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Oscillation of Emden–Fowler-Type Differential Equations with Non-Canonical Operators and Mixed Neutral Terms

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Abstract: The study of the symmetric properties of differential equations is essential for identifying effective methods for solving them. In this paper, we examine the oscillatory behavior of solutions of Emden–Fowler-type mixed non-linear neutral differential equations with both canonical and non-canonical operators. By utilizing integral conditions and the integral averaging method, we present new sufficient conditions to ensure that all solutions are oscillatory. Our results enhance and extend previous findings in the literature and are illustrated with suitable examples to demonstrate their effectiveness.

Keywords: Emden–Fowler equation; oscillation; mixed neutral; third order; Riccati technique

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1. Introduction

In this paper, we are concerned with Emden–Fowler-type differential equations with non-canonical operators and mixed neutral terms

$$(q_1(\iota)(q_2(\iota)|\mathcal{N}'(\iota)|^{\gamma-1}\mathcal{N}'(\iota))')' + p_1(\iota)|y(\varrho_1(\iota))|^{\alpha-1}y(\varrho_1(\iota)) + p_2(\iota)|y(\varrho_2(\iota))|^{\beta-1}y(\varrho_2(\iota)) = 0, \quad (1)$$

for $\iota \geq \iota_0$ where $\mathcal{N}(\iota) = y(\iota) + r(\iota)y(\tau(\iota))$. Throughout this paper, we will assume that the following conditions hold:

(A_1) $q_1(\iota), q_2(\iota), r(\iota) \in C([\iota_0, \infty), \mathbb{R}_+)$ and $0 < r(\iota) \leq r_1 < 1$, where $\mathbb{R}_+ = (0, \infty)$;
(A_2) $p_1(\iota), p_2(\iota) \in C([\iota_0, \infty), \mathbb{R}_+)$, α, β, γ are positive constants with $0 < \alpha < \gamma < \beta$;
(A_3) $\tau(\iota), \varrho_i(\iota) \in C([\iota_0, \infty), \mathbb{R}_+)$, $\tau(\iota) \leq \iota, \varrho_i(\iota) \leq \iota$, $\lim_{\iota\to\infty} \tau(\iota) = \lim_{\iota\to\infty} \varrho_i(\iota) = \infty$, where $\mathrm{i} = 1, 2$, and $\widetilde{\alpha} = \min\{\gamma, \alpha\}, \widetilde{\beta} = \min\{\gamma, \beta\}$.

By a solution of (1), we mean a function $y(\iota) : [T_y, \infty) \to \mathbb{R}$ such that $\mathcal{N}(\iota) \in C^3[T_y, \infty)$, $q_2(\iota)|\mathcal{N}'(\iota)|^{\gamma-1}\mathcal{N}'(\iota) \in C^2[T_y, \infty)$, $q_1(\iota)(q_2(\iota)|\mathcal{N}'(\iota)|^{\gamma-1}\mathcal{N}'(\iota)) \in C^1[T_y, \infty)$ and satisfies (1) on $[T_y, \infty)$. We will assume that every non-trivial solution $y(\iota)$ of (1) under consideration here is continuable to the right and satisfies $\sup\{|y(\iota)| : \iota \geq T\} > 0$ for all $T \geq T_y$. We suppose that (1) possesses such a solution. A non-trivial solution of (1) is called oscillatory if it has arbitrary large zeros on $[T_y, \infty)$, otherwise it is called non-oscillatory. Equation (1) is called oscillatory if all of its solutions are oscillatory.

In the present paper, we shall discuss the following three cases:

$$\int_{\iota_0}^{\infty} \frac{1}{q_1(\iota)} d\iota = \infty, \int_{\iota_0}^{\infty} \frac{1}{q_2(\iota)} d\iota = \infty, \tag{2}$$

$$\int_{\iota_0}^{\infty} \frac{1}{q_1(\iota)} d\iota < \infty, \int_{\iota_0}^{\infty} \frac{1}{q_2(\iota)} d\iota = \infty, \tag{3}$$

$$\int_{\iota_0}^{\infty} \frac{1}{q_1(\iota)} d\iota < \infty, \int_{\iota_0}^{\infty} \frac{1}{q_2(\iota)} d\iota < \infty. \tag{4}$$

The Emden–Fowler equations were developed in the middle of the 9th century from astrophysical ideas addressing gaseous dynamics. Emden–Fowler equations are one of the most significant classical topics in differential equation theory. Fowler investigated an equation to model multiple processes in fluid mechanics [1]. Since then, there has been an increase in interest in summarizing this equation and using it to explain different physical phenomena [2,3]. This interest extends to delay differential equations, particularly neutral-type equations. In this type of equation, the second- and highest-order derivative of the unknown function occurs both with and without delayed arguments. This type of equation has practical significance, because it simulates a variety of situations including electric networks, vibrating mass coupled to an elastic bar, etc. [4].

In this context, the study of the oscillations of the solutions to these problems is of specific importance; in particular, there has been a great deal of research on the oscillation of second- and higher-order Emden–Fowler neutral delay differential equations over the past several decades. To the best of our knowledge, the number of works devoted to the study of second- and higher-order neutral differential equations in non-canonical conditions is significantly less than the number of works addressing equations in the canonical case (see [5–25]).

Oscillation theory is an important area of research in mathematics and has numerous applications in various fields. In particular, the study of oscillations in neutral differential equations has received significant attention in recent years. The study of advanced differential equations, which contain both advanced and delayed arguments, is also an active area of research. In this context, the diffusive convection model has been widely used to study the oscillation behavior of solutions. Many studies have been conducted to investigate the oscillation of solutions to diffusive convection models and to derive sufficient conditions for oscillation, including [26–29].

As a result, several studies on the oscillation of various orders of certain differential equations in canonical and non-canonical form have been studied. As we have established, nearly all oscillation criteria described in the literature, such as [30,31] are specified for Emden–Fowler-type equations with mixed nonlinearities of second order. In 2007, Xu et al. [32] studied the oscillatory behavior of the second-order Emden–Fowler neutral delay differential equation in the form

$$(|\mathcal{N}'(\iota)|^{\gamma-1}\mathcal{N}'(\iota))' + q_1(\iota)|y(\iota-\varrho)|^{\alpha-1}y(\iota-\varrho) + q_2(\iota)|y(\iota-\varrho)|^{\beta-1}y(\iota-\varrho) = 0,$$

for $\iota \geq 0$, where $\mathcal{N}(\iota) = y(\iota) + p(\iota)y(\iota-\tau)$.

This motivated our current study, the principal goal of which is not just to investigate oscillations of (1) in both canonical and non-canonical operator cases mentioned above, but to derive new oscillation criteria for (1), also including the case where condition $0 < r(\iota) \leq r_1 < 1$ holds. This rest of the current paper has the following structure: In Section 2, we present some new results of oscillation of solutions of (1) under both canonical and non-canonical operators (2), (3) and (4). In Section 3, three examples are provided to illustrate the main results.

2. Main Results

In this section, we will present some new oscillation results for (1).

Theorem 1. *Suppose that conditions (A_1)–(A_3) and (2) hold. If there exists a $\psi \in C^1([\iota_0,\infty),\mathbb{R}_+)$, for some $\iota_1 \geq \iota_0$ and for $\iota_3 > \iota_2 > \iota_1$, one has*

$$\limsup_{\iota\to\infty}\int_{\iota_3}^{\iota}\psi(s)\left(p_1(s)(1-r(\varrho_1(s)))^{\tilde{\alpha}}q_2^{\tilde{\alpha}-1}(s)\left(\frac{\int_{\iota_2}^{\varrho_1(s)}\frac{\int_{\iota_1}^{v}\frac{1}{q_1(u)}du}{q_2(v)}dv}{\int_{\iota_1}^{s}\frac{1}{q_1(u)}du}\right)^{\tilde{\alpha}}\right.$$
$$\left.+p_2(s)(1-r(\varrho_2(s)))^{\tilde{\beta}}q_2^{\tilde{\beta}-1}(s)\left(\frac{\int_{\iota_2}^{\varrho_2(s)}\frac{\int_{1}^{v}\frac{1}{q_1(u)}du}{q_2(v)}dv}{\int_{\iota_2}^{s}\frac{1}{q_1(u)}du}\right)^{\widetilde{\beta}}-\frac{q_1(s)(\psi(s))^2}{4\psi(s)}\right)ds=\infty \tag{5}$$

and

$$\int_{\iota_0}^{\infty}\left(\frac{1}{q_2(\mathrm{v})}\int_{v}^{\infty}\frac{1}{q_1(\mathrm{u})}\int_{\mathrm{u}}^{\infty}(\mathrm{p}_1(\,\mathrm{s})+\mathrm{p}_2(\,\mathrm{s}))\mathrm{dsdu}\right)^{\frac{1}{\gamma}}\mathrm{dv}=\infty. \tag{6}$$

Then, every solution of Equation (1) is oscillatory or $\lim_{\iota\to\infty}y(\iota)=0$.

Proof. Assume that the solution $y(\iota)$ is an eventually positive solution of Equation (1). We examine the following two cases based on condition (2):

$(\mathcal{C}_{\mathcal{I}})$ $\mathcal{N}(\iota)>0$, $\mathcal{N}'(\iota)>0$, $(q_2(\mathcal{N}')^{\gamma})'(\iota)>0$, $(q_1(q_2(\mathcal{N}')^{\gamma})')'(\iota)<0$

$(\mathcal{C}_{\mathcal{II}})$ $\mathcal{N}(\iota)>0$, $\mathcal{N}'(\iota)<0$, $(q_2|\mathcal{N}'|^{\gamma-1}\mathcal{N}')'(\iota)>0$, $(q_1(q_2|\mathcal{N}'|^{\gamma-1}\mathcal{N}')')'(\iota)<0$, for $\iota\geq\iota_1$ is large enough.

First, assume $(\mathcal{C}_{\mathcal{I}})$ holds. Define the generalized Ricatti function $w(\iota)$ by

$$w(\iota)=\psi(\iota)\frac{q_1(\iota)(q_2(\iota)(\mathcal{N}'(\iota))^{\gamma})'}{q_2(\iota)(\mathcal{N}'(\iota))^{\gamma}}, \tag{7}$$

and $w(\iota)>0$ on $\iota\geq\iota_1$. Using $\mathcal{N}'(\iota)>0$, we have

$$y(\iota)\geq(1-r(\iota))\mathcal{N}(\iota). \tag{8}$$

Thus, for all $\iota\geq\iota_1$,

$$\begin{cases} y(\varrho_1(\iota))\geq\mathcal{N}(\varrho_1(\iota))(1-r(\varrho_1(\iota))) \\ y(\varrho_2(\iota))\geq\mathcal{N}(\varrho_2(\iota))(1-r(\varrho_2(\iota))). \end{cases} \tag{9}$$

Because

$$q_2(\iota)(\mathcal{N}'(\iota))^{\gamma}\geq\int_{\iota_1}^{\iota}\frac{q_1(s)(q_2(s)(\mathcal{N}'(s))^{\gamma})'}{q_1(s)}ds\geq q_1(\iota)(q_2(\iota)(\mathcal{N}'(\iota))^{\gamma})'\int_{\iota_1}^{\iota}\frac{1}{q_1(s)}ds, \tag{10}$$

we have

$$\left(\frac{q_2(\iota)\,(\mathcal{N}'(\iota))^{\gamma}}{\int_{\iota_1}^{\iota}\frac{1}{q_1(s)}ds}\right)'\leq 0. \tag{11}$$

Therefore, we obtain

$$\begin{aligned} \mathcal{N}(\iota)-\mathcal{N}(\iota_2) &= \int_{\iota_2}^{\iota}\frac{q_2(s)\mathcal{N}'(s)}{\int_{\iota_1}^{s}\frac{1}{q_1(u)}}\frac{\int_{\iota_1}^{s}\frac{1}{q_1(u)}du}{q_2(s)}ds \\ \mathcal{N}(\iota) &\geq \frac{q_2(\iota)\mathcal{N}(\iota)}{\int_{\iota_1}^{\iota}\frac{1}{q_1(u)}du}\int_{\iota_2}^{\iota}\frac{\int_{\iota_1}^{s}\frac{1}{q_1(u)}du}{q_2(s)}ds, \end{aligned} \tag{12}$$

for $\iota\geq\iota_2\geq\iota_1$. Differentiating (7), we have that

$$\begin{aligned} w'(\iota)=\psi'(\iota)\frac{q_1(\iota)(q_2(\iota)(\mathcal{N}'(\iota))^{\gamma})'}{q_2(\iota)(\mathcal{N}'(\iota))^{\gamma}}&+\psi(\iota)\frac{(q_1(\iota)(q_2(\iota)(\mathcal{N}'(\iota))^{\gamma}))'}{q_2(\iota)(\mathcal{N})'(\iota))^{\gamma}} \\ &-\psi(\iota)\frac{q_1(\iota)((q_2(\iota)(\mathcal{N})(\iota)\gamma^{\gamma}))^2}{(q_2(\iota)(\mathcal{N})'(\iota))^{\gamma})^2}. \end{aligned} \tag{13}$$

Using (1), (7), (9) and (A_3),

$$w'(\iota) \leq \frac{\psi'(\iota)}{\psi(\iota)}w(\iota) - \frac{w^2(\iota)}{\psi(\iota)q_1(\iota)} - \frac{\psi(\iota)}{q_2(\iota)(\mathcal{N}'(\iota))^{\tilde{\alpha}}}p_1(\iota)(1-r(\varrho_1(\iota)))^{\tilde{\alpha}}\mathcal{N}^{\tilde{\alpha}}(\varrho_1(\iota)) - \frac{\psi(\iota)}{q_2(\iota)(\mathcal{N}'(\iota))^{\tilde{\beta}}}p_2(\iota)(1-r(\varrho_2(\iota)))^{\tilde{\beta}}\mathcal{N}^{\tilde{\beta}}(\varrho_2(\iota)). \tag{14}$$

From (11) and (12), we obtain

$$\begin{aligned} w'(\iota) \quad \leq & \frac{\psi'(\iota)}{\psi(\iota)}w(\iota) - \psi(\iota)p_1(\iota)(1-r(\varrho_1(\iota)))^{\tilde{\alpha}}q_2^{\tilde{\alpha}-1}(\iota)\left(\frac{\int_{\iota_2}^{\varrho_1(\iota)}\frac{\int_{\iota_1}^{s}\frac{1}{q_1(u)}du}{q_2(s)}ds}{\int_{\iota_1}^{\iota}\frac{1}{q_1(u)}du}\right)^{\tilde{\alpha}} \\ & -\psi(\iota)p_2(\iota)(1-r(\varrho_2(\iota)))^{\tilde{\beta}}q_2^{\tilde{\beta}-1}(\iota)\left(\frac{\int_{\iota_2}^{\varrho_2(\iota)}\frac{\int_{\iota_1}^{s}\frac{1}{q_1(u)}du}{q_2(s)}ds}{\int_{\iota_1}^{\iota}\frac{1}{q_1(u)}du}\right)^{\tilde{\beta}} \\ & -\frac{w^2(\iota)}{\psi(\iota)q_1(\iota)}. \end{aligned} \tag{15}$$

Hence, we have

$$\begin{aligned} w'(\iota) \quad \leq & \frac{q_1(\iota)(\psi'(\iota))^2}{4\psi(\iota)} - \psi(\iota)p_1(\iota)(1-r(\varrho_1(\iota)))^{\tilde{\alpha}}q_2^{\tilde{\alpha}-1}(\iota)\left(\frac{\int_{\iota_2}^{\varrho_1(\iota)}\frac{\int_{\iota_1}^{s}\frac{1}{q_1(u)}du}{q_2(s)}ds}{\int_{\iota_1}^{\iota}\frac{1}{q_1(u)}du}\right)^{\tilde{\alpha}} \\ & -\psi(\iota)p_2(\iota)(1-r(\varrho_2(\iota)))^{\tilde{\beta}}q_2^{\tilde{\beta}-1}(\iota)(\frac{\int_{\iota_2}^{\varrho_2(\iota)}\frac{\int_{\iota_1}^{s}\frac{1}{q_1(u)}du}{q_2(s)}ds}{\int_{\iota_1}^{\iota}\frac{1}{q_1(u)}du})^{\tilde{\beta}}. \end{aligned} \tag{16}$$

Integrating (16) from ι_3 to ι, we obtain

$$\begin{aligned} & \int_{\iota_3}^{\iota}(\psi(s)(p_1(s)(1-r(\varrho_1(s)))^{\tilde{\alpha}}q_2^{\tilde{\alpha}-1}(s)\left(\frac{\int_{\iota_2}^{\varrho_1(s)}\frac{\int_{\iota_1}^{v}\frac{1}{q_1(u)}du}{q_2(v)}dv}{\int_{\iota_1}^{s}\frac{1}{q_1(u)}du}\right)^{\tilde{\alpha}} \\ & +p_2(s)(1-r(\varrho_2(s)))^{\tilde{\beta}}q_2^{\tilde{\beta}-1}(s)\left(\frac{\int_{\iota_2}^{\varrho_2(s)}\frac{\int_{1}^{v}\frac{1}{q_1(u)}du}{q_2(v)}dv}{\int_{\iota_2}^{s}\frac{1}{q_1(u)}du}\right)^{\tilde{\beta}} - \frac{q_1(s)(\psi(s))^2}{4\psi(s)}ds) < w(\iota_3), \end{aligned} \tag{17}$$

which contradicts the condition (5).

Next, assume $(\mathcal{C}_{\mathcal{II}})$ holds. Because $\mathcal{N}(\iota) > 0$ and $\mathcal{N}'(\iota) < 0$, then

$$\lim_{\iota\to\infty}\mathcal{N}(\iota) = l.$$

Claim $l = 0$. Suppose that $l > 0$. We have $l+\varepsilon > \mathcal{N}(\iota) > l$, for any $\varepsilon > 0$. Set $0 < \varepsilon < \frac{l(1-r)}{r}$. Then, we have

$$y(\iota) = \mathcal{N}(\iota) - r(\iota)y(\tau(\iota)) > l - r\mathcal{N}(\tau(\iota)) > l - r(l+\varepsilon) = \mu(l+\varepsilon) > \mu\mathcal{N}(\iota),$$

where $\mu = \frac{l-r(l+\varepsilon)}{l+\varepsilon} > 0$. Integrating (1) from ι to ∞, we have

$$(q_2(\iota)|\mathcal{N}'(\iota)|^{\gamma-1}\mathcal{N}'(\iota))' \geq \frac{1}{q_1(\iota)}\int_{\iota}^{\infty}(p_1(s)y^{\alpha}\varrho_1(s) + p_2(s)y^{\beta}\varrho_2(s))ds.$$

Integrating again from ι to ∞, we obtain

$$-\mathcal{N}'(\iota) \geq (\frac{1}{q_2(\iota)}\int_{\iota}^{\infty}\frac{1}{q_1(u)}\int_{u}^{\infty}(p_1(s)y^{\alpha}\varrho_1(s) + p_2(s)y^{\beta}\varrho_2(s))dsdu)^{\frac{1}{\gamma}}.$$

Using the inequality, integrating ι_1 to ∞, we obtain

$$\mathcal{N}(\iota_1) \geq \mu^{\frac{\alpha}{\gamma}}l^{\frac{\alpha}{\gamma}}\int_{\iota_1}^{\infty}(\frac{1}{q_2(v)}\int_{v}^{\infty}\frac{1}{q_1(u)}\int_{u}^{\infty}(p_1(s)+p_2(s))dsdu)^{\frac{1}{\gamma}}dvs..$$

This contradicts (6). Because $l = 0$ and $0 \leq y(\iota) \leq \mathcal{N}(\iota)$ implies $\lim_{\iota\to\infty} y(\iota) = 0$. □

Theorem 2. *Suppose that conditions (A_1)–(A_3) and (3) hold. If there exists a $\psi \in C^1([\iota_0,\infty),\mathbb{R}_+)$, for some $\iota_1 \geq \iota_0$ and for $\iota_3 > \iota_2 > \iota_1$, one has (5) and (6). If*

$$\begin{aligned}&\limsup_{\iota\to\infty} \int_{\iota_2}^{\iota} \Bigg\{ \delta(s)\Big(p_1(s)(1-r(\varrho_1(s)))^{\tilde{\alpha}}(q_2(s))^{\tilde{\alpha}-1}\Big(\int_{\iota_1}^{\varrho_1(s)} \tfrac{dv}{q_2(v)}\Big)^{\tilde{\alpha}}\Big) \\ &+p_2(s)(1-r(\varrho_2(s)))^{\tilde{\beta}}(q_2(s))^{\tilde{\beta}-1}(\int_{\iota_1}^{\varrho_2(s)} \tfrac{dv}{q_2(v)})^{\tilde{\beta}}) - \tfrac{1}{4q_1(s)\delta(s)} \Bigg\} ds = \infty,\end{aligned} \tag{18}$$

where

$$\delta(\iota) := \int_{\iota}^{\infty} \frac{1}{q_1(\mathrm{s})} \mathrm{ds}. \tag{19}$$

Then, every solution of Equation (1) is oscillatory or $\lim_{\iota\to\infty} y(\iota) = 0$.

Proof. Assume that the solution $y(\iota)$ is an eventually positive solution of Equation (1). Based on condition (3), there exist three possible cases $(\mathcal{C}_{\mathcal{I}})$, $(\mathcal{C}_{\mathcal{II}})$ (as in Theorem 1) and

$(\mathcal{C}_{\mathcal{III}})$ $\mathcal{N}(\iota) > 0$, $\mathcal{N}'(\iota) > 0$, $(q_2(\mathcal{N}')^{\gamma})'(\iota) < 0$, $(q_1(q_2(\mathcal{N}')^{\gamma})')'(\iota) < 0$, for $\iota \geq \iota_1$ is large enough.

Let us assume that case $(\mathcal{C}_{\mathcal{I}})$ and case $(\mathcal{C}_{\mathcal{II}})$ hold. Using the proof of Theorem 2, we may arrive at the conclusion of Theorem 1. Assume that case $(\mathcal{C}_{\mathcal{III}})$ holds. From $(q_1(\iota)(q_2(\iota)(\mathcal{N}'(\iota))^{\gamma})')' < 0$, thus, we obtain

$$q_1(s)(q_2(s)(\mathcal{N}'(s))^{\gamma})' \leq q_1(\iota)(q_2(\iota)(\mathcal{N}'(\iota))^{\gamma})', \quad \text{for } s \geq \iota \geq \iota_1. \tag{20}$$

Dividing (20) by $q_1(s)$ and integrating it from ι to l, we obtain

$$q_2(l)(\mathcal{N}'(l))^{\gamma} \leq q_2(\iota)(\mathcal{N}'(\iota))^{\gamma} + q_1(\iota)(q_2(\iota)(\mathcal{N}'(\iota))^{\gamma})' \int_{\iota}^{l} \frac{ds}{q_1(s)}. \tag{21}$$

Letting $l \to \infty$, we obtain

$$0 \leq q_2(\iota)(\mathcal{N}'(\iota))^{\gamma} + q_1(\iota)(q_2(\iota)(\mathcal{N}'(\iota))^{\gamma})' \int_{\iota}^{\infty} \frac{1}{q_1(s)} ds, \tag{22}$$

then

$$-\frac{q_1(\iota)(q_2(\iota)(\mathcal{N}'(\iota))^{\gamma})'}{q_2(\iota)(\mathcal{N}'(\iota))^{\gamma}} \int_{\iota}^{\infty} \frac{ds}{q_1(s)} \leq 1. \tag{23}$$

The Riccati function $\phi(\iota)$ is defined by

$$\phi(\iota) := \frac{q_1(\iota)(q_2(\iota)(\mathcal{N}'(\iota))^{\gamma})'}{q_2(\iota)(\mathcal{N}'(\iota))^{\gamma}}, \iota \geq \iota_1. \tag{24}$$

Then, $\phi(\iota) < 0$ for $\iota \geq \iota_1$. Hence, by (23) and (24), we have

$$-\delta(\iota)\phi(\iota) \leq 1. \tag{25}$$

Now, differentiating (24), we have

$$\begin{aligned}\phi'(\iota) &= \tfrac{(q_1(\iota)(q_2(\iota)(\mathcal{N}'(\iota))^{\gamma})')}{q_2(\iota)(\mathcal{N}'(\iota))^{\gamma}} - \tfrac{q_1(\iota)(q_2(\iota)(\mathcal{N}'(\iota))^{\gamma})'(q_2(\iota)(\mathcal{N}'(\iota))^{\gamma}))'}{(q_2(\iota)(\mathcal{N}'(\iota))^{\gamma})^2} \\ &\leq \tfrac{-p_1(\iota)(1-r(\varrho_1(\iota)))^{\alpha}\mathcal{N}^{\alpha}(\varrho_1(\iota)) - p_2(\iota)(1-r(\varrho_2(\iota)))^{\beta}\mathcal{N}^{\beta}(\varrho_2(\iota))}{q_2(\iota)(\mathcal{N}'(\iota))^{\gamma}} - \tfrac{\phi^2(\iota)}{q_1(\iota)}.\end{aligned} \tag{26}$$

Using (A_3), we obtain

$$\begin{aligned} \phi'(\iota) \quad &\leq -\frac{p_1(\iota)(1-r(\varrho_1(\iota)))^{\widetilde{\alpha}}\widetilde{\mathcal{N}^{\widetilde{\alpha}}}(\varrho_1(\iota))}{q_2(\iota)(\mathcal{N}'(\iota))^{\tilde{\alpha}}}\frac{q_2^{\widetilde{\alpha}}(\iota)}{q_2^{\tilde{\alpha}}(\iota)} - \frac{p_2(\iota)(1-r(\varrho_2(\iota)))^{\widetilde{\beta}}\widetilde{\beta}^{\widetilde{\beta}}(\varrho_2(\iota))}{q_2(\iota)(\mathcal{N}'(\iota))^{\tilde{\beta}}}\frac{q_2^{\widetilde{\beta}}(\iota)}{q_2^{\tilde{\beta}}(\iota)} \\ &\quad -\frac{\phi^2(\iota)}{q_1(\iota)}. \end{aligned} \tag{27}$$

In view of case $(\mathcal{C}_{\mathcal{III}})$, we see that

$$\mathcal{N}(\iota) \geq q_2(\iota)\mathcal{N}'(\iota)\int_{\iota_1}^{\iota}\frac{ds}{q_2(s)}. \tag{28}$$

Hence,

$$\left(\frac{\mathcal{N}(\iota)}{\int_{\iota_1}^{\iota}\frac{ds}{q_2(s)}}\right) \leq 0, \tag{29}$$

which implies

$$\frac{\mathcal{N}(\varrho_i(\iota))}{\mathcal{N}(\iota)} \geq \frac{\int_{\iota_1}^{\varrho_i(\iota)}\frac{ds}{q_2(s)}}{\int_{\iota_1\iota}^{\iota}\frac{ds}{q_2(s)}}, \tag{30}$$

where $i = 1,2$. By (24) and (27), (28) and (30), we obtain

$$\begin{aligned} \phi'(\iota) \quad &\leq -p_1(\iota)(1-r(\varrho_1(\iota)))^{\widetilde{\alpha}}(q_2(\iota))^{\widetilde{\alpha}-1}\left(\int_{\iota_1}^{\varrho_1(\iota)}\frac{ds}{q_2(s)}\right)^{\widetilde{\alpha}} \\ &\quad -p_2(\iota)(1-r(\varrho_2(\iota)))^{\widetilde{\beta}}(q_2(\iota))^{\widetilde{\beta}-1}\left(\int_{\iota_1}^{\varrho_2(\iota)}\frac{ds}{q_2(s)}\right)^{\widetilde{\beta}} - \frac{\phi^2(\iota)}{q_1(\iota)}. \end{aligned} \tag{31}$$

Multiplying (31) by $\delta(\iota)$ and integrating it from ι_2 to ι, we obtain

$$\begin{aligned} &\delta(\iota)\phi(\iota) - \delta(\iota_2)\phi(\iota_2) + \int_{\iota_2}^{\iota}\delta(s)(-p_1(s)(1-r(\varrho_1(s)))^{\widetilde{\alpha}}(q_2(s))^{\widetilde{\alpha}-1}\left(\int_{\iota_1}^{\varrho_1(s)}\frac{dv}{q_2(v)}\right)^{\widetilde{\alpha}} \\ &\quad -p_2(s)(1-r(\varrho_2(s)))^{\widetilde{\beta}}(q_2(s))^{\widetilde{\beta}-1}\left(\int_{\iota_1}^{\varrho_2(s)}\frac{dv}{q_2(v)}\right)^{\widetilde{\beta}})ds - \int_{\iota_2}^{\iota}\delta(s)\frac{\phi^2(s)}{q_1(s)}ds, \end{aligned} \tag{32}$$

which follows that

$$\begin{aligned} &\int_{\iota_2}^{\iota}\Bigg\{\delta(s)(p_1(s)(1-r(\varrho_1(s)))^{\widetilde{\alpha}}(q_2(s))^{\widetilde{\alpha}-1}(\int_{\iota_1}^{\varrho_1(s)}\frac{dv}{q_2(v)})^{\widetilde{\alpha}} \\ &+p_2(s)(1-r(\varrho_2(s)))^{\widetilde{\beta}}(q_2(s))^{\widetilde{\beta}-1}(\int_{\iota_1}^{\varrho_2(s)}\frac{dv}{q_2(v)})^{\widetilde{\beta}}) - \frac{1}{4q_1(s)\delta(s)}\Bigg\}ds \leq \delta(\iota_2)\phi(\iota_2) + 1, \end{aligned} \tag{33}$$

due to (25), which contradicts (18). □

Theorem 3. *Suppose that conditions (A_1)–(A_3) and (4) hold. If there exists a $\psi \in C^1([\iota_0,\infty),\mathbb{R}_+)$, for some $\iota_1 \geq \iota_0$ and for $\iota_3 > \iota_2 > \iota_1$, one has (5), (6) and (18). If*

$$\int_{\iota_1}^{\infty}\frac{1}{q_2(v)}\int_{\iota_1}^{v}\frac{1}{q_1(u)}\int_{\iota_1}^{u}p_1(s)\xi^{\alpha}(\varrho_1(s))\eta_1^{\alpha}(s)ds\,du\,dv = \infty \tag{34}$$

and

$$\int_{\iota_1}^{\infty}\frac{1}{q_2(v)}\int_{\iota_1}^{v}\frac{1}{q_1(u)}\int_{\iota_1}^{u}p_2(s)\xi^{\beta}(\varrho_2(s))\eta_2^{\beta}(s)ds\,du\,dv = \infty, \tag{35}$$

where

$$\eta_i(\iota) = 1 - p(\varrho_i(\iota))\frac{\xi(\tau(\varrho_i(\iota)))}{\xi(\varrho_i(\iota))} > 0, \quad \xi(\iota) = \int_\iota^\infty \frac{1}{q_2(s)|\mathcal{N}'(s)|^{\gamma-1}}ds. \tag{36}$$

Then, every solution of Equation (1) is oscillatory or $\lim_{\iota\to\infty} y(\iota) = 0$.

Proof. Assume that the solution $y(\iota)$ is an eventually positive solution of the Equation (1). By condition (4), there exist four possible cases $(\mathcal{C}_{\mathcal{I}})$, $(\mathcal{C}_{\mathcal{II}})$ and $(\mathcal{C}_{\mathcal{III}})$, (as those of Theorem 2) and

$(\mathcal{C}_{\mathcal{IV}})$ $\mathcal{N}(\iota) > 0$, $\mathcal{N}'(\iota) < 0$, $(q_2|\mathcal{N}'|^{\gamma-1}\mathcal{N}')'(\iota) < 0$, $(q_1(q_2|\mathcal{N}'|^{\gamma-1}\mathcal{N}')')'(\iota) < 0$, for $\iota \geq \iota_1, \iota_1$ is large enough.

We suppose that case $(\mathcal{C}_{\mathcal{I}})$, case $(\mathcal{C}_{\mathcal{II}})$, and case $(\mathcal{C}_{\mathcal{III}})$ hold. Using the proof of Theorem 2, we may arrive at the conclusion of Theorem 3.

Assume that case $(\mathcal{C}_{\mathcal{IV}})$ holds. Because $(q_2(\iota)|\mathcal{N}'(\iota)|^{\gamma-1}\mathcal{N}'(\iota))' < 0$, we obtain that

$$\mathcal{N}'(s) \leq \frac{q_2(\iota)|\mathcal{N}'(\iota)|^{\gamma-1}}{q_2(s)|\mathcal{N}'(s)|^{\gamma-1}}\mathcal{N}'(\iota), \quad s \geq \iota, \tag{37}$$

which implies that

$$\mathcal{N}(\iota) \geq -q_2(\iota)|\mathcal{N}'(\iota)|^{\gamma-1}\mathcal{N}'(\iota)\xi(\iota) \geq L\xi(\iota) \tag{38}$$

for some $L > 0$. By (38), we have

$$\left(\frac{\mathcal{N}(\iota)}{\xi(\iota)}\right)' \geq 0. \tag{39}$$

Using (8) and (39), we have

$$y(\iota) = \mathcal{N}(\iota) - r(\iota)y(\varrho_1(\iota)) \geq \mathcal{N}(\iota) - r(\iota)\mathcal{N}(\varrho_1(\iota)) \geq (1 - r(\varrho_1(\iota))\frac{\xi(\tau(\varrho_1(\iota)))}{\xi(\varrho_1(\iota))}), \tag{40}$$

$$y(\iota) = \mathcal{N}(\iota) - r(\iota)y(\varrho_2(\iota)) \geq \mathcal{N}(\iota) - r(\iota)\mathcal{N}(\varrho_2(\iota)) \geq (1 - r(\varrho_2(\iota))\frac{\xi(\tau(\varrho_2(\iota)))}{\xi(\varrho_2(\iota))}). \tag{41}$$

From (1), (36), (38), (40) and (41) we obtain

$$\begin{aligned}(q_1(\iota)(q_2(\iota)|\mathcal{N}'(\iota)|^{\gamma-1}\mathcal{N}'(\iota))')' + p_1(\iota)L^\alpha\xi^\alpha(\varrho_1(\iota))[\eta_1(\iota)]^\alpha \\ +p_2(\iota)L^\beta\xi^\beta(\varrho_2(\iota))[\eta_2(\iota)]^\beta \leq 0.\end{aligned} \tag{42}$$

Integrating (42) from ι_1 to ι, we have

$$\begin{aligned}q_1(\iota)(q_2(\iota)|\mathcal{N}'(\iota)|^{\gamma-1}\mathcal{N}'(\iota))' + L^\alpha \textstyle\int_{\iota_1}^\iota p_1(s)\xi^\alpha(\varrho_1(s))[\eta_1(\iota)]^\alpha ds \\ +L^\beta \textstyle\int_{\iota_1}^\iota p_2(s)\xi^\beta(\varrho_2(s))[\eta_2(\iota)]^\beta ds \leq 0.\end{aligned} \tag{43}$$

Integrating again, we obtain

$$\begin{aligned}q_2(\iota)|\mathcal{N}'(\iota)|^{\gamma-1}\mathcal{N}'(\iota) + L^\alpha \textstyle\int_{\iota_1}^\iota \frac{1}{q_1(u)}\int_{\iota_1}^u p_1(s)\xi^\alpha(\varrho_1(s))[\eta_1(s)]^\alpha dsdu \\ +L^\beta \textstyle\int_{\iota_1}^\iota \frac{1}{q_1(u)}\int_{\iota_1}^u p_2(s)\xi^\beta(\varrho_2(s))[\eta_2(s)]^\beta dsdu \leq 0.\end{aligned} \tag{44}$$

Integrating once again, we obtain

$$\mathcal{N}(\iota_1) \geq L^\alpha \int_{\iota_1}^\iota \frac{1}{q_2(v)}\int_{\iota_1}^v \frac{1}{q_1(u)}\int_{\iota_1}^u p_1(s)\xi^\alpha(\varrho_1(s))[\eta_1(s)]^\alpha ds\,du\,dvs. \tag{45}$$

$$+ L^\beta \int_{\iota_1}^\iota \frac{1}{q_2(v)}\int_{\iota_1}^v \frac{1}{q_1(u)}\int_{\iota_1}^u p_2(s)\xi^\beta(\varrho_2(s))[\eta_2(s)]^\beta ds\,du\,dvs. + \mathcal{N}(\iota), \tag{46}$$

which contradicts (34) and (42). □

Theorem 4. *Suppose that conditions* (A_1)*–*(A_3) *and* (4) *hold. If there exists a* $\psi \in C^1([\iota_0, \infty), \mathbb{R}_+)$*, for some* $\iota_1 \geq \iota_0$ *and for* $\iota_3 > \iota_2 > \iota_1$*, one has* (5)*,* (6) *and* (18)*. If*

$$\int_{\iota_1}^{\infty} \left(\frac{1}{q_2(v)} \int_{\iota_1}^{v} \frac{1}{q_1(u)} \int_{\iota_1}^{u} (p_1(s) + p_2(s)) ds du \right)^{\frac{1}{\gamma}} dv = \infty. \tag{47}$$

Then every solution of Equation (1) *is oscillatory or* $\lim_{\iota \to \infty} y(\iota) = 0$.

Proof. Assume that the solution $y(\iota)$ is an eventually positive solution of the Equation (1). By the convergent condition (4), it is natural to consider that four possible cases $(\mathcal{C}_{\mathcal{I}})$, $(\mathcal{C}_{\mathcal{II}})$, $(\mathcal{C}_{\mathcal{III}})$ and $(\mathcal{C}_{\mathcal{IV}})$ hold (as those of Theorem 3. By assuming the cases $(\mathcal{C}_{\mathcal{I}})$, $(\mathcal{C}_{\mathcal{II}})$ and $(\mathcal{C}_{\mathcal{III}})$ hold, the conclusion of Theorem 2 is derived. For the case $(\mathcal{C}_{\mathcal{IV}})$ when $\lim_{\iota \to \infty} Z(\iota) = l \geq 0$ (l is finite). Assume that $l > 0$. From (6) in Theorem 1, there exists a positive constant $\mu > 0$ such that $y(\iota) > \mu l$. Because the rest of the proof is similar to that of Theorem 3, we omit the details. □

3. Examples

The examples below demonstrate applications of some of the theoretical concepts discussed in the earlier sections.

Example 1. *Consider the Emden–Fowler-type neutral delay differential equation*

$$(\frac{1}{\iota}(\iota^{\frac{1}{2}}(y(\iota) + \frac{1}{2}y(\iota - \pi))')'))' + \frac{3}{8}\iota^{-\frac{5}{2}} y^{\frac{2}{3}}(\iota - \frac{3\pi}{2}) + \frac{1}{2}\iota^{-\frac{1}{2}} y^{\frac{5}{3}}(\iota - \frac{7\pi}{2}) = 0, \tag{48}$$

for $\iota \geq 1$*, where* $q_1(\iota) = \frac{1}{\iota}$, $q_2(\iota) = \iota^{\frac{1}{2}}$, $\tau(\iota) = \iota - \pi$, $\alpha = \frac{2}{3}$, $\beta = \frac{5}{3}$, $\gamma = 1$, $p_1(\iota) = \frac{3}{8}\iota^{-\frac{5}{2}}$, $p_2(\iota) = \frac{1}{2}\iota^{-\frac{1}{2}}$, $\varrho_1(\iota) = \iota - \frac{3\pi}{2}$, $\varrho_2(\iota) = \iota - \frac{7\pi}{2}$, $r(\iota) = \frac{1}{2}$, $\psi(s) = 1$*. Hence, all the conditions of Theorem* 1 *are satisfied. Therefore, every solution of* (48) *is oscillatory or tends to zero as* $\iota \to \infty$*.*

Example 2. *Consider the third-order Emden–Fowler-type differential equation*

$$(\frac{1}{\iota}(\iota^2(y(\iota) + r_1 y(\iota - 3))')')' + \iota^m y^{\frac{1}{2k+1}}(\iota - 1) + \iota^n y^{\frac{4k+1}{2k+1}}(\iota - 1) = 0, \tag{49}$$

for $\iota \geq 1$*. Here,* $q_1(\iota) = \frac{1}{\iota}, q_2(\iota) = \iota^2, \tau(\iota) = \iota - 3, \alpha = \frac{1}{2k+1}, \beta = \frac{4k+1}{2k+1}, \mathrm{k} > 0, \gamma = 1, \mathrm{p}_1(\iota) = \iota^{\mathrm{m}}$, $\mathrm{p}_2(\iota) = \iota^{\mathrm{n}}, \mathrm{m}, \mathrm{n} > 1$, $\varrho_1(\iota) = \varrho_2(\iota) = \iota - 1, 0 \leq r(\iota) \leq r_1 < 1, \psi(s) = 1$*. Hence, all the conditions of Theorems* 3 *and* 4 *are satisfied. Therefore, each solution of* (49) *is oscillatory and tends to zero as* $\iota \to \infty$*.*

Example 3. *Consider the Emden–Fowler of the third-order-type differential equation*

$$\begin{aligned} (\iota^3((\iota^{-2}+1)(y(\iota) + \tfrac{1}{3}y(\tfrac{\iota}{2}))')')' + \frac{\mathrm{C}_1\,\mathrm{m}(\,\mathrm{m}+3)(1-\mathrm{m})2^{\mathrm{m}/3}}{\iota^{(6-2m)/3}} y^{\frac{1}{3}}(\tfrac{\iota}{2}) \\ + \mathrm{C}_1 \frac{\mathrm{m}(1-\mathrm{m}^2)}{\iota^{m/2}} y^{\frac{3}{2}}(\iota) = 0, \end{aligned} \tag{50}$$

for $\iota \geq 1$*. Here,* $q_1(\iota) = \iota^3$, $q_2(\iota) = \iota^{-2} + 1$, $\tau(\iota) = \frac{\iota}{2}$, $\alpha = \frac{1}{3}$, $\beta = \frac{3}{2}$, $\gamma = 1$, $p_1(\iota) = \frac{C_1 m(m+3)(1-m)2^{m/3}}{\iota^{(6-2m)/3}}$, $p_2(\iota) = \mathrm{C}_1\mathrm{m}(1-\mathrm{m}^2)\iota^{-m/2}$, $\mathrm{m} > 1$, $\varrho_1(\iota) = \frac{\iota}{2}$, $\varrho_2(\iota) = \iota$, $0 \leq r(\iota) \leq r_1 < 1$, $\psi(s) = 1$*. It follows that condition* (5) *in Theorem* 1 *is not satisfied. It follows that there exists a non-oscillatory solution of* (50) *and in this case* $y(\iota) = \iota^{\mathrm{m}}$ *is such a solution.*

4. Conclusions

In this study, a new criterion was developed to test the oscillatory behavior of yjr solutions of an Emden–Fowler-type mixed non-linear neutral differential equation with both canonical and non-canonical operators (2), (3) and (4). This criterion is simple to apply, takes into consideration all of the variables, and may be used when $0 < r(\iota) \leq r_1 < 1$. Our results improve, unify, and extend some known results for differential equations with neutral terms. Suitable examples are given to illustrate effectiveness of our results. It would be of interest to suggest a different method to further investigate (1) assuming that the unbounded neutral coefficient $r(\iota)$.

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