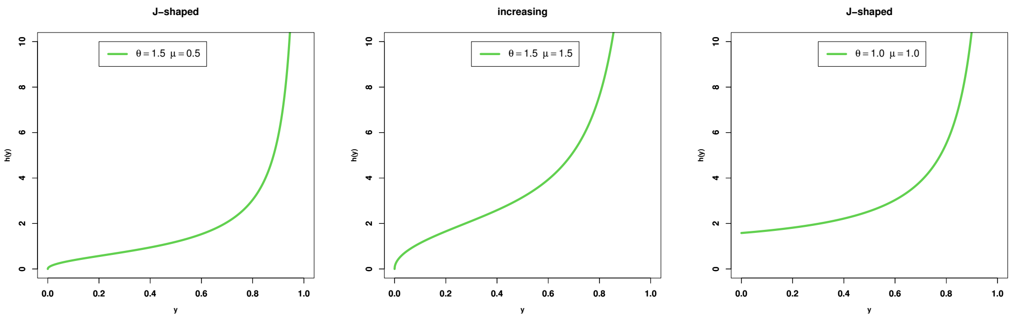

Figure 2. Plots of hrf for the KMKu model.

3. Useful Expansions

In this section, the expansions of the $f(y;\theta,\mu)$, $[f(y;\theta,\mu)]^{\lambda}$ and $[F(y;\theta,\mu)]^{h}$ for the KMKu model are derived to make the computation of the statistical features easy because we can write the pdf and cdf of the KMKu model as a linear combination of the Ku model. First, we can rewrite Equation (6) as

$$f(y;\theta,\mu) = \frac{\theta\mu e}{e-1} y^{\theta-1}\left(1-y^{\theta}\right)^{\mu-1} e^{-\left[1-\left(1-y^{\theta}\right)^{\mu}\right]}, \tag{8}$$

Using the exponential expansion $e^{-\beta y} = \sum_{i=0}^{\infty} \frac{(-1)^i \beta^i y^i}{i!}$ to Equation (8), we obtain,

$$f(y;\theta,\mu) = \frac{\theta\mu e}{e-1} y^{\theta-1} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\left(1-y^{\theta}\right)^{\mu-1}\left[1-\left(1-y^{\theta}\right)\right]^{i}. \tag{9}$$

Using the binomial expansion

$$(1-y)^{\beta-1} = \sum_{j=0}^{\infty} (-1)^j \binom{\beta-1}{j} y^j, \quad |y|<1, \tag{10}$$

by inserting Equation (10) in Equation (9), we obtain

$$f(y;\theta,\mu) = y^{\theta-1} \sum_{i,j=0}^{\infty} \varsigma_{i,j}\left(1-y^{\theta}\right)^{\mu+j-1}, \tag{11}$$

where $\varsigma_{i,j} = \frac{\theta\mu e}{e-1}\frac{(-1)^{i+j}}{i!}\binom{i}{j}$.

To obtain the expansion of $[f(y;\theta,\mu)]^{\lambda}$, then

$$[f(y;\theta,\mu)]^{\lambda} = \left(\frac{\theta\mu e}{e-1}\right)^{\lambda} y^{\lambda(\theta-1)}\left(1-y^{\theta}\right)^{\lambda(\mu-1)} e^{-\lambda\left[1-\left(1-y^{\theta}\right)^{\mu}\right]},$$

by applying the exponential expansion to the above equation, we obtain,

$$[f(y;\theta,\mu)]^{\lambda} = \left(\frac{\theta\mu e}{e-1}\right)^{\lambda} y^{\lambda(\theta-1)} \sum_{i=0}^{\infty} \frac{(-1)^i \lambda^i}{i}\left(1-y^{\theta}\right)^{\lambda(\mu-1)}\left[1-\left(1-y^{\theta}\right)^{\mu}\right]^{i}, \tag{12}$$

by applying the binomial expansion (10) in Equation (12), we obtain

$$[f(y;\theta,\mu)]^{\lambda} = y^{\lambda(\theta-1)} \sum_{i,j=0}^{\infty} \varpi_{i,j}\left(1-y^{\theta}\right)^{\lambda(\mu-1)+j}, \tag{13}$$

where $\varpi_{i,j} = \left(\frac{\theta\mu e}{e-1}\right)^{\lambda}\frac{(-1)^{i+j}\lambda^i}{i}\binom{i}{j}$. To obtain the expansion of $[F(y;\theta,\mu)]^{h}$, then

$$[F(y;\theta,\mu)]^h = \left(\frac{e}{e-1}\right)^h \left(1 - e^{-\left[1-(1-y^\theta)^\mu\right]}\right)^h,$$

by employing the binomial expansion $(1-y)^{\beta-1} = \sum_{k=0}^{\beta-1} (-1)^k \left(\begin{array}{c} \beta-1 \\ k \end{array}\right) y^k$, to the above equation, we obtain

$$[F(y;\theta,\mu)]^h = \left(\frac{e}{e-1}\right)^h \sum_{k=0}^{h} (-1)^k \left(\begin{array}{c} h \\ k \end{array}\right) e^{-k\left[1-(1-y^\theta)^\mu\right]},$$

again by employing the exponential expansion to the above equation, we obtain

$$[F(y;\theta,\mu)]^h = \left(\frac{e}{e-1}\right)^h \sum_{k=0}^{h} \sum_{m=0}^{\infty} \frac{(-1)^{k+m} k^m}{m!} \left(\begin{array}{c} h \\ k \end{array}\right) \left[1 - \left(1 - y^\theta\right)^\mu\right]^m,$$

again by using the binomial expansion (10) to the above equation, we obtain

$$[F(y;\theta,\mu)]^h = \sum_{k=0}^{h} \sum_{m,s=0}^{\infty} \Omega_{k,m,s} \left(1 - y^\theta\right)^{\mu s}, \tag{14}$$

where $\Omega_{k,m,s} = \left(\frac{e}{e-1}\right)^h \frac{(-1)^{k+m+s} k^m}{m!} \left(\begin{array}{c} h \\ k \end{array}\right) \left(\begin{array}{c} m \\ s \end{array}\right)$.

4. General Statistical Properties

The section discusses various essential characteristics of the KMku model, such as the quantile function (qf), raw and incomplete moments, probability-weighted moments, and order statistics.

4.1. Quantile Function

Quantile function (qf) is an important measure and is used to generate a random variate. Let $Y \sim \text{KMKu}(\theta,\mu)$ for $Y > 0$ and $\theta,\mu > 0$, then its qf, say Q(u), is given by in Equation (15)

$$Q(u) = \left(1 - \left(1 + \ln\left(1 - u\left(1 - e^{-1}\right)\right)\right)^{\frac{1}{\mu}}\right)^{\frac{1}{\theta}}, \tag{15}$$

where u is a uniform distribution. The median (M) of the KMKu model may be calculated by entering u = 0.5 in Equation (15) as

$$M = \left(1 - \left(1 + \ln\left(1 - 0.5\left(1 - e^{-1}\right)\right)\right)^{\frac{1}{\mu}}\right)^{\frac{1}{\theta}}.$$

4.2. Moments and Incomplete Moments

The wth moment of Y can be computed by using the next formula

$$\mu'_w = \int_0^1 y^w f(y;\theta,\mu) dy. \tag{16}$$

We acquire Equation (11) by entering it into Equation (16)

$$\mu'_w = \sum_{i,j=0}^{\infty} \varsigma_{i,j} \int_0^1 y^{w+\theta-1} \left(1 - y^\theta\right)^{\mu+j-1} dy,$$

Let $v = y^{\theta}$, then we can write the previous equation as

$$\mu'_w = \frac{1}{\theta} \sum_{i,j=0}^{\infty} \varsigma_{i,j} \int_0^1 v^{\frac{w}{\theta}} (1-v)^{\mu+j-1} dv,$$

then, the wth moment of the KMKu model is provided via

$$\mu'_w = \frac{1}{\theta} \sum_{i,j=0}^{\infty} \varsigma_{i,j} B\left(\frac{w}{\theta}+1,\ \mu+j\right), \tag{17}$$

where $B(.,.)$ is the B function. By putting $w = 1,2,3,4$ in (17), we obtain the first four moments for Y as

$$\mu'_1 = \frac{1}{\theta} \sum_{i,j=0}^{\infty} \varsigma_{i,j} B\left(\frac{1}{\theta}+1,\ \mu+j\right), \tag{18}$$

$$\mu'_2 = \frac{1}{\theta} \sum_{i,j=0}^{\infty} \varsigma_{i,j} B\left(\frac{w}{\theta}+1,\ \mu+j\right), \tag{19}$$

$$\mu'_3 = \frac{1}{\theta} \sum_{i,j=0}^{\infty} \varsigma_{i,j} B\left(\frac{3}{\theta}+1,\ \mu+j\right), \tag{20}$$

and

$$\mu'_4 = \frac{1}{\theta} \sum_{i,j=0}^{\infty} \frac{1}{i!} B\left(\frac{4}{\theta}+1,\ \mu+j\right). \tag{21}$$

The mean and variance (var) of the KMKu model are computed as

$$mean = \frac{1}{\theta} \sum_{i,j=0}^{\infty} \varsigma_{i,j} B\left(\frac{1}{\theta}+1,\ \mu+j\right),$$

and

$$Var(y) = \frac{1}{\theta} \sum_{i,j=0}^{\infty} \varsigma_{i,j} B\left(\frac{2}{\theta}+1,\ \mu(i+1)\right) - \left(\frac{1}{\theta} \sum_{i,j=0}^{\infty} \varsigma_{i,j} B\left(\frac{1}{\theta}+1,\ \mu+j\right)\right)^2.$$

The skewness (γ_1), kurtosis (γ_2) are $\gamma_1 = \frac{\mu_3^2}{\mu_2^3}$ and $\gamma_2 = \frac{\mu_4}{\mu_2^2}$ where $\mu_2 = \mu'_2 - (\mu'_1)^2$, $\mu_3 = \mu'_3 - 3\mu'_1\mu'_2 + 2(\mu'_1)^3$ and $\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4$.

By using Equation (12), Table 1 shows some numerical values of the first four moments μ'_1, μ'_2, μ'_3, μ'_4, the var, γ_1, γ_2 and coefficient of variation ($CV = \frac{\sqrt{var}}{\mu}$).

The pth incomplete moments for the KMKu model can be computed as

$$\eta_p(t) = \sum_{i,j=0}^{\infty} \varsigma_{i,j} \int_0^t y^{p+\theta-1} \left(1-y^{\theta}\right)^{\mu+j-1} dy.$$

Let $v = y^{\theta}$, then the pth incomplete moments for the KMKu model are

$$\eta_p(t) = \frac{1}{\theta} \sum_{i,j=0}^{\infty} \varsigma_{i,j} B_{t^{\theta}}\left(\frac{p}{\theta}+1,\ \mu+j\right),$$

where $B_y(.,.)$ is the incomplete B function.

Table 1. Some numerical values of moments for different values of μ and θ.

μ	θ	μ_1'	μ_2'	μ_3'	μ_4'	**var**	γ_1	γ_2	**CV**
0.6	0.6	0.4230	0.2870	0.2240	0.1870	0.1080	0.3180	1.7130	0.7760
	0.8	0.3480	0.2140	0.1560	0.1240	0.0920	0.6210	2.1050	0.8730
	1.0	0.2930	0.1640	0.1130	0.0860	0.0780	0.8650	2.5950	0.9540
	1.2	0.2510	0.1290	0.0840	0.0610	0.0660	1.0720	3.1370	1.0220
	1.4	0.2180	0.1030	0.0630	0.0440	0.0560	1.2520	3.7110	1.0810
	1.6	0.1920	0.0840	0.0490	0.0330	0.0470	1.4140	4.3040	1.1330
	1.8	0.1700	0.0690	0.0380	0.0240	0.0400	1.5600	4.9080	1.1790
	2.0	0.1520	0.0580	0.0300	0.0190	0.0350	1.6930	5.5200	1.2210
0.8	0.6	0.4880	0.3400	0.2680	0.2240	0.1020	0.0830	1.6660	0.6540
	0.8	0.4160	0.2640	0.1960	0.1560	0.0910	0.3530	1.8630	0.7270
	1.0	0.3610	0.2110	0.1470	0.1130	0.0810	0.5630	2.1490	0.7860
	1.2	0.3190	0.1720	0.1140	0.0840	0.0710	0.7350	2.4730	0.8350
	1.4	0.2840	0.1430	0.0900	0.0630	0.0620	0.8820	2.8140	0.8770
	1.6	0.2570	0.1210	0.0720	0.0490	0.0550	1.0090	3.1620	0.9130
	1.8	0.2330	0.1030	0.0590	0.0380	0.0480	1.1210	3.5100	0.9440
	2.0	0.2130	0.0890	0.0480	0.0300	0.0430	1.2210	3.8550	0.9710
1	0.6	0.5390	0.3840	0.3060	0.2570	0.0940	−0.0960	1.7240	0.5680
	0.8	0.4710	0.3090	0.2310	0.1860	0.0870	0.1550	1.7990	0.6270
	1.0	0.4180	0.2540	0.1800	0.1390	0.0790	0.3450	1.9650	0.6740
	1.2	0.3760	0.2130	0.1440	0.1060	0.0720	0.4970	2.1700	0.7120
	1.4	0.3420	0.1820	0.1170	0.0830	0.0650	0.6240	2.3910	0.7430
	1.6	0.3140	0.1570	0.0960	0.0660	0.0580	0.7320	2.6170	0.7700
	1.8	0.2900	0.1370	0.0800	0.0530	0.0530	0.8260	2.8430	0.7930
	2.0	0.2690	0.1200	0.0680	0.0430	0.0480	0.9090	3.0640	0.8140
1.2	0.6	0.5810	0.4230	0.3400	0.2870	0.0860	−0.2390	1.8310	0.5040
	0.8	0.5160	0.3480	0.2640	0.2140	0.0820	−0.0012	1.8190	0.5530
	1.0	0.4660	0.2930	0.2110	0.1640	0.0760	0.1760	1.9060	0.5920
	1.2	0.4250	0.2510	0.1720	0.1290	0.0700	0.3150	2.0360	0.6230
	1.4	0.3920	0.2180	0.1430	0.1030	0.0650	0.4300	2.1840	0.6480
	1.6	0.3640	0.1920	0.1210	0.0840	0.0590	0.5260	2.3390	0.6690
	1.8	0.3400	0.1700	0.1030	0.0690	0.0550	0.6090	2.4940	0.6880
	2.0	0.3190	0.1520	0.0890	0.0580	0.0500	0.6810	2.6470	0.7030
1.4	0.6	0.6160	0.4580	0.3700	0.3150	0.0780	−0.3590	1.9640	0.4540
	0.8	0.5550	0.3840	0.2940	0.2400	0.0760	−0.1290	1.8840	0.4960
	1.0	0.5070	0.3290	0.2400	0.1880	0.0720	0.0390	1.9140	0.5290
	1.2	0.4680	0.2860	0.2000	0.1510	0.0670	0.1700	1.9910	0.5550
	1.4	0.4350	0.2520	0.1690	0.1230	0.0630	0.2760	2.0900	0.5760
	1.6	0.4080	0.2250	0.1450	0.1020	0.0590	0.3640	2.1980	0.5940
	1.8	0.3840	0.2020	0.1260	0.0860	0.0550	0.4390	2.3080	0.6090
	2.0	0.3640	0.1830	0.1100	0.0730	0.0510	0.5040	2.4180	0.6220

4.3. Probability Weighted Moments

The (n,h)th probability weighted moments (PWMs) of Y following the KMKu model, symbolized as $\tau_{n,h}$, are formally computed from the next formula

$$\tau_{n,h} = \int_0^1 y^n\, f(y;\theta,\mu)\, F(y;\theta,\mu)^h dy. \tag{22}$$

By inserting the two Equations (11) and (14) into (22), then we obtain

$$\tau_{n,h} = \sum_{i,j=0}^{\infty}\sum_{k=0}^{h}\sum_{m,s=0}^{\infty} \Omega_{k,m,s}\varsigma_{i,j} \int_0^1 y^{n+\theta-1}\left(1-y^{\theta}\right)^{\mu(s+1)+j-1} dy.$$

Let $v = y^{\theta}$, then the PWMs of the KMKu model are

$$\tau_{n,h} = \frac{1}{\theta}\sum_{i,j=0}^{\infty}\sum_{k=0}^{h}\sum_{m,s=0}^{\infty} \Omega_{k,m,s}\varsigma_{i,j} B\left(\frac{n}{\theta}+1,\ \mu(s+1)+j\right). \tag{23}$$

4.4. Order Statistics

A sample of size n is taken from the KMKu model and, by the following definition, yielded the pdf of ith order statistics for the KMKu model

$$f_{i:n}(y;\theta,\mu) = \frac{1}{B(i,n-i+1)} f(y;\theta,\mu) \sum_{l=0}^{n-i}(-1)^l \binom{n-i}{l} F(y;\theta,\mu)^{i+l-1}. \tag{24}$$

By employing (11) and (14) into (24) and replacing h with $i+l-1$, then

$$f_{i:n}(y;\theta,\mu) = \sum_{l=0}^{n-i}\sum_{i,j=0}^{\infty}\sum_{k=0}^{i+l-1}\sum_{m,s=0}^{\infty} \Omega_{l,i,j,k,m,s} y^{\theta-1}\left(1-y^{\theta}\right)^{\mu(s+1)+j-1},$$

where $\Omega_{l,i,j,k,m,s} = \frac{(-1)^l}{B(i,n-i+1)}\binom{n-i}{l}\Omega_{k,m,s}\varsigma_{i,j}$. For more information see [20,21].

5. Different Types of Entropy

Entropy measurements are significant in investigations of dependability and risk. It has been employed in a variety of biological, medicinal, and physical applications.

5.1. Rényi Entropy

The essential shape of the distribution is measured using Rényi entropy (RE) [22], and it is provided by

$$R_{\lambda} = \frac{1}{1-\lambda}\log\left[\int_0^1 (f(y;\theta,\mu))^{\lambda}\, dy\right],\ \lambda > 0, \lambda \neq 1. \tag{25}$$

By employing (13) into (25), then

$$R_{\lambda} = \frac{1}{1-\lambda}\log\left[\sum_{i,j=0}^{\infty}\varpi_{i,j}\int_0^1 y^{\lambda(\theta-1)}\left(1-y^{\theta}\right)^{\lambda(\mu-1)+j} dy\right].$$

Let $v = y^{\theta}$, then

$$R_{\lambda} = \frac{1}{1-\lambda}\log\left[\frac{1}{\theta}\sum_{i,j=0}^{\infty}\varpi_{i,j}\int_0^1 v^{\lambda-\frac{\lambda+1}{\theta}-1}(1-v)^{\lambda(\mu-1)+j} dv\right].$$

The RE of the KMKu model is

$$R_\lambda = \frac{1}{1-\lambda}\log\left[\frac{1}{\theta}\sum_{i,j=0}^{\infty}\varpi_{i,j}B\left(\lambda-\frac{\lambda+1}{\theta},\lambda(\mu-1)+j+1\right)\right].$$

5.2. Havrda and Charvat Entropy

The Havrda and Charvat entropy (HCE) [23] measure is provided via

$$HC_\lambda = \frac{1}{2^{1-\lambda}-1}\left[\left(\int_0^1 (f(y;\theta,\mu))^\lambda\,dy\right)^{\frac{1}{\lambda}}-1\right],\ \lambda>0,\lambda\neq 1. \tag{26}$$

By employing (13) into (26), then

$$HC_\lambda = \frac{1}{2^{1-\lambda}-1}\left[\left(\frac{1}{\theta}\sum_{i,j=0}^{\infty}\varpi_{i,j}\int_0^1 y^{\lambda(\theta-1)}\left(1-y^\theta\right)^{\lambda(\mu-1)+j}dy\right)^{\frac{1}{\lambda}}-1\right].$$

After some computational, then the HCE of the KMKu model is

$$HC_\lambda = \frac{1}{2^{1-\lambda}-1}\left[\frac{1}{\theta}\sum_{i,j=0}^{\infty}\varpi_{i,j}B\left(\lambda-\frac{\lambda+1}{\theta},\lambda(\mu-1)+j+1\right)\right].$$

5.3. Tsallis Entropy

The Tsallis entropy (TE) [24] measure is provided by

$$T_\lambda = \frac{1}{\lambda-1}\left[1-\int_0^1 (f(y;\theta,\mu))^\lambda\,dy\right],\ \lambda>0,\lambda\neq 1. \tag{27}$$

By employing (13) into (27), then

$$T_\lambda = \frac{1}{\lambda-1}\left[1-\frac{1}{\theta}\sum_{i,j=0}^{\infty}\varpi_{i,j}\int_0^1 y^{\lambda(\theta-1)}\left(1-y^\theta\right)^{\lambda(\mu-1)+j}dy\right].$$

After some computational, then the TE of the KMKu model is

$$T_\lambda = \frac{1}{\lambda-1}\left[1-\frac{1}{\theta}\sum_{i,j=0}^{\infty}\varpi_{i,j}B\left(\lambda-\frac{\lambda+1}{\theta},\lambda(\mu-1)+j+1\right)\right].$$

5.4. Arimoto Entropy

The Arimoto entropy (AE) [25] measure is provided by

$$A_\lambda = \frac{\lambda}{1-\lambda}\left[\left(\int_0^1 (f(y;\theta,\mu))^\lambda\,dy\right)^{\frac{1}{\lambda}}-1\right],\ \lambda>0,\lambda\neq 1. \tag{28}$$

By employing (13) into (28), then

$$A_\lambda = \frac{\lambda}{1-\lambda}\left[\left(\frac{1}{\theta}\sum_{i,j=0}^{\infty}\varpi_{i,j}\int_0^1 y^{\lambda(\theta-1)}\left(1-y^\theta\right)^{\lambda(\mu-1)+j}dy\right)^{\frac{1}{\lambda}}-1\right].$$

After some computational, then the AE of the KMKu model is

$$A_\lambda = \frac{\lambda}{1-\lambda}\left[\left(\frac{1}{\theta}\sum_{i,j=0}^{\infty}\varpi_{i,j}B\left(\lambda - \frac{\lambda+1}{\theta}, \lambda(\mu-1)+j+1\right)\right)^{\frac{1}{\lambda}} - 1\right].$$

Table 2 shows some numerical values of the different types of entropy.

Table 2. Some numerical values of entropy at $\lambda = 0.5$ and $\lambda = 0.8$.

μ	θ	$\lambda = 0.5$				$\lambda = 0.8$			
		RE	**HCE**	**TE**	AE	**RE**	**HCE**	**TE**	**AE**
0.6	0.6	−0.0260	−0.1400	−0.0590	−0.0580	−0.0470	−0.1780	−0.1060	−0.1060
	0.8	−0.0420	−0.2240	−0.0950	−0.0930	−0.0760	−0.2870	−0.1720	−0.1710
	1.0	−0.0690	−0.3540	−0.1530	−0.1470	−0.1210	−0.4510	−0.2700	−0.2680
	1.2	−0.1000	−0.4960	−0.2170	−0.2050	−0.1700	−0.6290	−0.3780	−0.3740
	1.4	−0.1330	−0.6350	−0.2830	−0.2630	−0.2210	−0.8030	−0.4840	−0.4780
	1.6	−0.1660	−0.7650	−0.3470	−0.3170	−0.2700	−0.9690	−0.5850	−0.5770
	1.8	−0.1980	−0.8850	−0.4080	−0.3670	−0.3180	−1.1250	−0.6810	−0.6690
	2.0	−0.2310	−0.9950	−0.4660	−0.4120	−0.3630	−1.2700	−0.7710	−0.7550
0.8	0.6	−0.0091	−0.0500	−0.0210	−0.0210	−0.0160	−0.0620	−0.0370	−0.0370
	0.8	−0.0130	−0.0690	−0.0290	−0.0290	−0.0210	−0.0810	−0.0480	−0.0480
	1.0	−0.0270	−0.1480	−0.0620	−0.0610	−0.0450	−0.1730	−0.1030	−0.1030
	1.2	−0.0470	−0.2500	−0.1060	−0.1040	−0.0760	−0.2890	−0.1730	−0.1720
	1.4	−0.0700	−0.3590	−0.1550	−0.1490	−0.1100	−0.4130	−0.2470	−0.2450
	1.6	−0.0930	−0.4670	−0.2040	−0.1930	−0.1440	−0.5350	−0.3210	−0.3180
	1.8	−0.1170	−0.5710	−0.2520	−0.2360	−0.1770	−0.6520	−0.3920	−0.3880
	2.0	−0.1410	−0.6680	−0.2990	−0.2770	−0.2100	−0.7640	−0.4600	−0.4550
1	0.6	−0.0071	−0.0390	−0.0160	−0.0160	−0.0130	−0.0500	−0.0300	−0.0300
	0.8	−0.0020	−0.0110	−0.0045	−0.0045	−0.0032	−0.0120	−0.0073	−0.0073
	1.0	−0.0089	−0.0490	−0.0200	−0.0200	−0.0140	−0.0550	−0.0330	−0.0330
	1.2	−0.0220	−0.1180	−0.0490	−0.0490	−0.0340	−0.1290	−0.0770	−0.0770
	1.4	−0.0370	−0.1990	−0.0840	−0.0830	−0.0570	−0.2160	−0.1290	−0.1290
	1.6	−0.0550	−0.2850	−0.1220	−0.1180	−0.0810	−0.3070	−0.1830	−0.1830
	1.8	−0.0720	−0.3710	−0.1600	−0.1530	−0.1060	−0.3970	−0.2380	−0.2360
	2.0	−0.0900	−0.4530	−0.1980	−0.1880	−0.1300	−0.4850	−0.2900	−0.2880
1.2	0.6	−0.0130	−0.0710	−0.0300	−0.0290	−0.0220	−0.0860	−0.0520	−0.0510
	0.8	−0.0016	−0.0087	−0.0036	−0.0036	−0.0025	−0.0097	−0.0058	−0.0058
	1.0	−0.0029	−0.0160	−0.0066	−0.0066	−0.0045	−0.0180	−0.0100	−0.0100
	1.2	−0.0100	−0.0580	−0.0240	−0.0240	−0.0160	−0.0620	−0.0370	−0.0370
	1.4	−0.0210	−0.1160	−0.0490	−0.0480	−0.0320	−0.1230	−0.0730	−0.0730
	1.6	−0.0340	−0.1820	−0.0770	−0.0750	−0.0500	−0.1900	−0.1140	−0.1130
	1.8	−0.0480	−0.2510	−0.1070	−0.1040	−0.0680	−0.2600	−0.1550	−0.1550
	2.0	−0.0620	−0.3200	−0.1370	−0.1320	−0.0870	−0.3290	−0.1970	−0.1960

Table 2. *Cont.*

μ	θ	$\lambda = 0.5$				$\lambda = 0.8$			
		RE	**HCE**	**TE**	AE	**RE**	**HCE**	**TE**	**AE**
1.4	0.6	−0.0230	−0.1250	−0.0530	−0.0520	−0.0380	−0.1460	−0.0870	−0.0870
	0.8	−0.0071	−0.0390	−0.0160	−0.0160	−0.0110	−0.0410	−0.0250	−0.0250
	1.0	−0.0041	−0.0230	−0.0095	−0.0094	−0.0062	−0.0240	−0.0140	−0.0140
	1.2	−0.0078	−0.0430	−0.0180	−0.0180	−0.0120	−0.0460	−0.0270	−0.0270
	1.4	−0.0150	−0.0830	−0.0350	−0.0340	−0.0230	−0.0870	−0.0520	−0.0520
	1.6	−0.0250	−0.1330	−0.0560	−0.0550	−0.0360	−0.1370	−0.0820	−0.0820
	1.8	−0.0350	−0.1870	−0.0790	−0.0770	−0.0500	−0.1910	−0.1140	−0.1140
	2.0	−0.0460	−0.2430	−0.1030	−0.1010	−0.0650	−0.2460	−0.1470	−0.1460

6. Sampling Techniques

The most popular sampling methods are simple random sampling (SRS) and ranked set sampling (RSS). SRS is the most common method of data collection. In many applications (such as fisheries and medical research), where actual measurement of the variable of interest would be either time-consuming or expensive, ranking a number of sampling units without actually measuring them can be performed reasonably simply and affordably. To obtain more representative samples from the underlying population and boost the effectiveness of the statistical inference under these circumstances, rank-based sampling strategies may be used. McIntyre's initial suggestion of RSS is cited in references [26,27]. Numerous studies have shown that RSS-based statistical procedures are superior to their SRS scheme analogs, either numerically or theoretically. The initial ranking of n samples of size n for the one-cycle RSS looks like this:

$$\begin{array}{llll}
1. & \underline{Y_{1(1:n)s}}Y_{1(2:n)s} & \ldots, & Y_{1(n:n)s} \rightarrow Z_{1s} = Y_{1(1:n)s} \\
2. & Y_{2(1:n)s}\underline{Y_{2(2:n)s}} & \ldots & Y_{2(n:n)s} \rightarrow Z_{2s} = Y_{2(2:n)s} \\
 & \vdots & \ddots & \vdots \\
n. & Y_{n(1:n)s}Y_{n(2:n)s} & \ldots & \underline{Y_{n(n:n)s}} \rightarrow Z_{ns} = Y_n(n:n)s
\end{array}$$

where s cycles to produce a sample of size $N = ns$, and $Y_{j(i:n)s}$ denotes the ith order statistic from the jth SRS of size N. The resulting sample is called a one-cycle RSS and has the size $n = N$. It is represented by the symbol $\underline{Z} = (Z_{11}, Z_{21}, \ldots, Z_{n1})$. Z_{i1}. Under the premise of perfect judgement ranking, $Zi1$ has the same distribution as $Y_{i(i:n)s}$, the ith order statistic in a set of size n generated from the ith sample with pdf, see [28].

$$g_{(i)}(z) = \frac{n!}{(i-1)!(n-i)!}[F(z)]^{i-1}[1-F(z)]^{n-i}f(z), -\infty < z < \infty. \tag{29}$$

The cycle can be repeated s times until $N = sn$ units are quantified.

Following that, several extensions to the original RSS were proposed. Assessing the performance of some ranked set of sampling designs using a hybrid approach has been discussed by [29]; RSS with the application of modified Kies exponential distribution has been obtained by [30]; some properties and estimations under RSS of the generalized Bilal distribution has been introduced by [31]; an estimation of the exponential parameters of the Pareto distribution under ranked- and double-RSS designs has been obtained [32]; and Bayesian estimation using an MCMC method of system reliability for inverted Topp–Leone distribution based on RSS has been introduced by [33]. Contrarily, median RSS and MRSS [34] only consider the units that rank as the median for each set. While paired RSS [35] is a less expensive alternative that ranks fewer units, double RSS [36] is a more effective but also more expensive version of RSS that rates a higher number of sets in two ordering stages. For other probability distributions, see the recently proposed sampling technique in [37–44].

7. Maximum Likelihood Estimation

In this section, we obtain the parameter estimator of θ and μ based on SRS and RSS.

7.1. MLE Based on SRS

Here, using SRS, we first need to obtain the MLE of θ and μ. Let Y_i, $i = (1, \ldots, N)$ be an independent SRS drawn from $Y \sim KMKu(\theta, \mu)$. The likelihood function based on SRS is:

$$\begin{aligned} L(\theta, \mu) &= \prod_{i=1}^{N} f_Y(y_i) \\ &= \left(\frac{1}{e-1}\right)^N \theta^N \mu^N e^{\sum_{i=1}^{N}\left(1-y_i^\theta\right)^\mu} \prod_{i=1}^{N} y_i^{\theta-1}\left(1-y_i^\theta\right)^{\mu-1}. \end{aligned} \tag{30}$$

The ln-likelihood function of the observed samples is

$$\ell(\theta,\mu) = N\left[\ln\left(\frac{1}{e-1}\right) + \ln(\theta) + \ln(\mu)\right] + \sum_{i=1}^{N}\left(1-y_i^\theta\right)^\mu + (\theta-1)\sum_{i=1}^{N}\ln(y_i) + (\mu-1)\sum_{i=1}^{N}\ln\left(1-y_i^\theta\right). \tag{31}$$

The partial derivatives of (31) with respect to θ and μ to obtain the ln-likelihood equations are

$$\frac{\partial\ell(\theta,\mu)}{\partial\theta} = \frac{N}{\theta} - \mu\sum_{i=1}^{N} y_i^\theta \ln(y_i)\left(1-y_i^\theta\right)^{\mu-1} + \sum_{i=1}^{N}\ln(y_i) - (\mu-1)\sum_{i=1}^{N}\frac{y_i^\theta \ln(y_i)}{1-y_i^\theta}, \tag{32}$$

$$\frac{\partial\ell(\theta,\mu)}{\partial\mu} = \frac{N}{\mu} + \sum_{i=1}^{N}\left(1-y_i^\theta\right)^\mu \ln\left(1-y_i^\theta\right) + \sum_{i=1}^{N}\ln(1-y_i^\theta). \tag{33}$$

Two non-linear systems of equations that are differentiating (32) and (33) with respect to θ and μ, respectively, and equating each solution to zero must be solved concurrently in order to create MLE for SRS. One can use the R statistical programming language software to compute the desired MLEs θ and μ for any given data set by utilizing the "optim" function of the "stats" package, which employs the Nelder–Mead (NM) approach of maximization in the maximum likelihood computations.

7.2. MLE Based on RSS

Let $Y_{i(i:n_s)}, i = 1, \ldots, n_s, s = 1, \ldots, p$ be an s-cycle of RSS from the KMKu(θ, μ). We denote $Y_{i(i:n)s}$ by Y_{is} to simplify the notation. The function based on RSS by using Equation (29) is:

$$g_{i:n_s}(z_{is};\theta,\mu) = C_{i:n_s}F(z_{is};\theta,\mu)^{i-1}[1-F(z_{is};\theta,\mu)]^{n_s-i}f(z_{is};\theta,\mu), \tag{34}$$

where $C_{i:n_s} = \frac{n_s!}{(i-1)!(n_s-i)!}$. In view of (29), the likelihood function can be written as

$$\begin{aligned} L(\theta,\lambda;z_{is}) &= \prod_{s=1}^{p}\prod_{i=1}^{n_s} g_i(z_{is};\theta,\mu) \\ &= \prod_{s=1}^{p}\prod_{i=1}^{n_s} C_{i:n_s}\left(\frac{e}{e-1}\right)^{i-1}\left(1-e^{-1}e^{\left(1-z_{is}^\theta\right)^\mu}\right)^{i-1}\left\{1-\frac{e}{e-1}\left(1-e^{-1}e^{\left(1-z_{is}^\theta\right)^\mu}\right)\right\}^{n_s-i} \\ &\times \prod_{s=1}^{p}\prod_{i=1}^{n_s}\frac{1}{e-1}\theta\mu z_{is}^{\theta-1}\left(1-z_{is}^\theta\right)^{\mu-1} e^{\left(1-z_{is}^\theta\right)^\mu}. \end{aligned} \tag{35}$$

where $N = \sum_{s=1}^{p} n_s$ In this setting, the log-likelihood function of the KMKu distribution is given by

$$\begin{aligned}\ell(\theta,\mu;z_{is}) = &\sum_{s=1}^{p}\sum_{i=1}^{n_s}\ln(C_{i:n_s}) + \sum_{s=1}^{p}\sum_{i=1}^{n_s}(i-1)\ln\left(\frac{e}{e-1}\right) + \sum_{s=1}^{p}\sum_{i=1}^{n_s}(i-1)\ln\left(1-e^{-1}e^{\left(1-z_{is}^{\theta}\right)^{\mu}}\right)\\ &+\sum_{s=1}^{p}\sum_{i=1}^{n_s}(n_s-i)\ln\left\{1-\frac{e}{e-1}\left(1-e^{-1}e^{\left(1-z_{is}^{\theta}\right)^{\mu}}\right)\right\} + N\left[\ln\left(\frac{1}{e-1}\right)+log(\theta)+\ln(\mu)\right]\\ &+(\theta-1)\sum_{s=1}^{p}\sum_{i=1}^{n_s}\ln(z_{is}) + (\mu-1)\sum_{s=1}^{p}\sum_{i=1}^{n_s}\ln\left(1-z_{is}^{\theta}\right) + \sum_{s=1}^{p}\sum_{i=1}^{n_s}\left(1-z_{is}^{\theta}\right)^{\mu} \end{aligned} \tag{36}$$

The partial derivatives of $\ell(\theta,\mu;z_{is})$ associated with unknown parameters can be expressed as

$$\begin{aligned}\frac{\partial\ell(\theta,\mu;z_{is})}{\partial\theta} =&\mu\sum_{s=1}^{p}\sum_{i=1}^{n_s}(i-1)\frac{\ln(z_{is})z_{is}^{\theta}\left(1-z_{is}^{\theta}\right)^{\mu-1}e^{-1}e^{\left(1-z_{is}^{\theta}\right)^{\mu}}}{1-e^{-1}e^{\left(1-z_{is}^{\theta}\right)^{\mu}}} + \frac{N}{\theta} + \sum_{s=1}^{p}\sum_{i=1}^{n_s}\ln(z_{is})\\ &-\frac{e}{e-1}\mu\sum_{s=1}^{p}\sum_{i=1}^{n_s}(n_s-i)\frac{\ln(z_{is})z_{is}^{\theta}\left(1-z_{is}^{\theta}\right)^{\mu-1}e^{-1}e^{\left(1-z_{is}^{\theta}\right)^{\mu}}}{1-\frac{e}{e-1}\left(1-e^{-1}e^{\left(1-z_{is}^{\theta}\right)^{\mu}}\right)}\\ &-(\mu-1)\sum_{s=1}^{p}\sum_{i=1}^{n_s}\frac{\ln(z_{is})z_{is}^{\theta}}{1-z_{is}^{\theta}} - \mu\sum_{s=1}^{p}\sum_{i=1}^{n_s}\ln(z_{is})z_{is}^{\theta}\left(1-z_{is}^{\theta}\right)^{\mu-1}\end{aligned} \tag{37}$$

and

$$\begin{aligned}\frac{\partial l(\theta,\mu;x_{(ii)j})}{\partial\mu} =&-e^{-1}\sum_{s=1}^{p}\sum_{i=1}^{n_s}(i-1)\frac{\ln\left(1-z_{is}^{\theta}\right)\left(1-z_{is}^{\theta}\right)^{\mu}e^{\left(1-z_{is}^{\theta}\right)^{\mu}}}{1-e^{-1}e^{\left(1-z_{is}^{\theta}\right)^{\mu}}} + \frac{N}{\mu} + \sum_{s=1}^{p}\sum_{i=1}^{n_s}\ln\left(1-z_{is}^{\theta}\right)\\ &+\frac{1}{e-1}\sum_{s=1}^{p}\sum_{i=1}^{n_s}(n_s-i)\frac{\ln\left(1-z_{is}^{\theta}\right)\left(1-z_{is}^{\theta}\right)^{\mu}e^{\left(1-z_{is}^{\theta}\right)^{\mu}}}{1-\frac{e}{e-1}\left(1-e^{-1}e^{\left(1-z_{is}^{\theta}\right)^{\mu}}\right)}\\ &+\sum_{s=1}^{p}\sum_{i=1}^{n_s}\ln\left(1-z_{is}^{\theta}\right)\left(1-z_{is}^{\theta}\right)^{\mu}.\end{aligned} \tag{38}$$

By numerically solving the nonlinear equations $\frac{\partial\ell(\theta,\mu;z_{is})}{\partial\theta}=0$ and $\frac{\partial\ell(\theta,\mu;z_{is})}{\partial\mu}=0$ with respect to θ and μ, the MLEs can be obtained.

7.3. Asymptotic Confidence Interval

The Hessian matrix by the "optim" function yields the Fisher information matrix I of parameters θ and μ, which are the negative expectation of the last-second derivative of the log-likelihood function and θ and μ, respectively. The inverse Fisher information matrix describes the variance-covariance matrix. The asymptotic distribution of the MLE $(\hat{\theta},\hat{\mu})$ is also given as $n\rightarrow$ increases:

$$\begin{pmatrix}\hat{\theta}\\ \hat{\mu}\end{pmatrix}\sim\mathbb{N}\left[\begin{pmatrix}\theta\\ \mu\end{pmatrix},\begin{pmatrix}\hat{V}_{11} & \hat{V}_{12}\\ \hat{V}_{21} & \hat{V}_{22}\end{pmatrix}\right].$$

By inverting the Hessian matrix, the estimates $\hat{\theta}$ and $\hat{\mu}$ of theasymptotic variance-covariance matrix V are produced. The following equations provide two-sided $100(1-\alpha)\%$ confidence intervals (CI) for θ and μ:

$$Lower(\theta)=\hat{\theta}-Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{11}},\quad Upper(\theta)=\hat{\theta}+Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{11}},$$

and

$$Lower(\mu)=\hat{\mu}-Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{22}},\quad Upper(\mu)=\hat{\mu}+Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{22}},$$

respectively, where $Z_{\frac{\alpha}{2}}$ is the $\frac{\alpha}{2}$ percentile of the standard normal distribution.

8. Simulation

Some calculations are made in accordance with Monte Carlo simulation experiments using R packages with various combinations of sample sizes n and cycle size s as part of our rigorous effort to assess the effectiveness of the inference methods suggested in this paper. We generate a KMKu sample with the parameters (θ, μ): (0.2, 0.75), (0.2, 1.5), (0.2, 3), (0.5, 0.75), (0.5, 1.5), and (0.5, 3). It has been determined how well the resulting θ, and μ estimators perform in terms of their bias, corresponding mean squared error (MSE), relative efficiencies (REs) and coverage probability (CP) as follows:

$$Bias(\theta) = \frac{1}{N_I}\sum_{i=1}^{N_I}(\hat{\theta} - \theta),\ MSE(\mu) = \frac{1}{N_I}\sum_{i=1}^{N_I}(\hat{\mu} - \mu),$$

$$MSE(\theta) = \frac{1}{N_I}\sum_{i=1}^{N_I}(\hat{\theta} - \theta)^2,\ MSE(\mu) = \frac{1}{N_I}\sum_{i=1}^{N_I}(\hat{\mu} - \mu)^2,$$

$$RE1(\theta) = \frac{MSE_{SRS}(\theta)}{MSE_{RSS\ s=1}(\theta)},\ RE2(\theta) = \frac{MSE_{SRS}(\theta)}{MSE_{RSS\ s=3}(\theta)},\ RE3(\theta) = \frac{MSE_{RSS\ s=1}(\theta)}{MSE_{RSS\ s=3}(\theta)},$$

$$RE1(\mu) = \frac{MSE_{SRS}(\mu)}{MSE_{RSS\ s=1}(\mu)},\ RE2(\mu) = \frac{MSE_{SRS}(\mu)}{MSE_{RSS\ s=3}(\mu)},\ RE3(\mu) = \frac{MSE_{RSS\ s=1}(\mu)}{MSE_{RSS\ s=3}(\mu)}.$$

$$CP(\mu) = Mean\begin{cases} 1 & \text{if} \quad Lower(\mu) \le \hat{\mu} \le Upper(\mu) \\ 0 & \text{otherwise} \end{cases},$$

and

$$CP(\theta) = Mean\begin{cases} 1 & \text{if} \quad Lower(\theta) \le \hat{\theta} \le Upper(\theta) \\ 0 & \text{otherwise} \end{cases}.$$

Using a Monte Carlo simulation in R software (we used function "optim" in R-package "stats") with N =10,000 repeats for various set sizes, the number of cycles, and chosen parameter values, the performance of the estimations is compared. To generate a sample of RSS, we used function "rss" in R-package "RSSampling", which the "rss" function samples from a target population by using a ranked set sampling method.

Tables 3 and 4 present the results of the simulation investigation. As sample size increases, bias and MSE decrease. We see that bias and MSE values based on RSS are always less than those based on SRS. Additionally, as sample sizes are increased for all parameters, the MSE values based on the SRS and RSS techniques become lower. Figure 3 shows the MSE values of parameters based on SRS with different sample sizes and different parameter values, and this indicates that the MSE decreases when sample size increases. Figure 4 shows the heat map for MSE values of parameters based on RSS with different fixed values of sample sizes and parameters, and μ increases, the MSE for θ decreases, while the MSE for μ increases. In Figure 4, the X label is MSE for different methods of each parameter: CIP1SRS is MSE of SRS for θ when $\theta = 0.2$, CIP2SRS is MSE of SRS for μ when $\theta = 0.2$, etc., while the Y label is MSE with different sample sizes, where C1n7 is MSE when $\mu = 0.75$ and n = 7, C2n7 is MSE when $\mu = 1.5$ and n = 7, etc. The results of the simulation show that the RSS scheme performs better than the SRS method. We can also infer that the RSS technique is more effective than the SRS scheme for estimating the unknown parameters of the KMKu distribution. The results of RSS in simulation tables show that the RSS with a greater cycle, s = 3, is superior to the RSS with a single cycle.

Table 3. Bias, MSE and REs for parameters of KMKu with different cycles: $\theta = 0.2$ and 0.5.

			SRS		**RSS r = 1**		**RSS r = 3**				
θ, μ	**m**		**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**RE1**	**RE2**	**RE3**
0.2, 0.75	4	θ	0.1651	0.1474	0.0890	0.0346	0.0813	0.0296	426%	499%	117%
		μ	0.9047	3.1293	0.5787	1.3401	0.4815	1.0431	234%	300%	128%
	7	θ	0.0717	0.0349	0.0366	0.0087	0.0309	0.0084	403%	416%	103%
		μ	0.4008	0.8798	0.2021	0.2457	0.1974	0.2407	358%	366%	102%
	10	θ	0.0519	0.0147	0.0190	0.0036	0.0172	0.0033	408%	445%	109%
		μ	0.2775	0.3929	0.1120	0.0847	0.0962	0.0743	464%	529%	114%
	15	θ	0.0386	0.0093	0.0078	0.0015	0.0076	0.0013	634%	686%	108%
		μ	0.2331	0.3179	0.0512	0.0339	0.0511	0.0316	938%	1005%	107%
0.2, 1.5	4	θ	0.0712	0.0328	0.0340	0.0084	0.0305	0.0076	391%	432%	110%
		μ	0.8320	3.7420	0.6290	7.1765	0.3595	0.7037	52%	532%	1020%
	7	θ	0.0318	0.0093	0.0095	0.0023	0.0055	0.0021	408%	442%	109%
		μ	0.3885	0.8633	0.0996	0.1745	0.0900	0.1672	495%	516%	104%
	10	θ	0.0273	0.0060	0.0061	0.0013	0.0058	0.0011	474%	527%	111%
		μ	0.3359	0.6755	0.1135	0.1803	0.0940	0.1531	375%	441%	118%
	15	θ	0.0188	0.0040	0.0003	0.0004	0.0002	0.0003	1071%	1148%	107%
		μ	0.2567	0.4604	0.0175	0.0289	0.0131	0.0241	1595%	1910%	120%
0.2, 3	4	θ	0.0221	0.0077	0.0072	0.0022	0.0068	0.0020	354%	380%	107%
		μ	0.2639	0.8307	0.2024	0.8127	0.0940	0.3292	102%	252%	247%
	7	θ	0.0102	0.0028	0.0030	0.0006	0.0010	0.0006	469%	466%	99%
		μ	0.2096	1.3086	0.0326	0.0649	0.0269	0.0505	2016%	2591%	129%
	10	θ	0.0081	0.0019	0.0016	0.0005	0.0039	0.0005	393%	387%	99%
		μ	0.0895	0.4816	0.0851	0.4519	0.0818	0.3829	107%	126%	118%
	15	θ	0.0060	0.0014	−0.0003	0.0001	−0.0002	0.0001	1051%	1227%	117%
		μ	0.1332	0.5084	0.0139	0.0326	0.0077	0.0291	1558%	1746%	112%
0.5, 0.75	4	θ	0.4500	0.9750	0.2569	0.2633	0.2410	0.2403	370%	406%	110%
		μ	1.3133	6.3410	0.8140	2.7293	0.7887	2.6856	232%	236%	102%
	7	θ	0.1936	0.2332	0.0982	0.0600	0.0845	0.0587	389%	397%	102%
		μ	0.5368	1.7774	0.2345	0.3502	0.2327	0.3494	507%	509%	100%
	10	θ	0.1380	0.0989	0.0533	0.0264	0.0576	0.0261	375%	379%	101%
		μ	0.3433	0.7597	0.1324	0.1237	0.1111	0.0991	614%	767%	125%
	15	θ	0.0996	0.0604	0.0199	0.0094	0.0215	0.0090	644%	668%	104%
		μ	0.2526	0.3891	0.0527	0.0364	0.0523	0.0328	1067%	1185%	111%
0.5, 1.5	4	θ	0.2394	0.2725	0.1415	0.0892	0.1333	0.0847	306%	322%	105%
		μ	1.6074	7.5157	1.1325	4.4234	0.9699	3.9644	170%	190%	112%
	7	θ	0.1170	0.0834	0.0564	0.0261	0.0440	0.0244	319%	341%	107%
		μ	0.8210	2.8512	0.3655	0.7510	0.3359	0.6951	380%	410%	108%
	10	θ	0.0980	0.0532	0.0356	0.0145	0.0341	0.0142	367%	375%	102%
		μ	0.6881	2.4032	0.2709	0.4835	0.2478	0.4554	497%	528%	106%
	15	θ	0.0733	0.0360	0.0114	0.0056	0.0119	0.0056	643%	642%	100%
		μ	0.5825	3.0070	0.0978	0.1478	0.0937	0.1348	2034%	2231%	110%

Table 3. *Cont.*

			SRS		RSS r = 1		RSS r = 3				
θ, μ	m		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2	RE3
0.5, 3	4	θ	0.0923	0.0755	0.0420	0.0231	0.0424	0.0224	327%	337%	103%
		μ	1.1344	5.2244	0.7868	4.8975	0.6287	2.6572	107%	197%	184%
	7	θ	0.0478	0.0290	0.0101	0.0069	0.0034	0.0067	419%	432%	103%
		μ	0.7166	2.9233	0.1408	0.5447	0.1252	0.5316	537%	550%	102%
	10	θ	0.0461	0.0216	0.0108	0.0051	0.0106	0.0051	422%	425%	101%
		μ	0.7557	3.3001	0.2461	0.8321	0.2375	0.7869	877%	928%	106%
	15	θ	0.0255	0.0142	−0.0067	0.0013	0.0010	0.0012	1102%	1153%	105%
		μ	0.5203	2.6514	0.0383	0.1276	0.0345	0.1237	11482%	11847%	103%

Table 4. Confidence intervals and CP: $\theta = 0.2$ and 0.5.

			SRS			RSS r = 1			RSS r = 3		
θ, μ	m		Lower	Upper	CV	Lower	Upper	CV	Lower	Upper	CV
0.2, 0.75	4	θ	0.002	0.772	97.0%	0.026	0.487	96.3%	0.014	0.504	95.0%
		μ	0.106	3.388	93.3%	0.191	2.388	94.7%	0.233	2.626	94.3%
	7	θ	0.012	0.568	95.0%	0.077	0.396	96.0%	0.058	0.389	96.0%
		μ	0.165	3.126	97.0%	0.029	1.920	97.0%	3.029	1.870	95.3%
	10	θ	0.030	0.474	95.7%	0.108	0.334	93.3%	0.113	0.327	97.3%
		μ	0.030	2.093	95.3%	0.362	1.375	94.3%	0.350	1.372	95.0%
	15	θ	0.082	0.392	96.7%	0.134	0.286	96.0%	0.135	0.279	95.7%
		μ	0.126	1.768	96.3%	0.468	1.144	95.7%	0.472	1.119	94.7%
0.2, 1.5	4	θ	0.028	0.557	96.0%	0.070	0.397	96.0%	0.066	0.395	94.0%
		μ	0.339	4.062	95.3%	0.377	4.050	98.3%	0.305	3.203	96.3%
	7	θ	0.045	0.450	95.7%	0.114	0.316	94.7%	0.097	0.316	95.0%
		μ	0.002	3.387	97.7%	0.483	2.248	96.7%	0.469	2.131	96.7%
	10	θ	0.067	0.407	96.0%	0.128	0.296	93.7%	0.133	0.288	95.7%
		μ	0.046	3.159	94.7%	0.639	2.039	95.7%	0.659	1.983	97.0%
	15	θ	0.099	0.356	96.3%	0.151	0.257	95.7%	0.150	0.254	95.3%
		μ	0.338	2.623	94.7%	0.867	1.619	95.3%	0.835	1.636	96.0%
0.2, 3	4	θ	0.084	0.344	95.7%	0.127	0.282	94.7%	0.124	0.281	95.0%
		μ	1.613	4.815	95.0%	1.632	4.560	95.7%	1.718	3.399	99.0%
	7	θ	0.112	0.312	95.3%	0.158	0.246	94.7%	0.153	0.248	95.7%
		μ	1.395	4.209	95.7%	2.581	3.477	95.3%	2.486	3.508	96.0%
	10	θ	0.127	0.289	96.3%	0.162	0.240	95.0%	0.162	0.244	96.7%
		μ	1.768	4.049	94.3%	2.408	3.470	94.7%	2.106	3.122	95.3%
	15	θ	0.137	0.277	95.0%	0.173	0.228	96.7%	0.174	0.227	96.7%
		μ	2.232	3.964	95.0%	2.553	3.250	97.0%	2.552	3.053	96.0%

Table 4. *Cont.*

			SRS			RSS r = 1			RSS r = 3		
θ, μ	m		Lower	Upper	CV	Lower	Upper	CV	Lower	Upper	CV
0.5, 0.75	4	θ	0.0395	1.9725	0.9399	0.0310	1.2898	0.9533	0.0219	1.3072	0.9400
		μ	0.2752	4.4734	0.9333	0.1702	3.1423	0.9500	0.2963	3.1355	0.9433
	7	θ	0.0520	1.4757	0.9500	0.1898	1.0052	0.9600	0.1261	1.0117	0.9600
		μ	0.2907	3.9506	0.9567	0.0536	2.0574	0.9633	0.3216	2.2705	0.9700
	10	θ	0.0681	1.2080	0.9467	0.2446	0.8759	0.9400	0.2751	0.8347	0.9600
		μ	0.0534	2.5092	0.9667	0.2412	1.5481	0.9567	0.2790	1.4747	0.9567
	15	θ	0.2056	0.9832	0.9633	0.3339	0.7175	0.9600	0.3365	0.6988	0.9567
		μ	0.1256	1.7818	0.9567	0.4521	1.1662	0.9633	0.4691	1.1243	0.9467
0.5, 1.5	4	θ	0.0105	1.3514	0.9600	0.1852	1.0026	0.9600	0.1675	1.0249	0.9567
		μ	0.3629	6.5529	0.9600	0.4315	4.9020	0.9400	0.2164	5.2904	0.9500
	7	θ	0.1532	1.1011	0.9500	0.2781	0.8357	0.9533	0.2407	0.8216	0.9667
		μ	0.3760	5.2429	0.9567	0.2514	3.5801	0.9667	0.2006	3.3864	0.9533
	10	θ	0.1800	1.0181	0.9600	0.3095	0.7711	0.9367	0.3212	0.7557	0.9633
		μ	0.4251	4.8780	0.9400	0.5150	3.0744	0.9400	0.5552	2.9837	0.9633
	15	θ	0.2595	0.8825	0.9667	0.3661	0.6651	0.9500	0.3695	0.6479	0.9567
		μ	0.1953	3.7385	0.9533	0.8980	2.3224	0.9567	0.9243	2.2329	0.9633
0.5, 3	4	θ	0.1642	0.9694	0.9500	0.2860	0.7555	0.9433	0.2767	0.7550	0.9367
		μ	0.6024	8.7044	0.9733	1.1364	5.5158	0.9600	0.7555	5.8615	0.9867
	7	θ	0.2484	0.8595	0.9567	0.3547	0.6613	0.9467	0.3280	0.6623	0.9433
		μ	0.7034	7.0543	0.9733	1.6482	4.6778	0.9500	1.4562	4.6331	0.9600
	10	θ	0.2761	0.8144	0.9600	0.3773	0.6553	0.9567	0.3776	0.6505	0.9500
		μ	1.2968	6.0141	0.9900	1.6491	4.0989	0.9400	1.6483	4.8747	0.9467
	15	θ	0.3068	0.7480	0.9533	0.4253	0.5784	0.9533	0.4256	0.5736	0.9467
		μ	0.6290	5.1354	0.9867	2.3825	3.7132	0.9600	2.3523	3.7115	0.9633

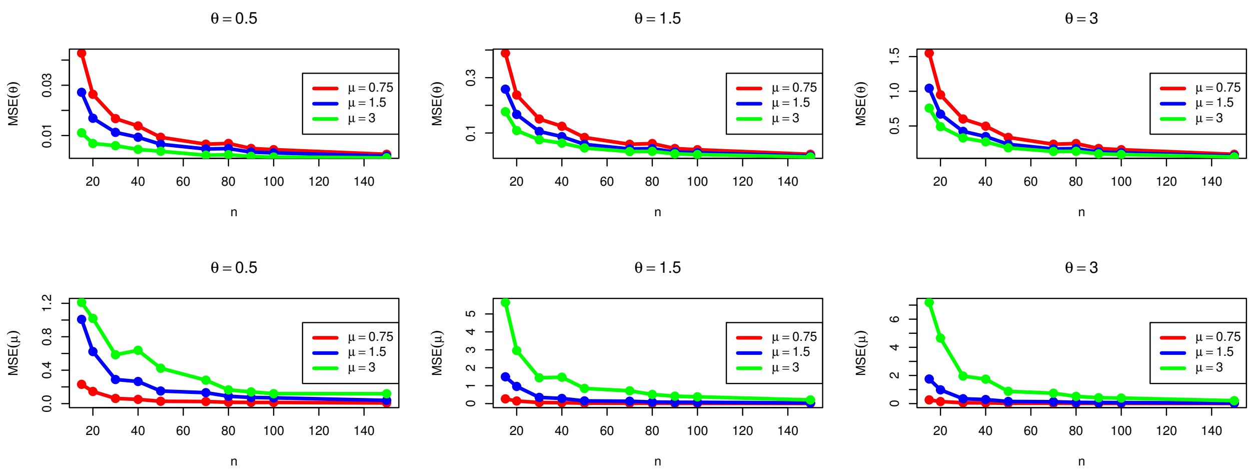

Figure 3. MSE of parameters based on SRS with different sample sizes.

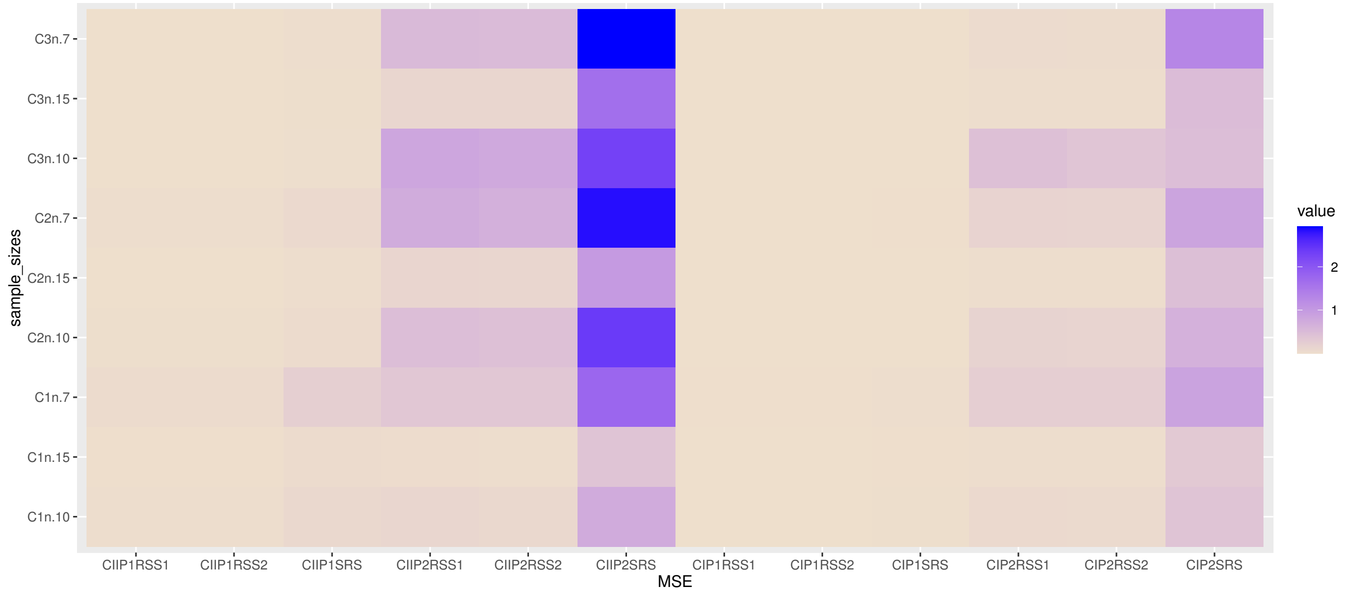

Figure 4. Heat-map of MSE for parameters based on RSS with different sample sizes.

9. Application

This section compares the KMKu distribution with well-known unit distributions in the literature by analyzing three data sets, one of which is connected to coronavirus data and the others to burr measurements on iron sheets. These models for comparison are (together with their pdfs for $(0 \leq y \leq 1)$): unit Weibull (UW), which was discussed by [45], Ku, B; Marshall–Olkin Ku (MOKu), which introduced by [8]; unit-Gompertz (UG), which was obtained by [46]; Marshall–Olkin-extended Topp–Leone (MOTL), which was discussed by [47]; and unit-exponentiated half logistic (UEHL), which was introduced by [48].

For all statistical models, we also calculate the goodness-of-fit statistics for the estimated different measures such as “Akaike Information Criteria (AIC), corrected AIC (CAIC), Bayesian information criterion (BIC), Hannan–Quinn information criterion (HQIC), Kolmogorov–Smirnov distance (KSS), p-value (PVKS), Cramer-von-Mises (WS), and Anderson–Darling (AS)”. The model with the smaller AIC, CAIC, BIC, HQIC, KSS, WS, and AS statistics and the higher PVKS of the goodness-of-fit statistics is typically considered to be the best one. It should be noted that the MLE approach was used to achieve all results.

We used SRS and RSS techniques with various set sizes and cycle counts to observe random samples of various sizes for analysis. We then computed MLEs of the parameters for the observed SRS and RSS with various cycle counts, and we compared the performance of the estimates.

Firstly: The COVID-19 data in question are mortality rates from the United Kingdom and span 82 days, from May 1 to July 16, 2021, as follows: 0.0023, 0.0023, 0.0023, 0.0046, 0.0065, 0.0067, 0.0069, 0.0069, 0.0091, 0.0093, 0.0093, 0.0093, 0.0111, 0.0115, 0.0116, 0.0116, 0.0119, 0.0133, 0.0136, 0.0138, 0.0138, 0.0159, 0.0161, 0.0162, 0.0162, 0.0162, 0.0163, 0.0180, 0.0187, 0.0202, 0.0207, 0.0208, 0.0225, 0.0230, 0.0230, 0.0239, 0.0245, 0.0251, 0.0255, 0.0255, 0.0271, 0.0275, 0.0295, 0.0297, 0.0300, 0.0302, 0.0312, 0.0314, 0.0326, 0.0346, 0.0349, 0.0350, 0.0355, 0.0379, 0.0384, 0.0394, 0.0394, 0.0412, 0.0419, 0.0425, 0.0461, 0.0464, 0.0468, 0.0471, 0.0495, 0.0501, 0.0521, 0.0571, 0.0588, 0.0597, 0.0628, 0.0679, 0.0685, 0.0715, 0.0766, 0.0780, 0.0942, 0.0960, 0.0988, 0.1223, 0.1343, 0.1781. This data have been cited in https://covid19.who.int/ (accesse on 1 February 2023). Table 5 discusses the MLE for different models with different measures of goodness-of-fit for COVID-19 data of the United Kingdom. Figure 5 shows estimated cdf with empirical cdf, estimated pdf with histogram probability of COVID-19 data of the United Kingdom and a P-P plot for KMKu. Figure 6 was obtained to check whether the estimators are maximum or not for parameters of the KMKu distribution based on COVID-19 data of the United Kingdom.

Table 5. MLE with different measures of goodness-of-fit: COVID-19 data of the United Kingdom.

Models		Estimates	SE	KSS	PVKS	AIC	BIC	CAIC	HQIC	WS	AS
KMKu	θ	1.3680	0.1123	0.0531	0.9749	−386.4656	−381.6522	−386.3137	−384.5331	0.0385	0.2875
	μ	62.0471	22.1058								
UW	α	0.0024	0.0003	0.0737	0.7639	−381.6038	−376.7903	−381.4519	−379.6713	0.0988	0.7080
	θ	4.3158	0.1099								
Ku	α	1.2399	0.1055	0.0597	0.9322	−384.6698	−379.8564	−384.5179	−382.7373	0.0601	0.4228
	θ	55.7476	18.3042								
UG	α	0.0181	0.0071	0.1082	0.2924	−363.6988	−358.8853	−363.5469	−361.7662	0.2941	1.9827
	θ	0.9759	0.0803								
UEHL	α	1.2515	0.1030	0.0574	0.9496	−385.0542	−380.2408	−384.9023	−383.1217	0.0555	0.3931
	θ	29.3838	9.3197								

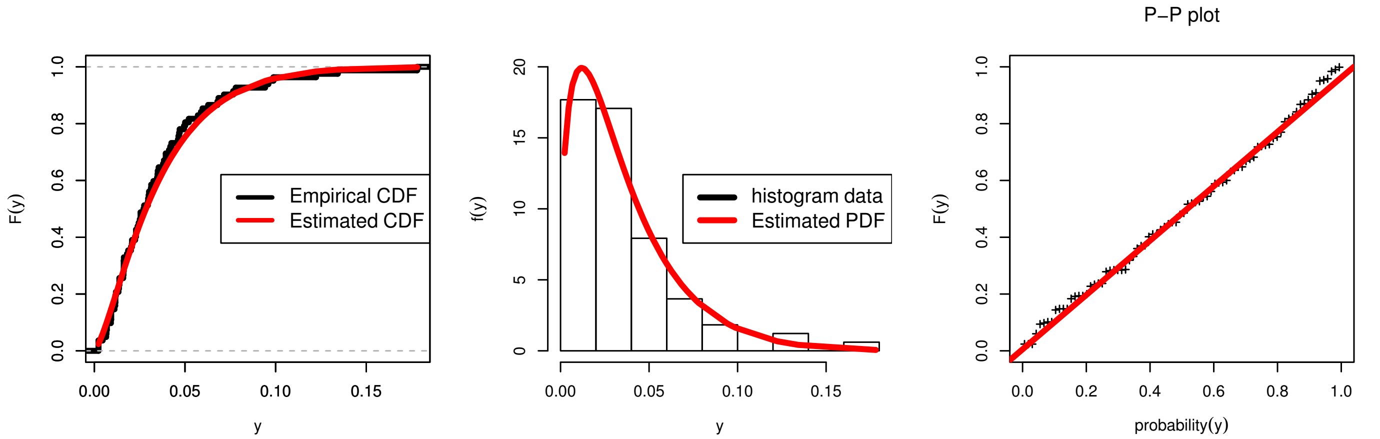

Figure 5. Estimated cdf, pdf and P-P plot for KMKu: COVID-19 data of the United Kingdom.

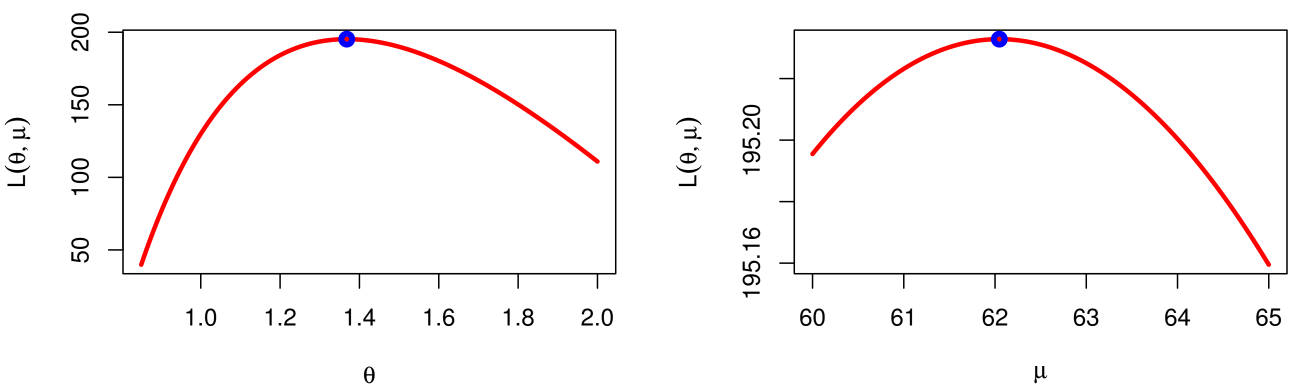

Figure 6. Profile MLE for KMKu parameters: COVID-19 data of the United Kingdom.

In the first set of data, we obtained the RSS data for when N = 50 with one cycle, as shown in Table 6, and we obtained the RSS data for when the size was n = 5, and the cycle was $s = 10$, as shown in Table 7. According to these data, the MLE based on SRS and RSS with the different cycles for COVID-19 data of the United Kingdom when n = 50 is shown in Table 8.

Table 6. RSS data with one cycle: three data sets.

Data	Observation									
1	i = 1	i = 2	i = 3	i = 4	i = 5	i = 6	i = 7	i = 8	i = 9	i = 10
	0.0023	0.0065	0.0069	0.0067	0.0069	0.0093	0.0162	0.0093	0.0116	0.0116
	i = 11	i = 12	i = 13	i = 14	i = 15	i = 16	i = 17	i = 18	i = 19	i = 20
	0.0159	0.0162	0.0161	0.0115	0.0162	0.0161	0.0187	0.0162	0.0239	0.023
	i = 21	i = 22	i = 23	i = 24	i = 25	i = 26	i = 27	i = 28	i = 29	i = 30
	0.0207	0.0138	0.0312	0.03	0.03	0.0255	0.023	0.0346	0.0297	0.0314
	i = 31	i = 32	i = 33	i = 34	i = 35	i = 36	i = 37	i = 38	i = 39	i = 40
	0.0346	0.0312	0.0394	0.0379	0.0588	0.0468	0.0468	0.0419	0.0379	0.0521
	i = 41	i = 42	i = 43	i = 44	i = 45	i = 46	i = 47	i = 48	i = 49	i = 50
	0.0425	0.0471	0.0588	0.0715	0.0679	0.0942	0.0628	0.0766	0.1343	0.1343
2	i = 1	i = 2	i = 3	i = 4	i = 5	i = 6	i = 7	i = 8	i = 9	i = 10
	0.01133	0.01796	0.02288	0.04829	0.02288	0.0385	0.0439	0.0507	0.0605	0.0605
	i = 11	i = 12	i = 13	i = 14	i = 15					
	0.0955	0.0818	0.1476	0.1099	0.1476					
3	i = 1	i = 2	i = 3	i = 4	i = 5	i = 6	i = 7	i = 8	i = 9	i = 10
	0.032	0.023	0.032	0.188	0.169	0.105	0.216	0.361	0.361	0.463
	i = 11	i = 12	i = 13	i = 14	i = 15					
	0.642	0.674	0.823	0.823	0.926					

Table 7. RSS data with different cycles: three data sets.

Data	Cycle	Observation				
		i = 1	**i = 2**	**i = 3**	**i = 4**	**i = 5**
1	s = 1	0.0093	0.0093	0.0464	0.0384	0.096
	s = 2	0.0065	0.0202	0.0225	0.0715	0.1223
	s = 2	0.0093	0.0093	0.0355	0.0255	0.096
	s = 3	0.0115	0.0187	0.0116	0.0464	0.1781
	s = 3	0.0115	0.0384	0.0187	0.0715	0.0942
	s = 4	0.0116	0.0115	0.0255	0.0314	0.1781
	s = 4	0.0115	0.0314	0.0202	0.0501	0.0942
	s = 5	0.0093	0.0161	0.0255	0.0384	0.0715
	s = 5	0.0161	0.0355	0.0255	0.0115	0.0501
	s = 6	0.0065	0.0115	0.0255	0.0501	0.1223
2		i = 1	i = 2	i = 3	i = 4	i = 5
	s = 1	0.0180	0.0385	0.0095	0.0231	0.1099
	s = 2	0.0074	0.0180	0.0180	0.0515	0.1099
	s = 3	0.0074	0.0212	0.0212	0.0385	0.1388
3		i = 1	i = 2	i = 3	i = 4	i = 5
	s = 1	0.032	0.105	0.463	0.169	0.752
	s = 2	0.023	0.395	0.169	0.311	0.823
	s = 3	0.023	0.032	0.127	0.255	0.823

Table 8. SRS and RSS with different cycles for COVID-19 data of the United Kingdom when n = 50.

		Estimates	**SE**	**Lower**	**Upper**
SRS	θ	1.3751	0.0201	1.3357	1.4146
	μ	56.2365	27.5437	2.2509	110.2221
RSS s = 1	θ	1.4676	0.0029	1.4619	1.4733
	μ	91.3018	13.2753	65.2823	117.3213
RSS s = 10	θ	1.2903	0.0100	1.2706	1.3100
	μ	43.7122	9.5155	25.0618	62.3627

The second data set relates to the newly discovered coronavirus epidemic in Turkey. This dataset has been cited in https://covid19.who.int/ (accesse on 1 February 2023). A total of 25 observations make up this data set, which covers the period from 27 March to 20 April and was calculated as the daily ratio of recoveries to confirm cases in Turkey. This indicates the daily proportion of people making a full recovery in all circumstances. The information was provided in data set II as follows: 0.0074 0.0095 0.0113 0.0150 0.0180 0.0212 0.0229 0.0231 0.0328 0.0385 0.0439 0.0464 0.0483 0.0507 0.0515 0.0568 0.0605 0.0648 0.0737 0.0818 0.0955 0.1099 0.1270 0.1388 0.1476. Table 9 presents the MLE for different models with different measures of goodness-of-fit for the COVID-19 data of Turkey. Figure 7 shows the estimated cdf with empirical cdf, the estimated pdf with histogram probability of COVID-19 data of Turkey, and the P–P plot for KMKu. Figure 8 was produced to check whether the estimators are maximums or not for the parameters of KMKu distribution based on COVID-19 data of Turkey.

Table 9. MLE with different measures of goodness of fit: COVID-19 data of Turkey.

Models		Estimates	SE	KSS	PVKS	AIC	BIC	CAIC	HQIC	WS	AS
KMKu	θ	1.5517	0.2452	0.1025	0.9315	−94.8552	−92.4174	−94.3097	−94.1791	0.0261	0.1989
	μ	55.3232	37.2191								
UW	α	0.0054	0.0031	0.1362	0.6923	−93.1603	−90.7226	−92.6149	−92.4842	0.0652	0.3860
	θ	4.1597	0.4182								
Ku	α	1.4164	0.2303	0.1022	0.9329	−94.6877	−92.2499	−94.1422	−94.0115	0.0278	0.2151
	θ	50.9406	31.3225								
B	α	1.7485	0.4553	0.1031	0.9284	−94.8924	−92.4547	−94.3470	−94.2163	0.0258	0.1944
	β	29.5605	8.8186								
MOKu	α	0.4536	0.6246	0.1047	0.9205	−92.9571	−89.3005	−91.8143	−91.9429	0.0262	0.1994
	β	1.6814	0.4880								
	θ	66.1138	51.4394								
UG	α	0.0167	0.0125	0.1601	0.4936	−89.1731	−86.7354	−88.6277	−88.4970	0.1241	0.7255
	β	1.1446	0.1796								
MOETL	α	0.0062	0.0046	0.1058	0.9147	−92.2698	−89.8320	−91.7243	−91.5936	0.0559	0.3477
	β	2.0660	0.2976								
UEHL	α	1.4306	0.2250	0.1026	0.9311	−94.6713	−92.2336	−94.1259	−93.9952	0.0274	0.2127
	β	26.8822	15.9639								

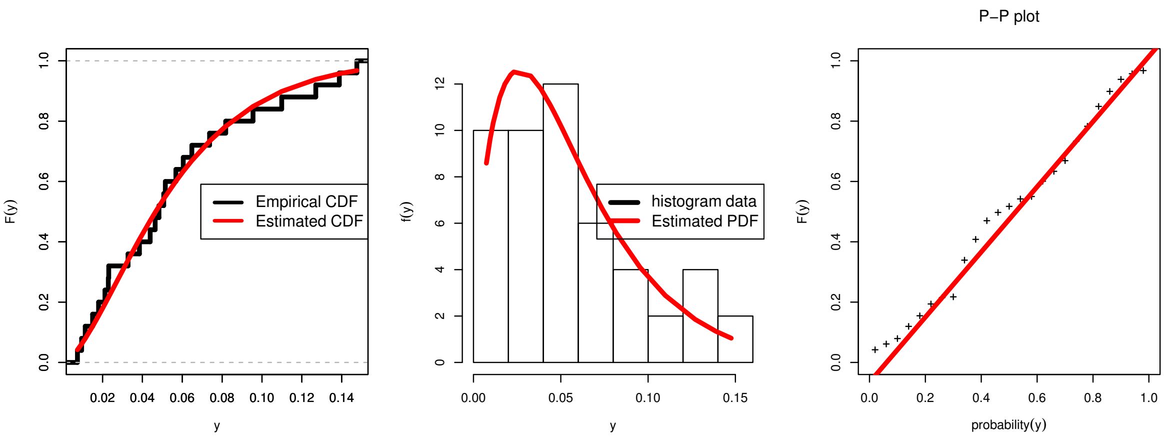

Figure 7. Estimated cdf, pdf and P-P plot for KMKu: COVID-19 data of Turkey.

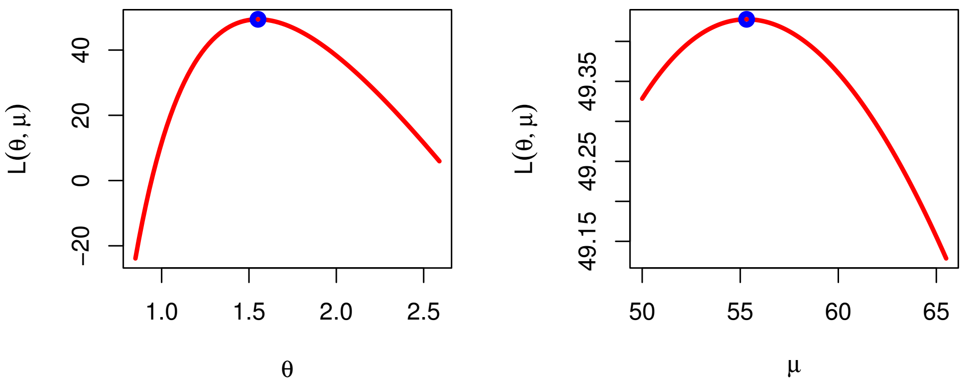

Figure 8. Profile MLE for KMKu parameters: COVID-19 data of Turkey.

In the second data set, we obtained the RSS data for when N = 15 with one cycle, as shown in Table 6, and we obtained the RSS data for when the size is n = 5 and the cycle is $s = 3$, as shown in Table 7. According to these data, the MLE based on SRS and RSS with the different cycles for COVID-19 data of Turkey when n = 15 is shown in Table 10.

Table 10. SRS and RSS with different cycles for COVID-19 data of Turkey.

		Estimates	SE	Lower	Upper
SRS	θ	1.6497	0.1006	1.4525	1.8468
	μ	111.3630	47.0753	19.0954	203.6306
RSS s = 1	θ	1.6652	0.0322	1.6022	1.7282
	μ	59.8835	21.3712	17.9959	101.7711
RSS s = 3	θ	1.2537	0.0313	1.1898	1.3176
	μ	39.7230	19.7096	1.0922	78.3539

The third data set included 30 measurements of polyester fibers' tensile strength, which has been discussed by [46]. The information is provided in data set III as follows: "0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.105, 0.127, 0.148, 0.169, 0.188, 0.216, 0.255, 0.277, 0.311, 0.361, 0.376, 0.395, 0.432, 0.463, 0.481, 0.519, 0.529, 0.567, 0.642, 0.674, 0.752, 0.823, 0.887, 0.926". Table 11 discusses MLE for different models with different measures of goodness-of-fit for data on the strength of polyester fibers. Figure 9 shows estimates cdf with empirical cdf, the pdf with histogram probability of data on the strength of polyester fibers, and the P–P plot for KMKu. Figure 10 was produced to check whether the estimators are maximums or not for parameters of KMKu distribution based on data on the strength of polyester fibers.

Table 11. MLE with different measures of goodness-of-fit: data III.

Models		Estimates	SE	KSS	PVKS	AIC	BIC	CAIC	HQIC	WS	AS
KMKu	θ	1.0826	0.2138	0.0569	0.9999	−3.0417	−0.2393	−2.5973	−2.1452	0.0151	0.1215
	μ	1.3797	0.3935								
Ku	α	0.9627	0.2017	0.0650	0.9987	−2.6221	0.1803	−2.1776	−1.7256	0.0183	0.1551
	θ	1.6084	0.4137								
B	α	0.9667	0.2238	0.0669	0.9979	−2.6101	0.1923	−2.1657	−1.7136	0.0184	0.1559
	β	1.6205	0.4107								
MOKu	α	0.4365	0.4732	0.0628	0.9992	−1.2087	2.9949	−0.2856	0.1361	0.0152	0.1224
	β	1.1872	0.3472								
	θ	1.2585	0.6458								
MOETL	α	1.0929	0.7021	0.0672	0.9978	−1.8272	0.9752	−1.3828	−0.9307	0.0203	0.1710
	β	1.0628	0.3883								

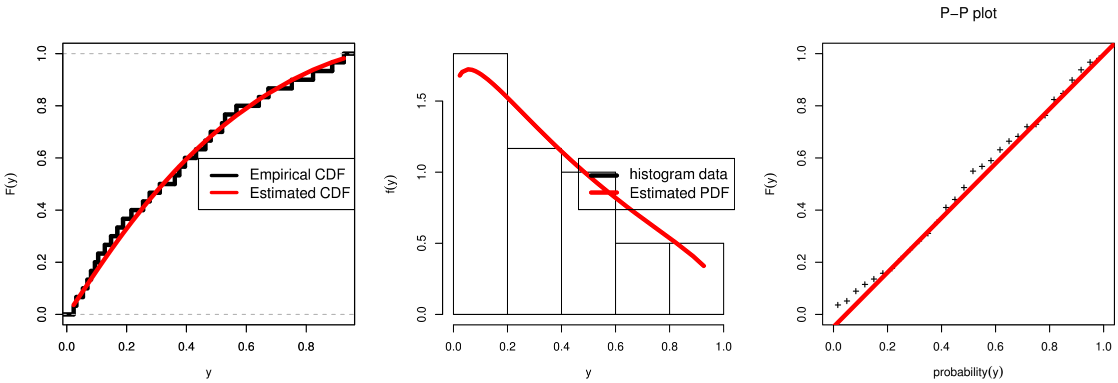

Figure 9. Estimated cdf, pdf and P–P plot for KMKu: data III.

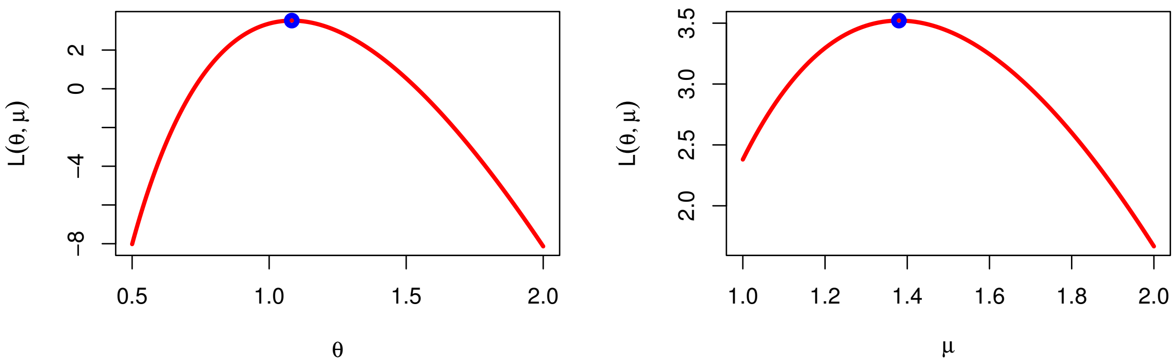

Figure 10. Profile MLE for KMKu parameters: data III.

In the third data set, we obtained the RSS data for when N = 15 with one cycle, as shown in Table 6, and we obtained the RSS data for when the size is n = 5 and the cycle is $s = 3$, as shown in Table 7. By these data, the MLE based on SRS and RSS with the different cycles for data on the strength of polyester fibers when n = 15 is shown in Table 12.

Table 12. SRS and RSS with different cycles for data set III.

		Estimates	SE	Lower	Upper
SRS	θ	0.8111	0.0552	0.7030	0.9192
	μ	1.1550	0.2076	0.7481	1.5619
RSS s = 1	θ	1.1444	0.0279	1.0897	1.1992
	μ	1.6753	0.1506	1.3801	1.9705
RSS s = 2	θ	0.8179	0.0288	0.7614	0.8744
	μ	1.4021	0.1488	1.0333	1.7709

By delivering the lowest AIC, BIC, CAIC, HQIC WS, and AS values in Tables 5, 9 and 11, the model fitted to KMKu is clearly outperforming other competitive models, such as UW, UG, Ku, B, UEHL, MOETL, and MOKu. This indicates that KMKu-based models provide more accurate and fitting information about these three real data sets. Figures 6, 8 and 10 show the parameters of the KMKu model have a maximum log-likelihood with the other parameters fixed. Tables 8, 10 and 12 make it evident that the model fitted using the RSS design is outperforming the other competing models by offering the lowest SE, indicating that RSS-based models are more accurate at representing the real model utilized in these numerical examples.

10. Concluding Remarks

In this article, we proposed and studied a new extension of the Ku model, which is called the KMKu model. The shape forms of the pdf for the new KMKu model for various values of parameters are is similar to the Ku model in that it is asymmetric. Some general statistical and computational features of the KMKu model, such as the qf, raw and incomplete moments, probability-weighted moments, and order statistics, were calculated. Four different measures of entropy were discussed. The maximum likelihood approach was employed to estimate the parameters for the KMKu model under simple and ranked set sampling. A simulation experiment is conducted to show that ranked set sampling is more efficient than simple random sampling. The KMKu has more flexibility than the Ku model and other well-known models, such as the UW model, Ku model, B model, MOKu model, UG model, MOTL model, and UEHL model. The KMKu model gives the best fit for the three real-world data sets.

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